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AN ANALYSIS OF STATICALLY INDETERMINATE TRUSSES HAVING MEMBERS STRESSED BEYOND THE PROPORTIONAL LIMIT

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SUMMARY

A procedure for analyzing statically indeterminate trusses in the plastic stress range is presented which is applicable to trusses having any number of redundant members. A numerical example is used to illustrate the procedure.

INTRODUCTION

Although the truss has generally been replaced by the stiffened sheet as the preferred structural form for the wing and fuselage, it still constitutes a useful configuration that finds application in aircraft. The elastic stress analysis of the truss with redundant members is a classical problem with which all structural engineers are familiar. Aeronautical engineers, however, are interested in the maximum strength of trusses and therefore a plastic analysis of the structure is necessary. Two somewhat different procedures for analyzing statically indeterminate trusses with members stressed beyond the proportional limit have recently appeared in the aeronautical literature (refs. 1 and 2). Reference 1 describes a procedure which in essence applies the principle of minimum complementary energy to plastic truss analysis but the method is limited to a single redundancy. The method of reference 2 is an extension of the relaxation technique of R. V. Southwell and, in general, requires the determination of more unknowns than the static redundancy of the truss.

Presented herein is an application of the principle of minimum complementary energy to the analysis of statically indeterminate trusses having members stressed beyond the proportional limit and having any number of redundant members. The number of unknown quantities to be determined is equal to the number of static redundancies. The use of the Ramberg and Osgood (ref. 3) analytical representation of the stress-strain curve that applies to aluminum alloys, magnesium, and stainless steel permits a concise analytic formulation of the problem for trusses made of these materials.

THEORY

Of the possible combinations of forces, in the members of a truss, which satisfy equilibrium among themselves and are in equilibrium with the external loads, the correct combination is the one that minimizes the complementary energy (see ref. 4 for a discussion of the method of complementary energy):

$$U = \sum_{k=1}^{m} \int_{0}^{P_k} e_k \, dP_k \qquad (1)$$

where

U complementary energy

m total number of members in truss

e change in length of a member

P force acting in the member

In addition to the usual assumptions of truss analysis, such as the assumption that the deformations are not sufficient to disturb the geometric relations between the members, the following assumptions are made:

(1) Any movement of a support is perpendicular to the reaction at the support.

(2) No forces are acting in the members before the application of the external loads.

(3) The final stresses in the members are on the stress-strain curve and therefore have not resulted from plastic stress reversal.

The integration indicated in equation (1) can be performed analytically through use of the Ramberg-Osgood representation of the stress-strain curve (ref. 3). This representation yields the following equation relating e and P:

$$e = \frac{PL}{AE} + \frac{3}{7}\frac{P_R L}{AE}\left(\frac{P}{P_R}\right)^n \qquad (2)$$

where

E elastic modulus

L length of the member

A cross-sectional area of the member

P_R force causing a stress in the member equal to the 0.7E secant yield stress

n shape parameter used by Ramberg and Osgood for the stress-strain curve

Relation (2) expresses the change in length as the sum of two terms: the first is recognized as the change in length that would occur if the member were elastic; the second term is the additional change in length caused by plastic deformation.

Substituting equation (2) into equation (1) and subsequently integrating yields

$$U = \sum_{k=1}^{m}\left[\frac{{P_k}^2 L_k}{2A_k E_k} + \frac{3}{7}\,\frac{{P_{R_k}}^2 L_k}{(n_k + 1)A_k E_k}\left(\frac{P_k}{P_{R_k}}\right)^{n_k+1}\right] \tag{3}$$

Let

$$P_k = P_{0k} + a_1 P_{1k} + \cdot\ \cdot\ \cdot + a_i P_{ik} + \cdot\ \cdot\ \cdot + a_j P_{jk} \tag{4}$$

where

P_{0k} statically determinate force in the kth member caused by the external loading when j redundant members are assumed missing

P_{ik} statically determinate force in the kth member caused by a unit tensile force in the ith redundant member when the applied loads and the forces in the other redundant members are assumed equal to zero

