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METHOD FOR CALCULATION OF COMPRESSIBLE LAMINAR BOUNDARY LAYER WITH AXIAL PRESSURE GRADIENT AND HEAT TRANSFER

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SUMMARY

A rapid and sufficiently accurate method, for most practical purposes, of determining laminar-boundary-layer characteristics in flow with a given free-stream Mach number and given velocity distribution at the edge of the boundary layer is presented. The method can be easily applied to flow with zero pressure gradient for any (constant) Prandtl number of the order of unity and any given temperature distribution along the wall. Numerical examples are given to illustrate the method and the satisfactory accuracy obtained. For flow in an axial pressure gradient, the method can be applied for a Prandtl number of unity and any given uniform wall temperature. The methods developed here are based on an application of the Kármán integral method to both the momentum and energy equations, in conjunction with a sixth-degree velocity profile and a seventh-degree stagnation-enthalpy profile. A single boundary-layer thickness and one of the coefficients in the thermal profile are the parameters in this two-parameter method.

INTRODUCTION

The aim of this report is to present a relatively simple method, sufficiently accurate for most practical purposes, of calculating the laminar-boundary-layer characteristics in the compressible flow over a given object with heat transfer at the wall.

The method is based on the extension of the Kármán-Pohlhausen method to sixth-degree velocity profiles and seventh-degree stagnation-enthalpy profiles. The use of sixth-degree velocity profiles is in accordance with the conclusions of reference 1, wherein it was found that such profiles can usually be expected to lead to results of adequate accuracy without much increase in computational work. Such profiles have been applied with satisfactory results for compressible flow over a flat plate with heat transfer (ref. 2) and for compressible flow in an axial pressure gradient without heat transfer (refs. 3 and 4).

The present analysis consists of two main parts. In the first a simple approximate general solution is developed for flow without axial pressure gradient, with any given Mach number (neglecting hypersonic effects) and given distribution of temperature along the wall. The Prandtl number is also left arbitrary, although it is required to be of the order of magnitude of unity. The coefficients of specific heat, as well as the Prandtl number, are assumed as constant, while the viscosity coefficient is assumed to vary linearly with the temperature, the proportionality factor being chosen in accordance with the Sutherland relation, as in references 1, 2, 3, 5, and 6. In the second part of this analysis, a simple approximate general solution is developed for flow in a pressure gradient with any given constant wall temperature, free-stream Mach number, and velocity distribution outside the boundary layer. The Prandtl number is, however, assumed to be unity.

A brief indication of the pertinent literature may be worth while. The flow without axial pressure gradient (e.g., flow over a flat plate) has, of course, already been treated to a considerable extent. Kármán and Tsien (ref. 7) derived solutions for a Prandtl number of 1, constant wall temperature, and viscosity coefficient proportional to a power of the temperature. Chapman and Rubesin (ref. 5) have obtained exact (numerical) solutions for variable wall temperature and a linear viscosity-temperature relation. Van Driest (ref. 8) has accurately calculated a series of curves showing skin friction, heat-transfer coefficients, and velocity and temperature profiles for a constant wall temperature, using the Sutherland viscosity-temperature relation. Klunker and McLean (ref. 9) have calculated the boundary-layer characteristics using the actual variation of Prandtl number, specific heat, and viscosity coefficient with temperature, while Young and Janssen (ref. 10) and Moore (ref. 11) have done likewise with the aid of a differential analyzer, Moore taking dissociation into account.

In each of the above references, the calculations, though exact, are fairly tedious, while, except for reference 5, the wall temperature is assumed as constant. The advantage of the method presented in the present report is that the calculations can be performed with relatively little difficulty for a prescribed wall-temperature distribution, Mach number, and Prandtl number. Although the results thus obtained will not be exact, they will usually be sufficiently accurate in practice.

In contrast with the literature on flow over a flat plate, the literature on compressible boundary-layer flow in a pressure gradient with heat transfer still appears to be limited. A good survey of work done on incompressible flow can be found in reference 12. Kalikhman (ref. 13) has treated the compressible case by applying the Kármán-Pohlhausen method to both the momentum and energy equations. Ferrari (ref. 14) has recently presented an accurate, but elaborate, method of calculating the boundary-layer characteristics for arbitrary Prandtl number and uniform wall temperature. The present analysis has, once again, the advantage of presenting a general approximate solution which involves relatively little computational difficulty. However, the Prandtl number is here restricted to unity.

The method of analysis used in this investigation merits some comment. As in references 13 and 15, stagnation-enthalpy, rather than temperature, profiles are used. The use of stagnation-enthalpy profiles leads to significant mathematical simplifications, while it can be shown, moreover, that for a Prandtl number of 1 the equations used here automatically lead to what are known to be exact integrals of the energy partial-differential equation. As has already been stated, the velocity profiles and stagnation-enthalpy profiles are respectively of sixth and seventh degree here. In reference 13, both profiles are of the more customary, but usually less accurate, fourth-degree type. Finally, it is of interest to observe that in the present analysis only a single boundary-layer thickness is used, instead of a dynamical and a thermal boundary-layer thickness, such as used in references 13 and 15. This, however, does not necessarily impose any undue restrictions on the thermal profiles, since the latter have here been permitted to contain an additional coefficient, not determined in advance by the boundary conditions. This additional coefficient, which must be determined by the differential equations, replaces the thermal boundary-layer thickness as a parameter. This explains why the stagnation-enthalpy profiles are here of one degree higher than the velocity profiles. For Prandtl numbers near unity, as is the case for air, it has been found (cf., for example, refs. 13, 15, and 16) that the ratio of dynamical to thermal boundary-layer thickness is close to unity. Consequently, the present analysis should yield sufficiently accurate results for such cases. The advantages of using a single boundary-layer thickness are that the equations thus obtained are somewhat simpler and that it appears somewhat easier to solve for the additional coefficient than for the second boundary-layer thickness, especially for flow without a pressure gradient.[1]

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[1]This is due primarily to the fact that in the two-thickness method, the ratio of the thicknesses appears in the equations in a complicated manner.

A,B	constants in equations (50) and (51)
A,D,φ_1	constants defined by equations (63)
a_j	coefficient of τ^j in velocity profile (eq. (20))
$\bar{a}_2$	constant "average" value of a_2
b_j	coefficient of τ^j in stagnation-enthalpy profile (eq. (21))
$\bar{b}_1$	constant "average" value of b_1
C	proportionality factor in temperature-viscosity relation (eqs. (9) and (10))
$\bar{C}$	constant "average" value of C

$C_1 = \int_0^\xi C\,d\xi$

$\bar{C}_1 = \int_0^1 C\,d\xi$

C_f	average skin-friction coefficient, $\frac{1}{x}\frac{\int_0^x (\mu\,\partial u/\partial y)_0\,dx}{(1/2)\rho_\infty {u_\infty}^2}$
$C_{f l}$	local skin-friction coefficient, $(\mu\,\partial u/\partial y)_0 \Big/ \left(\rho_\infty {u_\infty}^2\right)2$
c_j	coefficient of ξ^j in given wall-temperature distribution (eq. (A1))
c_p,c_v	specific heats at constant pressure and at constant volume, respectively
e_j	coefficient of ξ^j in equation (A5)

F_1, F_2, F_3 integrals appearing in equations (11) and (12) and defined thereafter (cf. also eqs. (61))

$\bar{F}_1, \bar{F}_2$ constant "average" values of F_1 and F_2

f_i functions of τ appearing in equation (A3) and defined there

$G_1(\xi) = H_o/H_1(\xi)$; for $N_{Pr} = 1$, $G_1 = T_o/T_e$ (cf. also eq. (46))

$G_2, G_3(\xi), G_4(\xi), G_5(\xi), G_6(\xi)$ parameters defined in equations (33), (38b), (45b), and (58)

H stagnation enthalpy, $(u^2/2) + c_pT$

H_e stagnation enthalpy corresponding to equilibrium wall temperature

h local heat-transfer coefficient (eq. (40))

k coefficient of heat conductivity

L characteristic length

M Mach number

m, j, l constants defined by equations (67a)

N_{Nu} Nusselt number

N_{Pr} Prandtl number, $\mu c_p/k$

p static pressure

q heat-transfer rate, $-(k\ \partial T/\partial y)_o$

R gas constant, $c_p - c_v$

R_L Reynolds number based on L, $\rho_\infty u_\infty L/\mu_\infty$

R_x Reynolds number based on x, $\rho_\infty u_\infty x/\mu_\infty$

S Sutherland constant; 216° R for air (cf. eq. (10))

T absolute temperature

T_e	equilibrium wall temperature for zero heat transfer
t	transformation variable, defined by equation (13) or (14)
u,v	velocity components in x- and y-directions, respectively
x,y	coordinates parallel and normal to surface, respectively
β_j	constant parameters, depending on N_{Pr}, and defined after equation (A4)
γ	ratio of specific heats, c_p/c_v; $\gamma = 1.4$ for air
δ,δ_t	boundary-layer thicknesses in xy- and xt-planes, respectively (cf. also eqs. (15) and (16))
η	recovery factor, defined by equation (42)

$\lambda = R_L(\delta_t/L)^2$

μ	coefficient of viscosity
ν	kinematic viscosity, μ/ρ
ξ	dimensionless distance along wall, x/L
ρ	mass density
τ	dimensionless variable, t/δ_t

Subscripts:

o	values at wall; for example, T_o
1	local values at outer edge of boundary layer; for example, T_1 and M_1
∞	values in undisturbed free stream; for example, M_∞ (Also, cf. footnote 2)

GENERAL EQUATIONS

The following equations describe the steady, two-dimensional, laminar-boundary-layer flow of a compressible gas along a slightly curved wall:

Momentum equation in the x-direction:

$$\rho u u_x + \rho v u_y = -p_x + \left(\mu u_y\right)_y \tag{1}$$

Momentum equation in the y-direction:

$$p_y = 0 \tag{2}$$

Continuity equation:

$$(\rho u)_x + (\rho v)_y = 0 \tag{3}$$

Equation of state:

$$p = \rho R T \tag{4}$$

Energy equation:

$$\rho u c_p T_x + \rho v c_p T_y = u p_x + \left(k T_y\right)_y + \mu u_y^{2} \tag{5}$$

Equation (2) implies that the static pressure within the boundary layer is the same as the static pressure in the potential flow just outside the boundary layer. Thus, for slightly curved walls

$$-p_x = -dp/dx = \rho_1 u_1 \left(du_1/dx\right) \tag{6}$$

where the subscript 1 denotes local conditions at the outer edge of the boundary layer. It may be noted that the assumption that the pressure in the boundary layer is equal to the potential flow pressure must be modified at high Mach numbers and very low Reynolds numbers (cf. ref. 17). The methods to be used in the analysis of such flows are as yet not clear. As is usual in aeronautical problems the coefficients of viscosity and heat conductivity are considered to be known functions of the temperature, while the coefficient of specific heat at constant pressure is assumed constant.

It is convenient in the method of analysis to be presented here to rewrite equation (5). If equation (1) is multiplied by u and added to equation (5), it is readily possible to put the resulting equation in the form

$$N_{Pr}\left(\rho u H_x + \rho v H_y\right) = \left\{\mu\left[H_y - \left(1 - N_{Pr}\right)\left(u^2/2\right)_y\right]\right\}_y \tag{7}$$

provided the Prandtl number is assumed constant. For most gases the actual change of Prandtl number with temperature is sufficiently small to justify this assumption, especially since the experimental determination of the heat conductivity appearing in the Prandtl number is subject to considerable error (cf. ref. 18).

