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CREEP BUCKLING OF COLUMNS

By Joseph Kempner and Sharad A. Patel

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SUMMARY

Formulas are presented for the determination of the creep deflection-time characteristics of an initially curved idealized H-section column. These results were obtained from closed-form solutions of the differential equation of bending (derived in NACA TN 3137) of a beam column whose creep properties are of a nonlinearly viscoelastic nature. The critical time (the time required for infinite deflections to develop) established by these solutions is tabulated and plotted for a wide range of the parameters involved.

INTRODUCTION

The effect of creep on the behavior of initially curved columns was previously investigated in reference 1, and the resulting deflection-time characteristics were obtained for several values of the parameters involved. It was found that every column whose material is subject to nonlinear creep — and this includes all columns made of structural metals such as aluminum, steel, titanium, and so forth, when subjected to high temperatures — buckles if the axially compressive load acts upon it for a sufficiently long time. This statement is true even though the compressive force is less than the static critical load which is defined as that load which would cause buckling instantaneously. Depending upon the ratio of the applied load to the static critical load, the initial deviation from straightness, and the creep properties of the metal, the time required for the development of infinite deflections — the so-called critical time — may be anywhere between a few seconds and a few years. It is of great importance to the structural designer of supersonic aircraft to know how much the critical time of his structure is.

In the present report the differential equations derived in reference 1 for the analysis of the behavior of idealized H-section columns are solved in closed form for integral values of the exponent in the power function defining the assumed creep law. These solutions are used for the determination of the deflection as a function of time of an initially curved column whose end load is less than the static critical load of the column. The critical time is calculated for a wide range of the exponent as a function of a parameter which includes the effect of end load and initial curvature. The results of the calculations are presented in tables and charts which enable the designer to determine the critical time once the parameters appearing in the basic uniaxial tensile or compressive creep law are determined experimentally at design temperature.

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SYMBOLS

A	area of idealized H-section
A,B_n,C_n	constants
E_1	effective elastic modulus
f_c	amplitude of time-dependent deflection (accrued for $t > 0$) divided by h
f_i	amplitude of initial deviation from straightness of unloaded column divided by h
f_{T_o}	amplitude at $t = 0$ of total deviation from x-axis of loaded column divided by h (x-axis is drawn between end points of column), $f_i/\left[1 - (\bar{\sigma}/\sigma_E)\right]$
h	distance between flanges of idealized H-section
I	moment of inertia of idealized H-section, $Ah^2/4$
I_1,I_2,J_1,J_2	integrals
L	column length
m	exponent in viscosity term
n	integer
r	integer

$$S = \sum_{n=1}^{(m-1)/2} (-1)^n \cos^{m-2}(n\pi/m)$$

t	time
t_{cr}	critical time
x	axial coordinate
z	amplitude for $t \geqq 0$ of total deviation from x-axis divided by h, $f_c + f_{T_o}$
ϵ	strain
ϵ_o	strain at $t = 0$, σ_o/E_1
λ	viscosity coefficient
σ	stress
$\overline{\sigma}$	average axial compressive stress
σ_E	static buckling stress, $\pi^2 E_1 I / AL^2$
σ_o	constant stress
τ	time parameter, $\left\{E_1(2\overline{\sigma})^m / \left[4\lambda(\sigma_E - \overline{\sigma})\right]\right\} t$
τ_{cr}	critical value of time parameter ($\tau \longrightarrow \tau_{cr}$ as $z \longrightarrow \infty$)
τ_1	time parameter corresponding to $z = 1/2$
Φ_1, Φ_2	functions of x

$(\dot{\ }) \equiv \partial(\ )/\partial t$

$(\ )' \equiv \partial(\ )/\partial x$

CREEP LAW

In the analysis of reference 1 the fundamental uniaxial tensile or compressive constant-stress creep curve is approximated by a straight line, the slope of which can be considered as the secondary creep rate of a real material (fig. 1). This idealization accounts for the actual secondary stage of creep (assumed to be of a viscous nature), approximates the initial elastic or elastoplastic stage and the primary creep stage, and ignores any final stage. The relation between the strain rate and the stress corresponding to the simplified creep curve is

$$\dot{\epsilon} = \left(\dot{\sigma}/E_1\right) + \left(\sigma^m/\lambda\right) \tag{1}$$

in which ϵ and σ, respectively, are the strain and stress, E_1 is the effective modulus (fig. 1), m and λ are parameters defining the viscous behavior of the material, and the dot over a symbol indicates differentiation with respect to time t. The three material parameters E_1, m, and λ are considered as constants for a given temperature and can be determined experimentally from conventional tensile or compressive creep tests. The corresponding relationship among stress, strain, and time for such tests can be found from equation (1) if the strain ϵ_o at $t = 0$ is taken as σ_o/E_1 (fig. 1). Thus

$$\epsilon/\epsilon_o = \left(E_1/\lambda\right)\sigma_o^{\,m-1} + 1 \tag{2}$$

DEFLECTIONS OF A COLUMN WITH INITIAL CURVATURE

Differential equations were derived in reference 1 defining the deflection-time characteristics of a simply supported idealized H-section column (figs. 2 and 3) whose material parameters are the same for tension and compression. The column is assumed to be loaded instantaneously at $t = 0$ with an axial load which remains constant for $t > 0$. The differential equations derived were simplified with the aid of the assumption that the sinusoidal shape of the centroidal axis of the unloaded column was maintained in the loaded column and only the amplitude varied. The differential equations so obtained were readily solved in integral form. Solutions to these equations are presented in table 1 in which the equations referred to are from reference 1.

