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A UNIVERSAL COLUMN FORMULA FOR LOAD AT WHICH YIELDING STARTS

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SUMMARY

An analysis is presented of the load at which yielding first occurs in actual columns, taking adequately into account all the factors which have an important effect upon this load. These factors include initial defects and the yielding limit of materials. Extensive tests were made to verify the assumed relation between the magnitudes of the defects and the known properties of columns. The results are expressed as a formula or chart applicable to all cases.

INTRODUCTION

Investigation of the buckling of columns began in 1744 with Euler's famous theory. Although a large amount of work has been done on this problem since that time, the amount of progress from the designer's standpoint seems surprisingly small. The classical stability studies initiated by Euler and later extended to cover various types of end conditions, variations in cross section, and so forth, consist in the determination of the conditions for neutral equilibrium, under infinitesimal displacement, of a perfectly homogeneous elastic column loaded along a perfectly straight elastic axis. Classical stability theories have been found to be satisfactory for predicting the ultimate strengths of "long," that is, very slender, columns. However, for medium or short columns the defects always present in actual columns and the limitations to the elastic behavior of actual materials, factors which are not considered in the idealized classical stability theories, become of great importance. For such columns, which include most practical applications, designers still rely upon empirical results expressed in the form of curves or formulas, each curve or formula being of limited applicability. These empirical results also determine the range of applicability of the classical stability theories and, hence, must be made use of even when applying these theories.

Buckling problems present certain difficulties by their very nature, but the case of the column is the simplest of such problems; and there seems to be no very good reason why a rational universal column theory should not be developed which would apply equally to all columns and take into account all the factors which actually have an important influence upon the results. Such a "theory" would, of course, like all theories, include a number of empirical factors or relations which would have to be determined from new or existing experiments; even the classical stability theories depend upon the empirically determined stress-strain relations of elastic materials. However, the amount of empirical information required to give such a theory universal applicability would be very small compared with what would be required by purely empirical methods. Such a "universal theory" might be somewhat inconvenient to use for design purposes in its complete form, but for the limited ranges for which present empirical methods apply it would certainly reduce to something of comparable simplicity. The theory could thus replace present design methods in these reduced forms even if it were impractical for direct use.

The advantages of such a development would go far beyond the mere replacement of one satisfactory design method by a no more satisfactory but more "elegant" method. For example, there is now no way to compare one set of empirical results with another set covering a different range. Yet, in many fields of engineering such comparisons can be made and prove of great value in bringing to light and making suitable allowance for errors and the effects of variations in testing technique and in the interpretations which different investigators put on test results, variations which always exist when tests are made and interpreted by different people at different times and places.

The main advantage of such a development would, however, be the same as appears in any field when empirical results are supplemented by adequate general theory. Experimental results are necessarily of limited range. Because of the number of variables involved, presently available data on columns - in spite of the great number of tests which have been made - cover only a small fraction of possible cases. Only an adequate theory can permit safe extrapolation, and the existence of such a theory should release designers from design limitations of which they may not even be aware.

Two general criteria are in common use for defining the static strength of the parts of machines and structures for design purposes. One is based upon the loads at which yielding of the material first starts; the other, upon the maximum loads which can be withstood. The first criterion seems logical to use as a basis for design of close fitting machine parts which "fail" insofar as serving their purpose is concerned if an appreciable permanent change of shape occurs. The second criterion seems the most logical to use in the design of structures for which the exact shape is of relatively little importance compared with the ultimate strength.

Since columns are important elements in both machines and structures there should evidently be not one but two column theories, one for the column load at which yielding starts (for which little information exists at present) and the other for the ultimate column strength. The present paper is intended to supply the first need, namely, a rational analysis, supported by tests of a special type, of the load at which yielding first occurs in actual columns of any type, taking adequately into account all the factors which have an important effect upon this load.

Although ultimate strengths will not be covered, it is of interest here to consider briefly the problem of developing an ultimate-strength theory. Up to the load at which yielding starts the action of a column is everywhere elastic. Between this load and the ultimate load, part of the column is in the elastic state and part in the plastic state (assuming that the material has some ductility; if not, the two loads coincide). It is not too difficult to analyze satisfactorily this elastic-plastic action for particular cases, and many such analyses have been made; but it is much more difficult to set up a general theory covering all columns, especially considering the widely varying behavior of different materials in the plastic range.

However, it seems to be general experience that the ultimate strength of long columns is only a little below the classical stability value, while the ultimate strength in the medium range is probably only a little above the load at which yielding starts. Only for very short columns, approaching something which would usually be thought of as "blocks" rather than columns, should the ultimate strength differ very greatly from some other known value. Hence, it may be possible to develop a sufficiently inclusive ultimate-strength theory by studying in a relatively approximate manner the small differences between the ultimate load and other known quantities. The difficulty, of course, is to choose the approximations so as to preserve reasonable fidelity over the great range of variables required to make such a theory truly "universal."

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SURVEY OF PREVIOUS WORK

Before detailing the present work some discussion should be made of previous efforts along these lines (refs. 1 to 9). While such work has shown promise, it has, in the authors' opinion, suffered from certain deficiencies which have largely vitiated its usefulness. The distinction between the load at which yielding starts and the ultimate load seems to have been given inadequate consideration. Theories have been derived for load at which yielding starts and the results of these theories have been compared with ultimate load data to determine the empirical factors defining the magnitude of expected defects. Where direct measurements of defects have been made, they have been confined to geometric crookedness; and other kinds of defects, which the present tests show to have as great an effect as crookedness, have been neglected.

The relations which have been assumed between the magnitudes of defects and the known properties of the columns also seem both unreasonable and founded upon inadequate data; it has usually been assumed that defect magnitude is a function of length only or of a cross-sectional dimension only or that it is a sum of independent functions of these dimensions, whereas certainly the effects of these dimensions are actually interdependent and other important factors influence the defects. Little thought has been given to putting results in convenient general form or to studying such matters as the effects of end conditions and variation in cross sections or of the less important components of the defects, all of which must be given adequate consideration before the generality of any theory can be considered to be established.

TESTS

Specimens

Because of the large amount of scatter to be expected in the quantities to be measured - the defects in columns - it was necessary to test a large number of specimens. All specimens were tested as columns, and measurements were taken of their deviation from straightness, initially and under load. These slender specimens of rectangular cross section were made of cold-rolled mild-steel bar stock, cold-rolled 2024-T3 (24S-T3) aluminum-alloy sheet, and cold-rolled 7075-T6 (75S-T6) aluminum-alloy sheet; all were of standard manufacture and cut and were handled carefully to avoid introducing any defects not already present. Although these specimens were in the long-column range, measurements of the second and third harmonics carried the data obtained into the medium-column range.

To eliminate questions regarding the artificial introduction or suppression of eccentricity at the ends, which may arise when hinged-end columns are tested - for instance, eccentricities can be introduced which add to or partially counteract initial curvatures - all columns were tested with built-in ends, as is the case in most practical applications. To simplify the tests and eliminate systematic errors due to friction in the measurement of end moments, the tests were made with 100-percent end fixity. Such tests, however, are subject to systematic errors due to deformations in the specimens or clamps at the point of clamping. To eliminate these errors the specimens were held in loading heads at some distance outside the points which were taken as the ends of the specimens, and small mirrors attached at these points detected any rotation, which was then brought to zero by rotation of the loading heads. While this system, of course, permits errors, it eliminates the systematic errors which might seriously affect the statistical information desired.

Description of Apparatus

Figure 1 shows a diagrammatic sketch of the loading apparatus and the optical system used for detecting rotations of the ends of the effective length of the specimen. The telescope is focused upon the image of the scale reflected through the back mirror and small mirror on the specimen. With the back mirror placed about 10 feet from the apparatus rotations of the small mirror of the order of 0.001° produce detectable shifts of the scale point seen against the telescope hairline.

In the photographs of figure 2, the specimen is shown at (a), with the small mirrors defining the effective length at (b), and with the end clamps in the loading heads at (c). The load can be measured by the dial gage (d) which measures the deflection of the flat springs (e); the working sections of these springs are machined down from a thicker stock, with fillets at the ends, which largely eliminates hysteresis. The screw (f) advances the loading head to adjust the axial load, while the screw crank (g) rotates the loading head about the axis (h) to bring rotations of the small mirrors to zero.

Deviations from straightness are measured by the micrometer screw (i) attached to the carriage (j) which moves upon a track formed of tightly stretched piano wires (k). The micrometer carries a silver-plated tip upon its end (l); when this tip touches the specimen an electrical circuit is completed. By using a galvanometer in this circuit, measurements can be made which are accurate to a fraction of a thousandth of an inch.

During measurement of the deviation from straightness in the initial no-load condition, in order to insure freedom from accidental end forces and moments, the specimens were held only at the center by a narrow clamp.

