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SHOCKS IN HELICAL FLOWS THROUGH ANNULAR CASCADES OF STATOR BLADES

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SUMMARY

A method is presented for calculating supersonic potential flows in annular cascades of blades by the method of characteristics. It is found that helical flows may be adjoined by helical shocks of uniform strength; these constitute a considerable addition to the class of simple flows available for designing cascades of lifting blades. It was also found that by selection of the proper variables, the derivatives of the velocity components, which occur in the characteristic equations, could be combined into an exact differential. This form of the equation facilitated computations. A flow and several cascade designs were computed.

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INTRODUCTION

In studying the flow of air through a compressor, an analytic description of the behavior of a particle on an "average" streamline or stream surface going through the compressor is quite useful. This type of analysis requires assumptions such as axial symmetry or flow on surfaces of revolution. While they frequently serve well to get a good over-all picture of the flow, in certain regions of the compressor these assumptions are generally untenable. These could be regions in which the major part of the loss is occurring, so that such two-dimensional pictures would be inadequate either for design control or investigation of the cause of these losses.

In order to construct compressors of maximum effectiveness and efficiency, it is desirable to have a description of the internal velocity field. As a step toward this ultimate objective, effort has been applied to the less general problem of obtaining potential flows of a perfect nonviscous fluid satisfying boundary conditions required in compressors. Without assumptions of certain symmetries in the flow, such an analysis involves the solution of differential equations in three independent variables. Although several methods for obtaining general three-dimensional potential flows in compressors have been described (refs. 1 and 2), the amount of numerical work required to get a specific solution is formidable.

Thus, since on the one hand present two-dimensional methods are frequently inadequate and on the other hand the existing three-dimensional methods are impractical with standard computing techniques, an attempt was made to find simple classes of flows, or close approximations thereto, which could be used with the boundary conditions required in a compressor. In particular, interest was focused on obtaining supersonic flows for the inlet region of a diffusing cascade of stator blades. From the class of uniform flows, a flow can be constructed through a straight cascade of lifting blades by joining such flows with plane shock surfaces which originate on the blade surfaces. The velocity potential is, in this case, a linear function of the cartesian coordinates. For an annular cascade, a free-vortex upstream flow results in shock surfaces which must be oriented for uniform shock strength in order for a velocity potential to exist. The simplest class of such surfaces are those linear in θ and z, which are the polar angle and axial distance, respectively, of the cylindrical coordinate system. These shock surfaces impose certain boundary conditions on the velocity components as functions of r (distance from polar axis) for the flow on the downstream side. It was found that the solutions which satisfy these boundary conditions are helical flows which have velocity components expressible as functions of two independent variables. The helical flows for annular cascades are therefore the counterparts of the uniform flows for straight cascades in that they represent the simplest class which contains the free-vortex upstream flow and which may be adjoined by uniform shocks, while the uniform (constant) flows represent the simplest class which contains the constant upstream flow and which may be joined by shocks.

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Because the upstream flow and the shock-surface normals involve only the variable r, a velocity potential of the form $\varphi = bz + c\theta + g(r)$ (where b and c are constants) might be tried; that is, the simplest class of potential flows with a radial component. However, the boundary conditions imposed by a shock of uniform strength give functional relations between the velocity components that cannot, in general, be satisfied by a flow of this class. Another approach was to try to adequately approximate the potential function in terms of arbitrary functions of r and then satisfy the boundary conditions exactly. This procedure resulted in inadequate approximations to the potential-flow equation. In the end, no potential flow or approximate potential flow, expressible in terms of functions of single variables, could be found which satisfied the required boundary conditions or approximations thereof.

Thus, while the three-dimensional problem is reduced without further assumption to a two-dimensional one, further reduction seems unlikely. Consequently, the problem was solved by the method of characteristics in two dimensions. In the mathematical sense, the flow is two-dimensional in that the flow problem involves two independent coordinates. In the physical sense, the flow is three-dimensional in that the velocity vectors fields are not identical for a set of surfaces as in the cartesian flow.

SYMBOLS

The following symbols are used in this report: (All velocities are referred to the stagnation sonic speed as the unit of velocity.)

A,B	constant coefficients in shock equation $F = 0$ and in coordinate transformation
a	local velocity of sound
C,D	constant coefficients in coordinate transformation
F,G	functions of coordinates which define shock surfaces
f	function of r in shock equation $F = 0$
h	component of stream function w, $h = h(r,s) = w + t$
J	Jacobian of transformation of coordinates, $J = \dfrac{\partial(x,y,z)}{\partial(r,s,t)}$
$K_{\pm}$	$\dfrac{1}{\sin\alpha}\left\{-\dfrac{\cos\beta}{\cos(\beta\mp\alpha)}\left[\sin^2\alpha+\left(\dfrac{A}{r\lvert\nabla s\rvert}\sin\beta-\dfrac{\Delta B}{r(\nabla s)^2}\dfrac{v_t}{U}\right)^2\right]+\sin(\beta\pm\alpha)\left[\dfrac{A^2\sin\beta}{r^2(\nabla s)^2}-\dfrac{2v_t\,AB\Delta}{Ur^2(\nabla s)^3}\right]+\dfrac{(\gamma-1)v_t^2B^2r^2}{2T^2R_t^4\Delta^2}\cos\alpha\right\}$
$\overline{M}$	Mach vector in $\nabla r,\nabla s$-plane ($\overline{M} = \overline{U}/a$)
M_n	Mach number normal to shock surface
$\overline{R}_r,\overline{R}_s,\overline{R}_t$	base vectors along helical coordinate curves, $\overline{R}_r = \dfrac{\nabla s\times\nabla t}{\nabla r\cdot\nabla s\times\nabla t}$, etc
r	cylindrical and helical coordinate (distance normal to axis of coordinate system)
s	helical coordinate, $A\theta + Bz$
T	corrected stagnation sonic speed, $T = \sqrt{1 - \frac{\gamma-1}{2}v_t^2/R_t^2}$
t	helical coordinate, $C\theta + Dz$

U magnitude of $\bar{U}$

$\bar{U}$ velocity component in $\nabla r, \nabla s$-plane

u integral of streamlines independent of t

V magnitude of $\bar{V}$

$\bar{V}$ velocity vector, $\bar{V} = V_r \nabla r + V_\theta r \nabla \theta + V_z \nabla z$ 3242

V_r, V_θ, V_z physical velocity components in cylindrical coordinate system

v_r, v_s, v_t $\bar{V} \cdot \bar{R}_r$, $\bar{V} \cdot \bar{R}_s$, $\bar{V} \cdot \bar{R}_t$, respectively

v^r, v^s, v^t $\bar{V} \cdot \nabla r$, $\bar{V} \cdot \nabla s$, $\bar{V} \cdot \nabla t$, respectively

W ratio of U to corrected stagnation sonic speed, $W = U/T$

w integral of streamlines linear in t

z axial distance of cylindrical coordinates (distance parallel to axis of coordinate system)

α Mach angle, $\alpha = \sin^{-1} \frac{a}{U} = \sin^{-1} \frac{1}{M} = \sin^{-1} \frac{a}{TW}$

β flow angle in $\nabla r, \nabla s$-plane measured from ∇r toward ∇s, $\beta = \tan^{-1} \left(\frac{v^s}{|\nabla s| v^r} \right)$