$a_1, \ldots a_j$ coefficients to be determined

When equation (4) is substituted into equation (3) and the resulting expression for the complementary energy is minimized by equating to zero its partial derivatives with respect to each a, there results a set of j nonlinear simultaneous equations which determine $a_1, \ldots a_j$. The general equation obtained by setting

$$\frac{\partial U}{\partial a_i} = 0 \qquad (i = 1, 2, \ldots j) \tag{5}$$

is

$$a_1 \sum_{k=1}^{m} \frac{P_{1k}P_{ik}L_k}{A_kE_k} + \ldots + a_i \sum_{k=1}^{m} \frac{P_{ik}^2L_k}{A_kE_k} + \ldots + a_j \sum_{k=1}^{m} \frac{P_{jk}P_{ik}L_k}{A_kE_k} =$$

$$- \sum_{k=1}^{m} \frac{P_{0k}P_{ik}L_k}{A_kE_k} -$$

$$\frac{3}{7} \sum_{k=1}^{m} \frac{P_{R_k}P_{ik}L_k}{A_kE_k} \left(\frac{P_{0k} + a_1P_{1k} + \ldots + a_iP_{ik} + \ldots + a_jP_{jk}}{P_{R_k}} \right)^{n_k} \tag{6}$$

$$(i = 1, 2, \ldots j)$$

With the exception of the second summation on the right-hand side, equation (6) is the same as would have resulted from an analysis of an elastic truss. The additional summation in equation (6) accounts for the effects due to plasticity.

SOLUTION OF THE NONLINEAR SIMULTANEOUS EQUATIONS

A description of a convenient procedure for solving the nonlinear simultaneous equations follows. (Experienced computers may prefer a method with which they are more familiar.)

The set of equations (6) may be written

$$[A] |a| = |K_E| + |F(a)| \tag{7}$$

or

$$|a| = [A]^{-1}|K_E| + [A]^{-1}|F(a)| \tag{8}$$

where

$[A]$ square matrix having as elements the summations appearing as coefficients of the a's in the left-hand sides of the equations

$[A]^{-1}$ inverse of $[A]$

$|a|$ column matrix of the a's

$|K_E|$ column matrix consisting of the first (elastic) terms in the right-hand sides of the equations

$|F(a)|$ column matrix of the second (plastic) terms in the right-hand sides of the equations

Were stress and strain related by the elastic modulus for all forces in all members, the solution would be

$$\left|a^{(0)}\right| = [A]^{-1}\left|K_E\right| \tag{9}$$

Equation (9) represents the elastic solution and may be used as a first approximation to the values for the a's which are to satisfy the nonlinear equations. The second approximation will then be

$$\left|a^{(1)}\right| = \left|a^{(0)}\right| + [A]^{-1}\left|F\left(a^{(0)}\right)\right| \tag{10}$$

and the rth approximation after the first will be

$$\left|a^{(r)}\right| = \left|a^{(0)}\right| + [A]^{-1}\left|F\left(a^{(r-1)}\right)\right| \tag{11}$$

Each set of values for the a's should be compared with the preceding values to determine whether or not continuance of the iterative procedure indicated by equation (11) is justified. For trusses having several redundant members, repeated application of equation (11) may yield varying rates of convergence or divergence for successive values of an a. If the iteration appears unsatisfactory, the procedure can usually be altered to produce rapid convergence.

NUMERICAL EXAMPLE

For illustration of the procedure, a truss having two redundant members will be considered. If the stress-strain curve for the material of each member is the same, and if the tensile and compressive portions of the curve are symmetrical about the origin, equation (6) yields:

$$\left.\begin{aligned} a_1 \sum_{k=1}^{m} \frac{{P_{1k}}^2 L_k}{A_k} + a_2 \sum_{k=1}^{m} \frac{P_{1k} P_{2k} L_k}{A_k} &= -\sum_{k=1}^{m} \frac{P_{0k} P_{1k} L_k}{A_k} - \\ &\frac{3}{7} \sum_{k=1}^{m} \frac{P_{R_k} P_{1k} L_k}{A_k} \left(\frac{P_{0k} + a_1 P_{1k} + a_2 P_{2k}}{P_{R_k}} \right)^n \\ a_1 \sum_{k=1}^{m} \frac{P_{1k} P_{2k} L_k}{A_k} + a_2 \sum_{k=1}^{m} \frac{{P_{2k}}^2 L_k}{A_k} &= -\sum_{k=1}^{m} \frac{P_{0k} P_{2k} L_k}{A_k} - \\ &\frac{3}{7} \sum_{k=1}^{m} \frac{P_{R_k} P_{2k} L_k}{A_k} \left(\frac{P_{0k} + a_1 P_{1k} + a_2 P_{2k}}{P_{R_k}} \right)^n \end{aligned}\right\} \quad (12)$$

For the statically indeterminate truss given in figure 1(a), the statically determinate load system is shown in figure 1(b) and the two redundant load systems are shown in figures 1(c) and 1(d). Lengths and areas of the truss members are summarized in table I, along with the statically determinate forces.