Before discussing a method of treatment of equations (1), (3), (4), (6), and (7) the mass density and coefficient of viscosity will be written explicitly in terms of the temperature. From equations (2) and (4) one obtains the equation

$$\rho/\rho_1 = T_1/T \tag{8}$$

For mathematical convenience the viscosity-temperature relation suggested in references 5 and 19 is extended here to the case of pressure gradient. Thus, within the boundary layer

$$\mu/\mu_\infty = C\left(T/T_\infty\right) \tag{9}$$

where

$$C = \left(T_o/T_\infty\right)^{1/2}\left(T_\infty + S\right)\Big/\left(T_o + S\right) \tag{10}$$

With this choice of C the approximation of equation (9) to the actual viscosity-temperature relation is good in the neighborhood of the wall, where the skin friction and heat-transfer properties are determined, and is, for most problems, considered satisfactory throughout the boundary layer. It may be pointed out that this viscosity-temperature relation is equivalent to using $\mu/\mu_1 = C\left(T/T_1\right)$ where C is similarly defined in terms of T_0 and T_1, if it is assumed that the ratio μ_1/μ_∞ is calculated by means of the exact Sutherland relation.

By integrating equations (1) and (7) with respect to t over the boundary-layer thickness $t = 0$ to $t = \delta_t$ and using the boundary conditions $u = v = 0$ at $t = 0$, together with smooth transition of the velocity and temperature profiles to their local main-stream values, the following differential-integral equations are obtained:

$$\left(F_1/2\right)\lambda' + \lambda\left\{F_1' + F_1\left(\log_e \rho_1\right)' + \left(\log_e u_1\right)'\left[F_1 + \left(1 + \frac{\gamma - 1}{2} M_1^2\right)F_2\right]\right\} =$$

$$C\left(\rho_\infty/\rho_1\right)\left(u_\infty/u_1\right)\left(T_1/T_\infty\right)\left[\left(u/u_1\right)_\tau\right]_0 \qquad (11)$$

$$\left(F_3/2\right)\lambda' + \lambda\left\{F_3' + F_3\left[\left(\log_e \rho_1\right)' + \left(\log_e u_1\right)'\right]\right\} =$$

$$C\left(\rho_\infty/\rho_1\right)\left(u_\infty/u_1\right)\left(T_1/T_\infty\right)\left(1/N_{Pr}\right)\left[\left(H/H_1\right)_\tau\right]_0 \qquad (12)$$

where

$$F_1 = \int_0^1 \left(u/u_1\right)\left[1 - \left(u/u_1\right)\right] d\tau$$

$$F_2 = \int_0^1 \left[\left(H/H_1\right) - \left(u/u_1\right)^2\right] d\tau$$

$$F_3 = \int_0^1 \left(u/u_1\right)\left[1 - \left(H/H_1\right)\right] d\tau$$

$\lambda = \left(\delta_t/L\right)^2\left(\rho_\infty u_\infty L/\mu_\infty\right)$

$\xi = x/L$

$\tau = t/\delta_t$

The prime denotes differentiation with respect to ξ, and t is the Dorodnitzyn variable defined by the transformation

$$t = \int_0^y \left(T_1/T\right) dy \tag{13}$$

or by the inverse

$$y = \int_0^t \left(T/T_1\right) dt \tag{14}$$

Moreover,

$$\delta_t = \int_0^\delta \left(T_1/T\right) dy \tag{15}$$

or

$$\delta = \int_0^{\delta_t} \left(T/T_1\right) dt \tag{16}$$

The quantities ρ_1/ρ_∞ and M_1 in equations (11) and (12) are related to u_1/u_∞, which is a function of ξ prescribed by the potential flow about the body in question. Thus, in accordance with the usual isentropic flow relations,[2]

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[2]In applying these relations and interpreting infinity conditions, the presence, in the supersonic case, of attached or detached bow waves should be considered. "Infinity" conditions should then be taken as conditions just behind such shocks on the streamline which in potential flow would form the body under consideration.

$$\rho_1/\rho_\infty = \left(T_1/T_\infty\right)^{\frac{1}{\gamma-1}} = \left\{1 + (\gamma - 1)\left(M_\infty^2/2\right)\left[1 - \left(u_1/u_\infty\right)^2\right]\right\}^{1/(\gamma-1)} \tag{17}$$

$$M_1 = \left(u_1/u_\infty\right)M_\infty\left\{1 + (\gamma - 1)\left(M_\infty^2/2\right)\left[1 - \left(u_1/u_\infty\right)^2\right]\right\}^{-1/2} \tag{18}$$

Approximate solutions to equations (11) and (12) can be found by assuming u/u_1 and H/H_1 as functions of τ which satisfy certain boundary conditions at $\tau = 0$ and $\tau = 1$ to be discussed in detail later. In these profiles one parameter in addition to λ is left undetermined after the boundary conditions are satisfied, and these two parameters are considered as the two unknown functions of ξ to be determined by equations (11) and (12).

In setting up the final differential equations, examination of equations (11) and (12) indicates that, in addition to selecting the profiles for u/u_1 and H/H_1 with their appropriate boundary conditions, one must prescribe $\left(u_1/u_\infty\right)(\xi)$, M_∞, and N_{Pr}. Moreover, the values of the fluid properties at infinity must also be specified. From the solutions to equations (11) and (12) all quantities in the $\xi\tau$-plane can easily be found; the transformation, equation (14), can be applied to obtain the corresponding quantities in the xy or physical plane. For this purpose equation (14) may be conveniently written as

$$y/L = \left(\delta_t/L\right)\int_0^\tau \left\{\left(H/H_1\right)\left[1 + (\gamma - 1)\left(M_1^2/2\right)\right] - (\gamma - 1)\left(M_1^2/2\right)\left(u/u_1\right)^2\right\} d\tau \tag{19}$$

The boundary conditions that the velocity and stagnation-enthalpy profiles are to satisfy will now be discussed. As indicated in the "Introduction" a sixth-degree polynomial in τ for the velocity and a seventh-degree polynomial for the stagnation enthalpy are suggested on the basis of the critical study of the use of integral methods in compressible-boundary-layer analysis presented in reference 1. Thus, let

$$u/u_1 = \sum_{j=0}^{6} a_j \tau^j \tag{20}$$

$$H/H_1 = \sum_{j=0}^{7} b_j \tau^j \tag{21}$$

All of the a_j and b_j coefficients except one are determined from boundary conditions at $\tau = 0$ and $\tau = 1$. In setting up the differential-integral equations (11) and (12), the boundary conditions which must be specified to obtain a unique and exact solution to the governing partial-differential equations have been applied. In the $\xi\tau$-plane these are:
At $\tau = 0$,

$$\left.\begin{aligned} u/u_1 &= v/u_1 = 0 \\ H/H_1 &= \left(H_0/H_1\right)(\xi) \end{aligned}\right\} \tag{22}$$

and at $\tau = 1$,

$$\left.\begin{aligned} u/u_1 &= H/H_1 = 1 \\ \left(u/u_1\right)_\tau &= \left(H/H_1\right)_\tau = 0 \end{aligned}\right\} \tag{23}$$

The boundary conditions specified here are the usual ones in heat-transfer problems. It is possible to specify instead of H_0/H_1, either $\left[\left(H/H_1\right)_\tau\right]_0$ or some relation between these two quantities as in the case of radiation cooling. In principle there is no loss in generality here, since H_0/H_1 is left arbitrary and nothing is specified with respect to $\left[\left(H/H_1\right)_\tau\right]_0$.

In addition to requiring that the assumed profiles satisfy boundary conditions (22) and (23), it is customary in the integral methods (cf., e.g., ref. 1) to select the profiles so that at the wall and at the edge of the boundary layer some of their derivatives have the same values as those which an exact solution to the original partial-differential equations under boundary conditions (22) and (23) would yield. For the profiles of equations (20) and (21) these additional conditions can be obtained from equations (1) and (7) and from differentiation of these equations with respect to t. In this way one obtains the following boundary conditions to supplement those of equations (22) and (23):
At $\tau = 0$,

$$C\left(T_1/T_\infty\right)\left(u/u_1\right)_{\tau\tau} = -\lambda\left(\rho_1/\rho_\infty\right)\left(H_o/H_1\right)\left[1 + (\gamma - 1)M_1{}^2/2\right]\left(u_1/u_\infty\right)' \tag{24}$$

$$\left(H_o/H_1\right)\left(u/u_1\right)_{\tau\tau\tau} = \left(u/u_1\right)_{\tau\tau}\left(H/H_1\right)_\tau \tag{25}$$

$$\left[1 + (\gamma - 1)\left(M_1{}^2/2\right)\right]\left(H/H_1\right)_{\tau\tau} = \left(1 - N_{Pr}\right)(\gamma - 1)M_1{}^2\left[\left(u/u_1\right)_\tau\right]^2 \tag{26}$$

$$\left(u_1/u_\infty\right)\lambda N_{Pr}\left(\rho_1/\rho_\infty\right)\left(u/u_1\right)_\tau\left(H_o/H_1\right)' = C\left(T_1/T_\infty\right)\left\{\left(H/H_1\right)_{\tau\tau\tau} - 3\left(1 - N_{Pr}\right)(\gamma - 1)M_1{}^2\left(u/u_1\right)_\tau\left(u/u_1\right)_{\tau\tau}\Big/\left[1 + (\gamma - 1)\left(M_1{}^2/2\right)\right]\right\} \tag{27}$$

At $\tau = 1$,

$$\left(u/u_1\right)_{\tau\tau} = \left(u/u_1\right)_{\tau\tau\tau} = \left(H/H_1\right)_{\tau\tau} = \left(H/H_1\right)_{\tau\tau\tau} = 0 \tag{28}$$

With equations (22) to (28) fourteen of the fifteen coefficients in the velocity and stagnation-enthalpy profiles of equations (20) and (21) can be determined in terms of λ, of the fifteenth coefficient (to be chosen as b_1), of the prescribed functions $\left(u_1/u_\infty\right)(\xi)$ and $\left(H_o/H_1\right)(\xi)$, and of the prescribed parameters M_∞ and N_{Pr}.

As explained in the "Introduction," it will be observed that only one boundary-layer thickness δ_t is here assumed and that thus no distinction is made between a velocity and a temperature boundary-layer thickness. This, however, does not impose an undue restriction on the shape of the temperature profiles, since the latter (see eq. (21)) are here permitted to retain an additional parameter to be determined, not by any boundary conditions, but by the differential equations (viz, eqs. (11) and (12)).

Examination of equations (24) to (28) indicates that considerable simplification is achieved in the boundary conditions, and therefore in the analysis, under two special cases: (a) The flow without axial pressure gradient $\left(u_1/u_\infty \equiv 1\right)$, which also corresponds to the practically interesting case of the supersonic flow over a thin wedge, as well as to the subsonic (or supersonic) flow over a flat plate and (b) the airfoil with pressure gradient, and constant wall temperature, in a fluid with Prandtl number equal to unity. Although the assumption of constant wall temperature does not appear to be exactly realized on practical wings, it appears desirable to investigate the boundary-layer characteristics under these assumptions since the pressure-gradient, heat-transfer case in a compressible flow has not yet been analyzed entirely satisfactorily from the standpoint of simultaneous accuracy and simplicity.

FLOW WITHOUT AN AXIAL PRESSURE GRADIENT

In this section a general solution based on the previously derived integral-differential equations will be obtained for the special case of flow without axial pressure gradient. The accuracy of this solution is then investigated by comparing it with more exact solutions.