In appendix A the integrals appearing in table 1 are evaluated in closed form for any even or odd integral value of m. Hence the following relations govern the deflection-time characteristics of the column analyzed for $\overline{\sigma} < \sigma_E$. For m an odd integer $(m > 1)$ with $0 < f_{T_o} < \infty$ and $f_{T_o} \leqq z \leqq \infty$,

$$\tau = \left(\frac{2^{m-2}}{m}\right)\left(\log\left(\frac{z}{f_{T_o}}\right) + \sum_{n=1}^{(m-1)/2} (-1)^n \cos^{m-2}\left(\frac{n\pi}{m}\right) \log\left\{\frac{\left[4z^2 + \tan^2\left(\frac{n\pi}{m}\right)\right]}{\left[4f_{T_o}{}^2 + \tan^2\left(\frac{n\pi}{m}\right)\right]}\right\}\right) \tag{3}$$

$$\tau_{cr} = \left(\frac{2^{m-2}}{m}\right)\left\{\log\left(\frac{1}{2f_{T_o}}\right) - \sum_{n=1}^{(m-1)/2} (-1)^n \cos^{m-2}\left(\frac{n\pi}{m}\right) \log\left[4f_{T_o}{}^2 + \tan^2\left(\frac{n\pi}{m}\right)\right]\right\} \tag{4}$$

For m an even integer $(m > 0)$ with $0 < f_{T_o} \leqq 1/2$ and $f_{T_o} \leqq z \leqq 1/2$,

$$\tau = \left(\frac{2^{m-2}}{m}\right)\left(\log\left(\frac{z}{f_{T_o}}\right) + \sum_{n=1}^{(m/2)-1} (-1)^n \cos^{m-2}\left(\frac{n\pi}{m}\right) \log\left\{\frac{\left[4z^2 + \tan^2\left(\frac{n\pi}{m}\right)\right]}{\left[4f_{T_o}{}^2 + \tan^2\left(\frac{n\pi}{m}\right)\right]}\right\}\right) \tag{5}$$

$$\tau_1 = \left(\frac{2^{m-2}}{m}\right)\left(\log\left(\frac{1}{2f_{T_o}}\right) + \sum_{n=1}^{(m/2)-1} (-1)^n \cos^{m-2}\left(\frac{n\pi}{m}\right) \log\left\{\frac{\left[1 + \tan^2\left(\frac{n\pi}{m}\right)\right]}{\left[4f_{T_o}{}^2 + \tan^2\left(\frac{n\pi}{m}\right)\right]}\right\}\right) \tag{6}$$

with $0 < f_{T_o} \leqq 1/2$ and $1/2 \leqq z \leqq \infty$,

$$\tau = \tau_1 + \left(\frac{2^{m-1}}{m}\right) \sum_{n=1,3,\ldots}^{m-1} (-1)^{(m-n-1)/2} \sin^{m-2}\left(\frac{n\pi}{2m}\right)\left\{\tan^{-1}\left[2z \tan\left(\frac{n\pi}{2m}\right)\right] - \left(\frac{n\pi}{2m}\right)\right\} \tag{7}$$

$$\tau_{cr} = \tau_1 + \left(2^{m-2}\pi/m^2\right) \sum_{n=1,3,\ldots}^{m-1} (-1)^{(m-n-1)/2}(m-n) \sin^{m-2}(n\pi/2m) \quad (8)$$

with $1/2 \leqq f_{T_o} < \infty$ and $f_{T_o} \leqq z \leqq \infty$,

$$\tau = \left(\frac{2^{m-1}}{m}\right) \sum_{n=1,3,\ldots}^{m-1} (-1)^{(m-n-1)/2} \sin^{m-2}\left(\frac{n\pi}{2m}\right) \left\{\tan^{-1}\left[2z \tan\left(\frac{n\pi}{2m}\right)\right] - \tan^{-1}\left[2f_{T_o} \tan\left(\frac{n\pi}{2m}\right)\right]\right\} \quad (9)$$

$$\tau_{cr} = \left(\frac{2^{m-1}}{m}\right) \sum_{n=1,3,\ldots}^{m-1} (-1)^{(m-n-1)/2} \sin^{m-2}\left(\frac{n\pi}{2m}\right) \left\{\left(\frac{\pi}{2}\right) - \tan^{-1}\left[2f_{T_o} \tan\left(\frac{n\pi}{2m}\right)\right]\right\} \quad (10)$$

If $m = 1$, the corresponding column is constructed of a linearly viscous material (Maxwell material) and from reference 1 its exact deflection-time behavior is governed by the relations (see also refs. 2, 3, and 4)

$$z = f_{T_o} e^{2\tau} \quad (11)$$

and

$$\tau_{cr} = \infty \quad (11a)$$

Equations (3) to (11a) define completely the deflection-time characteristics of columns whose properties have been previously defined. It may be noted that for all values of m greater than unity there exists a finite critical time which is defined as that time at which the column deflections increase without limit.

DISCUSSION

The results presented in equations (3) to (11a) can be used to determine the deflection as a function of time of columns whose properties are approximated by those described earlier. With the aid of these equations curves of deflection versus time for various values of the zero-time deflection parameter f_{T_0} can be computed. Such curves are presented in reference 1 for a wide range of values of f_{T_0} and for $m = 1, 2, 3, 4,$ and 5.

Since the critical time parameter τ_{cr} is a measure of the life span of a column, this quantity is perhaps the most significant parameter of the present analysis. Hence in table 2 and figures 4 and 5 τ_{cr} is given as a function of f_{T_0} for a wide range of the exponent m. The results for integral values of m were obtained with the aid of equations (4), (8), and (10). However, for large values of m it is more convenient to determine τ_{cr} from numerical integration of the pertinent relations given in table 1, than it is to apply the corresponding closed-form solutions. It may be noted that results for $m = 1.5$ and $m = 1.1$ are also included in table 2 and figure 4. These latter results were obtained by performing the integrations indicated in table 1(a) in closed form for $m = 1.5$ and numerically for $m = 1.1$. In order to obtain the actual critical time t_{cr} of a given column with known values of the applied stress $\overline{\sigma}$ and amplitude of initial deviation from straightness f_i, results of uniaxial tensile or compressive creep tests corresponding to the design temperature must be available. From such information the parameters E_1, m, and λ can be determined. If these parameters differ significantly for tension and compression tests, it is suggested that they be chosen to correspond to the compression tests, since for all total-deflection amplitudes $zh < h/2$ both flanges of the H-section column are in compression. In view of the simplifying assumptions regarding the creep law, shape of cross section, and affineness of the shapes of the unloaded and loaded column, the present results can be considered only as a first approximation in the analysis of actual columns.