Since some of the specimens were very flexible, the weights of the two ends of the specimen were balanced by overhead floats at the quarter points, as shown in figures 3(a) and 3(b). Measurements of deviation were made at the center line of the specimens at eight points along the length, as shown in figure 4.

RESULTS AND DISCUSSION

As mentioned previously, the inadequacy of classical stability analysis lies in the neglection of the limit to elastic action of actual materials and the defects always present in actual columns; the defects cause bending stresses to develop before the stability limit is reached, and these stresses combine with the direct stress (and with any initial stress which may be present at the critical point) to precipitate early yielding.

From the standpoint of column bending the important defects are geometric crookedness, lack of elastic homogeneity, and accidental eccentricity of loading. All of these have a similar effect in producing an initial deviation of the elastic axis of the bar from the straight line joining the points of application of the resultant axial loads, which is called herein the "load line."

In a perfectly homogeneous column the elastic axis, which defines the shape of the column for purposes of analysis by classical bending theory, would pass through the centers of gravity of cross sections and share the geometric crookedness of the outer surface. Because of elastic inhomogeneity from slag inclusions, gas bubbles, and so forth, and because of the variation in elastic properties in the axial direction due to the random orientation of the highly anisotropic crystals of which most engineering materials are composed, the true elastic axis will suffer an additional deviation from these centers of gravity, passing in effect through the centers of gravity of cross-sectional areas weighted according to the local stiffness in the axial direction. Eccentricity of loading shifts the load line and thus produces an additional deviation of the elastic axis from this line, as illustrated in an exaggerated manner in figure 5.

For purposes of this investigation all these causes of accidental deviation[1] can be lumped together. This total initial deviation of the elastic axis from the load line is designated by the symbol W (as

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[1]Lateral loading and built-in eccentricities also have similar effects, and it will be shown that they can be taken into consideration along with the defects; however, the latter are the main concern herein, since their evaluation is obviously the difficult problem.

distinguished from the movement under load w) and called herein simply the "deviation." The starting point of any general column theory must be the establishment of laws relating the magnitudes of the important constituents of the deviation to the characteristics of columns on which they depend.

Consider now the best way to measure the deviation and the characteristics of columns which affect it.

The deviation W will be some function of the distance x along the load line, a different function for each column. The most convenient way to describe this function is by the amplitudes of its harmonic components, and this proves also to be the most useful way to consider its effect upon the buckling process. In the tests, details of which are presented in appendix A, the amplitudes W_m of harmonic components of the deviation of half wave length l_m were measured over lengths of bar corresponding to one wave length of the component. This was done by testing lengths of the bars as columns and using an extension of Southwell's method (ref. 10) which had previously been developed in reference 11. A large number of lengths and thicknesses of bars were tested; the bars were made of three different standard materials processed by standard methods. As expected, the deviation components were found to depend very much upon the thickness and wave length, the components with larger wave lengths compared with the thickness averaging larger in amplitude than the shorter ones.

Experience has shown that, if a number of similar columns are tested which are as nearly identical in every way as it is possible to make them, their strengths will vary considerably, but quite definite average and limiting (that is, maximum and minimum) strengths can be determined. If the deviation components are measured, a corresponding variation (which is the chief cause of the variation in strength) will be found, and again quite definite average and limiting values can be determined for the amplitudes of each harmonic component. This is what is meant by "average" and "limiting" values of such quantities. The variations from the average represent true irreducible scatter, which can never be predicted. However, the average deviations can be allowed for, and the scatter in strength can be allowed for in a more rational and economical way than by blanket factors of safety by taking into consideration the maximum deviations which produce the minimum strengths.

If a series of related columns, identical except for a dimension or some other characteristic which can be varied continuously, is tested and the amplitudes W_m of deviation components are plotted against this characteristic, average and limiting curves can be determined, which describe the function by which the average and limiting values of W_m are related to this characteristic. If the relation between the average and limiting values of W_m and all the column characteristics which influence them can be determined, proper allowances can be made and uncertainty in design can be reduced to true scatter. Insofar as these factors are not determined and proper allowances are not made, the uncertainty regarding the effect of any characteristic is added to the true scatter.

The characteristics of columns upon which the deviation depends might be classified as follows: length and end conditions, size and shape of the cross section, the material, and the process by which the column is fabricated (which, of course, includes methods of straightening, if any, standards of inspection, etc.). The first two, length and end conditions, determine the wave lengths which are important in the buckling process and, hence, have a very important indirect influence upon the deviation; however, these characteristics are fully taken care of if the effect of the wave lengths of the deviation components upon their amplitudes is considered.

The shape of the cross section will usually be associated with the fabrication process, and this in turn is likely to depend upon the material; these three characteristics are thus closely associated. In general, it is impractical to vary these characteristics continuously or describe them by numbers. Hence, their effect upon W_m, while it may be real and important, cannot well be expressed analytically but can best be described and taken into account by a numerical coefficient, which is herein designated by C or K and whose value can be tabulated for important distinct combinations of these characteristics.

Finally, the size of the cross section can, like the wave length, be described by a number, and its effect upon W_m can theoretically be expressed analytically. For columns of a given shape of cross section (that is, for geometrically similar cross sections) the size of the cross section can be described equally well by any characteristic cross-sectional dimension, such as thickness t, distance from the neutral axis to the farthest fiber c, or radius of gyration ρ (all taken for the direction of buckling being investigated).

The desired functional relation thus should involve a numerical coefficient and three distances W_m, l_m, and, say, t. Since it must be dimensionally consistent there is no loss in generality if it involves only any two independent ratios between these distances, say W_m/t and l_m/t. It seems logical to try first a power-function relation between these two ratios, which can be expressed as

$$\frac{W_m}{t} = C\left(\frac{l_m}{t}\right)^n \qquad (1)$$

where C and n are to be determined. It seems likely that the exponent n depends upon broad probability factors and may be substantially constant for all columns.

Figures 6(a) to 6(c) show measured values of W_m/t and l_m/t plotted against each other on a logarithmic scale. Points labeled 1, 2, and 3 were obtained, respectively, from the magnitudes of the fundamental component and first two harmonics of the total deviation in the test bars. The plots show, as is to be expected, a great amount of scatter, but they also indicate a definite tendency for W_m/t to increase rapidly as l_m/t increases. The lines marked "max." describe the trend for the higher points. The lines marked "av." should have a somewhat steeper slope, corresponding to a larger value of n in equation (1), to fit the points best. However, these tests cover the range of wave lengths important for medium and long columns but not for short columns. The lines shown, when extrapolated into the short-column range, give results which are in line with the empirical curves and column formulas in common use, while steeper curves would be less conservative; in the absence of data on short columns it seems reasonable to use the relations given by the lines shown. These lines correspond to a value of 2 for the exponent n in equation (1) and values of C of about 0.00003 for the maximum lines and 0.000007 for the average lines. Even this value of n is larger than the values of 0 and 1 which were assumed (on the basis of practically no evidence) in the references previously cited, except for a recent paper (ref. 12) in which the value of 2 was proposed.

In the appendix B the following general formula is derived for the load upon a column at which yielding starts:

$$P = N - \sqrt{N^2 - P_y P_{cl}} \qquad (2)$$

where

$$2N = P_y + P_{cl}\left[1 + \left(cW_1/\rho^2\right)\right]$$

In this formula $P_y = AS_y$ is the cross-sectional area A times the yield stress S_y (which may be defined in any way desired and reduced to allow for initial stresses when this seems justified, as discussed in appendix B), P_{cl} is the buckling load given by classical stability theory (defined as in appendix B in case of a distributed load), P is the correspondingly defined load at which the stress S_y is reached at the most highly stressed point, and W_1 is the amplitude of that harmonic component of the deviation which has the same half wave length l_1 as the fundamental (longest) harmonic component of the buckling deflection predicted by classical stability theory. The length l_1 is what has been called the "reduced" or "equivalent hinged column" length, so that

$$P_{cl} = \pi^2 EA\rho^2/l_1^2 \tag{3}$$

To simplify the final results it is convenient to substitute for equation (1) the following equivalent relation:

$$\frac{cW_m}{\rho^2} = \left(\frac{K^{1/2} l_m}{\pi\rho}\right)^n$$

$$= \frac{Kl_m^2}{\pi^2\rho^2} \qquad (n = 2) \tag{4}$$

Using this with equations (2) and (3), the expression for load at which yielding starts becomes

$$P = N - \sqrt{N^2 - P_y P_{cl}} \tag{5}$$

where

$$2N = P_y + P_{cl}\left[1 + \left(\frac{KEA}{P_{cl}}\right)^{n/2}\right]$$

$$= P_y + P_{cl} + KEA \qquad (n = 2)$$

For some purposes it is more convenient to write this equation in terms of stresses. Dividing through by the cross-sectional area A gives

$$S = Q - \sqrt{Q^2 - S_y S_{cl}} \tag{6}$$

where

$$2Q = S_y + S_{cl}\left[1 + \left(\frac{KE}{S_{cl}}\right)^{n/2}\right]$$

$$= S_y + S_{cl} + KE \qquad (n = 2)$$

$S = P/A$ is the average stress at which yielding starts, and $S_{cl} = P_{cl}/A = \pi^2 E\rho^2/l_1^2$ is the average stress given by classical stability theory (that is, the stress at which instability would occur if the column were perfect).