General Case

In the case of the flow without axial pressure gradient and with general but constant Prandtl number and variable wall temperature the integral-differential equations (11) and (12) become:

$$\left(F_1/2\right)\lambda' + F_1'\lambda = C\left[\left(u/u_1\right)_\tau\right]_0 \tag{29}$$

$$\left(F_3/2\right)\lambda' + F_3'\lambda = \left(C/N_{Pr}\right)\left[\left(H/H_1\right)_\tau\right]_0 \tag{30}$$

Here $M_1 \equiv M_\infty$, $T_1 \equiv T_\infty$, and so forth. The subscript ∞ will be used to denote the free-stream conditions for this case.

It is convenient to consider λ and b_1 as the unknowns to be determined by equations (29) and (30). Then the velocity profile satisfying boundary conditions (22) to (28) is

$$u/u_1 = 2\tau - 5\tau^4 + 6\tau^5 - 2\tau^6 \tag{31}$$

while the b_j coefficients of the stagnation-enthalpy profile become:

$$\left.\begin{aligned}
b_0 &= H_0/H_\infty \equiv G_1 \\
b_2 &= 2G_2 \\
b_3 &= 2G_3\lambda \\
b_4 &= 35\left(1 - G_1\right) - 20G_2 - 8G_3\lambda - 20b_1 \\
b_5 &= -84\left(1 - G_1\right) + 40G_2 + 12G_3\lambda + 45b_1 \\
b_6 &= 70\left(1 - G_1\right) - 30G_2 - 8G_3\lambda - 36b_1 \\
b_7 &= -20\left(1 - G_1\right) + 8G_2 + 2G_3\lambda + 10b_1
\end{aligned}\right\} \tag{32}$$

where the G's are known functions of ξ defined by:

$$\left.\begin{aligned} G_1 &= \left(H_o/H_\infty\right)(\xi) \\ G_2 &= \left(1 - N_{Pr}\right)(\gamma - 1)M_\infty^2 \Big/ \left[1 + (\gamma - 1)\left(M_\infty^2/2\right)\right] \\ G_3 &= (1/6)N_{Pr}\, G_1'/C \end{aligned}\right\} \tag{33}$$

From the definition of F_1 and from equation (31) it is found that

$$F_1 = 985/9{,}009 \tag{34}$$

Similarly,

$$F_3 = (31/126)\left(1 - G_1\right) - (953/180{,}180)G_3\lambda - (302/9{,}009)G_2 - (821/12{,}012)b_1 \tag{35}$$

With the value of F_1 given by equation (34), $\lambda(\xi)$ can be found directly from equation (29). Thus with the requirement that $\lambda(0) = 0$,

$$\lambda = 4C_1/F_1 \tag{36}$$

where[3]

$$C_1 \equiv \int_0^\xi C \, d\xi$$

Equation (30), which gives $b_1(\xi)$, becomes:

$$F_3' - \left(F_1/4N_{Pr}\right)\left(C/C_1\right)b_1 + \left(F_3/2\right)\left(C/C_1\right) = 0 \tag{37}$$

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[3]It may be noted that in this analysis the temperature-viscosity proportionality factor C, unlike in reference 5, can be permitted to vary along the plate (i.e., $C = C(\xi)$) without any mathematical difficulty.

The solution of equation (37) satisfying the requirement that b_1 be finite at $\xi = 0$ is

$$b_1 = C_1^{-\beta_1} \int_0^{\xi} C_1^{\beta_1} G_4(\xi)\, d\xi \tag{38a}$$

where

$$\left.\begin{aligned} \beta_1 &= (1/2) + (985/2{,}463)\left(1/N_{Pr}\right) \\ G_4(\xi) &= (12{,}012/821)\Bigg((31/252)\left(1 - G_1\right)\left(C/C_1\right) - \\ &\quad G_1'\left\{(31/126) + \left(953N_{Pr}/29{,}550\right)\left[(3/2) - \left(C_1C'/C^2\right)\right]\right\} - \\ &\quad \left(953N_{Pr}/29{,}550\right)\left(C_1/C\right)G_1'' - (151/9{,}009)\left(C/C_1\right)G_2\Bigg) \end{aligned}\right\} \tag{38b}$$

From equation (38a) with any prescribed temperature distribution $G_1(\xi)$ at the wall, $b_1 = b_1(\xi)$ can be readily determined. From equations (32) the remaining b_j coefficients can then be found as functions of ξ and of the prescribed flow parameters, for example, N_{Pr} and M_∞. With the a_j and b_j coefficients and λ all determined, the velocity and stagnation-enthalpy distributions throughout the $\xi\tau$-plane are known. By application of equation (19), these distributions in the xy or physical plane are easily determined.

The average skin-friction coefficient C_f for the length L can be calculated from equations (31) and (36) as

$$C_f\sqrt{R_L} = 1.322\sqrt{\overline{C}_1} \tag{39}$$

where $\overline{C}_1 = \int_0^1 C\, d\xi$.

The local heat-transfer coefficient h is usually defined as

$$h \equiv -\left(kT_y\right)_o \Big/ \left(T_o - T_e\right) \tag{40}$$

Using solution (36) for λ, equation (40) yields:

$$h = -0.165\sqrt{R_L}\left(C\Big/\sqrt{C_1}\right)\left(k_1 b_1\right)\Big/L\left\{G_1 - \left[1 + (\gamma - 1)\left(M_1^2\eta/2\right)\right]\Big/\left[1 + (\gamma - 1)\left(M_1^2/2\right)\right]\right\} \tag{41}$$

where η is the recovery factor defined by the equation

$$T_e/T_1 = 1 + (\gamma - 1)\left(M_1^2\eta/2\right) = \left(H_e/H_1\right)\left[1 + (\gamma - 1)\left(M_1^2/2\right)\right] \tag{42}$$

The (nondimensional) Nusselt number N_{Nu} is thus

$$N_{Nu} \equiv hx/k_1 = -\frac{0.165\left(C\Big/\sqrt{C_1}\right)\sqrt{R_L}\,\xi b_1}{G_1 - \left[1 + (\gamma - 1)\left(M_1^2/2\right)\eta\right]\Big/\left[1 + (\gamma - 1)\left(M_1^2/2\right)\right]} \tag{43}$$

The value of η follows by setting $b_1 = 0$ and solving for the constant wall temperature, as determined by G_1. Equations (33), (38), and (42) with $\gamma = 1.4$ are thus found to imply:

$$\eta = 1 - 0.272\left(1 - N_{Pr}\right) \tag{44}$$

It should be pointed out that equation (44) yields for $N_{Pr} = 0.72$ a value of $\eta = 0.924$, which must be compared with the value of 0.845 given by the mathematically more exact method of Chapman and Rubesin. Although this result must be considered a consequence of the approximate method of analysis used here, calculations involving rates of heat transfer (such as Nusselt number) for given values of the wall temperature will not be directly affected by the equilibrium wall temperature and, therefore, will not be appreciably affected by any inaccuracy in η.[4] This is borne out by the numerical example and the comparison with an exact solution, given in a subsequent section here, with $N_{Pr} = 0.72$.

It may nevertheless be convenient for certain applications to express the wall temperature in terms of the predicted equilibrium temperature. With the additional simplification of replacing variable C by an average value $\overline{C}$ equation (43) becomes:

$$N_{Nu}\Big/\sqrt{R_x} = 0.297\sqrt{\overline{C}}\left[1 - \left(T_o/T_e\right)\right]^{-1} \xi^{-\beta_1} \int_0^{\xi} G_5 \, d\xi \tag{45a}$$

where

$$G_5 = \xi^{\beta_1 - 1}\left\{\left[1 - \left(T_o/T_e\right)\right] - \left[2 + (60,039/152,675)N_{Pr}\right]\left(T_o/T_e\right)'\xi - (40,026/152,675)N_{Pr}\left(T_o/T_e\right)''\xi^2\right\} \tag{45b}$$

In deriving equations (45a) and (45b), the following relation between G_1 and T_o/T_e (cf. eq. (42)) was used:

$$G_1 = \left(T_o/T_e\right)\left[1 + (\gamma - 1)\left(M_1^2/2\right)\eta\right]\Big/\left[1 + (\gamma - 1)\left(M_1^2/2\right)\right] \tag{46}$$

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[4]Because of the inaccuracy in η, it will be found, of course, that when the wall temperature has the same value as the true equilibrium temperature, the present equations will not lead exactly to a zero heat transfer at the wall. However, in such a case, it will be found that the calculated rate of heat transfer will be relatively small.

It should be observed that the recovery factor η does not appear in solution (45a), although T_o/T_e is prescribed.

If it is desired to calculate the actual rate of heat transfer q (in British thermal units per second per square foot) for a given ratio T_o/T_e from equation (45a), then the Nusselt number as determined by equation (45a) should be multiplied by $\left(k_1/x\right)\left(T_o - T_e\right)$, where T_e is the actual equilibrium wall temperature (as given, e.g., by exact theory or by experiment). In practice, however, the wall temperature T_o (or the ratio T_o/T_∞), rather than the ratio of wall temperature to equilibrium temperature, will usually be prescribed. In that case, the rate of heat transfer can be directly calculated (without any intermediate use at all of the recovery factor η) from the expression:

$$q \equiv -(k\ \partial T/\partial y)_o = -C\ \frac{k_\infty T_1}{L}\left(1 + \frac{\gamma - 1}{2} M_1^2\right)\frac{b_1}{\sqrt{\lambda}}\ \frac{T_1}{T_\infty}\sqrt{R_L} \tag{47}$$

The quantity b_1 can be calculated from equations (38a) and (38b) for flow over a flat plate, with the observation that

$$G_1 = \left(T_o/T_1\right)\left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{-1} \tag{48}$$

Equation (47) has been derived with the use of the following relation between temperature and stagnation-enthalpy profiles, resulting from the definition of the stagnation enthalpy H:

$$\frac{T}{T_1} = \frac{H}{H_1}\left(1 + \frac{\gamma - 1}{2} M_1^2\right) - \left(\frac{\gamma - 1}{2}\right) M_1^2 \left(\frac{u}{u_1}\right)^2 \tag{49}$$

In order to determine the velocity and temperature profiles in the physical (xy) plane, the inverse transformation given by equation (19) must be used to find the coordinate y associated with the variables ξ and τ, while the x-coordinate may be found directly from the definition of ξ.

If the wall-temperature distribution is prescribed as a polynomial in the distance ξ along the wall, then the Nusselt number and temperature profiles, following from equations (45a), (45b), and (49), can be directly calculated, without any quadratures, from the explicit expressions given in the appendix.

It is of interest to observe[5] the wall-temperature distribution required for zero heat transfer according to equations (45a) and (45b). Thus, by setting $G_5 = 0$, a second-order linear differential equation in T_o/T_e is obtained, whose general solution is:

$$\frac{T_o}{T_e} = 1 + A\xi^{-1/2} + B\xi^{-7.63/N_{Pr}} \tag{50}$$

where A and B are arbitrary constants. Equation (50) implies that zero heat-transfer conditions will be satisfied not only by a uniform wall temperature T_e but also by a wall temperature varying inversely as a certain power of the distance ξ along the flow. In particular, the heat-transfer rate vanishes for $1 - \left(T_o/T_e\right) \propto \xi^{-1/2}$ (regardless of the Prandtl number). This result was also obtained by Levy (ref. 20) for incompressible flow, using an analysis restricted to power-law wall-temperature distributions. The wall temperature according to this distribution would be infinite at $\xi = 0$, but Eckert (ref. 21) has reasoned that a heat-line source or sink placed at the region where $\xi = 0$ might result in a temperature distribution of this type. The case of upstream cooling such as analyzed in reference 22 is the physical counterpart of the abstraction of this line source or sink. Equation (50) indicates that there is another type of wall-temperature distribution for zero heat transfer, namely, $1 - \left(T_o/T_e\right) \propto \xi^{-7.63/N_{Pr}}$. This is not included in reference 20, where negative powers of ξ for $T_o(\xi)$ lower in absolute value than $7.63/N_{Pr}$ (approximately 10 for air) were assumed. It should be noted that the results obtained in the present analysis are valid for compressible flow (i.e., for any Mach number). Finally, it may be noted that temperature distributions of the form of equation (50) are not included in polynomial distributions, such as those to which reference 5 is restricted.