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APPENDIX A

EVALUATION OF INTEGRALS APPEARING IN DEFLECTION-TIME RELATIONSHIPS FOR INTEGRAL VALUES OF EXPONENT m

The integrals required for the evaluation of the various deflection-time relationships arising from the analysis of the creep deflections of an axially loaded initially curved bar are from table 1

$$I_1 = \int dz \Big/ \left\{ \left[(1/2) + z\right]^m - \left[(1/2) - z\right]^m \right\} \qquad \text{(A1)}$$

and

$$I_2 = \int dz \Big/ \left\{ \left[z + (1/2)\right]^m + \left[z - (1/2)\right]^m \right\} \qquad \text{(A2)}$$

For integral values of m, it is seen from table 1 that I_1 requires evaluation for both odd and even values of m, whereas I_2 need be determined only for even values of m.

In the ensuing calculations, the integrals I_1 and I_2 are replaced by

$$J_1 = \int dx/\Phi_1(x) \qquad \text{(A3)}$$

and

$$J_2 = \int dx/\Phi_2(x) \qquad \text{(A4)}$$

in which $\Phi_1(x) = \left[(1 + x)^m - (1 - x)^m\right]$, $\Phi_2(x) = \left[(x + 1)^m + (x - 1)^m\right]$, $x = 2z$, $J_1 = 2^{1-m}I_1$, and $J_2 = 2^{1-m}I_2$. Each of the required integrations will be performed with the method of partial fractions.

Determination of I_1 for Odd Values of m

If the denominator of the integral in equation (A3) is equated to zero, then the following relationship is obtained for the roots x of the resulting odd mth-order polynomial:

$$(1 + x)/(1 - x) = 1^{1/m} \tag{A5}$$

in which the required roots of unity are

$$1^{1/m} = e^{(2n\pi/m)i}, \quad n = 0, \ \pm1, \ \pm2, \ \ldots, \ \pm(m - 1)/2 \tag{A6}$$

Hence, substitution for $1^{1/m}$ from equation (A6) into equation (A5) and subsequent solution for x yield

$$x = i \tan (n\pi/m), \quad n = 0, \ \pm1, \ \pm2, \ \ldots, \ \pm(m - 1)/2 \tag{A7}$$

Thus, the denominator of equation (A3) can be factored as follows:

$$\Phi_1(x) = x\left[x^2 + \tan^2\left(\frac{\pi}{m}\right)\right]\left[x^2 + \tan^2\left(\frac{2\pi}{m}\right)\right] \ldots \left\{x^2 + \tan^2\left[\left(\frac{\pi}{m}\right)(m - 1)/2\right]\right\} \tag{A8}$$

The integrand of J_1 can be resolved into partial fractions to yield

$$1/\Phi_1(x) = (A/x) + \sum_{n=1}^{(m-1)/2}\left(B_n x + C_n\right)\Big/\left[x^2 + \tan^2(n\pi/m)\right] \tag{A9}$$

in which A, B_n, and C_n are constants. From equation (A9)

$$\Phi_1(x)\left\{(A/x) + \sum_{n=1}^{(m-1)/2}\left(B_n x + C_n\right)\Big/\left[x^2 + \tan^2(n\pi/m)\right]\right\} = 1 \tag{A10}$$

and hence A is determined from the condition that, since $\Phi_1(0) = 0$

$$\lim_{x\to 0} \Phi_1(x)(A/x) = \lim_{x\to 0} A\Phi_1'(x) = 1 \tag{A11}$$

in which the prime indicates differentiation with respect to x. Hence

$$A = 1/2m \tag{A12}$$

Similarly, since $\Phi_1[i \tan (n\pi/m)] = 0$,

$$\lim_{x \to i \tan (n\pi/m)} \Phi_1(x) \sum_{n=1}^{(m-1)/2} (B_n x + C_n)/[x^2 + \tan^2(n\pi/m)]$$

$$= \lim_{x \to i \tan (n\pi/m)} \Phi_1'(x)(B_n x + C_n)/2x$$

$$= 1 \tag{A13}$$

for each $n = 1, 2, \ldots, (m - 1)/2$. Thus

$$\{[B_n i \tan (n\pi/m) + C_n]/2i \tan (n\pi/m)\} \Phi_1'[i \tan (n\pi/m)] = 1 \tag{A14}$$

From equation (A7)

$$(1 \pm x) = e^{\pm(n\pi/m)i}/\cos (n\pi/m) \tag{A15}$$

and hence

$$\Phi_1'[i \tan (n\pi/m)] = 2m(-1)^n \cos^{2-m}(n\pi/m) \tag{A16}$$

Equations (A14) and (A16) yield

$$B_n = (1/m)(-1)^n \cos^{m-2}(n\pi/m) \tag{A17}$$

and

$$C_n = 0$$

From equations (A9), (A12), and (A17), the integrand of J_1 becomes

$$1/\Phi_1(x) = (1/2m)(1/x) + (1/m)\sum_{n=1}^{(m-1)/2}(-1)^n \cos^{m-2}(n\pi/m)\left\{x/\left[x^2 + \tan^2(n\pi/m)\right]\right\} \tag{A18}$$

Hence from equations (A1), (A3), and (A18), after performance of the indicated integration, the following expression is obtained for the indefinite integral I_1 for odd integral values of $m > 1$

$$I_1 = \left(2^{m-2}/m\right)\left\{\log 2z + \sum_{n=1}^{(m-1)/2}(-1)^n \cos^{m-2}(n\pi/m)\log\left[4z^2 + \tan^2(n\pi/m)\right]\right\} \tag{A19}$$