Equation (5) or (6) can readily be put into a form involving only three nondimensional ratios, say, $S/S_y = P/P_y$, $S_{cl}/S_y = P_{cl}/P_y$, and KE/S_y, as follows:

$$\frac{S}{S_y} = q - \sqrt{q^2 - \frac{S_{cl}}{S_y}} \tag{7}$$

where

$$2q = 1 + \frac{S_{cl}}{S_y} + \left(\frac{KE}{S_y}\right)^{n/2}\left(\frac{S_{cl}}{S_y}\right)^{1-\frac{n}{2}}$$

$$= 1 + \frac{S_{cl}}{S_y} + \frac{KE}{S_y} \qquad (n = 2)$$

These equations, or other equivalent forms, represent a true "universal theory" for column load at which yielding starts. Equation (7) can easily be put in chart form; figure 7 shows such a chart for the case $n = 2$, while figure 8 shows how such a chart would be affected by different values of n. These charts can be considered to be generalizations of the familiar chart of average stress versus slenderness ratio and cover the full range from zero to infinite slenderness ratio.

An interesting point brought out by these charts is that only with values of n less than 2 would the loads at which very long columns first yield approach the classical stability values. If $n = 2$ they approach values which are equal to $P_{cl}/\left[1 + \left(KE/S_y\right)\right]$. For values of n greater than 2 they would approach zero. It is common experience that ultimate loads of very long columns do approach the classical stability values, but it seems probable from the above that yielding starts at considerably lower values.

Calculations can readily be made from equation (5), (6), or (7) or charts such as figure 7, using values of K from tables, of which table I may be regarded as a first step; K may be regarded as a "roughness factor," measuring the general roughness of construction. It is a pure number, depending upon the associated factors of cross-sectional shape, material, and fabrication process; average and limiting values of K can eventually be determined for all the combinations of these factors of practical importance. This is a large order which, however, it will be quite practical to fill in a fairly inclusive manner by using the extensive column data in the literature, that is, by calculating the value of K required to make the theory fit such data; these calculations, however, will have to wait upon the extension of the theory to cover ultimate loads, since only ultimate-load data seem to be available.

It is expected that K will not differ widely for variations within broad categories such as might be described by the words "refined construction," "average construction," and "rough construction" and that a broad survey of available data, involving the determination of a single number to characterize each type of construction, could permit a consolidation of information, with the elimination of many discrepancies, and a final relatively simple tabulation from which engineers could choose values applying closely to any situation.

The values of K determined for the small range of column types which the present tests cover represent a start in this direction, but the main purpose of the tests was to check the general form of equation (1) and determine a reasonable value for the exponent n. As has been mentioned, n likely depends upon broad probability laws and is subject to little variation. The tests seem to bear out this view. Failure to use the most suitable value of n increases the gap between the limiting values of K; that is, the proper choice of n is a means of reducing unpredictable scatter to the minimum.

Illinois Institute of Technology,
Chicago, Ill., November 6, 1953.

APPENDIX A

DEVELOPMENT OF HARMONIC ANALYSIS

From the principles of harmonic analysis, harmonic components $D_m \cos m\pi x/l$ (where $m = 1, 2, 3, \ldots$) of a deviation $D(x)$ of the specimen will have amplitudes

$$D_m = (1/l) \int_0^{2l} D(x) \cos m\pi x/l \, dx$$

$$\approx \frac{1}{4}\left(a_1 \cos \frac{m\pi}{8} + b_1 \cos \frac{3m\pi}{8} + c_1 \cos \frac{5m\pi}{8} + d_1 \cos \frac{7m\pi}{8} + \right.$$

$$\left. d_2 \cos \frac{9m\pi}{8} + c_2 \cos \frac{11m\pi}{8} + b_2 \cos \frac{13m\pi}{8} + a_2 \cos \frac{15m\pi}{8}\right) \qquad (8)$$

In particular, the first three harmonic components will be

$$\left.\begin{aligned} D_1 &\approx 0.231\left[\left(a_1 + a_2 - d_1 - d_2\right) + 0.414\left(b_1 + b_2 - c_1 - c_2\right)\right] \\ D_2 &\approx 0.177\left(a_1 + a_2 - b_1 - b_2 - c_1 - c_2 + d_1 + d_2\right) \\ D_3 &\approx 0.231\left[0.414\left(a_1 + a_2 - d_1 - d_2\right) - \left(b_1 + b_2 - c_1 - c_2\right)\right] \end{aligned}\right\} \qquad (9)$$

These formulas permit the determination of the harmonic components of the deviation from straightness of the outer surface of the whole specimen, initially and under load. In the tests only the symmetrical components of the deviation such as those given by equation (9) were studied, since the nonsymmetrical buckling modes of a fixed-end column are less simple and easy to study by the present methods, and these symmetrical components covered as great a range as could have been covered by considering the nonsymmetrical modes. For components such as those given in equation (9) it makes no difference whether the distances a_1, b_1, c_1, . . . are measured from the load line or from any other parallel or nearly parallel straight line, since a linearly varying deviation contains no such components.

The following definitions are helpful in discussing the method used for determining the total deviation, including the part due to inhomogeneities:

W' geometric deviation or crookedness, that is, the initial deviation of the median line of the column from the load line

W'' nongeometric deviation (due to inhomogeneities); that is, the initial deviation of the elastic axis from the median line

$W = W' + W''$ total initial deviation of elastic axis from load line

w movement due to load

These are illustrated diagrammatically in figure 9 for the no-load and loaded condition of a fixed-end strut such as that used in the tests. General expressions for w and W (with similar expressions for W' and W'') can be taken as

$$\left.\begin{aligned} w &= w_o + v_o\,x/l + \sum_m w_m \cos m\pi x/l + \sum_p v_p \sin p\pi x/l \\ W &= W_o + V_o\,x/l + \sum_m W_m \cos m\pi x/l + \sum_p V_p \sin p\pi x/l \end{aligned}\right\} \tag{10}$$

The moment equilibrium equation of elementary bending theory is

$$-EI\,d^2w/dx^2 = M = M_o + S_o(x + l) + P(W + w) \tag{11}$$

and the boundary conditions are $x = \pm l$

$$w = dw/dx = 0$$

Substituting expressions (10) into these equilibrium and boundary conditions and using the relation $P_{cl} = 4\pi^2 EI/(2l)^2 = \pi^2 EI/l^2$ give

$$\begin{aligned} &\sum \left\{W_m - \left[\left(m^2 P_{cl}/P\right) - 1\right] w_m\right\} \cos m\pi x/l + \\ &\sum \left\{V_p - \left[\left(p^2 P_{cl}/P\right) - 1\right] v_p\right\} \sin p\pi x/l + \\ &M_o/P + S_o(x + l)\big/P + (W_o + w_o) + (V_o + v_o)x/l = 0 \end{aligned} \tag{12}$$

$$w_o \pm v_o + \sum w_m \cos m\pi \pm \sum v_p \sin p\pi = 0$$

$$v_o \pm m\pi \sum w_m \sin m\pi + p\pi \sum v_p \cos p\pi = 0$$

These relations are satisfied if, and in general only if, $m = 1, 2, 3 \ldots,$ $p = 1.43, 2.46, 3.47 \ldots$ and

$$\left.\begin{aligned} V_p - \left[\left(p^2 P_{cl}/P\right) - 1\right] v_p = 0 \\ W_m - \left[\left(m^2 P_{cl}/P\right) - 1\right] w_m = 0 \end{aligned}\right\} \tag{13}$$

Measurements of $a_1, b_1, c_1, \ldots$ and use of equations (9) in the no-load condition give W_m', while similar measurements and calculations under a load P give $W_m' + w_m$; subtraction of these gives w_m. Knowing P and calculating P_{cl} from the dimensions and modulus of elasticity of the material, the amplitude of the total deviation components W_m can then be obtained from the last equation in equations (13). In practice, however, it was found easier and more accurate to measure W_m' and two values $W_m' + w_{ma}$ and $W_m' + w_{mb}$ under two widely different loads P_a and P_b. The term P_{cl} can then be eliminated between the two relations

$$W_m = \left[\left(m^2 P_{cl}/P_a\right) - 1\right] w_{ma}$$

$$W_m = \left[\left(m^2 P_{cl}/P_b\right) - 1\right] w_{mb}$$

giving

$$W_m = \frac{w_{ma} w_{mb}\left(P_a - P_b\right)}{w_{ma} P_b - w_{mb} P_a} \tag{14}$$

With this formula for W_m, all measurements required are of the same type and only relative values are needed for the loads P_a and P_b.

