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CREEP BENDING AND BUCKLING OF NONLINEARLY VISCOELASTIC COLUMNS

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SUMMARY

Differential equations of bending of an idealized H-section beam column were derived for a nonlinearly viscoelastic material whose mechanical properties are analogous to a model consisting of a linear spring in series with a nonlinear dashpot whose strain rate is proportional to a power of the applied stress. The resulting constant stress or load creep curve consists of a straight line, the slope of which can be considered as the secondary creep rate of a real material.

The equations derived were used to obtain the creep-bending deflections of a beam in pure bending and of a column with initial sinusoidal deviation from straightness. The results of the analysis of the simple beam showed that the deflections vary linearly with time. The analysis of the deflections of the column, accomplished with the assumption that the original shape of the structure was maintained at all times, showed the existence of a finite critical time at which the deflections become indefinitely large. The critical time decreases rapidly with increasing axial compression and column inaccuracy.

INTRODUCTION

The need for methods predicting the behavior of structural components at high temperatures is becoming increasingly urgent, particularly in the fields of aircraft structures and propulsion. During the past 40 years considerable attention has been paid to the fundamental constant-stress and constant-load tensile creep behavior of materials. However, with few exceptions, it is only recently that results of the investigation of the creep behavior of beams and columns have been presented (see, e.g., refs. 1 to 9). It is the purpose of the present report to apply to the problem of the creep behavior of beams and columns a stress-strain-time relation which can be considered as a generalization of the relation obtained between the strain rate and stress for a Maxwell linearly viscoelastic model consisting of a spring connected in series with a dashpot (see fig. 1 and refs. 6 and 8 to 10). While the original Maxwell model consists of a spring and dashpot, in each of which the strain is linearly related to the stress, the model considered herein consists of a linear spring coupled with a nonlinear dashpot whose strain rate is proportional to a power of the applied stress. The corresponding stress-strain-time relation can be adapted to the creep characteristics of structural materials, such as aluminum or steel, by assuming that the usual creep curve indicated by the dashed curve in figure 2(a) can be replaced by the solid straight line also shown in the figure (see refs. 1 to 3). Hence, the idealized creep curve takes into account the actual secondary stage of creep, approximates the initial elastic or elastoplastic stage and the primary creep stage, and ignores the final stage. It is intended that the effective spring modulus, which defines the strain intercept of the idealized creep curve, together with parameters defining the action of the nonlinear dashpot, be determined from experiments at constant temperatures.

In order to simplify the calculations involved in the analysis of the creep behavior of beams and columns, the present investigation has been confined to the study of the behavior at constant temperature of an idealized H-beam, the cross section of which consists of two equal concentrated areas connected by a thin web of negligible bending resistance (fig. 3). Two differential equations for the determination of the deflections of a beam under combined axial and lateral loads were derived, one equation being applicable when the stresses in both flanges are in compression, the other when one flange is in compression and the other in tension. These equations were used to determine the deflection-time relations for a beam in pure bending and for a simply supported column whose axis before loading had the form of a half-sine wave.

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SYMBOLS

A total flange area of idealized H-section

E_1 effective elastic modulus

E_{1c}, E_{1t} effective moduli for compression and tension, respectively

$F = w/h$

$F_c = w_c/h$

$F_i = w_i/h$

$F_o = w_o/h$

F_{T_o}	total nondimensional deviation of axis of loaded column from x-axis at $t = 0$
f_c, f_i	amplitudes of F_c and F_i, respectively

$f_T = f_c + f_{T_o}$

$f_{T_o} = f_i/\left[1 - (\bar{\sigma}/\sigma_E)\right]$

h	distance between flanges of idealized H-section
I	moment of inertia of idealized H-section, $Ah^2/4$
L	length of beam
M	bending moment
M_o	constant bending moment
m	exponent in viscosity term
m_c, m_t	exponents for compression and tension, respectively
P	axially compressive load
t	time
t_{cr}	critical time
w	deflection due to loads
w_c	time-dependent deflection, accrued for $t > 0$
w_i	initial deviation from straightness
w_o	time-independent deflection due to loads at $t = 0$
w_{T_o}	total deviation of axis of loaded column from x-axis at $t = 0$
x	axial coordinate of beam

$z = f_c + f_{T_o}$

ϵ strain

ϵ_c, ϵ_t strains on concave and convex sides of beam, respectively (positive in compression)

ϵ_o strain at $t = 0$, σ_o/E_1

λ viscosity coefficient

λ_c, λ_t viscosity coefficients for compression and tension, respectively

ρ radius of curvature

σ stress

σ_c, σ_t stresses on concave and convex sides of beam, respectively

$\sigma_E = \pi^2 E_1 I/AL^2$

σ_o constant stress

$\bar{\sigma}$ average compressive stress, P/A

$\tau = \left\{E_1(2\bar{\sigma})^m/\left[4\lambda(\sigma_E - \bar{\sigma})\right]\right\} t$

τ_1 time parameter corresponding to $z = 1/2$

τ_{cr} critical value of time parameter

$(\dot{\ }) \equiv \partial(\ \ )/\partial t$

$(\ \ )_x \equiv \partial(\ \ )/\partial x$

DERIVATION OF DIFFERENTIAL EQUATIONS OF BENDING OF AN IDEALIZED H-SECTION

Stress-Strain-Time Law

The relation between the strain rate and the stress corresponding to the model previously described is

$$\dot{\epsilon} = \left(\dot{\sigma}/E_1\right) + \left(\sigma^m/\lambda\right) \tag{1}$$

in which ϵ and σ, respectively, are the strain and stress, E_1 is the effective modulus, m and λ are parameters defining the viscous behavior of the material, and the dot over a symbol indicates differentiation with respect to time t. The three parameters E_1, λ, and m are considered as constants for a given temperature and can be determined experimentally from conventional constant stress or load uniaxial tensile or compressive creep tests performed at constant temperatures. The corresponding relationship between stress, strain, and time for such tests can be found from equation (1) if the strain ϵ_o at $t = 0$ is taken as σ_o/E_1 (see fig. 2(a)). Thus

$$\epsilon/\epsilon_o = \left(E_1/\lambda\right)\sigma_o^{m-1}t + 1 \tag{2}$$

This equation is shown plotted in figure 2(b).

Adaptation of Stress-Strain-Time Law to Bending and Buckling Problems

When a beam whose cross section can be represented by the idealized H-section, shown in figure 3, is bent under the action of combined lateral and axial loads, the stresses in each of the flanges may be compressive, or the stress in one flange may be compressive and in the other tensile. Thus, in applying equation (1) to the bending problem, the following forms of this equation have been used:

$$\dot{\epsilon} = \left(\dot{\sigma}/E_{1c}\right) + \left(\sigma^{m_c}/\lambda_c\right) \tag{3}$$

for a flange under the action of compressive stress, and hence $\dot{\epsilon}$ and σ are positive in compression. The subscript c denotes parameters determined from conventional compressive creep tests. Similarly,

$$\dot{\epsilon} = -\left(\dot{\sigma}/E_{1t}\right) - \left(\sigma^{m_t}/\lambda_t\right) \tag{4}$$

for a flange under the action of tensile stresses, and hence $\dot{\epsilon}$ is positive in compression and σ is positive in tension. The subscript t refers to parameters determined from tensile creep tests.