γ ratio of specific heats

Δ $AD - BC$

δ $\frac{-2}{\gamma+1} \frac{1 - M_n^2}{M_n}$

ε angle of deflection of flow in $\nabla r, \nabla s$-plane through shock wave, $\varepsilon \equiv \beta_2 - \beta_1$ or $\varepsilon \equiv \beta_4 - \beta_3$

ζ_+, ζ_- slopes of characteristic curves in r,s-surface

θ polar angle of cylindrical coordinates

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ν	Prandtl-Meyer angle based on W, $\nu = \int \cot \alpha \frac{dW}{W}$
ξ,η	characteristic coordinates
ρ	density of gas (relative to stagnation value)
$\overline{\tau}$	leading-edge vector
Φ	potential function

Subscripts:

a,b,c, d,e,f	refer to fig. 1
h	hub
0	leading edge of blade
1	upstream side of $F = 0$
2	downstream side of $F = 0$
3	upstream side of $G = 0$
4	downstream side of $G = 0$

CONSTRUCTION OF SUPERSONIC DIFFUSOR FLOWS

Passage of Flow Through First Shock Surface

The flow upstream of the cascade is assumed to be a supersonic free-vortex flow; that is, $V_r = 0$, rV_θ = constant, V_z = constant, and the hub and the casing are circular cylinders. Also, the suction surface of the blade is initially a stream surface of the upstream flow. When the leading-edge wedge angle is not zero, an attached shock comes off the leading edge on the pressure side of the blade. This shock will intersect the hub, casing, and suction surface of the adjacent blade in certain curves.

In order that the flow downstream of the shock be a potential flow, the change of entropy across the shock must be constant and therefore the Mach number normal to the shock must be constant. If the shock surface is given by $F(r,\theta,z) = 0$, then F must satisfy

$$\frac{\overline{V}_1}{a_1} \cdot \frac{\nabla F}{|\nabla F|} = M_{n,1} = \text{constant} \tag{1}$$

on $F = 0$. Because the upstream velocity components are functions of r alone, the coefficients of the differential equation (1) for F do not involve the coordinates θ and z. A function F of the form

$$F = A\theta + Bz + f(r)$$

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will therefore satisfy equation (1), which reduces to

$$\left.\begin{aligned} \left(\frac{df}{dr}\right)^2 &= \left(\frac{\frac{AV_\theta}{r} + BV_z}{a_1 M_{n,1}}\right)^2 - \frac{A^2}{r^2} - B^2 \\ &\text{or} \\ \left(\frac{df}{dr}\right)^2 &= \left[\frac{(\overline{V}_1 \cdot \nabla s)^2}{a_1^2 M_{n,1}^2 (\nabla s)^2} - 1\right] (\nabla s)^2 \end{aligned}\right\} \tag{2}$$

where s is the quantity $A\theta + Bz$, and a is the sonic speed given by $a^2 = 1 - \frac{\gamma-1}{2} V^2$ (when the stagnation sonic speed is taken to be the unit of velocity). Thus, the class of shock surfaces obtained is

$$f(r) + s = 0 \tag{3}$$

in which f is the solution of equation (2). If it is required that the casing be a cylindrical surface of revolution, a reflected shock surface $G = 0$ results which intersects the surface $F = 0$ at the casing and which is so oriented that no radial deflection to the flow will result from passage through both shocks. The shock surface $G = 0$ will intersect the hub and the blade surfaces downstream of $F = 0$. The shapes of the hub and suction surfaces downstream of $F = 0$ are obtained by the condition that the shock $F = 0$ is not reflected at these surfaces.

Flow Field Downstream of First Shock

After calculation of the flow through the first shock ($F = 0$), the downstream flow field must be constructed. Helical flows will be shown to satisfy the boundary conditions on the shock, thus reducing the problem from a three-dimensional to a two-dimensional flow not restricted

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by the usual symmetries required of two-dimensional flows in cartesian, cylindrical, or spherical coordinates. The problem will be further reduced to ordinary differential equations by the introduction of characteristic coordinates, and then further simplified by the introduction of the Prandtl-Meyer angle variable.

To obtain the flow downstream of $F = 0$, consider the boundary conditions imposed by this surface. For the velocity downstream of $F = 0$,

$$\overline{V}_2 = \overline{V}_1 - a_1\delta_1 \frac{\nabla F}{|\nabla F|} \tag{4}$$

where

$$\delta_1 = \frac{-2}{\gamma+1} \frac{1 - M_{n,1}^2}{M_{n,1}}$$

and the signs of df/dr and the coefficients A and B are chosen to give $\nabla F \cdot \overline{V}_1 > 0$ and $V_{r,2} \frac{df}{dr} < 0$. In terms of a helical coordinate system given by

$$\left.\begin{aligned} r &= r \\ s &= A\theta + Bz \\ t &= C\theta + Dz \end{aligned}\right\} \tag{5}$$

the boundary conditions by equation (4) are

$$\left.\begin{aligned} \overline{V}_2 \cdot \overline{R}_r &= \overline{V}_1 \cdot \overline{R}_r - \frac{a_1\delta_1}{|\nabla F|} f' = \frac{-a_1\delta_1 f'}{|\nabla F|} \\ \overline{V}_2 \cdot \overline{R}_s &= \overline{V}_1 \cdot \overline{R}_s - \frac{a_1\delta_1}{|\nabla F|} = \frac{1}{\Delta}(DrV_\theta - CV_z) - \frac{a_1\delta_1}{|\nabla F|} \\ \overline{V}_2 \cdot \overline{R}_t &= \overline{V}_1 \cdot \overline{R}_t = \frac{1}{\Delta}(-BrV_\theta + AV_z) = \text{constant} \end{aligned}\right\} \tag{6}$$

where $\Delta \equiv AD - BC$, $\overline{R}_r = (\nabla s \times \nabla t)/(\nabla r \cdot \nabla s \times \nabla t)$, and so forth.

If there is a potential flow

$$\overline{V} = \nabla \Phi$$

with components

$$v_r = \overline{R}_r \cdot \overline{V} = \overline{R}_r \cdot \nabla\Phi = \frac{\partial\Phi}{\partial r}$$

$$v_s = \overline{R}_s \cdot \overline{V} = \frac{\partial\Phi}{\partial s}$$

$$v_t = \overline{R}_t \cdot \overline{V} = \frac{\partial\Phi}{\partial t}$$

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and if it is of the degenerate type having $\partial v_i/\partial t = 0$ (helical flow) where $i = r,s,t$, then

$$\frac{\partial v_t}{\partial s} = \frac{\partial v_s}{\partial t} = 0$$

$$\frac{\partial v_t}{\partial r} = \frac{\partial v_r}{\partial t} = 0$$

and v_t is therefore a constant. The boundary conditions (6) fulfill this requirement. Consequently, the equations for the flow downstream of $F = 0$ are written in r,s,t coordinates, and solutions are sought for which $\partial v_i/\partial t = 0$.