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[5]The observations in this paragraph were pointed out to the authors by Dr. M. Bloom.

Constant Wall Temperature and $N_{Pr} = 1$

A special simple case of the above solution for flow over a flat plate is of interest here. This is the case of constant wall temperature and Prandtl number equal to unity. This case will now be shown to give a particularly simple check on the accuracy of the method presented here.

It is well known that if the Prandtl number is assumed equal to unity then equations (1) and (7) imply for the flat-plate case that

$$H = A + Bu \tag{51}$$

where A and B are arbitrary constants. Equation (51) is an exact stagnation-enthalpy relation which satisfies the boundary conditions of constant wall temperature both with and without heat transfer. Therefore, one test of the approximate method presented above is its automatic reduction to this exact relation, if the conditions under which this relation is known to be exact are imposed. This test will now be applied.

For the case of constant wall temperature and a Prandtl number of unity, equations (38b) and (33) imply:

$$G_4 = (4,433/2,463)\left(1 - G_1\right)(1/\xi)$$

Hence, equation (38a) yields:

$$b_1 = 2\left(1 - G_1\right) \tag{52}$$

Furthermore, from equations (32),

$$\left.\begin{aligned} b_2 &= b_3 = 0 \\ b_4 &= -5\left(1 - G_1\right) \\ b_5 &= 6\left(1 - G_1\right) \\ b_6 &= -2\left(1 - G_1\right) \\ b_7 &= 0 \end{aligned}\right\} \tag{53}$$

When these b_j coefficients are substituted into the stagnation-enthalpy profile, one obtains

$$H/H_1 = G_1 + \left(1 - G_1\right)\left(2\tau - 5\tau^4 + 6\tau^5 - 2\tau^6\right) \tag{54}$$

and thus, by comparison with equation (31),

$$H/H_1 = G_1 + \left(1 - G_1\right)\left(u/u_1\right) \tag{55}$$

Equation (55) is exactly equation (51) in nondimensional form, with the boundary conditions that $H = H_o$ when $u = 0$ and $H = H_1$ when $u = u_1$. It is thus seen that the reduction to the exact energy equation for this special case is automatic and rigorous. It has already been shown in reference 1 that the agreement between the results of the integral method using a sixth-degree velocity profile in conjunction with equation (55) (i.e., for constant wall temperature and Prandtl number equal to unity) and the results of the exact solution of reference 5 is good. The skin friction and heat transfer are for all practical purposes exactly predicted and the laminar-boundary-layer stability limits are in good agreement. It can therefore be expected that the more general integral method presented here, valid for cases of heat transfer in a pressure gradient, will be accurate in more general cases, since the two integral methods are in complete agreement for the special case just considered.

Comparison With the More Exact Method of Chapman and Rubesin

(Variable Wall Temperature and $N_{Pr} \neq 1$)

In reference 5, Chapman and Rubesin have applied their analysis to the case of a variable, prescribed wall temperature. To simplify the numerical analysis the proportionality constant C was assumed by them to be equal to unity. The Prandtl number was assumed to be 0.72. In the notation used here the wall temperature was expressed there as

$$T_o/T_e = 1.25 - 0.83\xi + 0.33\xi^2 \tag{56}$$

Thus for equation (A1) (see appendix) $c_o = 1.25$, $c_1 = -0.83$, and $c_2 = 0.33$. The Nusselt number variation with ξ given by equation (A2) is shown in figure 1, along with the results of the more exact Chapman-Rubesin calculation. It can be seen that excellent agreement is realized.

It will be noted from equation (39) with $\overline{C}_1 = 1$ that the average skin friction obtained by the present equations is in excellent agreement with the exact value $C_f\sqrt{R_L} = 1.328$, given in reference 5.

In comparing the velocity and temperature profiles predicted by the two methods it must be kept in mind that for strict comparison the wall-temperature distribution should be precisely the same in the two methods. However, since Chapman and Rubesin assumed only the ratio of wall temperature to equilibrium temperature and since the two methods do not give precisely the same recovery factor, and hence not the same equilibrium temperature, it is necessary for this comparison to express the wall temperature assumed by Chapman and Rubesin, as well as that used in the present report, in terms of the free-stream temperature.

In figures 2 and 3 the comparisons of the temperature and velocity profiles predicted by the two methods are shown for three values of ξ and for two Mach numbers. It can be seen that excellent agreement is realized.

It may therefore be concluded from these numerical results that the momentum-and-energy integral method given here will yield good results with little computational difficulty when applied to the case of zero axial pressure gradient, variable wall temperature, and general but constant Prandtl number of the order of magnitude of unity.

From a practical point of view, it should be kept in mind that the solutions developed here are based on the viscosity-temperature relations (9) and (10), which are an approximation to the actual relation for air. Because of relations (9) and (10), the results obtained here, namely equations (39) and (45a), indicate that, for a fixed wall temperature, the skin-friction coefficient and the Nusselt number will be independent of Mach number. For the Sutherland viscosity-temperature relation, however, this will not be quite valid (cf. ref. 8).

FLOW WITH PRESSURE GRADIENT

For flow over a curved surface, such as an airfoil, where the local velocity distribution u_1/u_∞ outside the boundary layer is not constant, but may be considered as a given function of the distance x along the wall, the ordinary differential equations (11) and (12), unlike the case of zero pressure gradient, can no longer be easily solved for general Prandtl number and arbitrary distribution of temperature along the wall. For such flows, however, considerable mathematical simplifications occur when the Prandtl number is unity and the wall temperature is uniform (cf. especially eqs. (26) and (27)).

The assumption $N_{Pr} = 1$ has the additional theoretical advantage that, according to equation (44), it automatically leads to the known exact value of the recovery factor, namely, $\eta = 1$, for this value of the Prandtl number. In fact, the parameter G_1 now assumes particular physical significance, since for $N_{Pr} = 1$ equation (46) implies:

$$G_1 = T_o/T_e \tag{57}$$

Consequently, the case of $N_{Pr} = 1$ and uniform wall temperature will be treated now in detail. It will be shown that for such a case a relatively simple approximate solution of the equations can be obtained for any given velocity distribution outside the boundary layer.

General Approximate Solution for

$N_{Pr} = 1$ and T_o = Constant

For this case, with $\left(u_1/u_\infty\right)(\xi)$ arbitrary, the coefficients a_j and b_j in equations (20) and (21) for the velocity and stagnation-enthalpy profiles can, by virtue of boundary conditions (22) to (28), all be expressed in terms of a_2 and b_1, where b_1 remains arbitrary, while a_2 is given by

$$a_2 = -(1/2C)\left(T_1/T_\infty\right)^{\frac{2-\gamma}{\gamma-1}}\left(u_1'/u_\infty\right)G_1\left(1 + \frac{\gamma - 1}{2} M_1^2\right)\lambda = G_6(\xi)\lambda \tag{58}$$

where $G_6(\xi)$ is a given function. The profiles can then be given in terms of a_2 and b_1 as follows:

$$u/u_1 = \left(2\tau - 5\tau^4 + 6\tau^5 - 2\tau^6\right) + \left(a_2/5\right)\left[-2\tau + 5\tau^2 - 10\tau^4 + 10\tau^5 - 3\tau^6 + \left(b_1/6G_1\right)\left(-\tau + 10\tau^3 - 20\tau^4 + 15\tau^5 - 4\tau^6\right)\right] \tag{59}$$

$$H/H_1 = G_1 + \left(1 - G_1\right)\left(35\tau^4 - 84\tau^5 + 70\tau^6 - 20\tau^7\right) + b_1\left(\tau - 20\tau^4 + 45\tau^5 - 36\tau^6 + 10\tau^7\right) \tag{60}$$

With expressions (59) and (60), the following explicit expressions for F_1, F_2, and F_3 are obtained:

$$\left.\begin{aligned}
F_1 &= 0.1093 + 0.00211a_2 - 0.000622a_2^2 + 0.000412\left(b_1a_2/G_1\right) - \\
&\quad 0.0000095\left(b_1a_2/G_1\right)^2 - 0.000153\left(b_1a_2^2/G_1\right) \\
F_2 &= 0.395 - 0.500\left(1 - G_1\right) + 0.107b_1 + 0.0212a_2 - 0.00062a_2^2 + \\
&\quad 0.0028\left(b_1a_2/G_1\right) - 0.00015\left(b_1a_2^2/G_1\right) - 0.0000095\left(b_1a_2/G_1\right)^2 \\
F_3 &= \left(1 - G_1\right)\left[0.246 - 0.015a_2 - 0.00181\left(b_1a_2/G_1\right)\right] - \\
&\quad b_1\left[0.0683 - 0.00324a_2 - 0.00041\left(b_1a_2/G_1\right)\right]
\end{aligned}\right\} \tag{61}$$

With expressions (59), (60), and (61) inserted in equations (11) and (12), two ordinary differential equations for $\lambda(\xi)$ and $b_1(\xi)$ are obtained. Although these can be solved numerically for a given $\left(u_1/u_\infty\right)(\xi)$, the process may be tedious. A relatively simple general approximate solution of these equations will, therefore, be derived.

Equation (11) can be solved approximately for λ by assuming that F_1 and F_2 can be replaced there by constant "average" values $\overline{F}_1$ and $\overline{F}_2$ over the distance ξ. This is justified by the fact that the variable terms there, which are proportional to a_2 and b_1, are relatively small (cf. eqs. (61)). This is equivalent to replacing a_2 and b_1 by constant "average" values $\overline{a}_2$ and $\overline{b}_1$ for this purpose.

With equation (59) for the velocity profile and equation (58) for a_2, equation (11) then becomes the following linear ordinary differential equation in λ:

$$\left(\overline{F}_1/2\right)\lambda' + \lambda\left\{\overline{F}_1\left(\rho_1'/\rho_1\right) + \left(u_1'/u_1\right)\left[\varphi_1 + \frac{\gamma - 1}{2}M_1^2\left(\varphi_1 - \overline{F}_1\right)\right]\right\} =$$

$$2C\left(\rho_\infty/\rho_1\right)\left(T_1/T_\infty\right)\left(u_\infty/u_1\right) \tag{62}$$

where φ_1 is a constant, given by:

$$\left.\begin{aligned}
\varphi_1 &= \overline{F}_1 + \overline{F}_2 - \tfrac{1}{5}G_1 - \left(\overline{b}_1/60\right) \\
&= A + D\overline{b}_1 \\
A &= 0.3G_1 + 0.00438 + 0.0232\overline{a}_2 - 0.00124\overline{a}_2^2 \\
D &= 0.0905 + \left(0.0838 - 0.00458\overline{a}_2\right)\left(\overline{a}_2/30G_1\right)
\end{aligned}\right\} \tag{63}$$

With relations (17) and (18), the solution of equation (62) satisfying the condition $\lambda = 0$ or finite (if $u_1 = 0$ at $\xi = 0$) at the leading edge $(\xi = 0)$ is found to be:

$$\lambda = \left(4/\overline{F}_1\right)C\,\frac{\displaystyle\int_0^\xi \left(u_1/u_\infty\right)^{\left(\frac{2}{\overline{F}_1}\varphi_1 - 1\right)}\left(T_1/T_\infty\right)^{\frac{2\gamma-1}{\gamma-1} - \frac{\varphi_1}{\overline{F}_1}}d\xi}{\left(u_1/u_\infty\right)^{\frac{2}{\overline{F}_1}\varphi_1}\left(T_1/T_\infty\right)^{\frac{\gamma+1}{\gamma-1} - \frac{\varphi_1}{\overline{F}_1}}} \tag{64}$$

Equation (64) is similar in <u>form</u> to equations obtained for zero heat transfer in references 3 and 23 and for heat transfer, but with fourth-degree profiles and two boundary-layer thicknesses, in references 13 and 15.