Evaluated between the limits z and f_{T_o}, I_1 becomes

$$I_1\left(z,f_{T_o}\right) = \left(\frac{2^{m-2}}{m}\right)\left(\log\left(\frac{z}{f_{T_o}}\right) + \sum_{n=1}^{(m-1)/2}(-1)^n \cos^{m-2}\left(\frac{n\pi}{m}\right)\log\left\{\frac{\left[4z^2 + \tan^2\left(\frac{n\pi}{m}\right)\right]}{\left[4f_{T_o}{}^2 + \tan^2\left(\frac{n\pi}{m}\right)\right]}\right\}\right) \tag{A20}$$

Also

$$I_1\left(\infty,f_{T_o}\right) = \lim_{z\to\infty} I_1\left(z,f_{T_o}\right)$$

$$= \left(\frac{2^{m-2}}{m}\right)\left\{\log\left(\frac{1}{2f_{T_o}}\right) - \sum_{n=1}^{(m-1)/2}(-1)^n \cos^{m-2}\left(\frac{n\pi}{m}\right)\log\left[4f_{T_o}{}^2 + \tan^2\left(\frac{n\pi}{m}\right)\right]\right\} \tag{A21}$$

In the determination of equation (A21) use was made of the relationship

$$\sum_{n=1}^{(m-1)/2}(-1)^n \cos^{m-2}(n\pi/m) = -1/2 \tag{A22}$$

in which m is an odd integer greater than unity. This relation is derived in appendix B.

Determination of I_1 for Even Values of m

The present analysis is performed in a manner analogous to that of the preceding calculations. Since m is now an even integer the polynomial $\Phi_1(x)$ in equation (A3) is of order $m - 1$. The required $(m - 1)$ roots of $\Phi_1(x)$ are then

$$x = i \tan (n\pi/m), \quad n = 0, \pm 1, \pm 2, \ldots, \pm\left[(m/2) - 1\right] \tag{A23}$$

Since equations (A7) and (A23) differ only in the choice of values of n, the resulting expression for I_1 of the preceding section can be used in the present calculations provided the upper limit on the summation sign in equation (A19) is replaced by $(m/2) - 1$. Hence for even integral values of m

$$I_1 = \left(2^{m-2}/m\right)\left\{\log 2z + \sum_{n=1}^{(m/2)-1} (-1)^n \cos^{m-2}(n\pi/m) \log \left[4z^2 + \tan^2(n\pi/m)\right]\right\} \tag{A24}$$

Also

$$I_1\left(z,f_{T_o}\right) = \left(\frac{2^{m-2}}{m}\right)\left(\log\left(\frac{z}{f_{T_o}}\right) + \sum_{n=1}^{(m/2)-1} (-1)^n \cos^{m-2}\left(\frac{n\pi}{m}\right) \log\left\{\frac{\left[4z^2 + \tan^2\left(\frac{n\pi}{m}\right)\right]}{\left[4{f_{T_o}}^2 + \tan^2\left(\frac{n\pi}{m}\right)\right]}\right\}\right) \tag{A25}$$

$$I_1\left(\frac{1}{2}, f_{T_o}\right) = \left(\frac{2^{m-2}}{m}\right)\left(\log\left(\frac{1}{2f_{T_o}}\right) + \right.$$

$$\left.\sum_{n=1}^{(m/2)-1} (-1)^n \cos^{m-2}\left(\frac{n\pi}{m}\right) \log\left\{\frac{\left[1 + \tan^2\left(\frac{n\pi}{m}\right)\right]}{\left[4f_{T_o}{}^2 + \tan^2\left(\frac{n\pi}{m}\right)\right]}\right\}\right) \quad \text{(A26)}$$

Determination of I_2 for Even Values of m

If the denominator of equation (A4) is equated to zero, the following relation is satisfied by the roots x of $\Phi_2(x)$:

$$(x + 1)/(x - 1) = (-1)^{1/m} \quad \text{(A27)}$$

in which

$$(-1)^{1/m} = e^{(n\pi/m)i}, \quad n = \pm 1, \pm 3, \ldots, \pm(m - 1) \quad \text{(A28)}$$

Solution of equation (A27) together with equation (A28) yields

$$x = -i \cot (n\pi/2m), \quad n = \pm 1, \pm 3, \ldots, \pm(m - 1) \quad \text{(A29)}$$

Hence

$$1/\Phi_2(x) = \sum_{n=1,3,5,\ldots}^{m-1} \left(B_n x + C_n\right)/\left[x^2 + \cot^2(n\pi/2m)\right] \quad \text{(A30)}$$

Since

$$x \pm 1 = -ie^{\pm(n\pi/2m)i}\Big/\sin (n\pi/2m) \quad \text{(A31)}$$

and

$$\Phi_2'\left[-i \cot (n\pi/2m)\right] = 2mi^{1-m}(-1)^{(n-1)/2} \sin^{2-m}(n\pi/2m) \tag{A32}$$

$$B_n = 0$$

and

$$C_n = (1/m)(-1)^{(m-n-1)/2} \cos (n\pi/2m) \sin^{m-3}(n\pi/2m) \tag{A33}$$

Therefore from equations (A2), (A4), (A30), and (A33) for even integral values of m

$$I_2 = \left(2^{m-1}/m\right) \sum_{n=1,3,5,\ldots}^{m-1} (-1)^{(m-n-1)/2} \sin^{m-2}(n\pi/2m) \tan^{-1}\left[2z \tan (n\pi/2m)\right] \tag{A34}$$

$$I_2\left(z,f_{T_o}\right) = \left(\frac{2^{m-1}}{m}\right) \sum_{n=1,3,5,\ldots}^{m-1} (-1)^{(m-n-1)/2} \sin^{m-2}\left(\frac{n\pi}{2m}\right)\left\{\tan^{-1}\left[2z \tan\left(\frac{n\pi}{2m}\right)\right] - \tan^{-1}\left[2f_{T_o} \tan\left(\frac{n\pi}{2m}\right)\right]\right\} \tag{A35}$$

$$I_2\left(\infty,f_{T_o}\right) = \left(2^{m-1}/m\right) \sum_{n=1,3,5,\ldots}^{m-1} (-1)^{(m-n-1)/2} \sin^{m-2}(n\pi/2m)\left\{(\pi/2) - \tan^{-1}\left[2f_{T_o} \tan (n\pi/2m)\right]\right\} \tag{A36}$$

$$I_2\left(z,\frac{1}{2}\right) = \left(\frac{2^{m-1}}{m}\right) \sum_{n=1,3,5,\ldots}^{m-1} (-1)^{(m-n-1)/2} \sin^{m-2}\left(\frac{n\pi}{2m}\right)\left\{\tan^{-1}\left[2z \tan\left(\frac{n\pi}{2m}\right)\right] - \left(\frac{n\pi}{2m}\right)\right\} \tag{A37}$$

$$I_2(\infty,1/2) = \left(2^{m-2}\pi/m^2\right) \sum_{n=1,3,5,\ldots}^{m-1} (-1)^{(m-n-1)/2}(m-n)\ \sin^{m-2}(n\pi/2m) \tag{A38}$$