The continuity equation is

$$\frac{1}{\rho}\nabla\cdot(\rho\overline{V}) = \nabla\cdot\overline{V} + \overline{V}\cdot\frac{\nabla\rho}{\rho} = 0$$

Because

$$\rho = \text{constant} \cdot \left(1 - \frac{\gamma-1}{2} V^2\right)^{\frac{1}{\gamma-1}}$$

then

$$\nabla\cdot\overline{V} - \frac{1}{2a^2}\overline{V}\cdot\nabla V^2 = 0$$

The formula for the divergence (ref. 3), in the case $\partial/\partial t = 0$, reduces to

$$\nabla\cdot\overline{V} = \frac{1}{J}\left[\frac{\partial}{\partial r}\,(J\nabla r\cdot\overline{V}) + \frac{\partial}{\partial s}\,(J\nabla s\cdot\overline{V})\right]$$

where J is the Jacobian of the transformation:

$$J \equiv \frac{\partial(x,y,z)}{\partial(r,s,t)} = \frac{1}{\nabla r\cdot\nabla s\times\nabla t} = \frac{r}{\Delta} \tag{7}$$

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The second term of the continuity equation is

$$-\frac{1}{2a^2}\left[v^r\,\frac{\partial}{\partial r}\,(v_r v^r + v_s v^s + v_t v^t) + v^s\,\frac{\partial}{\partial s}\,(v_r v^r + v_s v^s + v_t v^t)\right]$$

where

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$$\left.\begin{aligned} v^r &\equiv \nabla r\cdot\overline{V} = \nabla r\cdot(v_r\nabla r + v_s\nabla s + v_t\nabla t) = v_r \\ v^s &\equiv \nabla s\cdot\overline{V} = v_s(\nabla s)^2 + v_t(\nabla s\cdot\nabla t) \\ v^t &\equiv \nabla t\cdot\overline{V} = v_s(\nabla s\cdot\nabla t) + v_t(\nabla t)^2 \end{aligned}\right\} \tag{8}$$

Because v_t is constant, the equation of continuity may be reduced to

$$\left[(v^r)^2 - a^2\right]\frac{\partial v_r}{\partial r} + 2v^r v^s\,\frac{\partial v_r}{\partial s} + \left[(v^s)^2 - a^2(\nabla s)^2\right]\frac{\partial v_s}{\partial s} - \frac{v^r}{r}\left[a^2 + r^2(\overline{V}\cdot\nabla\theta)^2\right] = 0 \tag{9}$$

where derivatives of v^r and v^s have been eliminated, and the equation for zero vorticity

$$\frac{\partial v_r}{\partial s} - \frac{\partial v_s}{\partial r} = 0 \tag{10}$$

has also been employed.

If characteristic coordinates ξ and η are introduced, then the equations for r and s in terms of ξ and η are

$$\frac{\partial s}{\partial \xi} = \zeta_+\,\frac{\partial r}{\partial \xi} \qquad \frac{\partial s}{\partial \eta} = \zeta_-\,\frac{\partial r}{\partial \eta} \tag{11}$$

where ζ satisfies the quadratic equation

$$\left[a^2 - (v^r)^2\right]\zeta^2 + 2v^r v^s \zeta + \left[a^2(\nabla s)^2 - (v^s)^2\right] = 0 \tag{12}$$

(ref. 4) with two roots

$$\zeta_{\pm} = \frac{-v^r v^s \pm a|\nabla s|\sqrt{(v^r)^2 + (v^s/|\nabla s|)^2 - a^2}}{a^2 - (v^r)^2} \tag{13}$$

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The characteristics flow equations are simplified somewhat by utilization of the following relation from equation (12):

$$\zeta_+\zeta_- = \frac{a^2(\nabla s)^2 - (v^s)^2}{a^2 - (v^r)^2}$$

and the form which results is

$$\left.\begin{aligned}
\frac{\partial v_r}{\partial \xi} + (\zeta_-)\frac{\partial v_s}{\partial \xi} + \frac{v^r}{r}\left[\frac{a^2 + r^2\,(Dv^s - Bv^t)^2/\Delta^2}{a^2 - (v^r)^2}\right]\frac{\partial r}{\partial \xi} = 0 \\
\frac{\partial v_r}{\partial \eta} + (\zeta_+)\frac{\partial v_s}{\partial \eta} + \frac{v^r}{r}\left[\frac{a^2 + r^2\,(Dv^s - Bv^t)^2/\Delta^2}{a^2 - (v^r)^2}\right]\frac{\partial r}{\partial \eta} = 0
\end{aligned}\right\} \tag{14}$$

Equations (11) and (14) are to be solved in a region downstream of $F = 0$, with the boundary conditions on $F = 0$ given by equations (6). Present experience by the authors in the use of these equations indicates that, when the trapezoidal rule of integration is used with finite differences, an integral converges more rapidly when the velocity increments given by the first two terms are combined into an exact differential. This can be accomplished by introducing the Prandtl-Meyer expansion angle ν and the direction angle β as the variables defining the flow. The velocity vector $\overline{V}$ may be expressed in terms of the set of unit orthogonal vectors ∇r, $\nabla s/|\nabla s|$, and $R_t/|R_t|$ as

$$\overline{V} = v^r\nabla r + \frac{v^s}{|\nabla s|}\frac{\nabla s}{|\nabla s|} + \frac{v_t}{|R_t|}\frac{R_t}{|R_t|}$$

The velocity components v^r and $v^s/|\nabla s|$ give a vector $\overline{U}$, where

$$\overline{U} = U\left(\cos\beta\ \nabla r + \sin\beta\ \frac{\nabla s}{|\nabla s|}\right)$$

Therefore,

$$v^r = U\cos\beta,\quad v^s/|\nabla s| = U\sin\beta$$

Since $v_r = v^r$ and

$$v^s = \nabla s\cdot\overline{V} = \nabla s\cdot(v_r\nabla r + v_s\nabla s + v_t\nabla t) = v_s(\nabla s)^2 + v_t(\nabla s\cdot\nabla t)$$

then

$$v_r = U\cos\beta$$

$$v_s = \frac{U\sin\beta}{|\nabla s|} - v_t\,\frac{\nabla s\cdot\nabla t}{(\nabla s)^2}$$

The characteristic slopes $\zeta_{\pm}/|\nabla s|$ involve not only U and β, but also $a^2 = 1 - \frac{\gamma-1}{2}\left(U^2 + v_t^2/R_t^2\right)$ and therefore the coordinate r by means of R_t^2. However, if a modified stagnation sonic speed T is introduced where $T^2 = a^2 + \frac{\gamma-1}{2}U^2$ and the velocity is referred to this speed, then an exact differential may be constructed. Let W be the ratio of velocity U to the modified stagnation sonic speed

$$W \equiv \frac{U}{\sqrt{1 - \frac{\gamma-1}{2}\,v_t^2/R_t^2}}$$

When the Mach angle α is introduced, where

$$\sin\alpha \equiv \frac{a}{U} = \frac{\sqrt{1 - \frac{\gamma-1}{2}W^2}}{W}$$

there are obtained

$$\frac{\zeta_{\pm}}{|\nabla s|} = \tan\ (\beta \mp \alpha) \tag{13a}$$

and

$$ds = |\nabla s|\ \tan\ (\beta \mp \alpha)\ dr \tag{11a}$$

in place of equations (13) and (11). With the definition

$$\nu \equiv \int \cot\ \alpha\ \frac{dW}{W}$$

the flow equations (14) become

$$d\nu \pm d\beta + K_{\pm}\frac{dr}{r} = 0 \tag{14a}$$

where

$$K_{\pm} \equiv \frac{1}{\sin\ \alpha}\left\{\frac{-\cos\ \beta}{\cos(\beta \mp \alpha)}\left[\sin^2\alpha + \left(\frac{A}{r|\nabla s|}\sin\ \beta - \frac{\Delta B}{r(\nabla s)^2}\frac{v_t}{U}\right)^2\right] + \right.$$

$$\left.\sin\ (\beta \pm \alpha)\left[\frac{A^2}{r^2(\nabla s)^2}\sin\ \beta - \frac{2v_t\ AB\Delta}{Ur^2(\nabla s)^3}\right] + \frac{(\gamma-1)v_t^2B^2r^2}{2T^2R_t^4\Delta^2}\cos\ \alpha\right\}$$

For the method of characteristics to be useable, α must be real, and therefore $U > a$. That is, the velocity component in the $\nabla r, \nabla s$-surface must be supersonic regardless of the magnitude of v_t.