Equation (3) is valid for $\sigma \geqq 0$ and positive in compression, whereas equation (4) applies when $\sigma \geqq 0$ and positive in tension.

Stress-Load and Strain-Load Relations

If σ_c and σ_t, respectively, are the stresses on the concave and convex side of an idealized H-beam loaded in the plane containing the web (see fig. 3) and if a moment M and an axially compressive load P are applied as indicated in figures 3 and 4, then

$$\left.\begin{aligned} \sigma_c &= (P/A) + (2M/Ah) \\ \sigma_t &= (P/A) - (2M/Ah) \end{aligned}\right\} \qquad (5)$$

in which σ_c and σ_t are considered positive in compression, and

$$\left.\begin{aligned} \sigma_c &= (2M/Ah) + (P/A) \\ \sigma_t &= (2M/Ah) - (P/A) \end{aligned}\right\} \qquad (6)$$

in which σ_c is positive in compression, while σ_t is positive in tension. In equations (5) and (6) $A/2$ is the area of each flange and h is the distance between flanges.

When both flanges are in compression, the strain in each flange can be related to the applied loads with the aid of equations (3) and (5). Thus

$$\left.\begin{aligned} \dot{\epsilon}_c &= (1/E_{1c})(\partial/\partial t)\Big[(P/A) + (2M/Ah)\Big] + (1/\lambda_c)\Big[(P/A) + (2M/Ah)\Big]^{m_c} \\ \dot{\epsilon}_t &= (1/E_{1c})(\partial/\partial t)\Big[(P/A) - (2M/Ah)\Big] + (1/\lambda_c)\Big[(P/A) - (2M/Ah)\Big]^{m_c} \end{aligned}\right\} \qquad (7)$$

in which $\dot{\epsilon}_c$ and $\dot{\epsilon}_t$, respectively, are the strain rates on the concave and the convex sides of the beam, positive in compression.

Similarly, when one flange is in compression and the other in tension

$$\left.\begin{aligned} \dot{\epsilon}_c &= (1/E_{1c})(\partial/\partial t)\left[(2M/Ah) + (P/A)\right] + (1/\lambda_c)\left[(2M/Ah) + (P/A)\right]^{m_c} \\ \dot{\epsilon}_t &= -(1/E_{1t})(\partial/\partial t)\left[(2M/Ah) - (P/A)\right] - (1/\lambda_t)\left[(2M/Ah) - (P/A)\right]^{m_t} \end{aligned}\right\} \tag{8}$$

Thus equations (7) are applicable when $P/A \geqq 2M/Ah$ and equations (8), when $P/A \leqq 2M/Ah$.

Differential Equations of Bending of a Beam Column

Since the strains ϵ_c and ϵ_t, considered positive in compression, are related to the radius of curvature ρ of the centroidal axis of the beam by the equation

$$(\epsilon_c - \epsilon_t)/h = 1/\rho \tag{9}$$

the differential equations of bending are readily obtained. Thus, if $P/A \geqq 2M/Ah$, that is, if both flanges are in compression, equations (7) and (9), together with the condition that for small deflections ρ is related to the deflection w by the relation

$$1/\rho = -w_{xx} \tag{9a}$$

where x is the axial coordinate of a point on the beam axis and a subscript x indicates differentiation with respect to x, yield

$$-h(\partial/\partial t)(w_{xx}) = (4/E_{1c}Ah)(\partial M/\partial t) + (1/\lambda_c)\left\{\left[(P/A) + (2M/Ah)\right]^{m_c} - \left[(P/A) - (2M/Ah)\right]^{m_c}\right\} \tag{10}$$

If $P/A \leqq 2M/Ah$, corresponding to one flange in compression and the other in tension, then from equations (8) and (9)

$$-h(\partial/\partial t)(w_{xx}) = (1/E_{1c})(\partial/\partial t)\left[(2M/Ah) + (P/A)\right] + (1/E_{1t})(\partial/\partial t)\left[(2M/Ah) - (P/A)\right] + (1/\lambda_c)\left[(2M/Ah) + (P/A)\right]^{m_c} + (1/\lambda_t)\left[(2M/Ah) - (P/A)\right]^{m_t} \tag{11}$$

Furthermore, if the parameters have the same values in tension and compression, the subscript c can be dropped in equation (10), and equation (11) becomes

$$-h(\partial/\partial t)(w_{xx}) = (4/E_1 Ah)(\partial M/\partial t) + (1/\lambda)\left\{\left[(2M/Ah) + (P/A)\right]^m + \left[(2M/Ah) - (P/A)\right]^m\right\} \tag{11a}$$

Thus, equations (10) and (11) are the differential equations applicable to the analysis of the creep-deflection behavior of an idealized H-section beam column, the material of which behaves according to equations (3) and (4).

APPLICATIONS OF DIFFERENTIAL EQUATIONS OF BENDING

Beam Under Pure Bending

Equation (11) can be readily applied to the problem of the determination of the creep deflections of an idealized H-section beam under the action of constant end couples M_o. Under such conditions this equation reduces to

$$-h\dot{w}_{xx} = (1/\lambda_c)(2M_o/Ah)^{m_c} + (1/\lambda_t)(2M_o/Ah)^{m_t} \tag{12}$$

The related boundary and initial conditions for a beam of length L with the origin of the axial coordinate x at one end of the beam are that, at $x = 0$ and L, $\dot{w} = 0$ and, at $t = 0$, w is

$$w_o = \left[(1/E_{1c}) + (1/E_{1t})\right](M_o L^2/4I)(x/L)\left[1 - (x/L)\right] \tag{13}$$

which is obtained from a simple "elastic" analysis. The moment of inertia I of the cross section of the beam with respect to the center line normal to the web (see fig. 3) is equal to $Ah^2/4$.

Hence, the solution of equation (12) can be expressed as

$$w - w_o = \left[(1/\lambda_c)(2M_o/Ah)^{m_c} + (1/\lambda_t)(2M_o/Ah)^{m_t}\right](L^2/2h)(x/L)\left[1 - (x/L)\right]t \tag{14}$$

If the material properties of the beam are the same for compression and tension, equation (14) can be expressed as

$$w/w_o = (E_1/\lambda)(2M_o/Ah)^{m-1}t + 1 \tag{14a}$$

or

$$w/w_o = (E_1/\lambda)\sigma_o^{m-1}t + 1 \tag{14b}$$

in which σ_o is the absolute value of the constant stress $2M_o/Ah$ acting in each flange. It may be noted that equation (14b) can also be obtained from solution of equations (2), (9), and (9a). Because the flange stresses are constant and equal in magnitude, the ratio of the deflection at any time t and the deflection at $t = 0$ is the same as the ratio of the strain at any time t and the strain at $t = 0$ of a bar under the action of constant tensile or compressive stresses (see fig. 2(b) and compare eqs. (2) and (14b)).