The stress-strain curve used was arbitrarily chosen and has a 0.7E secant yield stress of 40.5 kips per square inch and a value of 6.56 for n.

The sums obtained from table II are used (as indicated below the table) to determine the elastic solution, which is the first approximation. The sums required for the second approximation are given in table III, and the application of equation (10) to obtain the second approximation is indicated below table III.

Further iterations indicated an oscillatory divergence of the coefficients a_1 and an oscillatory convergence of a_2. Convergence was obtained when averages of successive values of the coefficient were used in the iteration. The final values for a_1 and a_2 are used in table IV; convergence is indicated when successive solutions of equation (11) result in identical values.

CONCLUDING REMARKS

The analytical expression for the stress-strain curve used in the present method for analyzing statically indeterminate trusses having members stressed above the proportional limit has a form which permits the analysis to be divided into two parts. The first part, or elastic solution, is simply an application of the familiar principle of least work to an elastic truss. The second part of the procedure is a routine iterative process which accounts for the plastic deformations of the truss and yields a correction to the elastic solution.

The method is applicable to a truss having any number of redundant members. The number of unknown quantities to be determined is equal to the number of redundant members in the truss, and the evaluation of these unknown quantities determines directly the force acting in each member of the truss. When the forces are known, the deformations can readily be found.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., September 16, 1952.

REFERENCES

1. Brooks, Philip D.: A Method for Solving Statically Indeterminate Trusses and for Finding Deflections When Members of the Truss Are Stressed Above the Proportional Limit. Northrop Aircraft, Inc., Apr. 10, 1947.

2. Steinbacher, F. R., Gaylord, C. N., and Rey, W. K.: Method for Analyzing Indeterminate Structures Stressed Above Proportional Limit. NACA TN 2376, 1951.

3. Ramberg, Walter, and Osgood, William R.: Description of Stress-Strain Curves by Three Parameters. NACA TN 902, 1943.

4. Westergaard, H. M.: On the Method of Complementary Energy and Its Application to Structures Stressed Beyond the Proportional Limit, to Buckling and Vibrations, and to Suspension Bridges. Paper No. 2145, Trans. A.S.C.E., vol. 107, 1942, pp. 765-793; discussion, pp. 794-803.

TABLE I.- DIMENSIONS AND STATICALLY DETERMINATE FORCES USED IN NUMERICAL EXAMPLE

Member	Length	Area	P_0	P_1	P_2
a	30	0.25	7.5	-0.6	0
b	50	.20	12.5	1.0	0
c	50	.20	0	1.0	0
d	30	.25	-15.0	-0.6	0
e	40	.20	0	-0.8	-0.8
f	30	.25	7.5	0	-0.6
g	50	.20	0	0	1.0
h	50	.20	-12.5	0	1.0
i	30	.25	0	0	-0.6
j	40	.20	10.0	0	-0.8

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TABLE II.- COMPUTATIONS FOR ELASTIC SOLUTION

(FIRST APPROXIMATION)

Member	$\frac{P_1^2L}{A}$	$\frac{P_1P_2L}{A}$	$\frac{P_2^2L}{A}$	$\frac{P_0P_1L}{A}$	$\frac{P_0P_2L}{A}$
a	43.2	0	0	-540	0
b	250.0	0	0	3125	0
c	250.0	0	0	0	0
d	43.2	0	0	1080	0
e	128.0	128.0	128.0	0	0
f	0	0	43.2	0	-540
g	0	0	250.0	0	0
h	0	0	250.0	0	-3125
i	0	0	43.2	0	0
j	0	0	128.0	0	-1600
$\sum$	714.4	128.0	842.4	3665	-5265

$$[A]^{-1} = \begin{vmatrix} 714.4 & 128.0 \\ 128.0 & 824.4 \end{vmatrix}^{-1} = \begin{vmatrix} 0.00143895 & -0.00021864 \\ 0.00021864 & 0.00122031 \end{vmatrix}$$

$$\left|a(0)\right| = [A]^{-1} \begin{vmatrix} -3665 \\ 5265 \end{vmatrix}$$

$$\begin{vmatrix} a_1(0) \\ a_2(0) \end{vmatrix} = \begin{vmatrix} -6.4249 \\ 7.2262 \end{vmatrix}$$