For trapezoidal integration, the following formulas are used to calculate conditions at point c (fig. 1(a)), where c is connected with point a by a curve of η = constant and with point b by a curve ξ = constant:

$$\left.\begin{aligned}
\nu_c - \nu_a + \beta_c - \beta_a + [K_+]_{c,a}\ (\log\ r_c - \log\ r_a) &= 0 \\
s_c - s_a &= \Big[|\nabla s|\ \tan\ (\beta - \alpha)\Big]_{c,a}\ (r_c - r_a) \\
\nu_c - \nu_b - \beta_c + \beta_b + [K_-]_{c,b}\ (\log\ r_c - \log\ r_b) &= 0 \\
s_c - s_b &= \Big[|\nabla s|\ \tan\ (\beta + \alpha)\Big]_{c,b}\ (r_c - r_b)
\end{aligned}\right\} \tag{15}$$

where the square brackets indicate the average of values designated by subscripts. The four equations will determine the four quantities ν_c, β_c, s_c, r_c. Iteration is required for accurate values of the K's.

Continuation of Flow Field Through Reflected Shock

To continue the solution downstream, the condition that the casing be a cylindrical surface ($v_r = 0$) is applied. The calculations up to this point cover the region ABC (fig. 1(b)), which is bounded by the first shock and the two characteristic curves ξ = constant (AB) and η = constant (BC). The required reflected shock is again assumed to satisfy equation (1); that is, if the shock is given by $G(r,s,t) = 0$, then G satisfies

$$\overline{M} \cdot \frac{\nabla G}{|\nabla G|} = M_{n,3} = \text{constant} \tag{16}$$

on the surface $G = 0$. However, since the coordinate t is not explicitly involved in any of the coefficients of the derivatives of G in the homogeneous differential equation (16), it is possible to find a function $G(r,s)$ of r and s only, which will satisfy the condition (16). Furthermore, since on the boundary

$$\overline{V}_4 = \overline{V}_3 - a_3\delta_3 \frac{\nabla G}{|\nabla G|}$$

that is,

$$v_{r,4} = \overline{R}_r \cdot \overline{V}_4 = v_{r,3} - a_3\delta_3 \frac{\partial G}{\partial r}\bigg/|\nabla G| = \text{a function of } r \text{ or } s$$

$$v_{s,4} = \overline{R}_s \cdot \overline{V}_4 = v_{s,3} - a_3\delta_3 \frac{\partial G}{\partial s}\bigg/|\nabla G| = \text{a function of } r \text{ or } s$$

$$v_{t,4} = \overline{R}_t \cdot \overline{V}_4 = v_{t,3} = \text{constant}$$

the boundary conditions are compatible with the assumption of a helical flow in the field downstream of the shock surface $G = 0$. That is, v_r and v_s are functions of r and s alone, and v_t is constant. As before, the problem is a two-dimensional one in the r,s-plane (containing ∇r, ∇s).

A typical situation which arises in the calculation is shown schematically in figure 1(b). The velocities are supposed known on a curve def which crosses the shock wave; the problem is to extend the solution first to the point P and subsequently to a new region containing the point P and the extended portion eP of the shock curve. The normal Mach number may be regarded as known, since it is constant and equal to the value determined at the tip (point C, fig. 1(b)) from the velocity in the $\nabla r, \nabla s$-plane and the deflections of the flow at the shock.

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In addition to the shock equations, there are also available equations relating the velocity parameter ν, the direction angle β, and the position of the point P with those of the points d and f, which are located on the characteristics curve η = constant. For trapezoidal integration these equations are

$$\left.\begin{aligned} \nu_3 - \nu_d - \beta_3 + \beta_d + [K_-]_{3,d}(\log r_3 - \log r_d) &= 0 \\ \nu_4 - \nu_f - \beta_4 + \beta_f + [K_-]_{4,f}(\log r_4 - \log r_f) &= 0 \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} s_3 - s_d &= \Big[|\nabla s| \tan (\beta + \alpha)\Big]_{3,d}(r_3 - r_d) \\ s_4 - s_f &= \Big[|\nabla s| \tan (\beta + \alpha)\Big]_{4,f}(r_4 - r_f) \end{aligned}\right\} \tag{18}$$

where the subscripts 3 and 4 refer to the point P upstream and downstream of the shock surface, respectively. The double subscript and square brackets refer to an average between the points. It is necessary to locate d and f in order to obtain by interpolation the values of the quantities required. By utilizing the slope of the shock and the curve η = constant, there are obtained the intersections

$$\left.\begin{aligned} \log r_d &= \log r_e + \frac{[r|\nabla s|\tan(\beta+\alpha)]_{3,d} - [r|\nabla s|\tan(\beta+\theta w)]_{3,e}}{[r|\nabla s|\tan(\beta+\alpha)]_{3,d} - [r|\nabla s|\tan(\beta-\alpha)]_{d,e}}(\log r_3 - \log r_e) \\ \log r_f &= \log r_g + \frac{[r|\nabla s|\tan(\beta+\alpha)]_{4,f} - [r|\nabla s|\tan(\beta+\theta w)]_{3,e}}{[r|\nabla s|\tan(\beta+\alpha)]_{4,f} - [r|\nabla s|\tan(\beta-\alpha)]_{g,f}}(\log r_3 - \log r_e) \end{aligned}\right\} \tag{19}$$

where θw is the wave angle of the shock and $\tan (\beta + \theta w)$ is the slope of the shock wave. By subtraction of the characteristic equations (17) and utilization of

$$\beta_4 = \beta_3 + \varepsilon \tag{20}$$

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where ε is the deflection angle of the flow passing through the shock, there results

$$\nu_3 - \nu_4 + \varepsilon = (\nu_d - \beta_d) - (\nu_f - \beta_f) + [K_-]_{3,d}(\log r_d - \log r_3) -$$

$$[K_-]_{4,f}(\log r_f - \log r_4) \tag{21}$$

Equation (21) may be solved in conjunction with the condition of constant normal Mach numbers and the shock relations by the following process: (1) Assume values of M_3; then find values for ν_3 and α_3 from supersonic-flow tables. (2) Calculate $\theta w = \sin^{-1}(M_{n,3}/M_3)$. (3) From shock tables and values of M_3 and θw, find values for ε, M_4, and ν_4. (4) The correct solution is that for which $\nu_3 - \nu_4 + \varepsilon$ is equal to the value calculated by equation (21).