Column With Initial Curvature

The deflection-time characteristics of an H-section column, whose centroidal axis initially deviates from a straight line and whose defining material parameters are the same for tension and compression, can be investigated with the aid of equations (10) and (11a). For a simply supported column with a constant axially compressive end load P, the moment M is simply

$$M = P(w + w_i) \tag{15}$$

in which w is the deflection due to loads and w_i represents the initial deviation from straightness (see fig. 5). Thus, for $P/A \geqq 2M/Ah$ and hence, from equation (15), for $w + w_i \leqq h/2$, equation (10) becomes

$$\dot{F}_{xx} + (\bar{\sigma}A/E_1 I)\dot{F} + \left[(2\bar{\sigma})^m/\lambda h^2\right]\left\{\left[(1/2) + F + F_i\right]^m - \left[(1/2) - F - F_i\right]^m\right\} = 0 \tag{16}$$

in which

$$P/A = \bar{\sigma}$$

$$w/h = F$$

$$w_i/h = F_i$$

Equation (16) is applicable to the present problem so long as both flanges are in compression.

Similarly, for $w + w_i \geqq h/2$, that is, for the flange on the concave side in compression and that on the convex side in tension, equation (11a) yields

$$\dot{F}_{xx} + (\bar{\sigma}A/E_1 I)\dot{F} + \left[(2\bar{\sigma})^m/\lambda h^2\right]\left\{\left[F + F_i + (1/2)\right]^m + \left[F + F_i - (1/2)\right]^m\right\} = 0 \tag{17}$$

It may be noted that equations (16) and (17) are identical if m is an odd integer.

Since the deflection due to loads consists of the time-independent deflection w_o, obtained at $t = 0$, and the time-dependent deflection w_c, accrued for $t > 0$, F can be expressed as

$$F = F_c + F_o \tag{18}$$

Hence,

$$F + F_i = F_c + F_{T_o} \tag{19}$$

in which F_{T_o} represents the total deviation of the axis of the loaded column from the x-axis at $t = 0$ and $F_c = w_c/h$. Hence, equations (16) and (17) become for $F_c + F_{T_o} \leqq 1/2$

$$\dot{F}_{c_{xx}} + (\bar{\sigma}A/E_1 I)\dot{F}_c + \left[(2\bar{\sigma})^m/\lambda h^2\right]\left\{\left[(1/2) + F_c + F_{T_o}\right]^m - \left[(1/2) - F_c - F_{T_o}\right]^m\right\} = 0 \tag{20}$$

and for $F_c + F_{T_o} \geq 1/2$

$$\dot{F}_{c_{xx}} + (\bar{\sigma}A/E_1 I)\dot{F}_c + \left[(2\bar{\sigma})^m/\lambda h^2\right]\left\{\left[F_c + F_{T_o} + (1/2)\right]^m + \left[F_c + F_{T_o} - (1/2)\right]^m\right\} = 0 \qquad (21)$$

If m is unity, then (since eq. (1) reduces to the stress-strain-time law for a Maxwell element) equations (20) and (21) are applicable to the analysis of the behavior of the corresponding linearly viscoelastic column (see refs. 6, 8, and 9). If the ratio of the initial deviations from straightness and the distance between flanges is taken as

$$F_i = f_i \sin(\pi x/L) \qquad (22)$$

in which L is the length of the beam and the coordinate x is measured from a support (see fig. 5), the deflections of the column can be expressed as

$$f_c/f_{T_o} = e^{2\tau} - 1 \qquad (23)$$

where f_c is the amplitude of the deflection F_c accrued for $t > 0$, f_{T_o} is the amplitude of the total deviation from the x-axis of the loaded column at $t = 0$, and

$$\tau = \left\{E_1\bar{\sigma}\Big/\left[2\lambda(\sigma_E - \bar{\sigma})\right]\right\}t$$

$$\sigma_E = \pi^2 E_1 I/AL^2$$

The amplitude of the total nondimensional midspan deflection measured from the x-axis is

$$f_T = f_c + f_{T_o}$$

Equation (23) shows that for $\bar{\sigma} < \sigma_E$ the creep deflections of the linearly viscoelastic column become infinitely large only for correspondingly large values of time (refs. 8 and 9).

For values of m other than unity, the collocation method can be applied to the solution of equations (20) and (21). This method will be used to investigate the midspan deflection of the column under the assumption that the shape of the original deviation from straightness F_i is maintained during the bending process. Hence, if F_i is as defined in equation (22), then F_c is assumed to be given by the equation

$$F_c = f_c \sin (\pi x/L) \tag{24}$$

Corresponding to equation (22), the total deviation of the axis of the loaded column from the x-axis at $t = 0$ is obtained from an "elastic" analysis as

$$F_{T_o} = f_{T_o} \sin (\pi x/L) \tag{25}$$

in which

$$f_{T_o} = f_i \Big/ \left[1 - (\bar{\sigma}/\sigma_E)\right]$$

Substitution of F_c and F_{T_o}, respectively, from equations (24) and (25) into equations (20) and (21) and evaluation of the resulting expressions at $x = L/2$ yield the following equations for the nondimensional midspan deflection f_c: For $f_c + f_{T_o} \leq 1/2$

$$df_c/d\tau - \left\{\left[(1/2) + f_c + f_{T_o}\right]^m - \left[(1/2) - f_c - f_{T_o}\right]^m\right\} = 0 \tag{26}$$

and for $f_c + f_{T_o} \geq 1/2$

$$df_c/d\tau - \left\{\left[f_c + f_{T_o} + (1/2)\right]^m + \left[f_c + f_{T_o} - (1/2)\right]^m\right\} = 0 \tag{27}$$

in which the nondimensional time variable τ is defined as

$$\tau = \left\{E_1(2\bar{\sigma})^m \Big/ \left[4\lambda\left(\sigma_E - \bar{\sigma}\right)\right]\right\}t \tag{28}$$