To obtain a general approximate solution for $b_1(\xi)$, λ' can first be eliminated from equations (11) and (12). Assuming $F_1' = F_3' = 0$ for this purpose and using equation (58), the following equation is thus obtained:

$$b_1\left(\frac{F_1}{F_3} + \frac{a_2}{30G_1}\right) = 2 + 2\left[F_2 - (1/5)G_1\right]\frac{a_2}{G_1} \tag{65}$$

Noting, in advance, that b_1 will usually have a value approximately equal to $2\left(1 - G_1\right)$, b_1^2 can be replaced by $2\left(1 - G_1\right)b_1$ in the relatively small term $0.0000095\left(b_1 a_2/G_1\right)^2$ appearing in F_1 and F_2 and in the small term $0.00041 b_1^2 a_2/G_1$ appearing in F_3. Expressions (61) for F_1, F_2, and F_3 can then be written as linear functions of b_1, with coefficients as functions of G_1 and a_2. By substituting these expressions into equation (65), the following quadratic equation in b_1 is obtained:

$$mb_1^2 + jb_1 - l = 0 \tag{66}$$

where

$$\left.\begin{aligned}
m &= \frac{2a_2}{G_1} \times 10^{-4}\left[63.90 - 4.496a_2 + 2.705\frac{a_2}{G_1} - 0.2096\frac{a_2^2}{G_1} + 0.01452\left(\frac{a_2}{G_1}\right)^2 + 0.0056\frac{a_2^3}{G_1} - \right. \\
&\qquad \left. 0.00052\frac{a_2^3}{G_1^2} - 0.00018\left(\frac{a_2}{G_1}\right)^3\right] \\
j &= 0.24602 + 0.07917a_2 - 0.005868a_2^2 + \frac{2a_2}{G_1} \times 10^{-4}\left[-284.5 + 42.26a_2 - 2.067a_2^2 + \right. \\
&\qquad \left. 0.0463a_2^3 - 7.906\frac{a_2}{G_1} + 0.9086\frac{a_2^2}{G_1} + 0.0467\left(\frac{a_2}{G_1}\right)^2 - 0.0232\frac{a_2^3}{G_1} - 0.00284\frac{a_2^3}{G_1^2}\right] \\
l &= 2\left(1 - G_1\right)\left(0.24602 - 0.01496a_2\right)\left[1 + \frac{a_2}{G_1}\left(-0.10495 + 0.3G_1 + 0.02116a_2 - 0.0006216a_2^2\right)\right]
\end{aligned}\right\} \tag{67a}$$

If terms which will ordinarily be negligible are rejected, equations (67a) can be slightly reduced, thus:

$$\left.\begin{aligned}
m &= \frac{2a_2}{G_1} \times 10^{-4}\left[63.90 - 4.496a_2 + \frac{a_2}{G_1}\left(2.705 - 0.2096a_2\right)\right] \\
j &= 0.24602 + a_2\left(0.07917 - 0.005868a_2\right) + \frac{2a_2}{G_1} \times 10^{-4} \times \\
&\quad \left[-284.5 + \left(42.26 - 2.067a_2\right)a_2 - \left(7.906 - 0.9086a_2\right)\frac{a_2}{G_1}\right] \\
l &= 2\left(1 - G_1\right)\left(0.24602 - 0.01496a_2\right)\left\{1 + \frac{a_2}{G_1}\left[-0.10495 + 0.3G_1 + \right.\right. \\
&\quad \left.\left. a_2\left(0.02116 - 0.0006216a_2\right)\right]\right\}
\end{aligned}\right\} \quad (67b)$$

The solution of equation (66) is:

$$b_1 = -\frac{j}{2m} \pm \left[\left(\frac{j}{2m}\right)^2 + \frac{l}{m}\right]^{1/2} \quad (68)$$

(The physically appropriate root will in general be that which is closer to the value $2\left(1 - G_1\right)$.)

Equations (64) and (68) represent a simple approximate solution of equations (11) and (12) for $N_{Pr} = 1$ and uniform wall temperature $\left(G_1 = \text{Constant}\right)$. Their application will involve at most numerical integration.

In applying equation (64), a reasonable "average" constant value for a_2, for any given $\left(u_1/u_\infty\right)(\xi)$, G_1, and M_∞, can usually be obtained by considering equation (58) for $a_2C/G_1\lambda$ and equation (64) for λ/C. An "average" value $\overline{b_1}$ for b_1, to be used in evaluating φ_1 (eq. (63)), can be obtained by considering equation (68) for b_1. For objects with sharp leading edges, for which $\lambda = 0$ at $\xi = 0$, it will ordinarily be found that $b_1 \approx 2(1 - G_1)$ very roughly. After $\lambda(\xi)$ has been found from equation (64), $b_1(\xi)$ can be easily determined from equation (68).[6]

With $\lambda(\xi)$ and $b_1(\xi)$ determined, the boundary-layer characteristics can all be straightforwardly calculated. The local skin friction will be:

$$c_{fl} \equiv \frac{(\mu \partial u/\partial y)_o}{\frac{1}{2}\rho_\infty u_\infty^2} = 4\left[1 - (a_2/5) - (b_1 a_2/60G_1)\right]\left(C/\sqrt{\lambda}\right)\left(T_1/T_\infty\right)\left(u_1/u_\infty\right)R_L^{-1/2} \tag{69}$$

The Nusselt number, giving heat-transfer properties at the wall, will be

$$N_{Nu} \equiv \frac{(k\partial T/\partial y)_o L}{k_\infty\left(T_o - T_e\right)} = \left(C/\sqrt{\lambda}\right)\left[b_1/\left(1 - G_1\right)\right]\left(T_1/T_\infty\right)\sqrt{R_L} \tag{70}$$

The velocity and temperature profiles follow from equations (59) and (60), in conjunction with equation (49) for T/T_1, and equation (19) for transforming to the physical plane.

In accordance with conclusions reached in references 1, 2, and 3, where the methods applied were essentially the same as in the present investigation (viz, sixth-degree profiles, together with a solution of the form of eq. (64)), it may be expected that the results obtained by using the equations developed in this section will be sufficiently accurate for most practical purposes, including stability calculations. However, for flow over a blunt-nosed object (i.e., "stagnation flow") or for determination of the separation point in an adverse pressure gradient, these equations may have to be modified to yield still greater accuracy (cf. ref. 3, where such modifications are shown for zero heat transfer). For stagnation flow $\left(u_1/u_\infty = k\xi\right)$, in fact, fourth-degree profiles give satisfactory

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[6]In equations (67a) and (67b), the exact expression (58) for $a_2(\xi)$ should be used and not any "average" value.

accuracy (ref. 3). Consequently, stagnation flows with heat transfer could be calculated by the method of reference 15, where such profiles are used for this case.

It should, finally, be observed that equations (64) and (68) have been based on the assumption that F_1, F_2, and F_3 are approximately constant, or vary slowly, along the flow. Regarding F_1 and F_3, this can be interpreted physically as an assumption that the ratio of the momentum thickness to the boundary-layer thickness, and the ratio of the thermal momentum thickness to the boundary-layer thickness, will be approximately constant along the flow in the xt-plane. This approximation has been justified by the relatively small variable a_2 terms (replaced here by terms in $\bar{a}_2$) in expressions (61). In ordinary cases, the a_2 terms will actually be found to be sufficiently small, either individually or collectively, so that the approximating assumptions made here will be satisfied. In any doubtful case, however, one can in general easily check, a posteriori, the validity of the approximate solutions developed here by computing the variable a_2 terms with the approximate solution obtained and ascertaining whether these terms have a sufficiently small net effect, relative to the other terms, to be replaced by constant "average" values. Assumptions corresponding essentially to the use of constant "average" values of F_1 and F_2 have already been applied in previous analyses (refs. 3 and 15).

CONCLUSIONS

From the analysis presented herein for the laminar compressible boundary layer with heat transfer in flow with and without an axial pressure gradient, the following conclusions can be drawn:

1. The boundary-layer characteristics for the flow over a flat plate can be easily determined with sufficient accuracy from the equations developed here, for a given constant Prandtl number (of the order of magnitude of unity) and a given wall-temperature distribution.

2. For flow with a pressure gradient, the boundary-layer characteristics can also be easily determined with sufficient accuracy by the equations developed here, provided the Prandtl number is unity and the wall temperature is uniform. Here, the velocity distribution outside the boundary layer and the free-stream Mach number are considered as prescribed.

Some modifications of the equations may, for greater accuracy, be necessary for flow over a blunt-nosed object and for determination of any possible separation point.

Polytechnic Institute of Brooklyn,
Brooklyn, N. Y., December 11, 1952.

APPENDIX

POLYNOMIAL WALL-TEMPERATURE DISTRIBUTION

If the ratio T_o/T_e is represented as a polynomial in ξ, that is if

$$T_o/T_e = \sum_{j=0}^{M} c_j \xi^j \tag{A1}$$

then equation (45a) yields:

$$N_{Nu} \Big/ \sqrt{R_x} = 0.297\left(1 - \sum_{j=0}^{M} c_j \xi^j\right)^{-1}\Bigg((1/\beta_1)(1 - c_o) - \sum_{j=1}^{M} c_j \xi^j (j + \beta_1)^{-1}\Big\{1 + 2j + (60{,}039/152{,}675) j N_{Pr}\left[1 + 13{,}342(j - 1)/20{,}013\right]\Big\}\Bigg) \tag{A2}$$

With the wall temperature specified as in equation (A1), equation (19) together with the use of equations (31), (32), (36), and (49) leads to the following explicit expression for $y(\xi,\tau)$:

$$(y/2x)\sqrt{R_x} = \sqrt{9{,}009\overline{C}/985}\Bigg(\left[\tau - f_1 + (2f_2/\beta_1)\right] + (\gamma - 1)\left(M_1{}^2/2\right)\left(\beta_2 f_2 - f_3 + \beta_3 f_4\right) + \left[1 + (\gamma - 1)\left(M_1{}^2/2\right)\eta\right] \sum_{j=0}^{M} c_j \xi^j \Big\{f_1 + \beta_4 f_5 j - \left[2f_2/(j + \beta_1)\right]\left[1 + \beta_5 j + (40{,}026/152{,}675) j^2\right]\Big\}\Bigg) \tag{A3}$$

where

$$f_1 = \tau\left[1 - 7\tau^4 + 14\tau^5 - 10\tau^6 + (5/2)\tau^7\right]$$

$$f_2 = (4{,}433/9{,}852)\tau^2\left[2 - 16\tau^3 + 30\tau^4 - (144/7)\tau^5 + 5\tau^6\right]$$

$$f_3 = -\left(\tau^3/18\right)\left[-24 + 126\tau^2 - 192\tau^3 + (828/7)\tau^4 - 27\tau^5 - 50\tau^6 + 108\tau^7 - (1{,}008/11)\tau^8 + 36\tau^9 - (72/13)\tau^{10}\right]$$

$$f_4 = \left(\tau^3/3\right)\left[2 - 12\tau^2 + 20\tau^3 - (90/7)\tau^4 + 3\tau^5\right]$$

$$f_5 = \left(\tau^4/20\right)\left[10 - 32\tau + 40\tau^2 - (160/7)\tau^3 + 5\tau^4\right]$$

For convenience, the f functions and their first derivatives with respect to time, denoted by f_i', have been calculated and are plotted in figures 4 and 5.