By iteration a convergence might be reached in which equation (21) and the shock relations are first solved only roughly for values of ν_3, ν_4, β_3, and β_4. Solutions are then obtained for r_d and r_f, followed by interpolation for the required variables at those points. The process is then repeated with refined values of the coefficients. However, this process will often fail to converge for weak shocks because the condition of constant normal Mach number and the condition on $\nu_3 - \nu_4 + \varepsilon$ are nearly the same. Therefore, the solution is nearly indeterminate and will fluctuate widely from one approximation to the next. That these two conditions are not suitable for determination of the shock wave may be seen from the following considerations. When the shock is weak, $\nu_3 - \nu_4 \approx \varepsilon$, and equation (21) gives a value for $\nu_3 - \nu_4 + \varepsilon \approx 2\varepsilon$. Also, a condition of constant normal Mach number corresponds to constant pressure ratio. Graphs of shock-wave solutions will show that, for assigned values of the pressure ratio and ε, the incoming Mach number is nearly indeterminate.

Since for weak shocks the solution is practically indeterminate, an alternate procedure was used, which involved the assumption of a value for one parameter such as β_4. The value of ν_4 may then be determined from the second equation (17). By using the approximation $\nu_3 = \nu_4 + \varepsilon$ in equation (21), a value is then determined for ε; this value is substituted into equation (20) to determine β_3 and $\nu_3 = \nu_4 + \varepsilon$. Tables are used to find M_3 from ν_3, and the wave angle is then calculated from $\theta w = \sin^{-1}(M_3/M_{n,3})$. The results should be checked for consistency with the shock tables. If the agreement is not close enough, a new value for β_4 is assumed.

It has been found expedient in the solution to fix a value for r_3 and to permit s_3 and the points d and f to fluctuate from one iteration to the next, because the locations of d and f vary only slightly thus permitting rapid convergence and because the size of the network is controlled.

When the solution for the shock is started at the casing, the situation is somewhat different in that $\beta = 90^{\circ}$, but a value for ν is not available. However, if the point P is chosen close enough to C that variation in parameters may be neglected from C to the point on the casing corresponding to f, no great increase in computation will result because the size of the interval on the shock curve can be rapidly increased after the solution is begun. Such a procedure is valid, because it can be shown that it results in a solution continuous at the point C. 3242

BLADE AND HUB SURFACES

After the flow field has been obtained, the hub and blade surfaces may be computed by using the property that they are stream surfaces and therefore contain the velocity vector $\overline{V}$. Because the equation of continuity

$$\nabla \cdot (\rho \overline{V}) = 0$$

is satisfied, there exist two independent stream functions u,w which satisfy the equations

$$\nabla u \cdot \overline{V} = \nabla w \cdot \overline{V} = 0$$

These functions may be identified by examining the expanded form of the continuity equation

$$\frac{\partial}{\partial r} (\rho J v^{r}) + \frac{\partial}{\partial s} (\rho J v^{s}) + \frac{\partial}{\partial t} (\rho J v^{t}) = 0$$

where $J \; (= r/\Delta = (\nabla r \cdot \nabla s \times \nabla t)^{-1})$ is the Jacobian of the transformation of coordinates. Because all the quantities $\rho J v^{i}$ are functions of r and s only, then the last derivative is zero and the remainder of the equation indicates the existence of a potential function $u(r,s)$, which is the stream function of the flow. Then,

$$\frac{\partial u}{\partial r} = \rho J v^{s} \qquad \frac{\partial u}{\partial s} = - \rho J v^{r}$$

or

$$du = \rho J (v^{s} dr - v^{r} ds)$$

or

$$\nabla u = \rho J(v^s \nabla r - v^r \nabla s) = \rho \overline{V} \times \overline{R}_t$$

The surface u = constant is therefore the solution of

$$\frac{dr}{v^r} = \frac{ds}{v^s}$$

With the first solution assumed known, then the second stream function w can be found such that

$$\rho \overline{V} = \nabla u \times \nabla w$$

For an arbitrary path element dR,

$$\rho\ dR \times \overline{V} = \nabla u\ dR \cdot \nabla w - \nabla w\ dR \cdot \nabla u = \nabla u\ dw - \nabla w\ du$$

whereas for a path element on the surface u = constant, there results

$$dw = \frac{\rho A \cdot dR \times \overline{V}}{A \cdot \nabla u} = \frac{\rho A \cdot dR \times \overline{V}}{\rho A \cdot \overline{V} \times \overline{R}_t} = -\frac{dR \cdot A \times \overline{V}}{\overline{R}_t \cdot A \times \overline{V}}$$

where A is an arbitrary vector such that $A \cdot \nabla u \neq 0$. If, for example, $A = \overline{R}_r$, then

$$dw = -\ dt + \frac{v^t}{v^s}\ ds = -\ dt + \frac{v^t}{v^r}\ dr$$

where, in the course of the integration, u is to be regarded as constant. In integrated form,

$$w + t = h(u,s) \equiv \int_u \frac{v^t}{v^s}\ ds$$

The coefficients of integration may be exchanged for those computed in the field, so that

$$u = \frac{1}{\Delta} \int \rho\ Ur\ (|\nabla s|\ \sin \beta\ dr - \cos \beta\ ds)$$

Since u may be calculated by integration on any path, the integrations are made on characteristic curves; these curves were chosen because the data have been computed on them. Therefore,

$$ds = |\nabla s|\ \tan\ (\beta \pm \alpha)\ dr$$

and

$$u = \pm \frac{1}{\Delta} \int \rho \, Ur|\nabla s| \, \frac{\sin \alpha \, dr}{\cos (\beta \mp \alpha)} = \pm \frac{1}{\Delta} \int \frac{a^{\frac{\gamma+1}{\gamma-1}} r|\nabla s| \, dr}{\cos (\beta \mp \alpha)}$$

The upper sign is used for integration on curves of η = constant and the lower, for curves of ξ = constant. The sonic speed is given by 3242

$$\frac{a^2}{T^2} = \left(1 - \frac{\gamma-1}{2} W^2\right) = \frac{1}{1 + \frac{\gamma-1}{2} M^2}$$

where

$$T \equiv \sqrt{1 - \frac{\gamma-1}{2} v_t^2/R_t^2}$$

and $M = 1/\sin \alpha$. That is, a/T is taken as the ratio of sonic speed to stagnation sonic speed, which is consistent with the values of ν, α, or M.