Equation (26) is applicable if both flanges of the column are in compression, whereas equation (27) applies when one of the flanges is in tension. The solution of equation (26) can be written as

$$\tau = \int_{f_{T_o}}^{z} dz \Big/ \left\{ \left[(1/2) + z\right]^m - \left[(1/2) - z\right]^m \right\} \tag{29}$$

in which $z = f_c + f_{T_o}$, and hence $f_{T_o} \leqq z \leqq 1/2$. Thus equation (29) is valid provided $0 \leqq \tau \leqq \tau_1$, where τ_1 corresponds to $z = 1/2$ and hence to that time at which one flange is completely unstressed. If $f_{T_o} \leqq 1/2$, then equation (29) must be applied until $\tau = \tau_1$, whereupon the solution to equation (27) becomes applicable. This solution is

$$\tau = \tau_1 + \int_{1/2}^{z} dz \Big/ \left\{ \left[z + (1/2)\right]^m + \left[z - (1/2)\right]^m \right\} \tag{30}$$

for $1/2 \leqq z \leqq \infty$ and $\tau_1 \leqq \tau \leqq \tau_{cr}$, where τ_{cr} is defined as the critical time and corresponds to infinite deflections. Thus, for $f_{T_o} \leqq 1/2$,

$$\tau_{cr} = \tau_1 + \int_{1/2}^{\infty} dz \Big/ \left\{ \left[z + (1/2)\right]^m + \left[z - (1/2)\right]^m \right\} \tag{31}$$

and from equation (29)

$$\tau_1 = \int_{f_{T_o}}^{1/2} dz \Big/ \left\{ \left[(1/2) + z\right]^m - \left[(1/2) - z\right]^m \right\} \tag{32}$$

If $f_{T_o} \geqq 1/2$, only one flange is in compression for all values of τ, and hence equation (27) applies throughout the duration of loading. Therefore, for $f_{T_o} \geqq 1/2$

$$\tau = \int_{f_{T_o}}^{z} dz \Big/ \left\{ \left[z + (1/2)\right]^m + \left[z - (1/2)\right]^m \right\} \tag{33}$$

and $0 \leqq \tau \leqq \tau_{cr}$, where

$$\tau_{cr} = \int_{f_{T_o}}^{\infty} dz \Big/ \left\{ \left[z + (1/2)\right]^m + \left[z - (1/2)\right]^m \right\} \tag{34}$$

The integrals in equations (29) to (34) can be evaluated readily for integral values of m. If m is an even integer (or fractional) all six equations are required for a complete range of f_{T_o} and τ. If m is an odd integer, since equations (26) and (27) are then identical, the evaluation of the six equations reduces to the evaluation of equations (33) and (34) for all values of f_{T_o} and τ. Thus τ_1 no longer is of any significance. A summary of equations (29) to (34) is given in table 1 and the corresponding results of integration for $m = 1, 2, 3, 4,$ and 5 are given in table 2. The integrations indicated in equations (29) to (34) are performed in reference 11 in terms of m, where m is any integer. It may be noted from comparison of equation (23) with the results in table 2 that, for $m = 1$, the collocation method yields the exact results of equation (23).

The expressions in table 2 were used to obtain the deflection-time curves shown in figures 6(a) to 6(d). These curves relate the nondimensional time-dependent midspan deflection f_c (accrued for $\tau > 0$) to the time variable τ and the initial total deviation from the x-axis of the midspan of the loaded column f_{T_o}. In figure 6(a) the curve designated $m = 2$, $0 \leqq \tau \leqq \tau_1$, or $m = 1$, $0 \leqq \tau \leqq \infty$, is valid for the case in which $m = 2$ and both flanges are in compression, as well as for the entire time range for $m = 1$ (see table 2). In table 3 the time to failure, characterized by τ_{cr}, and the time to zero stress in the "tension" flange, represented by τ_1, are listed for the m and f_{T_o} values considered in figures 6(a) to 6(d).

The curves in figures 6(a) to 6(d) show that for the same value of the exponent $m > 1$ the creep deflections become large much more rapidly for large values of f_{T_o} than for small values of this parameter. The result is, of course, that the critical time parameter τ_{cr} decreases as f_{T_o} increases (see table 3). These results are as expected, since large values of f_{T_o} indicate either large initial deviations of the column from straightness or large values of the average axial compressive stress $\overline{\sigma} < \sigma_E$. From table 3, as well as from a comparison of the locations of the asymptotes shown in figures 6(a) to 6(d), it is seen that for large values of f_{T_0} the critical time parameter τ_{cr} decreases with increasing m, whereas for small values of f_{T_0} this parameter increases with m. In this respect it may be noted, however, that since τ_{cr} is proportional to $\bar{\sigma}^m$ it is not a true measure of the actual critical time t_{cr} (eq. (28)). Thus, it is probable that for real materials the actual critical time t_{cr} decreases monotonically with increasing m, since t is inversely proportional to $\bar{\sigma}^m$.

DISCUSSION

Differential equations of bending of an idealized H-section beam column were derived for a nonlinearly viscoelastic material whose mechanical behavior is analogous to a model consisting of a spring connected in series with a dashpot. The spring was assumed to be elastic with a modulus corresponding to the strain intercept of the projection of the secondary region of a conventional tensile or compressive creep curve, whereas the dashpot was considered as nonlinear with a strain rate proportional to a power of the stress. Thus, the conventional creep curve is represented by a straight line, the slope of which is the secondary creep rate of the actual creep curve. The general equations, one which applies when both flanges are in compression and the other when one flange is in compression and the other in tension, are applicable to materials whose mechanical properties for tension differ from those for compression.

The results of the analysis of the creep deflections of a beam under pure bending showed that the deflections vary linearly with time. This is to be expected, since the stress in each flange remains constant with time, and hence each flange behaves in a manner analogous to a bar under constant stress. If the material properties of the beam are the same for tension and compression, the ratio of the deflection at any time t to the deflection at $t = 0$ is identical to the ratio of the strain at the same time t to the strain at $t = 0$ of a bar under the action of a constant stress whose magnitude is that of the stress in either flange of the beam (see eqs. (2) and (14b) and fig. 2(b)).

The nonlinear differential equations for the bending of a simply supported H-section column with initial sinusoidal deviation from straightness were solved with the aid of the assumption that the shape of the loaded column remained sinusoidal. The material properties of the beam were assumed to be the same in tension and compression. In contrast with the results presented in references 8 and 9 for linearly viscoelastic columns, the nonlinearly viscoelastic columns investigated in the present paper possessed "critical times." Thus for stresses less than the Euler stress the deflections of an initially crooked column become infinitely large in a finite time. In figures 6(a) to 6(d) curves are presented showing the effect of the initial deviation from straightness of the loaded column on the time behavior of the additional deflections obtained with time. As expected, for a given exponent in the power law for viscosity the creep deflections increased more rapidly with time for the larger values of amplitude of the initial deviation from straightness than for the smaller values. Thus, the critical time decreases very rapidly with increasing initial deviation from straightness and end load.