The temperature profiles in this case can be determined from the explicit expression:

$$T/T_1 = \left[1 - f_1' + \left(2f_2'/\beta_1\right)\right] + (\gamma - 1)\left(M_1^2/2\right)\left(\beta_2 f_2' - f_3 + \beta_3 f_4'\right) + \left[1 + (\gamma - 1)\left(M_1^2/2\right)\eta\right]\sum_{j=0}^{M} c_j\xi^j\left\{f_1' + \beta_4 f_5' j - \left[2f_2'/\left(j + \beta_1\right)\right]\left[1 + \beta_5 j + (40{,}026/152{,}675)j^2\right]\right\} \tag{A4}$$

where

$$\beta_1 = (1/2) + (985/2{,}463)\left(1/N_{Pr}\right)$$

$$\beta_2 = \left[1 - (1{,}208/4{,}433)\left(1 - N_{Pr}\right)\right]\left(2/\beta_1\right)$$

$$\beta_3 = 2\left(1 - N_{Pr}\right)$$

$$\beta_4 = 2N_{Pr}/3F_1 = (6{,}006/985)N_{Pr}$$

$$\beta_5 = 2 + (20{,}013/152{,}675)N_{Pr}$$

The velocity profiles may be found from equations (31) and (A3).

It may be noted that, if desired, the ratio T_o/T_1 may be prescribed as a polynomial in ξ, thus:

$$T_o/T_1 = \sum_{j=0}^{M} e_j \xi^j \tag{A5}$$

As can be seen by comparison of equations (A1) and (A5) in connection with equations (46) and (48), the results given by equations (A3) and (A4) can still be used, provided that one substitutes

$$c_j = e_j\left[1 + (\gamma - 1)\left(M_1{}^2/2\right)\eta\right]^{-1} \tag{A6}$$

there.

REFERENCES

1. Libby, Paul A., Morduchow, Morris, and Bloom, Martin: Critical Study of Integral Methods in Compressible Laminar Boundary Layers. NACA TN 2655, 1952.

2. Bloom, Martin: The Calculation of Boundary Layer Stability on a Flat Surface With Heat Transfer Using an Integral Solution for the Mean Flow. PIBAL No. 179, Contract NAw-5809, NACA and Polytechnic Inst. of Brooklyn, June 1951.

3. Morduchow, Morris, and Clarke, Joseph H.: Method for Calculation of Compressible Laminar Boundary-Layer Characteristics in Axial Pressure Gradient With Zero Heat Transfer. NACA TN 2784, 1952.

4. Weil, Herschel: Effects of Pressure Gradient on Stability and Skin Friction in Laminar Boundary Layers in Compressible Fluids. Jour. Aero. Sci., vol. 18, no. 5, May 1951, pp. 311-318.

5. Chapman, D. R., and Rubesin, M. W.: Temperature and Velocity Profiles in the Compressible Laminar Boundary Layer With Arbitrary Distribution of Surface Temperature. Jour. Aero. Sci., vol. 16, no. 9, Sept. 1949, pp. 547-565.

6. Low, George M.: Simplified Method for Calculation of Compressible Laminar Boundary Layer With Arbitrary Free-Stream Pressure Gradient. NACA TN 2531, 1951.

7. Von Kármán, Th., and Tsien, H. S.: Boundary Layer in Compressible Fluids. Jour. Aero. Sci., vol. 5, no. 6, Apr. 1938, pp. 227-232.

8. Van Driest, E. R.: Investigation of Laminar Boundary Layer in Compressible Fluids Using the Crocco Method. NACA TN 2597, 1952.

9. Klunker, E. B., and McLean, F. Edward: Laminar Friction and Heat Transfer at Mach Numbers From 1 to 10. NACA TN 2499, 1951.

10. Young, George B. W., and Janssen, Earl: The Compressible Boundary Layer. Jour. Aero. Sci., vol. 19, no. 4, Apr. 1952, pp. 229-236, 288.

11. Moore, L. L.: A Solution of the Laminar Boundary-Layer Equations for a Compressible Fluid With Variable Properties, Including Dissociation. Jour. Aero. Sci., vol. 19, no. 8, Aug. 1952, pp. 505-518.

12. Boelter, L. M. K., Grossman, L. M., Martinelli, R. C., and Morrin, E. H.: An Investigation of Aircraft Heaters. XXIX - Comparison of Several Methods of Calculating Heat Losses From Airfoils. NACA TN 1453, 1948.

13. Kalikhman, L. E.: Heat Transmission in the Boundary Layer. NACA TM 1229, 1949.

14. Ferrari, Carlo: Velocity and Temperature Distribution Through the Laminar Boundary Layer in Supersonic Flow. Jour. Aero. Sci., vol. 19, no. 1, Jan. 1952, pp. 39-47, 65.

15. Morduchow, Morris: On Heat Transfer Over a Sweat-Cooled Surface in Laminar Compressible Flow With a Pressure Gradient. Jour. Aero. Sci., vol. 19, no. 10, Oct. 1952, pp. 705-712.

16. Fluid Motion Panel of the Aeronautical Research Committee and Others (S. Goldstein, ed.): Modern Developments in Fluid Dynamics. Vol. II. The Clarendon Press (Oxford), 1938, p. 627.

17. Lees, Lester, and Probstein, Ronald F.: Hypersonic Viscous Flow Over a Flat Plate. Aero. Eng. Lab. Rep. No. 195, Contract AF-33(038)-250, WADC and Princeton Univ., Apr. 20, 1952.

18. Loeb, Leonard B.: The Kinetic Theory of Gases. McGraw-Hill Book Co., Inc., 1927, p. 217.

19. Chapman, Dean R.: Laminar Mixing of a Compressible Fluid. NACA Rep. 958, 1950. (Supersedes NACA TN 1800.)

20. Levy, Solomon: Heat Transfer to Constant-Property Laminar Boundary-Layer Flows With Power-Function Free-Stream Velocity and Wall-Temperature Variation. Jour. Aero. Sci., vol. 19, no. 5, May 1952, pp. 341-348.

21. Eckert, E. R. G.: Developments in Convective Heat-Transfer Research. Proc. Second Midwestern Conf. Fluid Mechanics (Ohio State Univ., Mar. 17-19, 1952), Bull. No. 149, Eng. Exp. Station, Ohio State Univ., Sept. 1952, pp. 309-318.

22. Libby, Paul A.: The Heat Insulating Properties of the Laminar Compressible Boundary Layer. Presented at Eighth Int. Cong. Theor. and Appl. Mech. (Istanbul, Turkey), Aug. 1952.

23. Rott, Nicholas, and Crabtree, L. F.: Simplified Laminar Boundary-Layer Calculations for Bodies of Revolution and for Yawed Wings. Jour. Aero. Sci., vol. 19, no. 8, Aug. 1952, pp. 553-565.

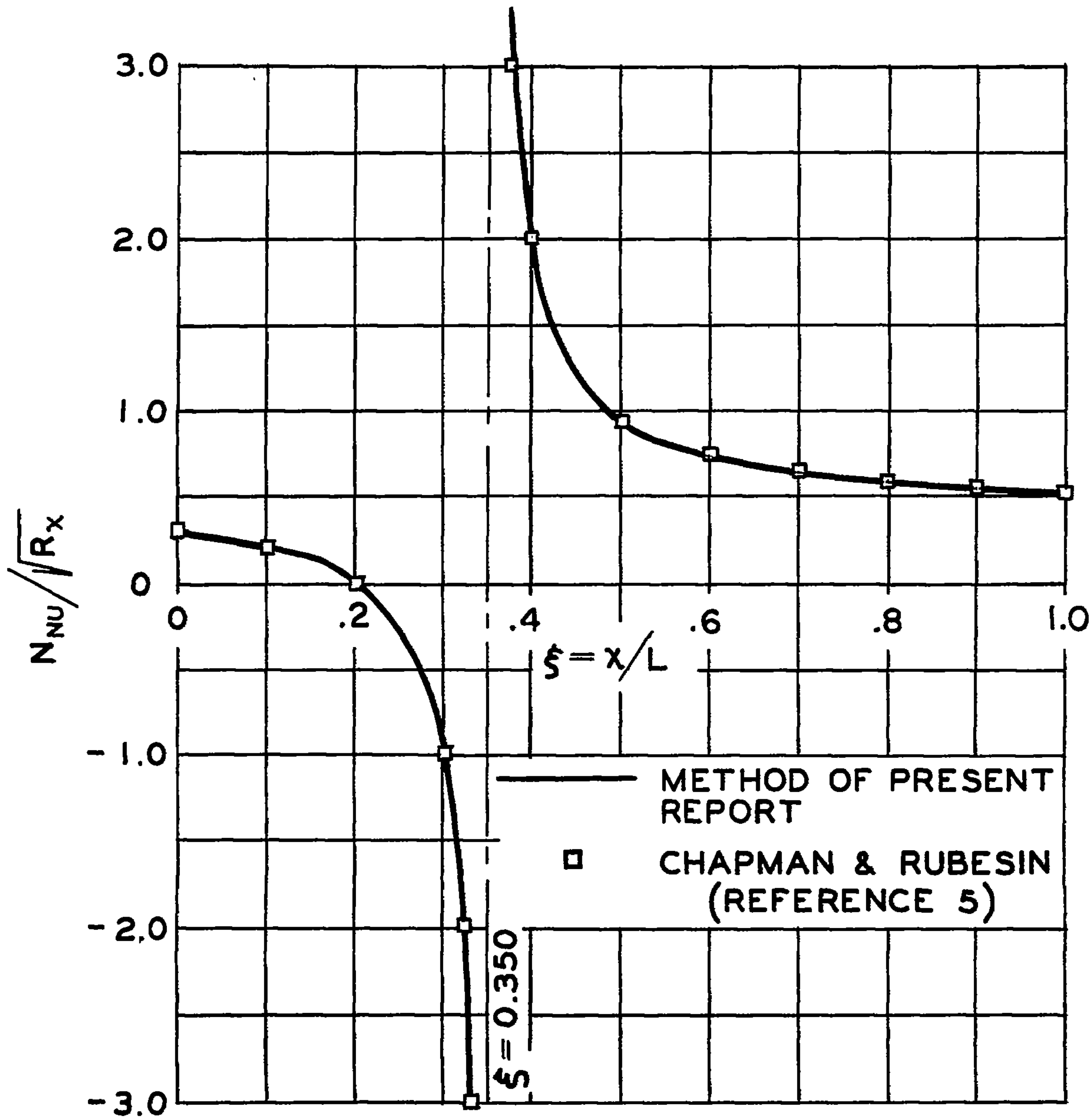


Figure 1.- Nusselt number distribution $N_{Nu}/\sqrt{R_x}$ versus ξ (eq. (A2)). Flat plate; variable wall temperature; $N_{Pr} = 0.72$.