APPENDIX B

DERIVATION OF THE RELATION $\sum_{n=1}^{(m-1)/2} (-1)^n \cos^{m-2}(n\pi/m) = -1/2$

FOR ODD INTEGRAL VALUES OF m

In appendix A $I_1(\infty, fT_o)$ was determined with the aid of the relationship

$$S = \sum_{n=1}^{(m-1)/2} (-1)^n \cos^{m-2}(n\pi/m)$$

$$= -1/2 \tag{B1}$$

in which m is any odd integer greater than unity. Equation (B1) will now be shown to be valid. From reference 5, page 69,

$$\cos^{m-2}(n\pi/m) = 2^{3-m} \sum_{r=0}^{(m-3)/2} \binom{m-2}{r} \cos\left[(m - 2r - 2)(n\pi/m)\right] \tag{B2}$$

in which the binomial coefficient (ref. 5, p. 19) is defined as

$$\binom{m-2}{r} = \binom{m-2}{m-2-r} = \frac{(m - 2)!}{r!(m - 2 - r)!} \tag{B3}$$

Equations (B1) and (B2) yield

$$S = 2^{3-m} \sum_{n=1}^{(m-1)/2} \sum_{r=0}^{(m-3)/2} (-1)^n \binom{m-2}{r} \cos\left[(m - 2r - 2)(n\pi/m)\right] \tag{B4}$$

With the order of summations interchanged, equation (B4) becomes

$$S = 2^{3-m} \sum_{r=0}^{(m-3)/2} \binom{m-2}{r} \sum_{n=1}^{(m-1)/2} (-1)^n \cos\left[(m - 2r - 2)(\pi/m)n\right] \tag{B5}$$

From reference 5, page 82,

$$\sum_{n=1}^{(m-1)/2} (-1)^n \cos\left[(m - 2r - 2)(\pi/m)n\right] =$$

$$-(1/2)\left\{1 - (-1)^{(m-1)/2} \cos\left[(m - 2r - 2)(\pi/2)\right]/\cos\left[(m - 2r - 2)(\pi/2m)\right]\right\} \tag{B6}$$

Since m is odd, $m - 2r - 2$ is also odd, and therefore equation (B6) reduces to

$$\sum_{n=1}^{(m-1)/2} (-1)^n \cos\left[(m - 2r - 2)(\pi/m)n\right] = -1/2 \tag{B7}$$

Equation (B7) together with equation (B5) yields

$$S = -2^{2-m} \sum_{r=0}^{(m-3)/2} \binom{m-2}{r} \tag{B8}$$

From reference 5, page 19,

$$\sum_{r=0}^{m-2} \binom{m-2}{r} = 2^{m-2} \tag{B9}$$

The number of terms under the summation sign in equation (B9) is $m - 1$ which, since m is odd, is an even number. From equations (B3) it can be seen that the binomial coefficients are symmetric. Hence the sum of the first $(m - 1)/2$ terms under the summation sign of equation (B9) is $(1/2)\left(2^{m-2}\right) = 2^{m-3}$. In order to obtain this sum from equation (B9), let $r = 0, 1, \ldots, (m - 3)/2$. The number of terms considered is then $(1/2)(m - 3) + 1 = (m - 1)/2$. Consequently

$$\sum_{r=0}^{(m-3)/2} \binom{m-2}{r} = 2^{m-3} \tag{B10}$$

and, therefore, equation (B8) yields

$$S = -1/2$$

which was to be proved.

REFERENCES

1. Kempner, Joseph: Creep Bending and Buckling of Nonlinearly Visco-elastic Columns. NACA TN 3137, 1953.

2. Freudenthal, Alfred M.: The Inelastic Behavior of Engineering Materials and Structures. John Wiley & Sons, Inc., 1950.

3. Kempner, Joseph, and Hoff, N. J.: Behavior of a Linear Viscoelastic Column. Appendix II of "Structural Problems of Future Aircraft" by N. J. Hoff, Proc. Third Anglo-American Aero. Conf. (Brighton, England), R.A.S., 1951.

4. Kempner, Joseph: Creep Bending and Buckling of Linearly Viscoelastic Columns. NACA TN 3136, 1953.

5. Adams, Edwin P., and Hippisley, R. L.: Smithsonian Mathematical Formulae and Tables of Elliptic Functions. Second reprint, Smithsonian Institution, 1947.