In figure 10 values of the ratio W_m/t obtained for the 2024-T3 specimens, are plotted against l_m/t, where $l_m = l/m$ is the half wave length of each harmonic component. Points labeled 1, 2, and 3 give, respectively, the magnitudes of the fundamental component and first two harmonics of the total deviation, in bars of length $2l$. This information is needed in setting up the theory for the buckling of bars of length $2l$; the fundamental component is by far the most important component, but the higher harmonic components have some effect upon the bending stresses produced; and this effect must be evaluated (considered in appendix B) before it can safely be disregarded.

It was also desired to use the information obtained regarding the size of the higher harmonics in order to extend the data regarding the size of the fundamental components into the range of shorter columns.

This, however, cannot be done directly; that is, the average and limiting magnitudes of the third harmonics of bars of length $2l$ are not necessarily the same as those of the fundamental components of bars of one-third this length. If a bar of length $2l$ is divided into three sections and the fundamental components are determined for each section, then the algebraic averages of their three magnitudes should be the same as that of the third harmonic found from the original bar. In many cases, however, the fundamental components of the short sections will be of opposite sign and will cancel each other as far as the third harmonic of the original bar is concerned. For the purpose of extending the data regarding fundamental components into the range of shorter bars, the absolute values of the fundamental components of fractions of the bars are needed. These values could be obtained for the geometric deviation W' merely by using known data to make separate harmonic analyses for each fraction of the bar. By the same principles as those expressed in equation (8) the average of the absolute values of the fundamental component of each half or third of a bar is

$$\left.\begin{aligned} D_2' &\approx 0.177\left(\left|a_1 - b_1 - c_1 + d_1\right| + \left|a_2 - b_2 - c_2 + d_2\right|\right) \\ D_3' &\approx 0.231\Big[\left|0.414(a_1 - c_1) - (b_1 - c_1)\right| + \\ &\quad 0.414\left|c_1 + c_2 - d_1 - d_2\right| + \left|0.414(a_2 - c_2) - (b_2 - c_2)\right|\Big] \end{aligned}\right\} \quad (15)$$

The inaccuracy of harmonic analyses based upon so few points is probably made up for by the fact that each value of D_2' or D_3' represents an average for two or three bar lengths. Of course, this averaging process also eliminates some scatter, but the scatter of values obtained from such a limited number of points would probably be misleading.

The second and third harmonics of the geometric deviations of all the columns tested were calculated by equations (9) and (15). The values for W_2' obtained from equation (15) averaged 1.2 times those obtained from equation (9), while the values of W_3' averaged 2.0 times those obtained from equation (9). Figures 11(a) to 11(c) are plotted from the data obtained from equation (15); thus, although the numbers 1, 2, and 3 indicate the source of the data, all the points can be taken as representing the magnitudes of the fundamental components of the geometric deviation of bars. Formulas (15) could not be used in calculating the total deviation, because the end conditions of sections of a bar under load are obviously not those of fixed ends. However, there is no reason to believe that the algebraic averages and the averages of absolute values of fundamental components of sections of a bar would have a different average ratio for bars under load (if the sections had been tested as separate bars) than for bars under no load. Hence, the values for total deviation components W_2 were multiplied by 1.2

and those for W_3 by 2.0 in plotting the charts of figures 6(a) to 6(c) and it is considered that these charts therefore show, to good approximation, the magnitudes of fundamental components of the total deviation.

Comparison of the values of the total deviation given by figures 6(a) to 6(c) with values of the geometric deviation given by figures 11(a) to 11(c) does not reveal very much difference in slope and in average values and not very much difference in the scatter. From this, the important conclusion may be drawn that the much easier measurement of geometric deviations will hereafter be sufficient and should give results which are representative of total deviations. However, this result is in no sense due to the effects of inhomogeneities being small - as a matter of fact, values of $W_m'' = W_m - W_m'$ proved to be as large on the average as W_m and W_m', as is indicated by figure 12 for the column made of 7075-T6 aluminum alloy; similar results were obtained with the other two materials. The reason why, in spite of this, there is so little difference between average and limiting values of W and of W' is that the deviations caused by inhomogeneities W'' are as often in the opposite direction and subtract from those due to geometric curvature as they are in the same direction; hence, these deviations have little effect upon the average values and not very much upon limiting values. However, it would have been impossible to predict this result in advance or to have verified it without an experimental program similar to the present one.

APPENDIX B

THEORETICAL DERIVATION

Simple Case

Consider first the simplest case, namely, that of a uniform column hinged at both ends and of such proportions that only buckling in one plane, taken as that of the paper, need be considered. Figure 13 shows the elastic axis of such a hinged-end column of length l, loaded by an axial force P, and with initial total deviation W and movement under load w. Neglecting the weight or other lateral loading (which can be considered by adding the corresponding deflection to W, as discussed later), the equilibrium is given by

$$-EI\, d^2w/dx^2 = P(W + w) \tag{16}$$

and the end conditions are

$$x = 0, l$$

$$w = d^2w/dx^2 = 0$$

These relations can be satisfied if

$$\left.\begin{aligned} w &= \sum_m w_m \sin m\pi x/l \\ W &= \sum_m W_m \sin m\pi x/l \end{aligned}\right\} \tag{17}$$

and this expression for W is sufficiently general to represent (that is, converge to) any possible deviation shape. Substituting expressions (17) into equation (16) and using $P_{cl} = \pi^2 EI/l^2$ gives

$$\sum \left\{W_m - \left[\left(m^2 P_{cl}/P\right) - 1\right] w_m\right\} \sin m\pi x/l = 0$$

which is satisfied, in general, only if

$$w_m = W_m \Big/ \left[\left(m^2 P_{cl}/P\right) - 1\right] \tag{18}$$

It is general experience that yielding will first occur in such a case because of a combination of the direct stress P/A and the bending stress at an extreme fiber of the middle cross section. This will occur when the yield-point stress in compression

$$S_y = \frac{P}{A} + \frac{Mc}{I}$$

$$= \frac{P}{A} - Ec\left(\frac{d^2w}{dx^2}\right)_{x=l/2}$$

$$= \frac{P}{A} + \frac{\pi^2 Ec}{l^2} \sum m^2 w_m \sin m\pi/2$$

Using relation (18) this becomes

$$S_y = \frac{P}{A} + \frac{\pi^2 Ec}{l^2} \frac{W_1}{(P_{cl}/P) - 1} \sum \frac{m^2(W_m/W_1) \sin m\pi/2}{\left[(m^2 P_{cl}/P) - 1\right] / \left[(P_{cl}/P) - 1\right]} \tag{19}$$

Multiplying through by A gives

$$P_y = P + \frac{\pi^2 EI}{l^2} \frac{cW_1/\rho^2}{(P_{cl}/P) - 1} \left\{ 1 - \frac{9W_3/W_1}{\left[(9P_{cl}/P) - 1\right] / \left[(P_{cl}/P) - 1\right]} \cdot \cdot \cdot \right\} \tag{20}$$

Now, from figure 10 the harmonic components of the deviation of a column are on the average related about as

$$\left.\begin{aligned} \frac{W_m}{t} &= C\left(\frac{l_m}{t}\right)^3 = C\left(\frac{l}{mt}\right)^3 \\ &\text{or} \\ \frac{m^3 W_m}{t} &= C\left(\frac{l}{t}\right)^3 \end{aligned}\right\} \tag{21}$$

so that, on the average,

$$|W_m| \approx |W_1|/m^3$$

$$|W_3| \approx |W_1|/27$$

Equation (19) then becomes

$$P_y = P + P_{cl} \frac{cW_1/\rho^2}{(P_{cl}/P) - 1}\left\{1 \pm \frac{(P_{cl}/P) - 1}{3\left[(9P_{cl}/P) - 1\right]} \ . \ . \ .\right\} \qquad (22)$$

The bending stress will increase and approach infinity as P approaches P_{cl} so that yielding must always occur before P reaches this value. For values of P between 0 and P_{cl} the second term in the braces of equation (22) never exceeds 1/27. For practical struts its maximum value would be considerably less, and further terms of the series would be much smaller. Hence, in this case the effect of the higher harmonics upon the bending stress can be neglected; in any case they would only affect the scatter, since they are as likely as not to cause bending of opposite sign from the fundamental component, as suggested by the $\pm$ sign in equation (22).

Neglecting all but the first term in the braces of equation (22) and solving for P give equation (2), which has previously been discussed. It might be pointed out here that in applying relation (4) to the case of hinged struts the values of K found from figure 6 should probably be multiplied by 1.2, since, as discussed previously, values higher by this amount, on the average, would probably have been obtained had the fundamental component been measured over half the length (by testing hinged-end columns) instead of over the lengths actually tested. This factor has been included in making up table I, so that the values given in this table are suitable for hinged-end columns.