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TABLE 1.- INTEGRALS REQUIRED FOR DETERMINATION OF τ AS A FUNCTION OF z

(a) m even integer (or fractional)

f_{T_o}	z	τ	τ_1	τ_{cr}
$0 < f_{T_o} \leq \frac{1}{2}$	$f_{T_o} \leq z \leq \frac{1}{2}$	[a] $\int_{f_{T_o}}^{z} \frac{dz}{\left(\frac{1}{2} + z\right)^m - \left(\frac{1}{2} - z\right)^m}$	[b] $\int_{f_{T_o}}^{1/2} \frac{dz}{\left(\frac{1}{2} + z\right)^m - \left(\frac{1}{2} - z\right)^m}$	
	$\frac{1}{2} \leq z \leq \infty$	[c] $\tau_1 + \int_{1/2}^{z} \frac{dz}{\left(z + \frac{1}{2}\right)^m + \left(z - \frac{1}{2}\right)^m}$		[d] $\tau_1 + \int_{1/2}^{\infty} \frac{dz}{\left(z + \frac{1}{2}\right)^m + \left(z - \frac{1}{2}\right)^m}$
$\frac{1}{2} \leq f_{T_o} < \infty$	$f_{T_o} \leq z \leq \infty$	[e] $\int_{f_{T_o}}^{z} \frac{dz}{\left(z + \frac{1}{2}\right)^m + \left(z - \frac{1}{2}\right)^m}$		[f] $\int_{f_{T_o}}^{\infty} \frac{dz}{\left(z + \frac{1}{2}\right)^m + \left(z - \frac{1}{2}\right)^m}$

(b) m odd integer

f_{T_o}	z	τ	τ_{cr}
$0 < f_{T_o} < \infty$	$f_{T_o} \leq z \leq \infty$	[e] $\int_{f_{T_o}}^{z} \frac{dz}{\left(z + \frac{1}{2}\right)^m + \left(z - \frac{1}{2}\right)^m}$	[f] $\int_{f_{T_o}}^{\infty} \frac{dz}{\left(z + \frac{1}{2}\right)^m + \left(z - \frac{1}{2}\right)^m}$

[a]Eq. (29).
[b]Eq. (32).
[c]Eq. (30).
[d]Eq. (31).
[e]Eq. (33).
[f]Eq. (34).

TABLE 2.- SOLUTION OF CREEP-BUCKLING EQUATIONS FOR $m = 1, 2, 3, 4$, AND 5

(a) m even integer

f_{T_0}	z	Solution	τ_1	τ_{cr}
		$m = 2,\ t/\tau = \lambda(\sigma_E - \bar{\sigma})/(E_1\bar{\sigma}^2)$		
$0 < f_{T_0} \leqq \frac{1}{2}$	$f_{T_0} \leqq z \leqq \frac{1}{2}$	$z = f_{T_0}e^{2\tau}$	$\frac{1}{2}\log\frac{1}{2f_{T_0}}$	
	$\frac{1}{2} \leqq z \leqq \infty$	$z = \frac{1}{2}\tan\left(\tau - \tau_1 + \frac{\pi}{4}\right)$		$\tau_1 + \frac{\pi}{4}$
$\frac{1}{2} \leqq f_{T_0} < \infty$	$\frac{1}{2} \leqq z \leqq \infty$	$z = \frac{1}{2}\tan\left[\tau + \tan^{-1}(2f_{T_0})\right]$		$\frac{\pi}{2} - \tan^{-1}(2f_{T_0})$
		$m = 4,\ t/\tau = \lambda(\sigma_E - \bar{\sigma})/(4E_1\bar{\sigma}^4)$		
$0 < f_{T_0} \leqq \frac{1}{2}$	$f_{T_0} \leqq z \leqq \frac{1}{2}$	$z = \frac{f_{T_0}e^{2\tau}}{4f_{T_0}^2(1 - e^{2\tau}) + 1}$	$\frac{1}{2}\log\left(\frac{1}{2} + \frac{1}{8f_{T_0}^2}\right)$	
	$\frac{1}{2} \leqq z \leqq \infty$	$\tau = \tau_1 + \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{3 - 2\sqrt{2}}}\left[\tan^{-1}\frac{2z}{\sqrt{3 - 2\sqrt{2}}} - \tan^{-1}\frac{1}{\sqrt{3 - 2\sqrt{2}}}\right] - \frac{1}{\sqrt{3 + 2\sqrt{2}}}\left[\tan^{-1}\frac{2z}{\sqrt{3 + 2\sqrt{2}}} - \tan^{-1}\frac{1}{\sqrt{3 + 2\sqrt{2}}}\right]\right\}$		$\tau_1 + \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{3 - 2\sqrt{2}}}\tan^{-1}\sqrt{3 - 2\sqrt{2}} - \frac{1}{\sqrt{3 + 2\sqrt{2}}}\tan^{-1}\sqrt{3 + 2\sqrt{2}}\right]$
$\frac{1}{2} \leqq f_{T_0} < \infty$	$\frac{1}{2} \leqq z \leqq \infty$	$\tau = \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{3 - 2\sqrt{2}}}\left[\tan^{-1}\frac{2z}{\sqrt{3 - 2\sqrt{2}}} - \tan^{-1}\frac{2f_{T_0}}{\sqrt{3 - 2\sqrt{2}}}\right] - \frac{1}{\sqrt{3 + 2\sqrt{2}}}\left[\tan^{-1}\frac{2z}{\sqrt{3 + 2\sqrt{2}}} - \tan^{-1}\frac{2f_{T_0}}{\sqrt{3 + 2\sqrt{2}}}\right]\right\}$		$\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{3 - 2\sqrt{2}}}\tan^{-1}\frac{\sqrt{3 - 2\sqrt{2}}}{2f_{T_0}} - \frac{1}{\sqrt{3 + 2\sqrt{2}}}\tan^{-1}\frac{\sqrt{3 + 2\sqrt{2}}}{2f_{T_0}}\right]$

TABLE 2.- SOLUTION OF CREEP-BUCKLING EQUATIONS FOR $m = 1, 2, 3, 4$, AND 5 - Concluded

(b) m odd integer $\left(0 < f_{T_0} < \infty,\ f_{T_0} \leqq z \leqq \infty\right)$

Solution	τ_{cr}
$m = 1,\ t/\tau = 2\lambda(\sigma_E - \bar{\sigma})/(E_1\bar{\sigma})$	
$z = f_{T_0} e^{2\tau}$	∞
$m = 3,\ t/\tau = \lambda(\sigma_E - \bar{\sigma})/(2E_1\bar{\sigma}^3)$	
$z = f_{T_0}\left[\dfrac{3e^{3\tau}}{4f_{T_0}^{\ 2}(1 - e^{3\tau}) + 3}\right]^{1/2}$	$\dfrac{1}{3}\log\left(1 + \dfrac{3}{4f_{T_0}^{\ 2}}\right)$
$m = 5,\ t/\tau = \lambda(\sigma_E - \bar{\sigma})/(8E_1\bar{\sigma}^5)$	
$\tau = \dfrac{8}{5}\log\dfrac{z}{f_{T_0}} - \left(\dfrac{2}{5} + \dfrac{1}{\sqrt{5}}\right)\log\dfrac{4z^2 + 5 - 2\sqrt{5}}{4f_{T_0}^{\ 2} + 5 - 2\sqrt{5}} - \left(\dfrac{2}{5} - \dfrac{1}{\sqrt{5}}\right)\log\dfrac{4z^2 + 5 + 2\sqrt{5}}{4f_{T_0}^{\ 2} + 5 + 2\sqrt{5}}$	$\left(\dfrac{2}{5} + \dfrac{1}{\sqrt{5}}\right)\log\left(4f_{T_0}^{\ 2} + 5 - 2\sqrt{5}\right) + \left(\dfrac{2}{5} - \dfrac{1}{\sqrt{5}}\right)\log\left(4f_{T_0}^{\ 2} + 5 + 2\sqrt{5}\right) - \dfrac{8}{5}\log\left(2f_{T_0}\right)$