After contours of u = constant are established, then at appropriate points on the contours r, a, and α are determined and the following integration is made:

$$h(u,s) = \int \frac{v^t}{v^s} \, ds = v_t \int \frac{ds/|\nabla s|}{R_t^2 \, U \sin \beta} - \int \frac{R_s \cdot R_t}{R_t^2} \, ds$$

with

$$U = a/\sin \alpha = TW = \frac{TM}{\sqrt{1 + \frac{\gamma-1}{2} M^2}}$$

The leading edge of the blade is defined by a curve that lies in the surface $F = 0$ and on which r assumes a set of values $r_h \leq r \leq r_t$, where r_h is the inner wall radius. Consequently, r may be used as a parameter for the curve

$$s_0 = s_0(r_0)$$

$$t_0 = t_0(r_0)$$

where the subscript 0 indicates the leading edge. However, the condition that $F = 0$ establishes a condition on s_0, and the leading edge is given by

$$s_0 = -f(r_0)$$

$$t_0 = t_0(r_0)$$

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When $h(r,s)$ is calculated, the integration on surfaces of constant u may be started from the curve $F = 0$ with a boundary value of $h = 0$, so that $h(r, -f(r)) = 0$, and consequently $h_0 = 0$ and $w_0 = h_0 - t_0 = -t_0(r_0)$. A sequence of values of r_0 will result in a sequence for w_0, s_0 and therefore u_0, so that the the following functional relation is given between w and u, which defines the stream surface in question:

CI-3 back

$$w = -t_0(r_0)$$

$$u = u(r_0, s_0) = u_0(r_0, -f(r_0))$$

A section of this surface at constant t is obtained by assuming values of h, which in turn define $w = h - t$. The stream-surface relation between w and u then gives the correspondence between h and u, thus determining the desired curve. A section at constant r is obtained by assuming values for s, which then define h, u, w, and $t = h - w$. A section at z = constant is determined by substituting the value of t in terms of h and $w(u)$

$$t = h - w(u)$$

in the following relation between z, s, and t:

$$Cs - At = (BC - AD)z$$

Then,

$$Cs - Ah + Aw(u) = (BC - AD)z$$

expresses a relation between r and s that, with

$$A\theta = s - Bz$$

determines the section.

If $A = 0$ and z is constant, then s = constant, $t = \theta = h(r,\theta) - w(u)$ gives the desired relation between r and θ for constant z.

Finally, the factors involved in the selection of the function $t_0(r_0)$ are considered. One factor of importance in fabrication is the wedge angle of the blade as measured between the intersections of the blade surfaces in the region near the leading edge with plane elements normal to the leading edge. These angles are obtainable directly as the deflection angles for oblique shocks where the gas velocities are components normal to the leading edge. If the leading-edge vector is given by 3242

$$\bar{\tau} = R_r + R_s \frac{ds}{dr} + R_t \frac{dt}{dr}$$

the condition that $\bar{\tau}\cdot\nabla F = 0$ determines $ds/dr = -f'(r)$. Then,

$$\bar{\tau} = R_r - f'(r)R_s + R_t \frac{dt}{dr}$$

The magnitude of the velocity required is then $|\bar{\tau}\times\bar{V}|/|\tau|$, and the effective Mach number and wave angle are $\dfrac{|\bar{\tau}\times\bar{V}|}{a|\tau|}$ and $\sin^{-1}\left(\dfrac{M_{n,1}a|\tau|}{|\bar{\tau}\times\bar{V}|}\right)$, respectively. The deflection angle can then be evaluated from standard tables as a function dt/dr and from the initial shock surface.

EXAMPLE

As an example, a flow field was calculated for the following conditions: The free-vortex upstream flow was determined by the assumption of a Mach number of 2.0 at an angle of 45° with respect to the axial direction and lying in a cylindrical surface $r = 1.0$. The curve representing the intersection of the initial shock with the surface $r = 1.0$ was determined by setting $s = \theta$. The shock strength was also assumed by setting $M_{n,1} = 1.0927$.

With no loss in generality, $C = 0$, $D = 1$. The formula for $K_\pm$ reduces to

$$K_\pm = \pm \tan (\beta \mp \alpha)$$

In constructing the network of points not on a shock, the first calculation gave accurate values of ν_c, β_c, r_c, θ_c when the assumptions that $[K_+]_{c,a} \approx [K_+]_a$ and $[K_-]_{c,b} \approx [K_-]_b$ were used. However, the coefficients $[K_+]_c$, $[K_-]_c$, $\tan(\beta_c - \alpha_c)$, and so forth, were calculated and one iteration executed as a check. This calculation was based on a net with an increment in r of approximately 0.05 between adjacent computed points of a characteristic curve. When a point for extension of the shock curve $G = 0$ was calculated ($F = 0$ was calculated by a direct integration), the first iteration gave an accurate answer and a second was required for a check. If good estimates are made on the variation of the coefficients, the first iteration could serve as a check on the first integration.

The net of characteristic curves and the initial and reflected shocks are drawn in figure 2 in the surface normal to the diffuser axis and viewed in the direction of flow. The Mach number contours resulting from the velocity component normal to the surface downward through the sheet and the calculated r,θ components are also shown. These contours are, of course, discontinuous at the shocks.

Stream functions are shown in figure 3. The values of u have been divided by u_h, which is the value of u for the entire flow in the region $0.7 \leq r \leq 1.0$ upstream of the first shock, in order to indicate more clearly the equal increments into which the flow has been divided. Similarly, the contours of h = constant are shown with values of h divided by a displacement of z equal to 0.06556 times the tip radius. If a blade is assumed to pass through the shock surface $F = 0$ at $z = 0$, then the h contours are blade pressure-surface sections at constant z, spaced at axial-distance increments of 0.06556 times the tip radius. If 20 blades are assumed, then at $r = 1.0$ the preceding blade is encountered at $\theta = -18^{\circ}$, which corresponds to $h/0.06556 = -4.8$. These sections were computed and then shifted 18° to indicate the pressure and suction surface of a blade at the same z-location rather than the channel between two blades. The hub surface to the right indicates the hub at the suction surface.

A different shape for the blade can be obtained by assuming $w = w(u)$ instead of $w = 0$. For example, if the pressure surface of the blade is to contain a radial line, then, when $z = 0$, θ must be zero. Therefore, on the initial curve, $w(u)$ is established by

$$w(u) = h(u,\theta) - z = h(u,0)$$

From this equation, h can be determined for each pair of values u,z, and consequently the value of θ can be determined from the value of $h(u,\theta)$ and of u. Curves of this type are shown in figure 4, where the blade sections are taken at equally spaced values of z. The suction side of the blade is computed as for the previous example (fig. 3), with the assumption of 20 blades.

EVALUATION OF METHOD

Calculation of helical flows by characteristics requires more time than for cartesian flows, but the time involved is not prohibitive. The main advantage of the method is that it gives rigorously a more general class of flows than have been heretofore available by two-dimensional methods. It is now possible in a larger variety of circumstances to give a qualitative description of actual flows in three-dimensional machines and to broaden the approach to the design problem. As a design method, it has the disadvantage of not permitting the description of the blade shape, hub, and casing surfaces in advance of the calculation. However, once a flow is computed, a variety of blade shapes may be computed, depending upon the assumed spacing and leading-edge orientation. These blade shapes include swept and tilted blades, thus permitting a new degree of freedom to the designer.