Before finishing with this case some discussion might be made of the effect of initial stresses and lateral loads or built-in eccentricities. Initial stresses distributed on a microscopic scale (due presumably to yielding under previous small loads caused by stress concentrations around crystals and inclusions) can probably be neglected, like these stress concentrations themselves, since such effects are very local and scattered and probably have no significant effect on over-all shape. However, in cases where significant initial compressive stress S_i in the axial direction is known to be present on the outer fibers of the column (as may sometimes be the case because of rolling or other fabrication processes) the stress S_i should evidently be added to the right-hand side of equation (19). This is the same thing as substituting a "reduced yield stress" $S_y' = S_y - S_i$ for S_y, and this seems to be the simplest way to allow for such effects.

Deflections due to known lateral loads and known built-in eccentricity add to or (just as likely) subtract from those already considered. Let the amplitude of the harmonic component of the total deviation due to these causes, having the same shape as the fundamental component of the buckling shape predicted by classical stability theory, be U_1, which corresponds to the amplitude W_1 in equations (4) and (5) due to accidental causes. Then, if K_{max}' is a modified value to be used instead of K_{max} to allow for the effect of U_1, there result

$$\frac{cW_1}{\rho^2} = \left(\frac{K_{max}^{1/2} l_1}{\pi\rho}\right)^n$$

and

$$\frac{c(W_1 + U_1)}{\rho^2} = \left[\frac{\left(K_{max}'\right)^{1/2} l_1}{\pi\rho}\right]^n$$

Eliminating W_1 and solving for K_{max}' give

$$K_{max}' = \left(K_{max}^{n/2} + \frac{\pi^n c U_1 \rho^{n-2}}{l_1^n}\right)^{2/n}$$

$$= K_{max} + \frac{\pi^2 c U_1}{l_1^2} \qquad (n = 2) \qquad (23)$$

Thus, K_{av} should evidently not be changed unless $U_1 > W_1$, in which case K_{av} should be figured from U_1 instead of W_1.

General Case

The foregoing results were derived for the special case of uniform hinged-end columns. It is easy to show that figures 7 and 8 and the equations from which they are derived apply to <u>any</u> column when $KE/S_y = 0$ (that is, when there is no deviation of the elastic axis from the load line) provided that P_{cl} or S_{cl} is defined as the classical stability limit for the column in question. This is true because when $P_{cl} > P_y$ yielding evidently will occur as soon as $P = P_y$, while if $P_{cl} < P_y$ elastic buckling will occur first but will immediately result in infinite deflections and, hence, infinite bending stresses and yielding, so that $P_{cl} = P$.

It is not the purpose of this paper to discuss classical stability limits, solutions for which can be found in the literature for a great variety of columns. The interest here is in the effects of defects and of yielding of the material, and it remains to be demonstrated that these effects, as exemplified by the lowering of the curves in figures 7 or 8 as KE/S_y increases, are the same for all columns. It will be shown that with certain simple modifications they probably are. Because different questions arise in different cases regarding such matters as the point in the column where yielding will first occur, it would be difficult to set up a general solution covering all types of columns. Part of the demonstration will therefore have to be restricted to discussion of specific cases; in doing this an attempt will be made to span as far as possible the range covered by actual columns.

That the foregoing results apply approximately to all types of columns can be shown by the following reasoning. It is well known that, whatever the complications - variations in sections, end conditions (including negative fixity), elastic support, and so forth - the equilibrium equation for any perfect strut can be satisfied by an infinite number of deflection shapes or "buckling modes," each associated with a particular value of the load. Let $w = w_1 f(x)$ represent such a buckling mode where $f(x)$ defines the shape and w_1, the magnitude of the movement, and let P_b be the associated buckling load, that is, the load at which equilibrium can exist when the column without defects is deflected in this shape. Now compare the equilibrium equations (representing the equilibrium of external moments and internal resisting moments at every section) for this perfect strut and for the same strut with an initial deviation $W = W_1 f(x)$ having the same shape but a given fixed magnitude defined by W_1.

Then, the term in the equilibrium equation representing the moment of the axial force will be $P_b w = P_b w_1 f(x)$ for the perfect strut, where w_1 can have any value. For the strut with initial deviation the corresponding term will be $P(W + w) = P(W_1 + w_1)f(x)$, where W_1 is given but either the load P or w_1 is to be determined. All the other terms will be identical in the two equations. Hence, $f(x)$ will also be a solution for the second equation (satisfying the same boundary conditions) and the following relation must exist between the coefficients of the above terms:

$$P(W_1 + w_1) = P_b w_1$$

Solving for w_1

$$w_1 = W_1 \Big/ \left[(P_b/P) - 1\right] \tag{24}$$

One way to describe the above result is that, if any column has an initial deviation in the shape of one of its buckling modes, then a movement of this same shape with a magnitude given by equation (24) will occur under an axial load P; this movement tends to infinity if P approaches P_b, the buckling load corresponding to this mode. If the column has an initial deviation consisting entirely of components of such shapes, then corresponding movements given by equation (24) will occur for each of these components and will superpose (assuming that the total movements are small).

The next question is whether any possible deviation of a column can be separated wholly into components having the shapes of the buckling modes of the column. It would be easy to answer this question if buckling modes were represented by normal functions, like the "normal modes" of vibration of an elastic body. Buckling-mode functions are not necessarily normal to each other (consider, for example, the symmetrical modes of a uniform fixed-end column), but they nonetheless appear to have the property that any possible deviation of a column can be decomposed into components having the shapes of the buckling modes.

Now, for any end-loaded column (and for any column with loads applied between the ends, provided that P and P_b are defined as the axial load on the critical cross section due to loads distributed in the prescribed manner), yielding will occur when

$$S_y = \frac{P}{A} - Ec_{cr}w_1\left(\frac{d^2f}{dx^2}\right)_{cr} = \frac{P}{A} - Ec_{cr}\frac{W_1}{(P_{cl}/P) - 1}\left(\frac{d^2f}{dx^2}\right)_{cr} \tag{25}$$

where A_{cr}, c_{cr}, and $(d^2f/dx^2)_{cr}$ are, respectively, the area, distance to the farthest fiber, and curvature at the critical cross section (where yielding first occurs), P_{cl} is the lowest of the values of P_b, and $f(x)$ and W_1 are, respectively, the corresponding buckling shape and the magnitude of the corresponding component of the initial deviation. Only this component of the initial deviation is considered in equation (25). The effects of the other components were considered in the case of a simple hinged-end column and found to be negligible, but this must be reconsidered in other specific cases; in any case these effects will be negligible when P_{cl}/P is close to unity, since the primary term considered in equation (25) then "blows up" while other terms remain small.

Now, it is general experience that $f(x)$, the shape of a buckling mode, is always either a harmonic function (say a sine function but with nodes not necessarily at the ends) or close to such a function. If it is such a function, equation (25) corresponds exactly to the previous derivation and gives the same results. If $f(x)$ is not such a function (as in the case of variable cross sections or loads distributed along the length) it can certainly be represented closely by

$$f(x) = \sin \pi x/l_1 + a \sin 2\pi x/l_1 + b \sin 3\pi x/l_1 \qquad (26)$$

where l_1 has the same meaning as in its previous use and where the values of a and b are limited by the fact that the curvature may come to zero (because of large local bending stiffness) in some part of the primary wave but cannot reverse in sign. Considering these limitations, it is possible to calculate limiting values of $(d^2f/dx^2)_{cr}$, where the critical section is taken as that at which the curvature d^2f/dx^2, and, hence, usually the bending stress, is a maximum. These limiting values are from $-\pi^2/l_1^2$ to $-2\pi^2/l_1^2$, while the value $-\pi^2/l_1^2$ would be required to conform to the previous derivation. It can then be concluded that, if only that component of the initial deviation which has the shape of the buckling mode is considered in calculating stresses, the results obtained for simple hinged-end columns can be applied to all columns provided that the values of K obtained for simple columns are multiplied by factors ranging from 1 to 2. Changes of K of this magnitude, of course, produce much smaller changes in P or S, as can be seen in figure 7 or 8.

Extreme Cases

Now consider some extreme cases more closely. Considering first the effect of end conditions, at one extreme there is the case of a column free at one end and fixed at the other; this can be considered to be half of an equivalent hinged-end column consisting of the column and its reflection in the plane normal to the load line at the fixed end. If l is taken as the length of the equivalent hinged-end column the entire derivation given previously applies to this case.