TABLE 3.- VALUES OF τ_1 AND τ_{cr} CORRESPONDING TO VALUES OF m AND f_{T_o} OF FIGURES 6(a) TO 6(d)

m = 2			m = 3
f_{T_o}	τ_1	τ_{cr}	τ_{cr}
0.01	1.956	2.741	2.974
.05	1.152	1.937	1.902
.10	.8045	1.590	1.444
.20	.4582	1.244	.9943
.50	0	.7854	.4621
.80	------	.5586	.2586
1.00	------	.4637	.1865
2.00	------	.2450	.05728
m = 4			m = 5
f_{T_o}	τ_1	τ_{cr}	τ_{cr}
0.01	3.565	3.891	5.612
.10	1.282	1.608	1.985
1.00	-----	.09112	.04767

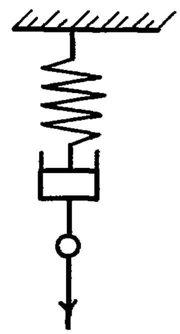
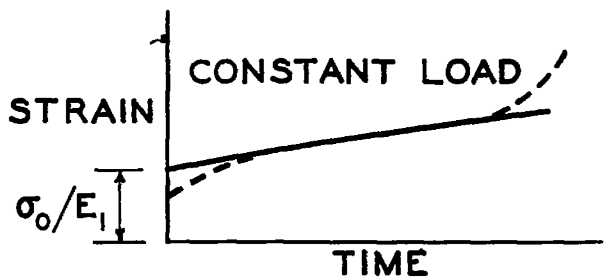
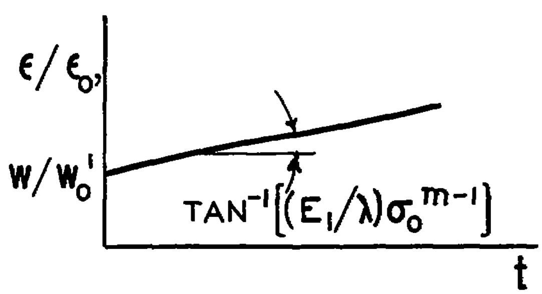


Figure 1.- Maxwell model.



(a) Experimental.



(b) Idealized.

Figure 2.- Experimental and idealized creep curves.

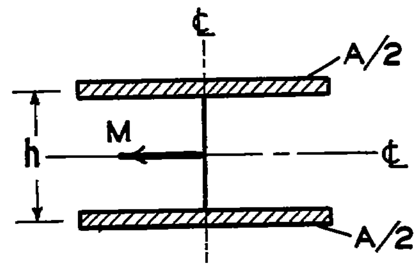


Figure 3.- Cross section of idealized H-section beam.

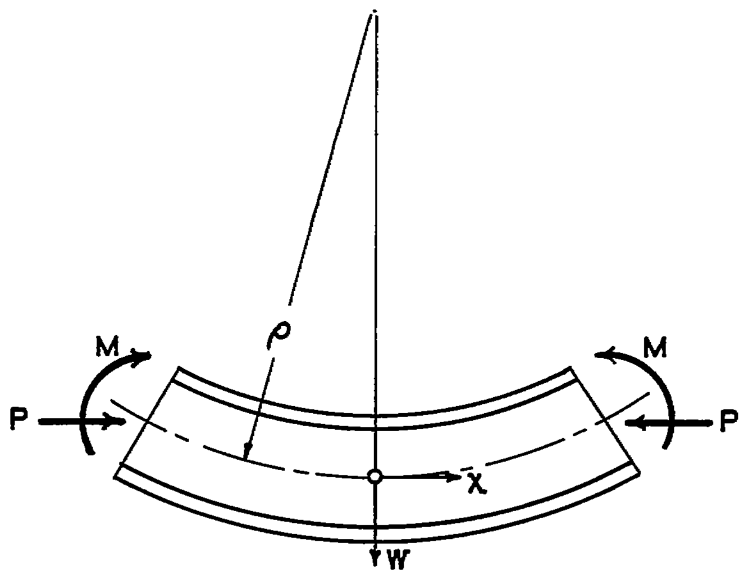


Figure 4.- Portion of bent beam column.