TABLE 1.- INTEGRALS REQUIRED FOR DETERMINATION OF τ AS A FUNCTION OF z

[Reproduced from ref. 1]

(a) m even integer (or fractional)

f_{T_o}	z	τ	τ_1	τ_{cr}
$0 < f_{T_o} \leq \frac{1}{2}$	$f_{T_o} \leq z \leq \frac{1}{2}$	[a] $\int_{f_{T_o}}^{z} \frac{dz}{\left(\frac{1}{2} + z\right)^m - \left(\frac{1}{2} - z\right)^m}$	[b] $\int_{f_{T_o}}^{1/2} \frac{dz}{\left(\frac{1}{2} + z\right)^m - \left(\frac{1}{2} - z\right)^m}$	
	$\frac{1}{2} \leq z \leq \infty$	[c] $\tau_1 + \int_{1/2}^{z} \frac{dz}{\left(z + \frac{1}{2}\right)^m + \left(z - \frac{1}{2}\right)^m}$		[d] $\tau_1 + \int_{1/2}^{\infty} \frac{dz}{\left(z + \frac{1}{2}\right)^m + \left(z - \frac{1}{2}\right)^m}$
$\frac{1}{2} \leq f_{T_o} < \infty$	$f_{T_o} \leq z \leq \infty$	[e] $\int_{f_{T_o}}^{z} \frac{dz}{\left(z + \frac{1}{2}\right)^m + \left(z - \frac{1}{2}\right)^m}$		[f] $\int_{f_{T_o}}^{\infty} \frac{dz}{\left(z + \frac{1}{2}\right)^m + \left(z - \frac{1}{2}\right)^m}$

(b) m odd integer

f_{T_o}	z	τ	τ_{cr}
$0 < f_{T_o} < \infty$	$f_{T_o} \leq z \leq \infty$	[e] $\int_{f_{T_o}}^{z} \frac{dz}{\left(z + \frac{1}{2}\right)^m + \left(z - \frac{1}{2}\right)^m}$	[f] $\int_{f_{T_o}}^{\infty} \frac{dz}{\left(z + \frac{1}{2}\right)^m + \left(z - \frac{1}{2}\right)^m}$

[a]Eq. (29).
[b]Eq. (32).
[c]Eq. (30).
[d]Eq. (31).
[e]Eq. (33).
[f]Eq. (34).

TABLE 2.- VALUES OF τ_{cr} FOR $1.1 \leq m \leq 14$ AND $0.01 \leq f_{T_o} \leq 1.00$

f_{T_o} \ m	1.1	1.5	2	3	4	5	6	7
0.01	7.165	3.178	2.741	2.974	3.891	5.612	8.502	13.73
.02	6.918	2.852	2.395	2.512	3.198	4.505	6.759	10.58
.05	6.471	2.420	1.937	1.902	2.286	3.053	4.352	6.483
.07	6.306	2.261	1.769	1.679	1.955	2.529	3.402	5.048
.10	6.134	2.093	1.590	1.444	1.608	1.985	2.628	3.624
.20	5.795	1.765	1.244	.994	.969	1.042	1.191	1.417
.30	5.597	1.571	1.041	.745	.643	.609	.610	.635
.40	5.456	1.433	.897	.579	.449	.379	.359	.313
.50	5.344	1.323	.785	.462	.325	.248	.200	.167
.60	5.252	1.231	.695	.375	.242	.169	.125	.0940
.70	5.175	1.153	.619	.310	.185	.119	.0800	.0557
.80	5.108	1.087	.559	.259	.144	.0858	.0534	.0344
.90	5.049	1.031	.507	.219	.115	.0633	.0371	.0220
1.00	4.997	.982	.464	.187	.0911	.0477	.0259	.0145
f_{T_o} \ m	8	9	10	11	12	13	14	
0.01	22.56	37.96	64.91	112.6	197.4	349.4	623.6	
.02	17.05	28.18	47.35	80.71	139.2	242.3	425.3	
.05	9.943	15.68	25.11	40.80	67.03	111.2	185.8	
.07	7.499	11.47	17.79	27.97	44.48	71.36	115.2	
.10	5.131	8.018	11.07	16.59	25.11	38.33	58.95	
.20	1.716	2.169	2.754	3.560	4.598	6.021	7.940	
.30	.667	.745	.828	.932	1.058	1.213	1.400	
.40	.292	.291	.287	.288	.290	.295	.304	
.50	.143	.125	.111	.100	.0909	.0833	.0769	
.60	.0733	.0583	.0469	.0386	.0319	.0265	.0223	
.70	.0398	.0291	.0215	.0161	.0122	.0094	.0072	
.80	.0227	.0153	.0105	.0073	.0051	.0036	.0025	
.90	.0134	.0085	.0053	.0035	.0023	.0015	.00097	
1.00	.0083	.0049	.0029	.0017	.0011	.00065	.00040	

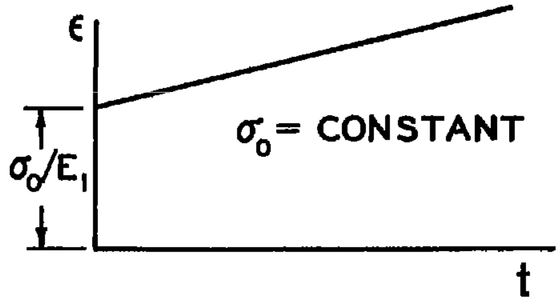

Figure 1.- Idealized creep curve.

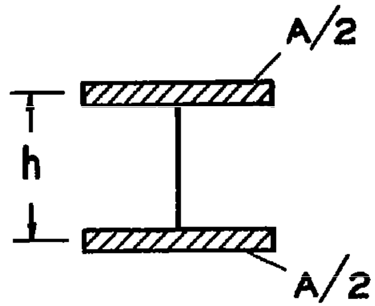

Figure 2.- Idealized H-section.

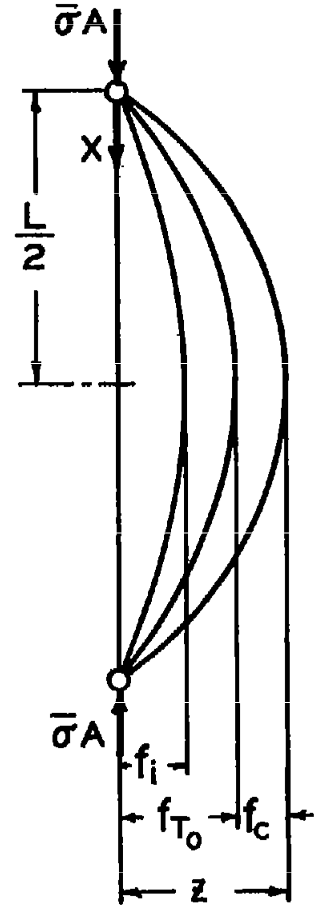

Figure 3.- Deflections of simply supported column.