At the other extreme, the case of a fixed-end column has been studied previously for a different purpose. The critical section in this case will be at one end, where the maximum bending due to the deviation components having the shape of the primary (symmetrical) buckling mode and the first antisymmetrical mode will add to each other; there will always be one side of one end where these and the direct stress are all of the same sign, but the stress due to the next symmetrical mode will be as likely to subtract from this, as to add to it. Then, using equations (10) and (13), yielding will occur when

$$S_y = \frac{P}{A} - Ec\left(\frac{d^2w}{dx^2}\right)_{x=0,2l}$$

$$= \frac{P}{A} + \frac{\pi^2 Ec}{l^2}\,\frac{W_1}{(P_{cl}/P) - 1}\left\{1 + \frac{(1/1.43)\left[(P_{cl}/P) - 1\right]}{\left[1.43^2(P_{cl}/P) - 1\right]} \pm \frac{(1/2)\left[(P_{cl}/P) - 1\right]}{\left[2^2(P_{cl}/P) - 1\right]}\ .\ .\ .\right\} \qquad (27)$$

The maximum value of the quantity in the braces occurs when $P_{cl}/P \longrightarrow \infty$ and is 1.34 ± 0.13. From this it may be concluded that, to be on the safe side, the values of K_{av} obtained for simple columns should be increased by a factor of 1.34 and those of K_{max}, by a factor of 1.47. However, for $P_{cl}/P \longrightarrow 1$ the factor would be unity; for $P_{cl}/P = 2$ these factors would be 1.23 and 1.30, respectively, and so on.

Consider next the effects of nonuniformity of cross section. Using figure 13 let the moment of inertia of cross sections be

$$I(x) = I_o \frac{\sin \pi x/l + \sum a_p \sin p\pi(x/l)}{\sin \pi x/l + \sum p^2 a_p \sin p\pi(x/l)} \quad (p = 2, 3, 4, \ldots) \tag{28}$$

which, with suitable values of a_p, can readily describe any practical variation of stiffness; with one even term a_2, this expression can describe a wide variety of unsymmetrical variations, or with one odd term a_3, of symmetrical variations of stiffness. Then, the equilibrium equation

$$-EI(x)\ d^2w/dx^2 = P(W + w) \tag{29}$$

and boundary conditions

$$x = 0, l$$

$$w = d^2w/dx^2 = 0$$

for hinged ends are exactly satisfied if

$$\left.\begin{aligned} w &= w_1\left[\sin \pi x/l + \sum a_p \sin p\pi x/l\right] \\ W &= W_1\left[\sin \pi x/l + \sum a_p \sin p\pi x/l\right] \end{aligned}\right\} \tag{30}$$

Using expressions (30), equation (29) is satisfied if

$$(\pi^2/l^2)EI_o w = P(W + w) \tag{31}$$

For $W_1 = 0$ (perfect column) this would become

$$\pi^2 EI_o/l^2 = P_{cl} \tag{32}$$

so from equation (31)

$$w_1 = W_1\left[(P_{cl}/P) - 1\right]$$

Yielding will then first occur when

$$\left.\begin{aligned} S_y &= \frac{P}{A} - Ec_{cr}\left(\frac{d^2w}{dx^2}\right)_{cr} \\ &= \frac{P}{A} + \frac{\pi^2 Ec_{cr}}{l^2} \frac{W_1}{(P_{cl}/P) - 1}\left[\sin \pi x/l + \sum p^2 a_p \sin p\pi x/l\right]_{cr} \\ P_y &= P + P_{cl} \frac{c_{cr}W_1 A/I_o}{(P_{cl}/P) - 1}\left[\sin \pi x/l + \sum p^2 a_p \sin p\pi x/l\right]_{cr} \end{aligned}\right\} \tag{33}$$

Since, from equation (32), I_o evidently corresponds to $I = A\rho^2$ of a uniform column, it seems very reasonable to assume that $c_{cr}W_1A/I_o$ will have about the same average and limiting values as are found for cW_1/ρ^2 in a uniform column. The equations and charts obtained for uniform columns should then apply to nonuniform columns if the values of K found for uniform columns are multiplied by the value of the quantity in brackets in equation (33).

Choosing x to maximize the expression in the brackets, it is found that this quantity may have values as high as 2 for the extreme cases contemplated in the previous discussion of the general case. However, it is found that this quantity never differs greatly from unity for variations which would be used in practical columns and, in fact, may be a little less than unity. For columns symmetrical about the middle, with a ratio of stiffness in the center to stiffness at the end of 2:1, this quantity is about 0.9; for a stiffness ratio of 3 this quantity is about unity. The effects of end conditions and of the other components of the initial deviation (which are not considered in the above discussion) should not be very different from those for uniform columns, and so it may be concluded that the results obtained for uniform columns can be applied to practical nonuniform columns also.

Next, consider the effect of intermediate loading. For simplicity the extreme case was studied of a hinged-end column with axial loads P applied at the end $x = 0$, and at the middle $x = l/2$ of figure 13. Then, a good approximation to the buckling mode should be

$$w = w_1\left[\sin \pi x/l + a \sin 2\pi x/l\right] \tag{34a}$$

where w_1 and a are to be determined by energy considerations. Let the component of the deviation of the same shape be

$$W = W_1\left[\sin \pi x/l + a \sin 2\pi x/l\right] \tag{34b}$$

Then, the total energy change during a small change in w_1 is zero:

$$dw_1 \frac{\partial}{\partial w_1} \frac{EI}{2} \int_0^l \left(\frac{d^2w}{dx^2}\right)^2 dx - \frac{P}{2} \int_0^{l/2} \left\{ \left[\frac{d}{dx}\left(W + w + w \frac{dw_1}{w_1}\right)\right]^2 - \right.$$

$$\left. \left[\frac{d}{dx}(W + w)\right]^2 \right\} dx = 0 \tag{35}$$

with a similar equation for a small change in a. Using equations (34a) and (34b), carrying out the integrations, and solving the equations simultaneously give

$$\left.\begin{aligned} a &= 0.066 \\ 1.9 \frac{\pi^2 EI}{l^2} w_1 &= P(W + w) \end{aligned}\right\} \tag{36}$$

from which $P_{cl} = 1.9\pi^2 EI/l^2$ is obtained. Yielding occurs when

$$S_y = \frac{P}{A} - Ec\left(\frac{d^2w}{dx^2}\right)_{cr}$$

$$= \frac{P}{A} + \frac{\pi^2 Ec}{l^2} \frac{W_1}{(P_{cl}/P) - 1}\left[\sin \frac{\pi x}{l} + 0.26 \sin \frac{2\pi x}{l}\right]_{cr} \tag{37}$$

The value of the quantity in brackets in equation (37) is about 1.12. Thus, it appears that the results obtained for end-loaded columns will apply closely to columns axially loaded at intermediate points, although a small increase in the value of K might be advisable. As mentioned previously, P and P_{cl} must be taken as the axial loads on the critical section (where yielding first occurs) due to the load system distributed in the prescribed manner. This is necessary in order for the term P/A in equation (37) to represent correctly the direct stress on the critical section. In the case just considered the critical section was at $x \approx 0.4l$ (between the loads) so that this condition was easily satisfied. In case of a distributed load the location of the critical section can probably be estimated sufficiently closely and should be at about the above location for any antisymmetrical distribution of load such as a uniformly distributed load.

As a final example, consider the case of a column on an elastic foundation. If the hinged-end column of figure 13 has an elastic support of β (force per unit length per unit deflection) the equilibrium and boundary conditions can be satisfied by the same expressions (eqs. (17)) for w and W as were used for the simple column. In a previous paper (ref. 11) it is shown that in such a case

$$\left(\frac{m^2\pi^2 EI}{l^2} + \frac{l^2\beta}{m^2\pi^2}\right)w_m = P(W_m + w_m) \tag{38}$$

where the expression in the parentheses on the left side represents the buckling stress P_m corresponding to the buckling-mode shape $\sin m\pi x/l$, as can be seen by letting $W_m = 0$ in equation (38). Buckling will occur with the number of half waves $m = m'$ corresponding to the smallest value of P_m; that is,

$$P_{cl} = \left(\frac{m^2\pi^2 EI}{l^2} + \frac{l^2\beta}{m^2\pi^2}\right)_{min} = \left(\frac{m^2\pi^2}{l^2}\sqrt{\frac{EI}{\beta}} - \frac{l^2}{m^2\pi^2}\sqrt{\frac{\beta}{EI}}\right)_{min}\sqrt{\beta EI} \tag{39}$$