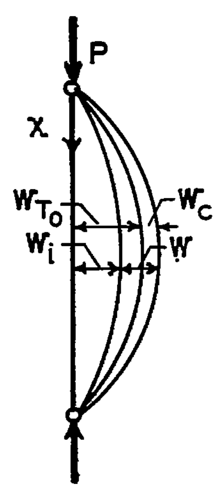
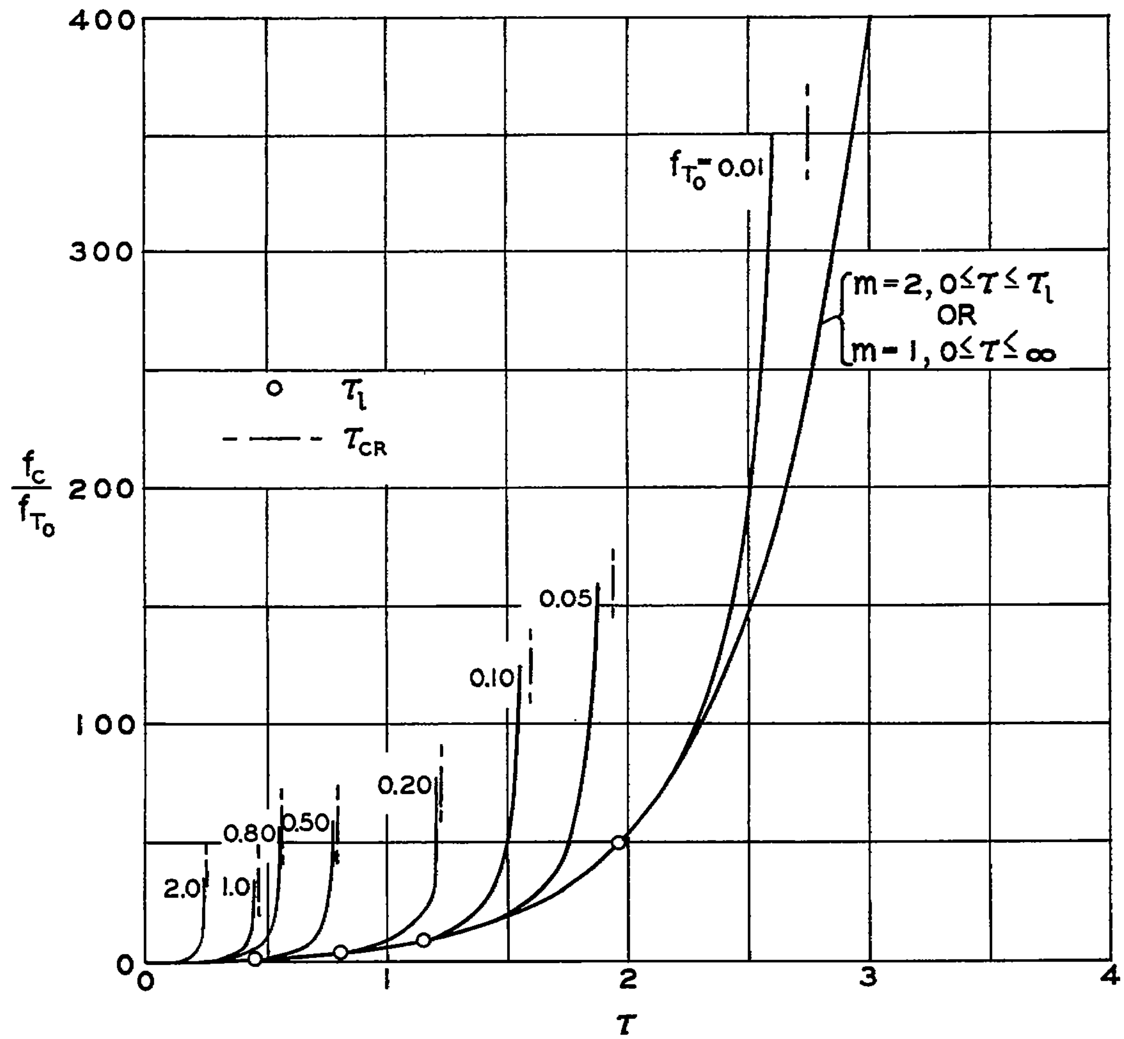
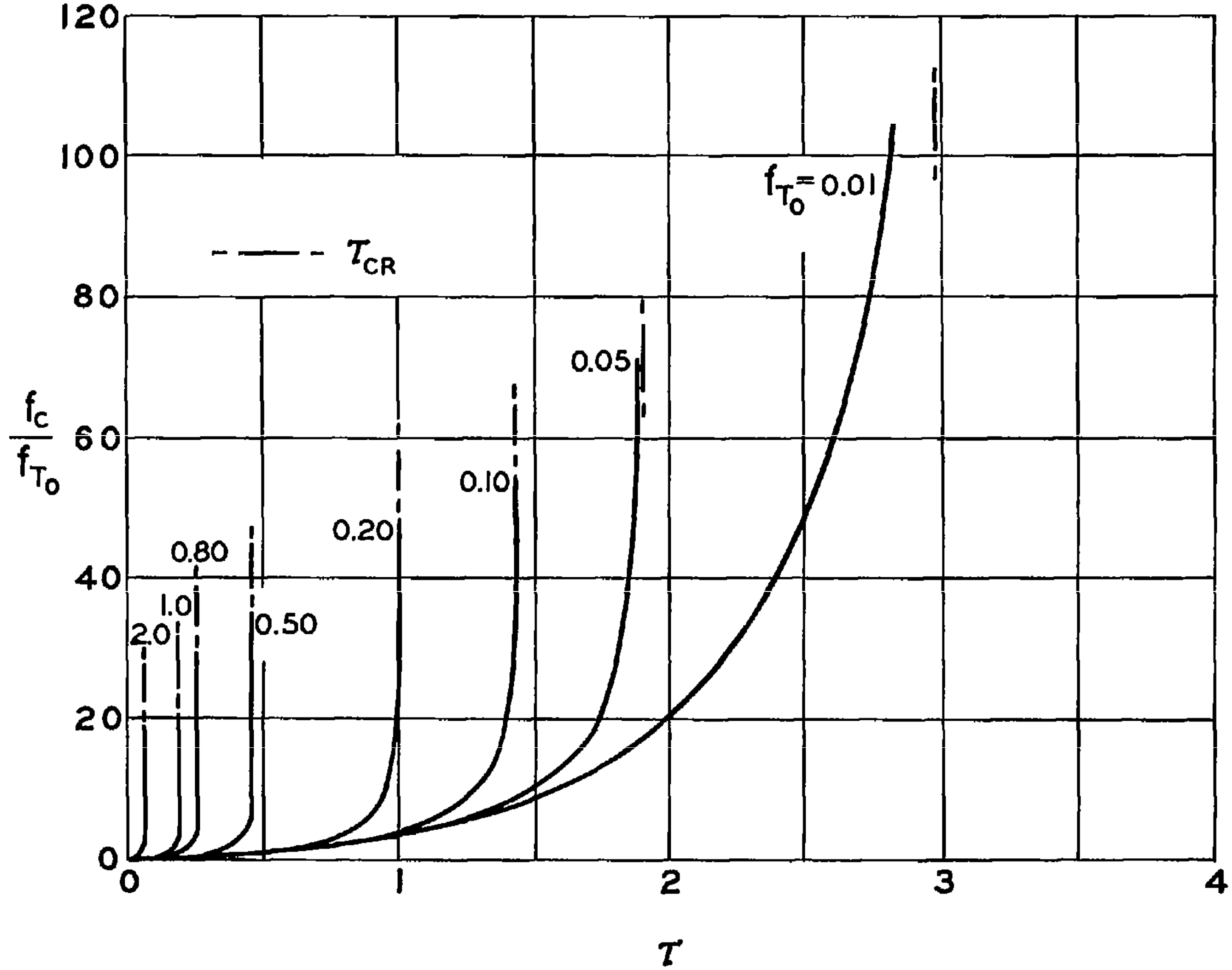


Figure 5.- Deflections of simply supported column.