TABLE III.- COMPUTATIONS FOR SECOND APPROXIMATION

(1) Member	(2) P_R (a)	(3) $\frac{P_R P_1 L}{A}$	(4) $\frac{P_R P_2 L}{A}$	(5) $a_1 P_1$	(6) $a_2 P_2$	(7) $\frac{P}{P_R}$	(8) $\left(\frac{P}{P_R}\right)^{6.56}$	(9) (3) × (8)	(10) (4) × (8)
a	10.125	-729	0	3.8549	0	1.1215	2.1214	-1546.5	0
b	8.100	2025	0	-6.4249	0	.7500	.1515	306.8	0
c	-8.100	-2025	0	-6.4249	0	.7932	.2188	-443.0	0
d	-10.125	729	0	3.8549	0	1.1007	1.8771	1368.4	0
e	-8.100	1296	1296	5.1399	-5.7810	.0791	------	--------	------
f	10.125	0	-729	0	-4.3357	.3125	.0005	0	-.4
g	8.100	0	2025	0	7.2262	.8921	.4729	0	957.7
h	-8.100	0	-2025	0	7.2262	.6511	.0599	0	-121.3
i	-10.125	0	729	0	-4.3357	.4282	.0038	0	2.8
j	8.100	0	-1296	0	-5.7810	.5209	.0139	0	-18.0
							$\sum$	-314.3	820.8
							$\frac{3}{7} \times \sum$	-134.7	351.8

[a]Sign chosen to give positive $\frac{P}{P_R}$.

$$\left|a^{(1)}\right| = \left|a^{(0)}\right| + \left[A\right]^{-1} \begin{vmatrix} 134.7 \\ -351.8 \end{vmatrix}$$

$$\begin{vmatrix} a_1^{(1)} \\ a_2^{(1)} \end{vmatrix} = \begin{vmatrix} -6.4249 \\ 7.2262 \end{vmatrix} + \begin{vmatrix} 0.2707 \\ -0.4588 \end{vmatrix} = \begin{vmatrix} -6.1542 \\ 6.7674 \end{vmatrix}$$