A restriction on shock-wave orientation for cascade design is evident on comparison with the class of uniform flows. These uniform flows may be joined by plane shock surfaces of any orientation provided that the normal Mach number is supersonic. The helical flows, on the other hand, when joined by shocks of uniform strength, generally give rise to flows with a variable helical covariant velocity component v_s in addition to the radial component. Shock surfaces are therefore limited to the class containing the vectors $\bar{R}_t$ normal to both ∇s and ∇r. That is, the coordinate s must not be changed as long as v_s is not constant. When v_s is reduced to a constant, then a new class of shocks may be used and the variable s may be modified. In the case of the annular cascade this does not represent a limitation in design procedure, because the uniform flows are inapplicable and the helical flows are required.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, September 14, 1954

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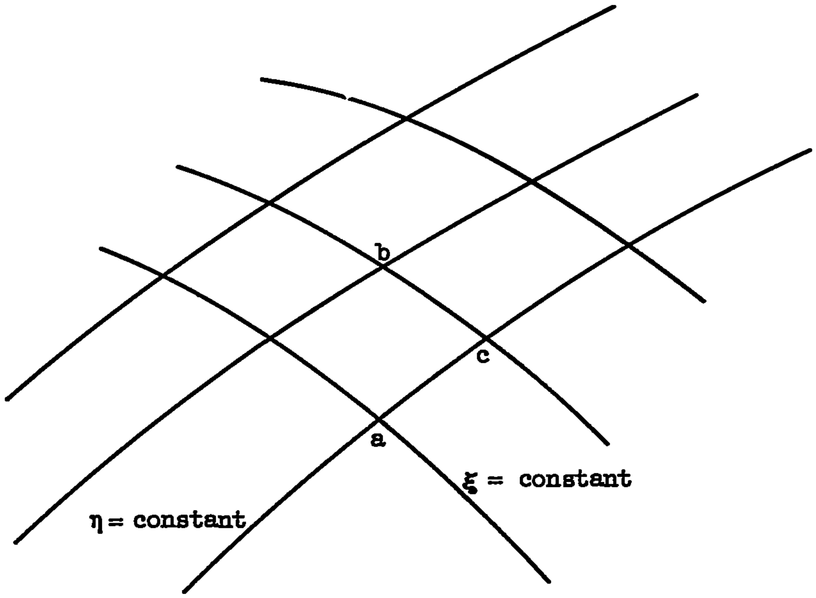
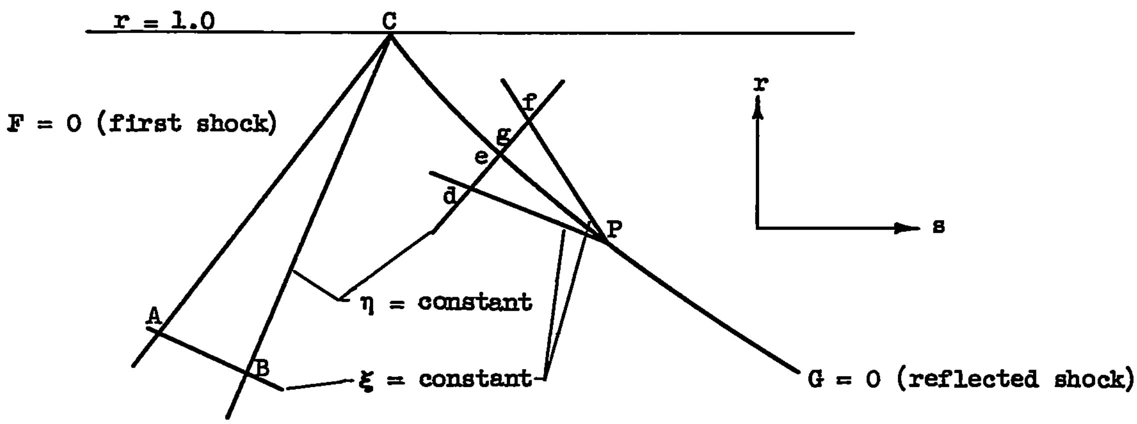


(a) Interior point in continuous flow region.



(b) Boundaries and shock waves.

Figure 1. - Notation for points in flow field.

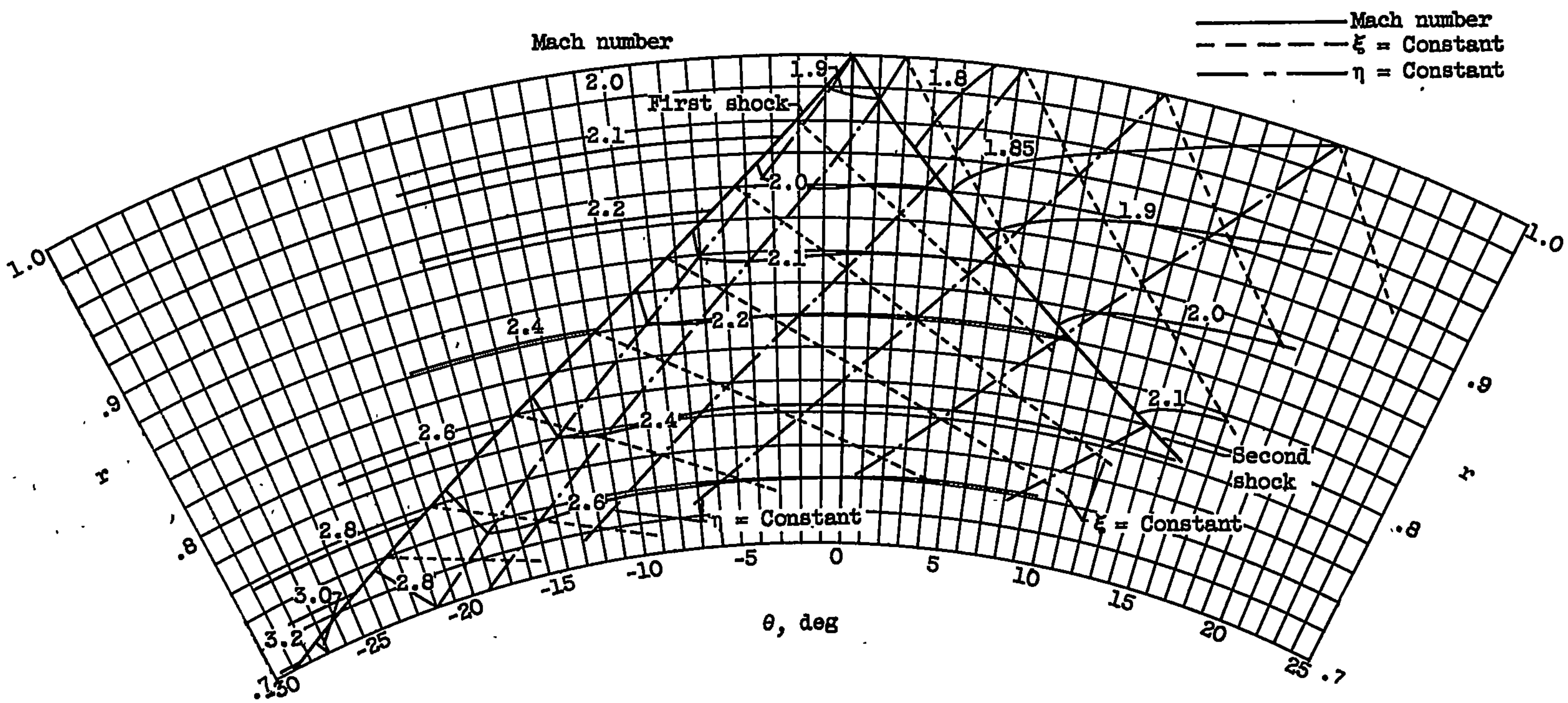


Figure 2. - Characteristic curves, shock curves, and constant Mach contours.