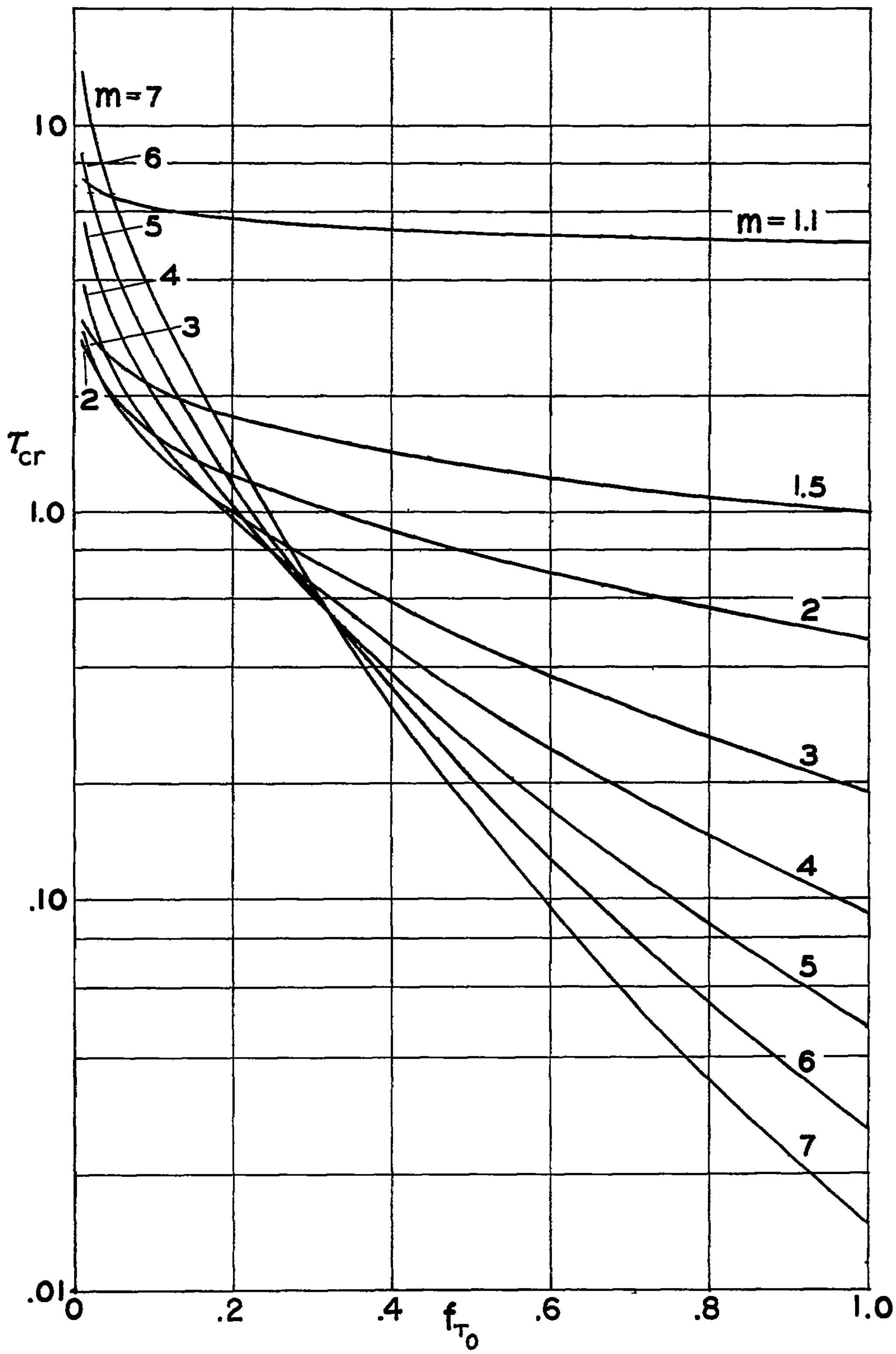

Figure 4.- Critical time parameter for $1.1 \leqq m \leqq 7$ and $0.01 \leqq f_{T_0} \leqq 1.0$.