The smallest value which the expression in the parentheses on the right side can have is 2, which occurs when

$$\frac{m^2\pi^2}{l^2}\sqrt{\frac{EI}{\beta}} = \frac{(m')^2\pi^2}{l^2}\sqrt{\frac{EI}{\beta}} = 1 \tag{40}$$

for which

$$P_{cl} = 2\sqrt{\beta EI}$$

Assuming for simplicity that the length is such that m' given by equation (40) is an integer, and considering only three components of the initial deviation W, corresponding to m', $m' + 1$, and $m' - 1$, equations (17), (38), (40), and (21) lead to the following condition for yielding:

$$\begin{aligned} S_y &= \frac{P}{A} - Ec\left(\frac{d^2w}{dx^2}\right)_{cr} \\ &= \frac{P}{A} + \frac{\pi^2 Ec}{(l/m)^2}\frac{W_m}{(P_{cl}/P) - 1}\left(\sin\frac{m'\pi x}{l} \pm \vphantom{\frac{1}{1}}\right. \\ &\quad \frac{\left(\frac{m'}{m'+1}\right)\left(\frac{P_{cl}}{P} - 1\right)\left[\sin(m'+1)\pi x/l\right]}{\left\{\left[\frac{(m'+1)^2\pi^2 EI}{l^2} + \frac{l^2\beta}{(m'+1)^2\pi^2}\right]\Big/P\right\} - 1} \pm \\ &\quad \left.\frac{\left(\frac{m'}{m'-1}\right)\left(\frac{P_{cl}}{P} - 1\right)\left[\sin(m'-1)\pi x/l\right]}{\left\{\left[\frac{(m'-1)^2\pi^2 EI}{l^2} + \frac{l^2\beta}{(m'-1)^2\pi^2}\right]\Big/P\right\} - 1}\right) \end{aligned} \tag{41}$$

For $P_{cl}/P \longrightarrow 1$ the value of the largest term in parentheses in equation (41) is unity and the results derived for simple struts then apply exactly to the present case. The expression has its largest value if $P_{cl}/P \longrightarrow \infty$ when it becomes

$$\sin \frac{m'\pi x}{l} \pm \frac{2 \frac{m'}{m' + 1} \sin \frac{(m' + 1)\pi x}{l}}{\left(\frac{m' + 1}{m'}\right)^2 + \left(\frac{m'}{m' + 1}\right)^2} \pm \frac{2 \frac{m'}{m' - 1} \sin \frac{(m' - 1)\pi x}{l}}{\left(\frac{m' - 1}{m'}\right)^2 + \left(\frac{m'}{m' - 1}\right)^2} \qquad (42)$$

This is a function of m' and can be evaluated without great difficulty. It is found that points can always be found where stresses due to the first and second, or the first and third, terms have the same sign as the direct stress (whichever combination gives the largest value is chosen to determine K_{av}), but the stress corresponding to the remaining term will then be as likely to subtract from as to add to this amount (and so may be used to determine K_{max}).

Using these findings it is determined that for $P_{cl}/P \longrightarrow \infty$ the value of K_{av} for simple columns should be multiplied by approximately $2 - \left[1/(m')^2\right]$ and the value of K_{max}, by $3 - \left[2/(m')^2\right]$, where m' is the number of half waves in which the column will buckle according to classical stability theory. Thus, a column with a mild elastic support, such that the buckling shape is still one half wave, would require no correction for K. However, a column which is so long that it buckles in many waves but is at the same time so strongly supported that P is small compared with the classical stability limit (that is, the column is in the short-column range) may require corrections by a factor as high as 2 for K_{av} and 3 for K_{max}, because there is only a slight difference between the classical stability load and the loads corresponding to neighboring buckling modes; and, hence, the corresponding components of the deviation contribute a good deal to the total bending stress.

REFERENCES

1. Jasinski, Felix S.: La Flexion des pièces comprimées. Memoirs et doc., Ann. ponts et chaussées, sér. 7, t. VIII, no. 39, 1894, pp. 233-364.

2. Marston, A.: Theory of the Ideal Column. Trans. Am. Soc. Civil Eng., vol. 39, June 1898, pp. 108-113.

3. Jensen, C.: The Quebec Bridge Disaster. Engineering, vol. LXXXV, no. 2205, Apr. 3, 1908, pp. 433-434.

4. Lilly, W. E.: The Strength of Columns. Trans. Am. Soc. Civil Eng., vol. 76, Dec. 1913, pp. 258-274.

5. Kayser, H.: Buckling Tests of Double-Frame Trusses. Bautechnik, vol. 8, no. 12, Mar. 18, 1930, pp. 200-210.

6. Salmon, E. H.: Steel Column Research. Trans. Am. Soc. Civil Eng., vol. 95, 1931, pp. 1258-1267.

7. Timoshenko, S.: Working Stresses for Columns in Thin-Walled Structures. Jour. Appl. Mech., vol. 1, no. 4, Oct.-Dec. 1933, pp. 173-177. (Calculations by D. H. Young, pp. 174-175.)

8. Van den Broek, J. A.: Rational Column Analysis. Eng. Jour., vol. 24, no. 12, Dec. 1941, pp. 570-583.

9. Van den Broek, J. A.: Column Formula for Materials of Variable Modulus. Eng. Jour., vol. 28, no. 12, Dec. 1945, pp. 772-777, 783.

10. Southwell, R. V.: On the Analysis of Experimental Observations in Problems of Elastic Stability. Proc. Roy. Soc. (London), ser. A, vol. 135, no. 828, Apr. 1, 1932, pp. 601-616.

11. Donnell, L. H.: On the Application of Southwell's Method for the Analysis of Buckling Data. Timoshenko Sixtieth Anniversary Vol., The Macmillan Co., 1938, pp. 27-38.

12. Donnell, L. H., and Wan, C. C.: Effect of Imperfections on Buckling of Thin Cylinders and Columns Under Axial Compression. Jour. Appl. Mech., vol. 17, no. 1, Mar. 1950, pp. 73-83.

TABLE I

VALUES OF ROUGHNESS FACTOR K

[Values are for simple columns but may be used for other cases as discussed in appendix B]

	K_{max}	K_{av}
Standard cold-rolled steel bar stock	0.00015	0.00003
Strips cut from standard flat sheets of 2024-T3 and 7075-T6 aluminum alloy	.00019	.00004
Columns of "refined" construction (tentative values)	.00015	.00003
Columns of "average" construction (tentative values)	.00040	.00010