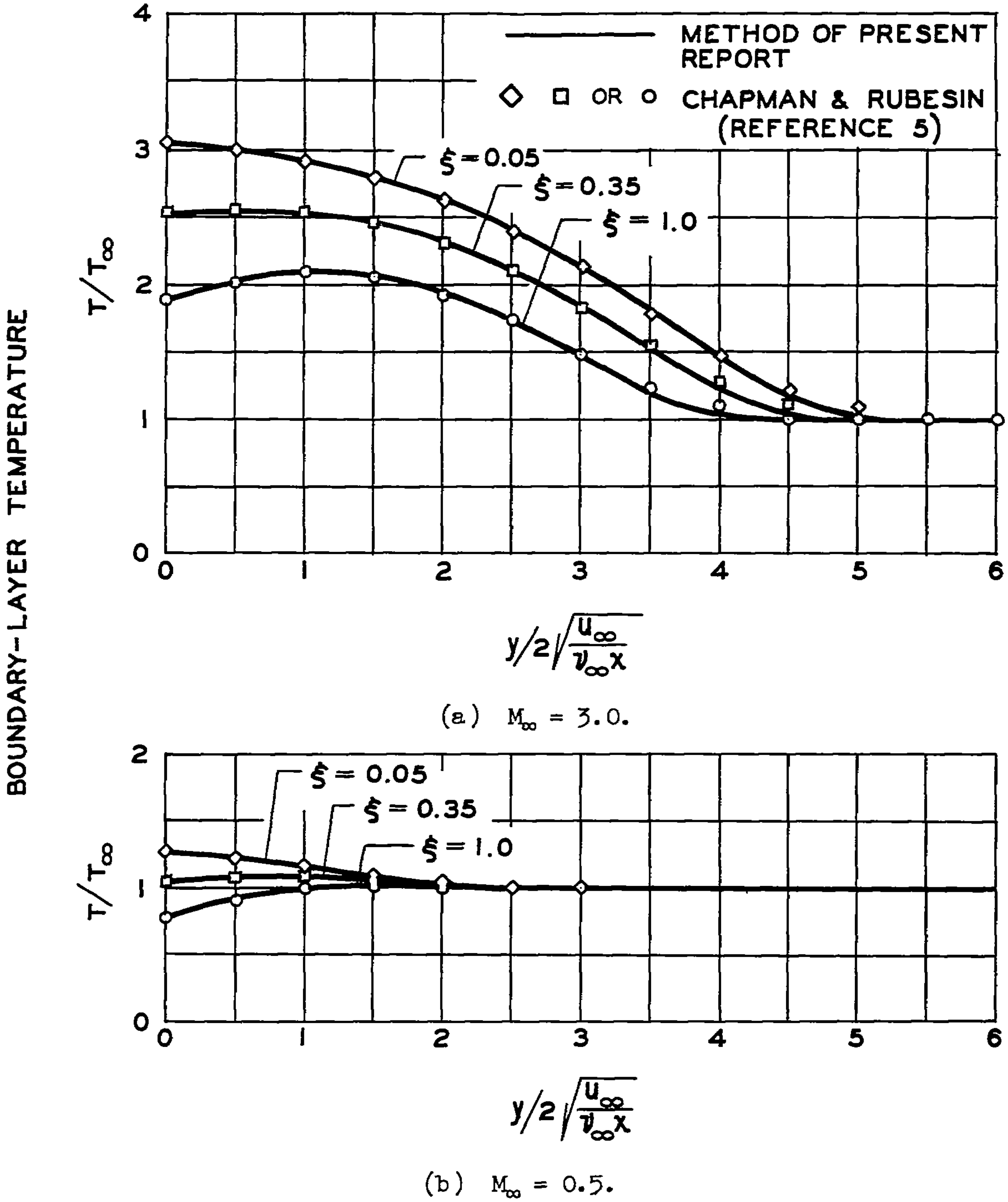


Figure 2.- Comparison of temperature profiles in boundary layer. Variable wall temperature; $N_{Pr} = 0.72$.

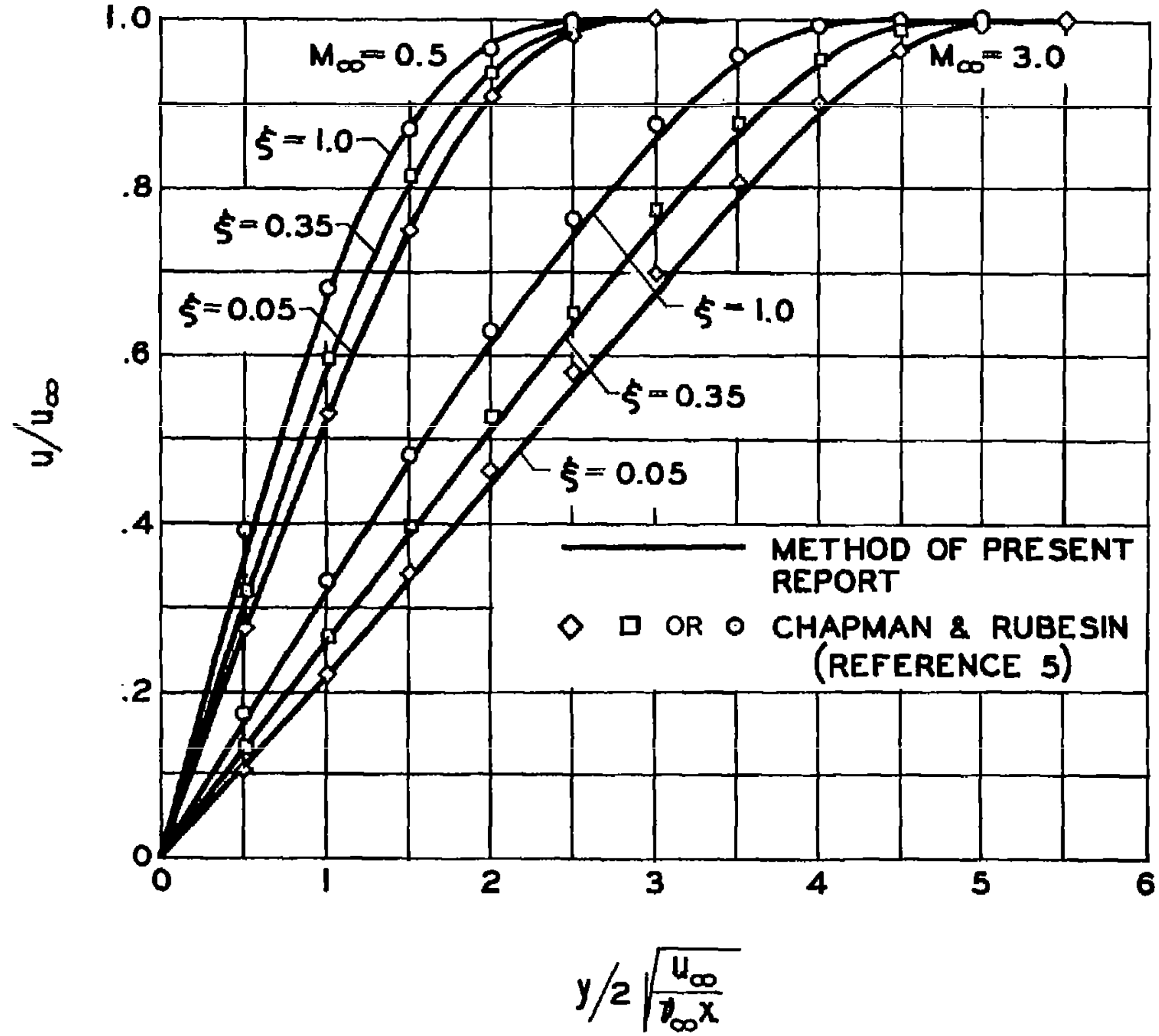
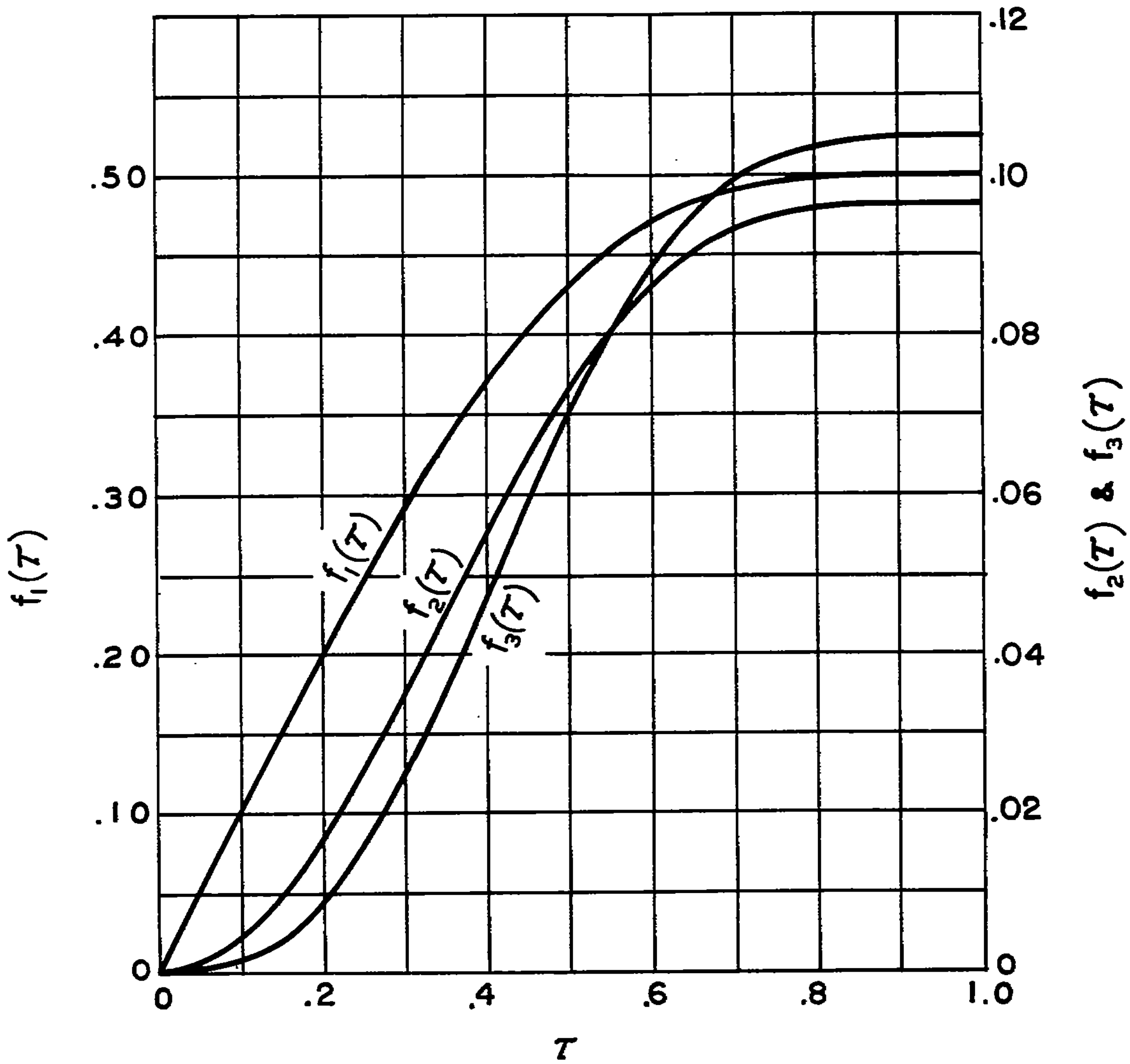
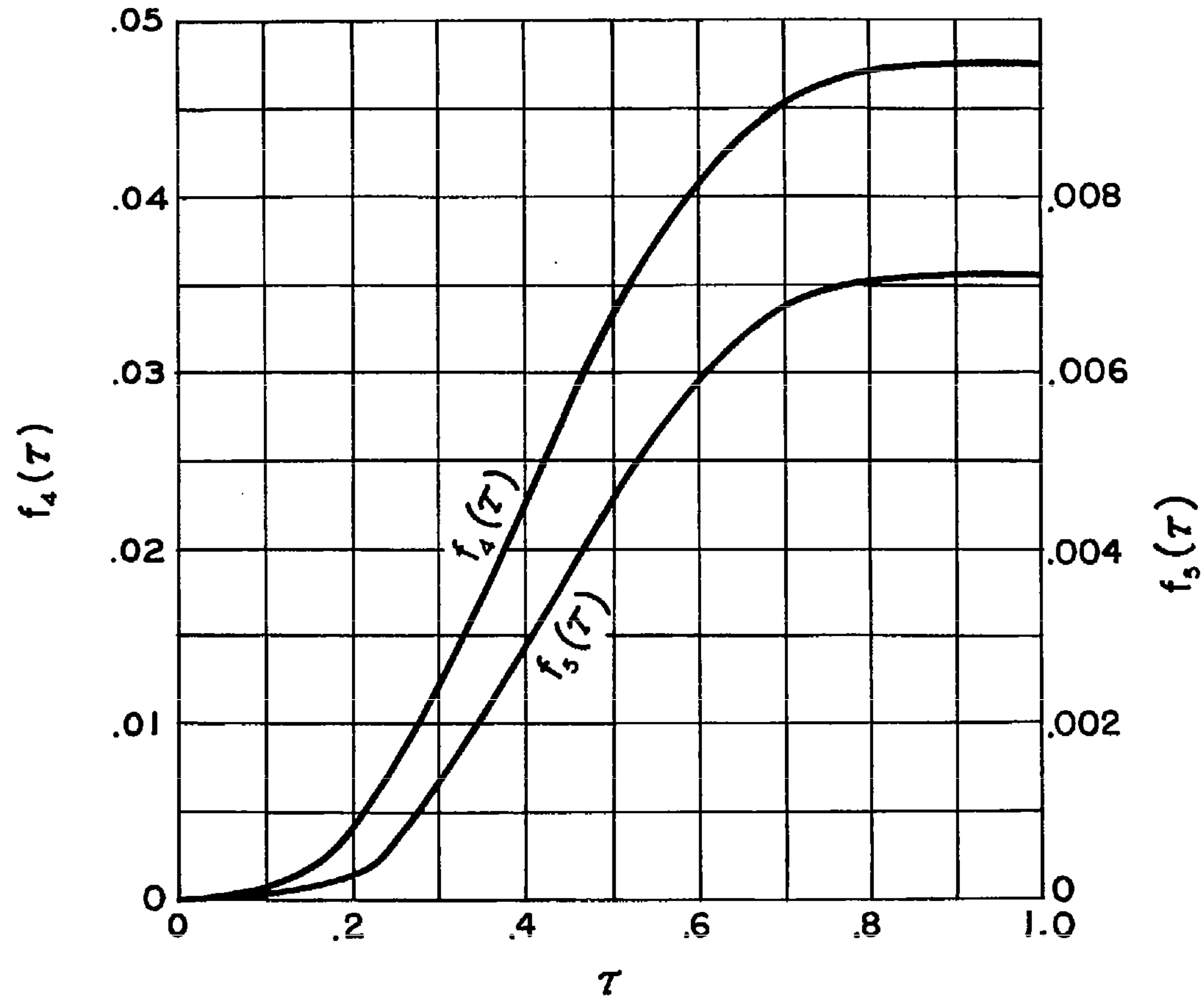