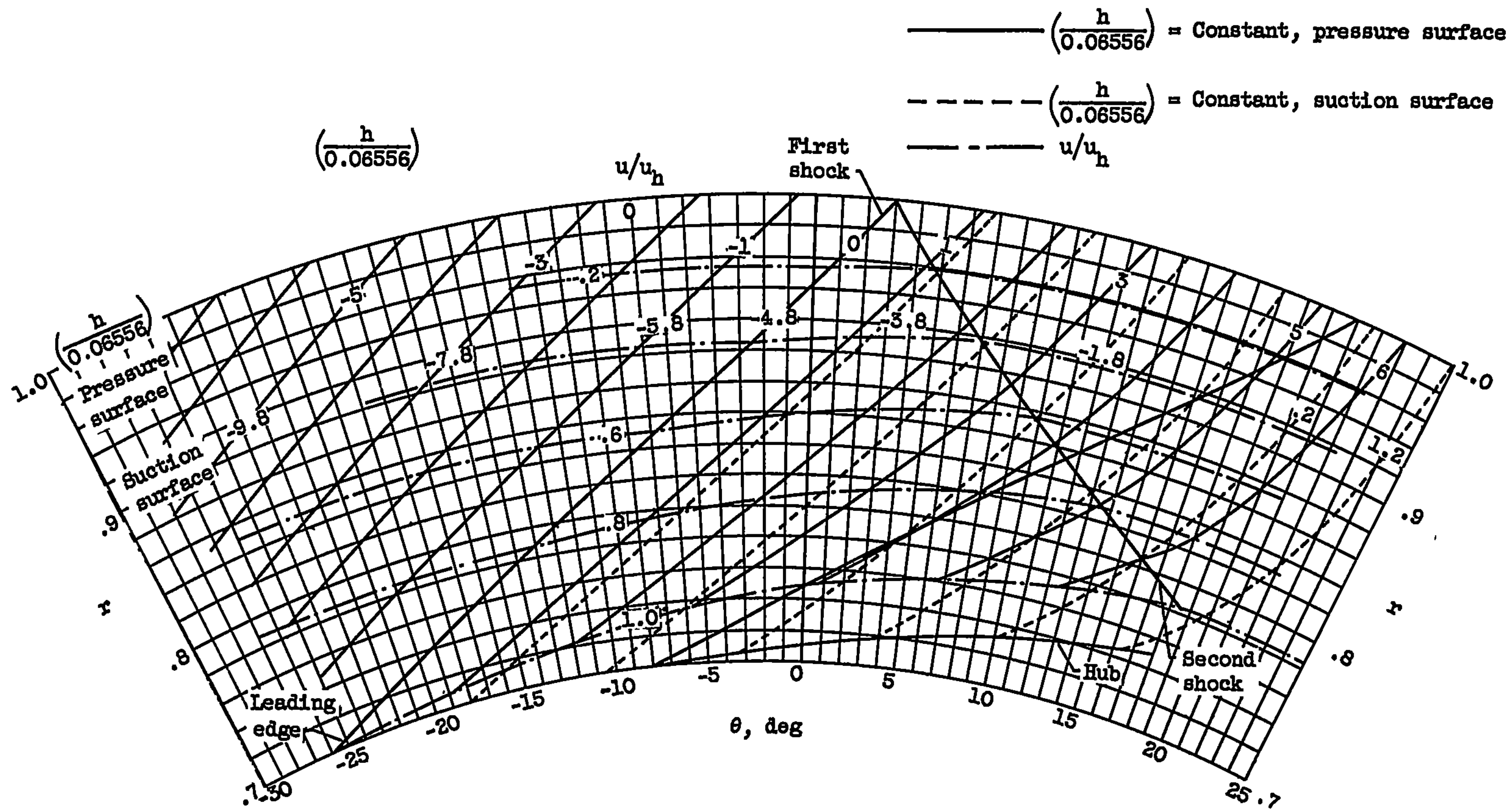


Figure 3. - Stream functions u, h and axial sections of blades 18° apart passing through z = 0 on initial shock.

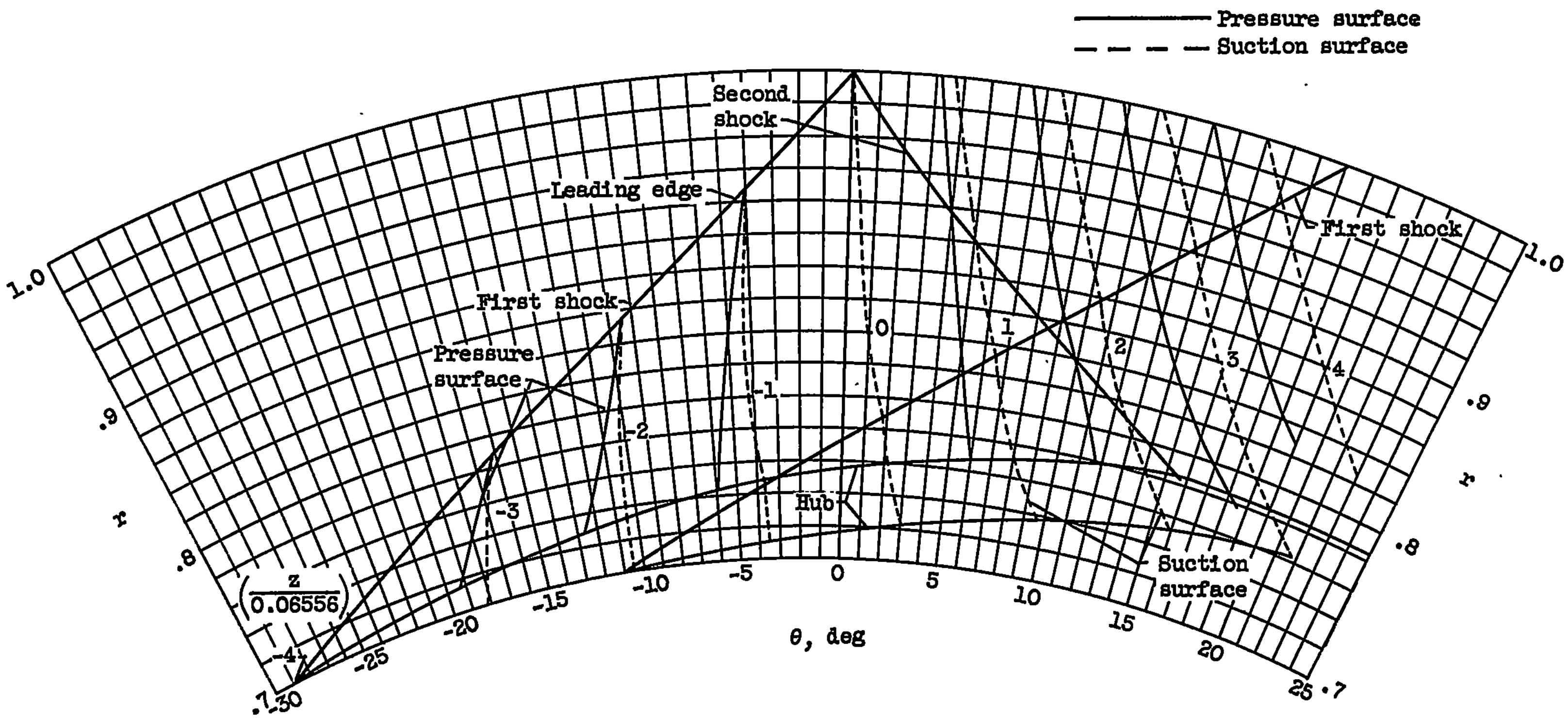


Figure 4. - Axial sections of blades 18° apart and passing through $\theta = 0$, $z = 0$.

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