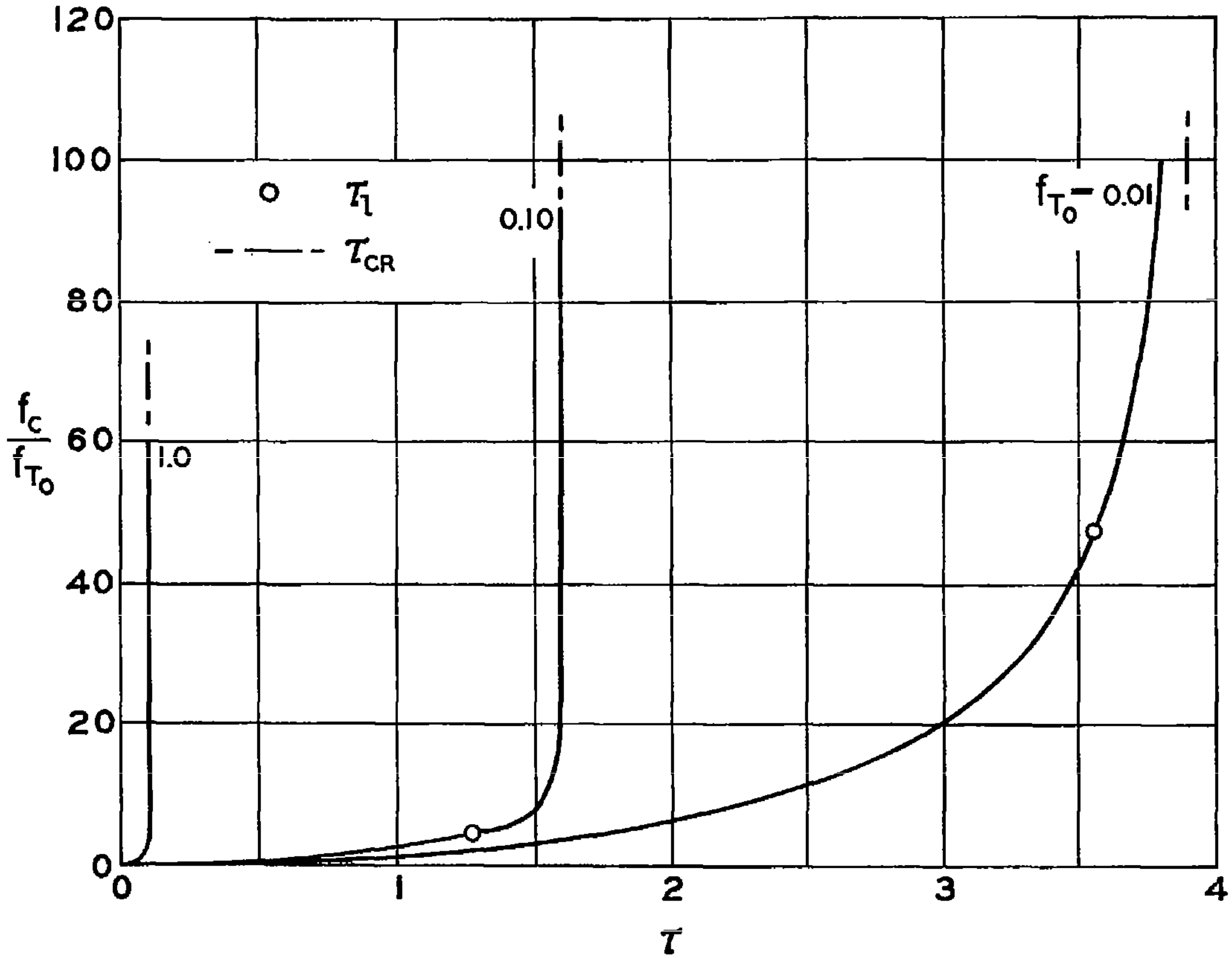
(a) m = 2.

Figure 6.- Variation of deflections accrued for $t > 0$ with time parameter τ for $m = 2, 3, 4$, and 5 and several values of initial-deflection parameter f_{τ_0}.



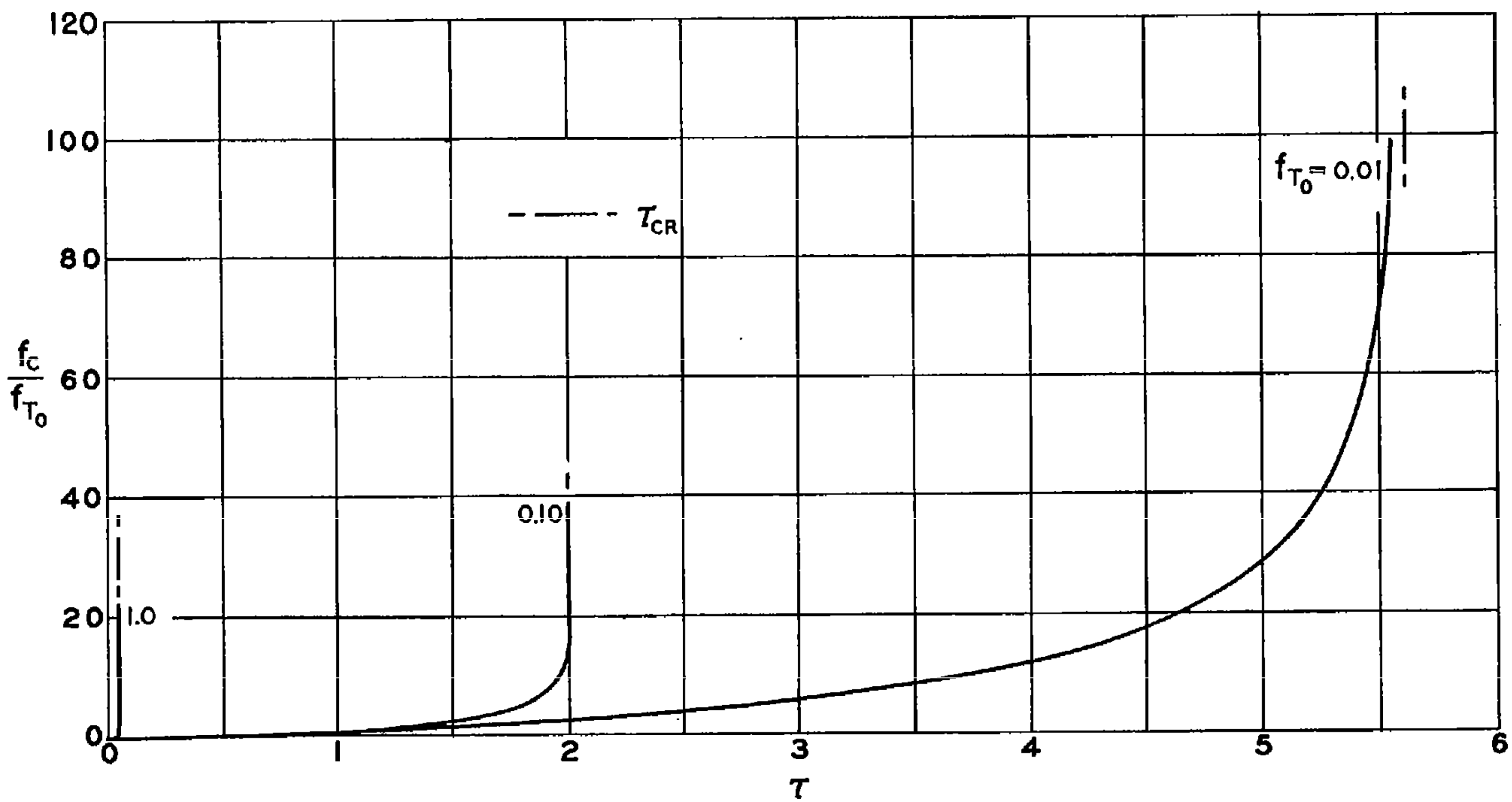
(b) m = 3.

Figure 6.- Continued.



(c) m = 4.

Figure 6.- Continued.



(d) m = 5.

Figure 6.- Concluded.

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