Figure 3.- Comparison of velocity profiles in boundary layer. Variable wall temperature; $N_{Pr} = 0.72$.



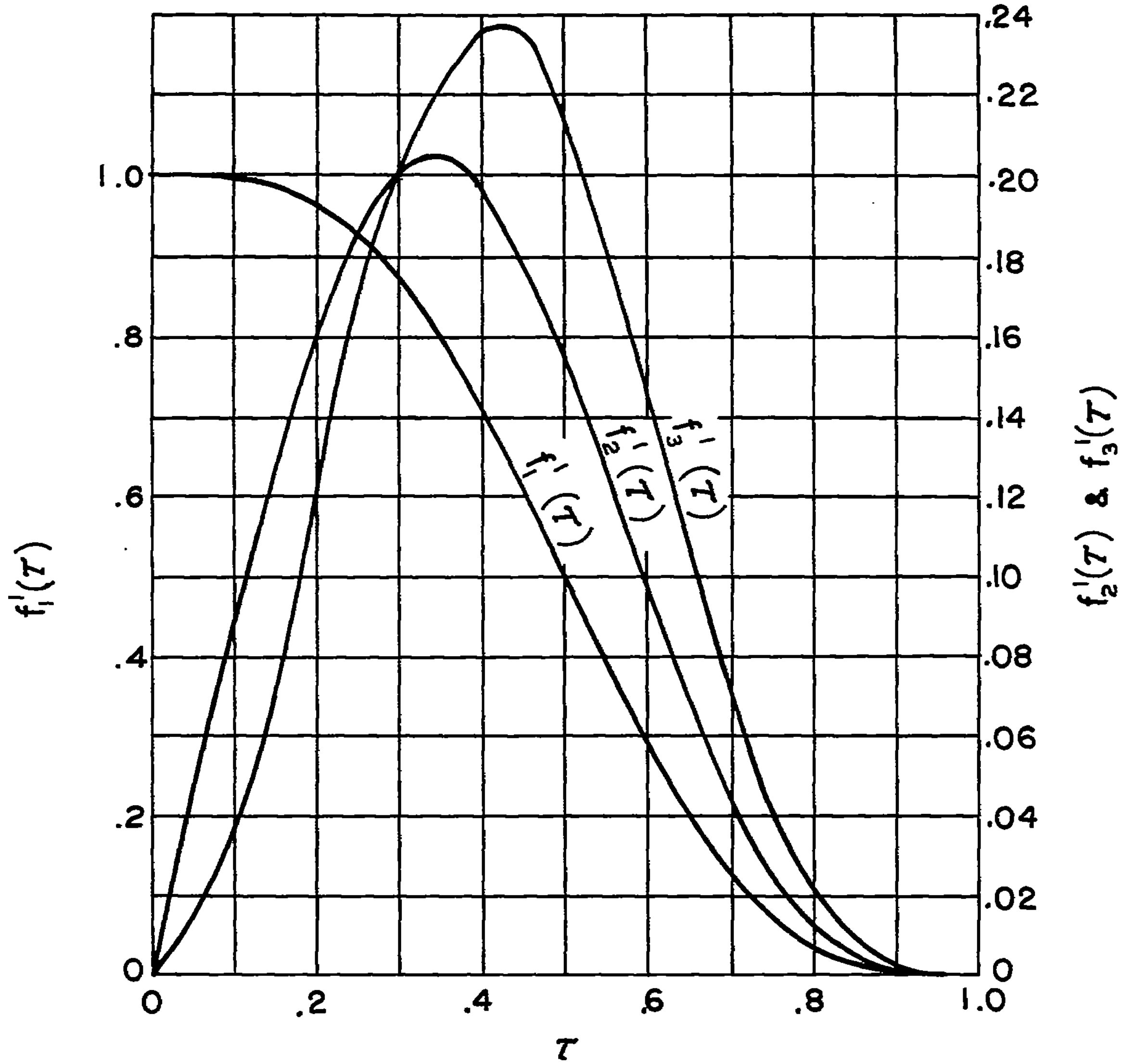
(a) $f_1(\tau)$, $f_2(\tau)$, and $f_3(\tau)$.

Figure 4.- Variation of f functions with τ.



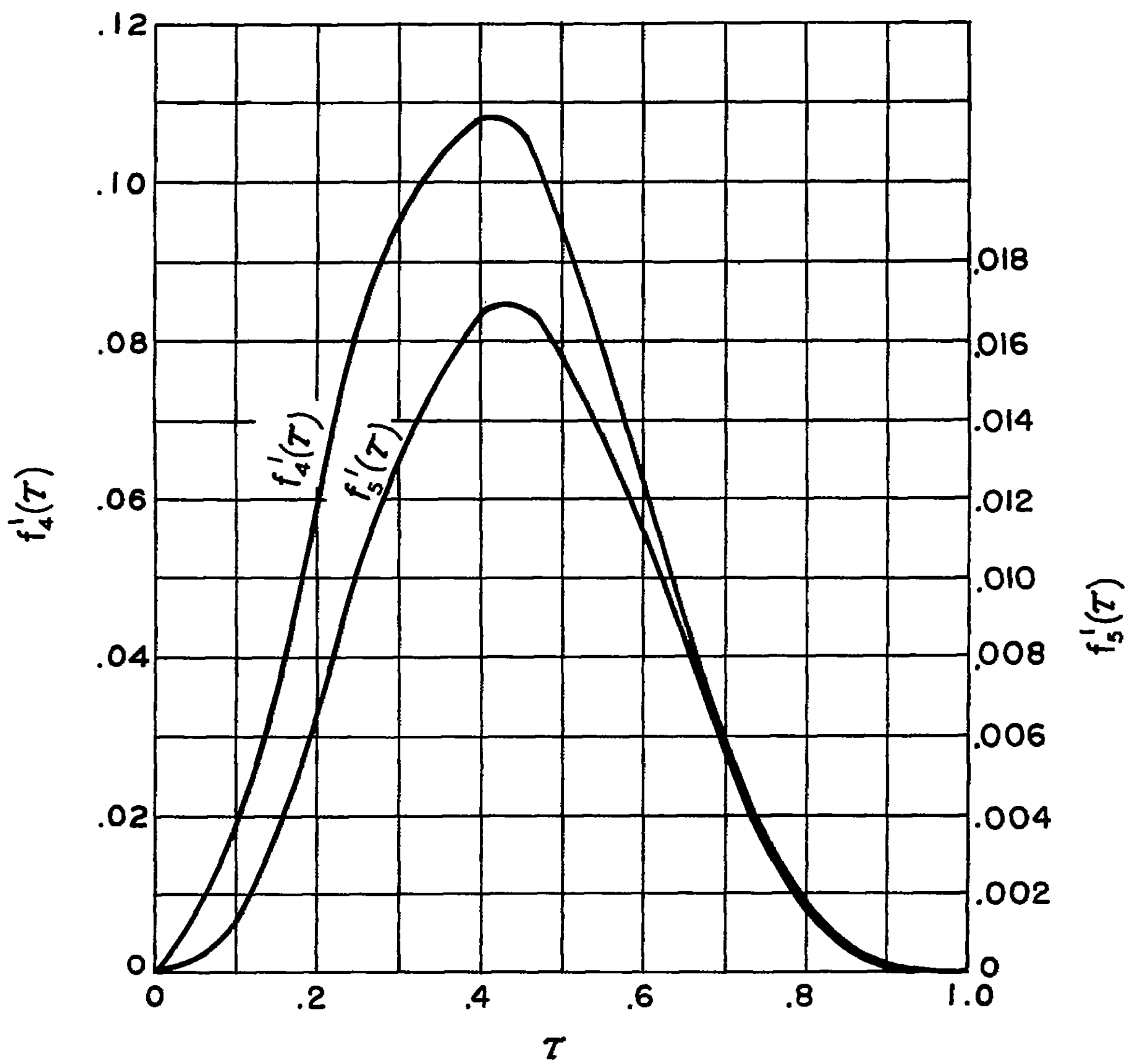
(b) $f_4(\tau)$ and $f_5(\tau)$.

Figure 4.- Concluded.



(a) $f_1'(\tau)$, $f_2'(\tau)$, and $f_3'(\tau)$.

Figure 5.- Variation of first derivatives of f functions with τ.



(b) $f_4'(\tau)$ and $f_5'(\tau)$.

Figure 5.- Concluded.