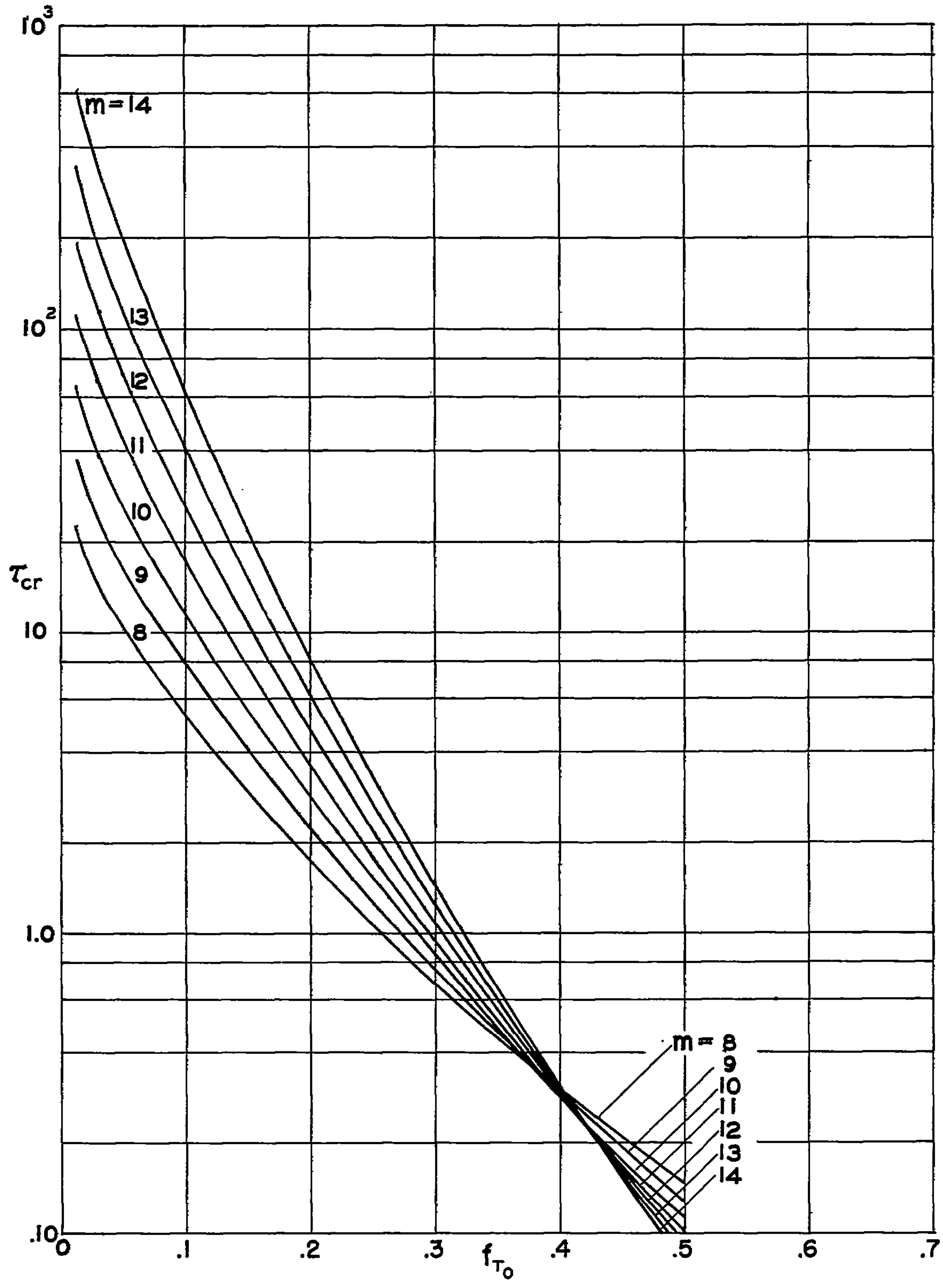

Figure 5.- Critical time parameter for $8 \leqq m \leqq 14$ and $0.01 \leqq f_{T_0} \leqq 1.0$.

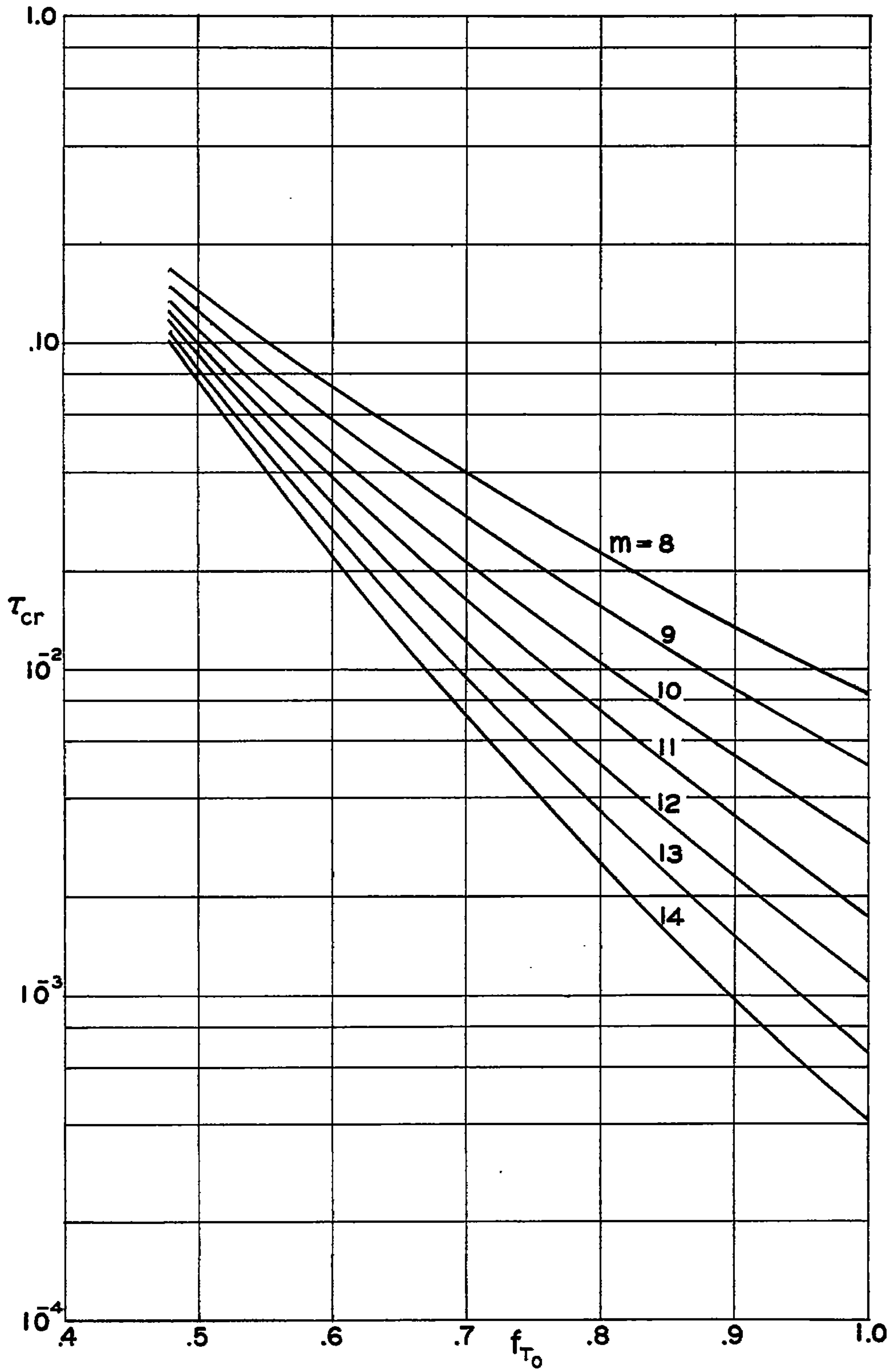

Figure 5.- Concluded.

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