TABLE IV.- COMPUTATIONS FOR FINAL APPROXIMATION

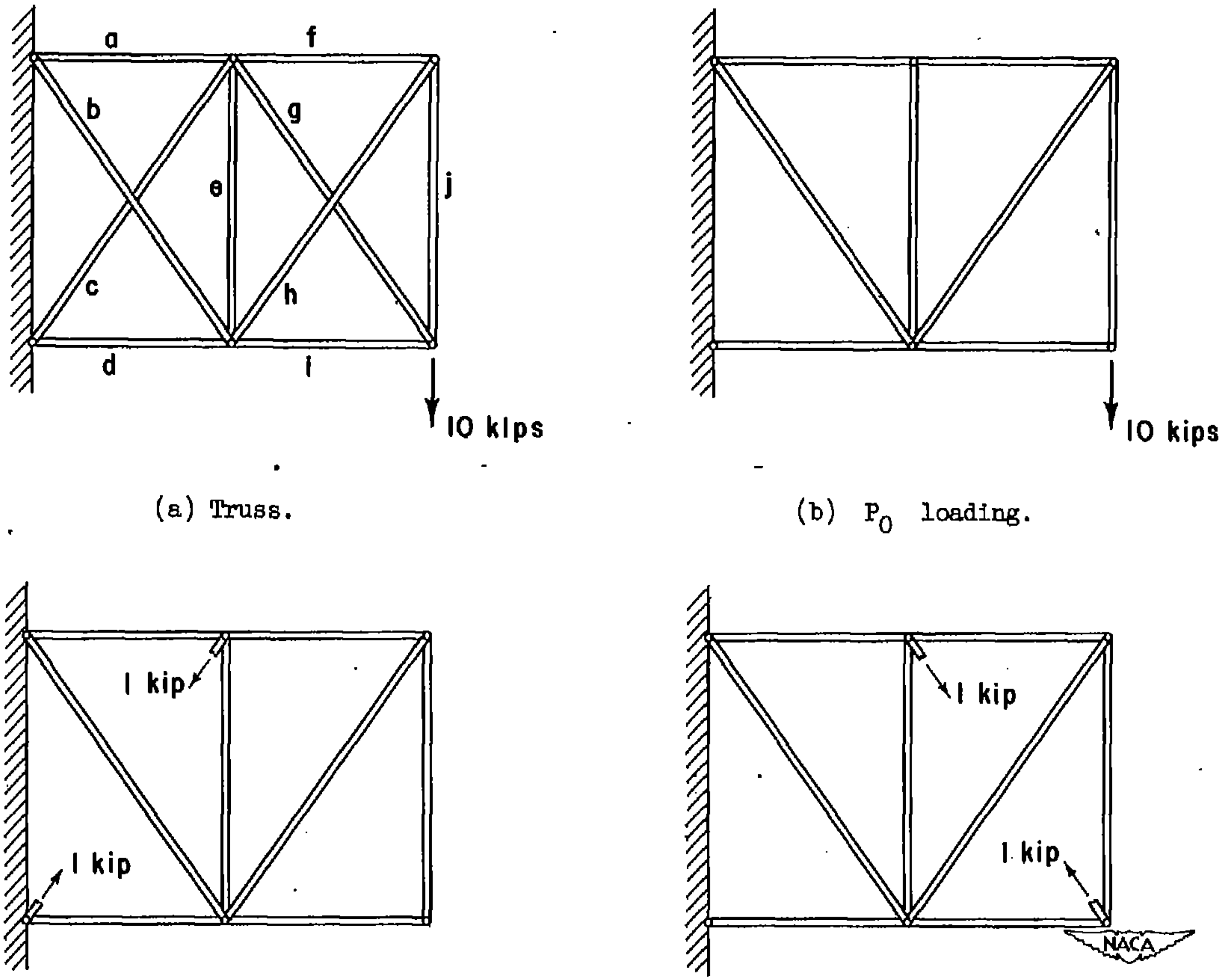
(1) Member	(5) a_1P_1	(6) a_2P_2	(7) $\frac{P}{P_R}$	(8) $\left(\frac{P}{P_R}\right)^{6.56}$	(9) (3) × (8)	(10) (4) × (8)
a	3.7855	0	1.1146	2.0378	-1485.5	0
b	-6.3092	0	.7643	.1715	347.2	0
c	-6.3092	0	.7789	.1942	-393.2	0
d	3.7855	0	1.1076	1.9550	1425.2	0
e	5.0474	-5.5489	.0619	------	-------	------
f	0	-4.1617	.3297	.0007	0	-.5
g	0	6.9362	.8563	.3615	0	732.0
h	0	6.9362	.6869	.0851	0	-172.4
i	0	-4.1617	.4110	.0029	0	2.1
j	0	-5.5489	.5495	.0197	0	-25.5
				$\sum$	-106.3	535.7
				$\frac{3}{7} \times \sum$	-45.56	229.6

$$a_1^{(r-1)} = -6.3092$$

$$a_2^{(r-1)} = 6.9362$$

$$\left| a^{(r)} \right| = \begin{vmatrix} -6.4249 \\ 7.2262 \end{vmatrix} + [A]^{-1} \begin{vmatrix} 45.56 \\ -229.6 \end{vmatrix}$$

$$\begin{vmatrix} a_1^{(r)} \\ a_2^{(r)} \end{vmatrix} = \begin{vmatrix} -6.4249 \\ 7.2262 \end{vmatrix} + \begin{vmatrix} 0.1157 \\ -0.2900 \end{vmatrix} = \begin{vmatrix} -6.3092 \\ 6.9362 \end{vmatrix}$$



(a) Truss.

(b) P_0 loading.

(c) P_1 loading.

(d) P_2 loading.

Figure 1.- Truss and load systems used in numerical example.

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