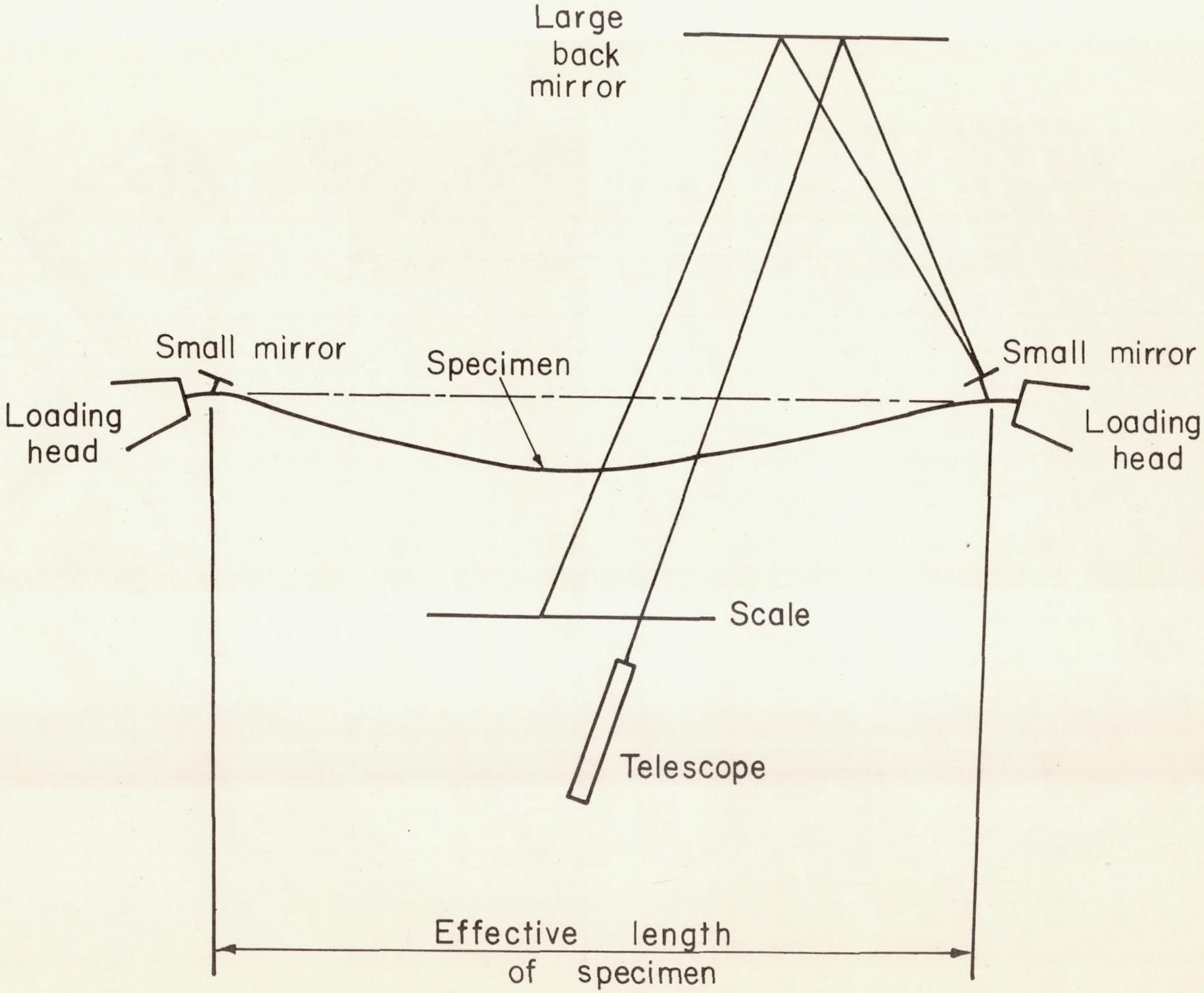
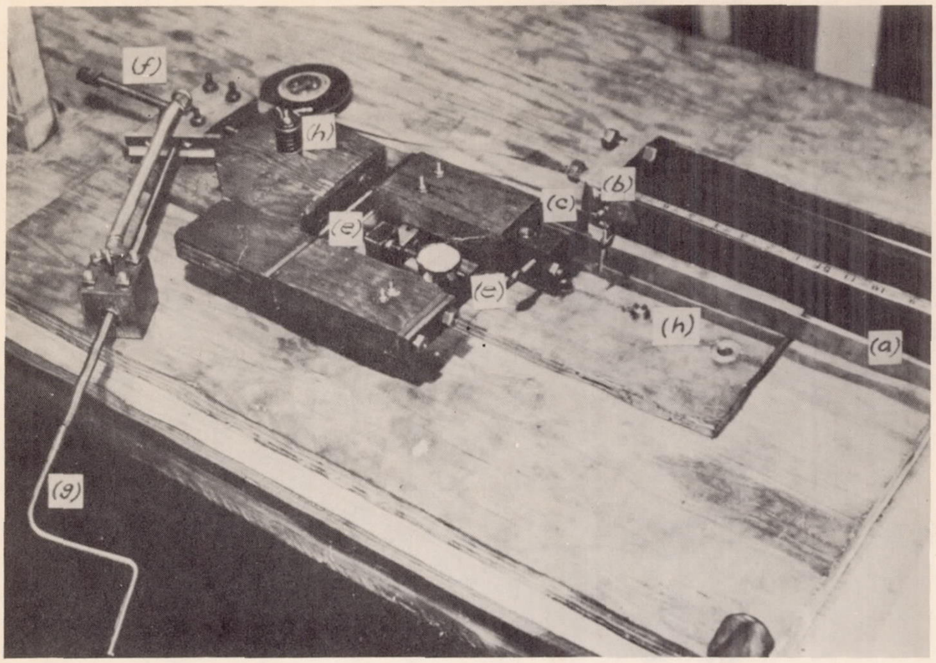
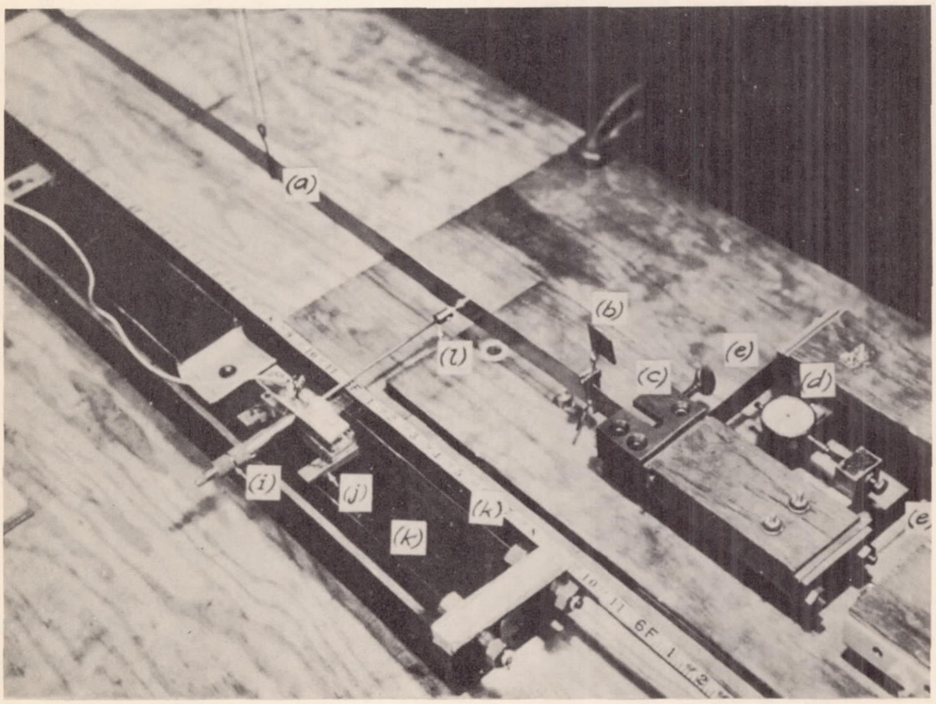


Figure 1.- Sketch of loading apparatus and optical system.

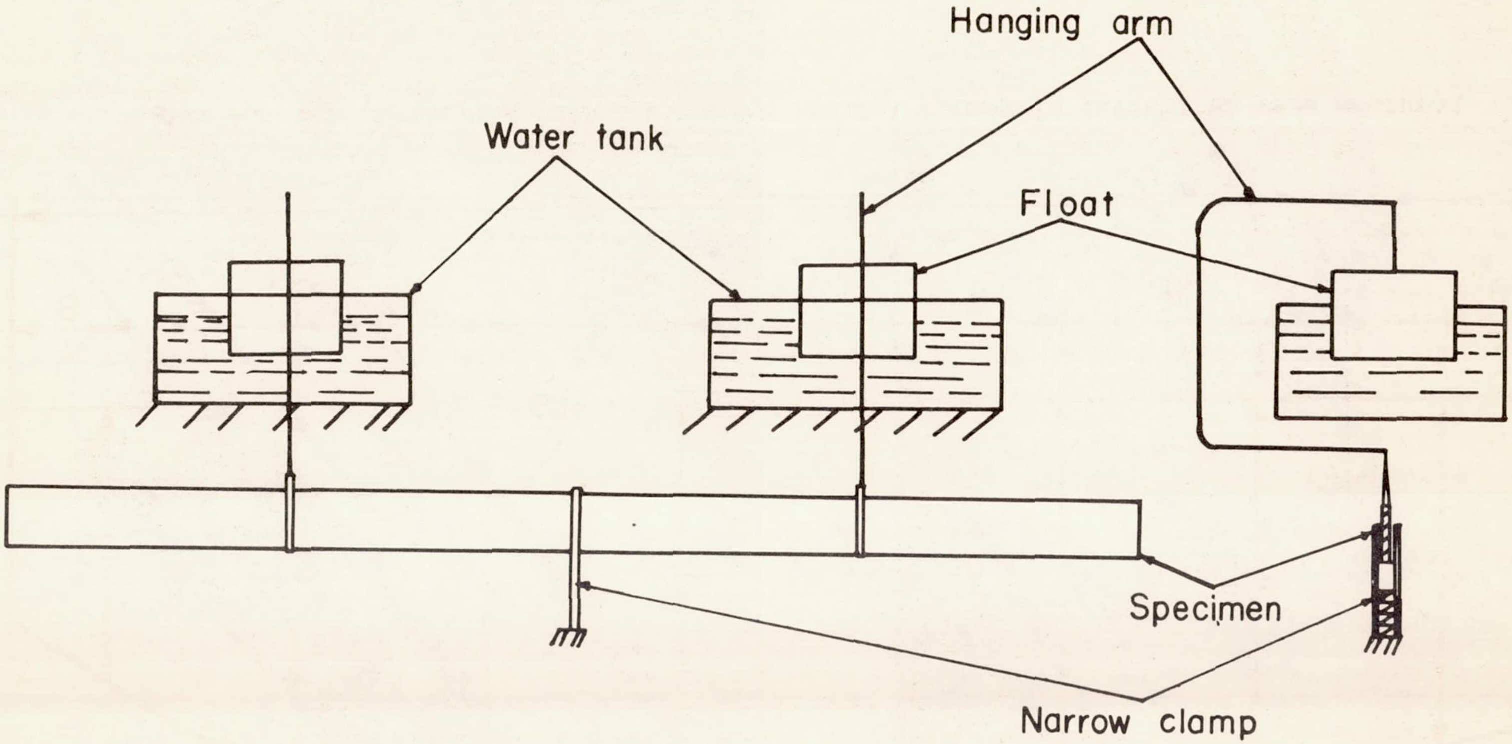


(a) Loading head.



(b) Deflection-measuring unit. L-89308

Figure 2.- Photographs of test apparatus.



(a) Front view

(b) Side view.

Figure 3.- Sketch showing method of balancing weight of slender specimen.

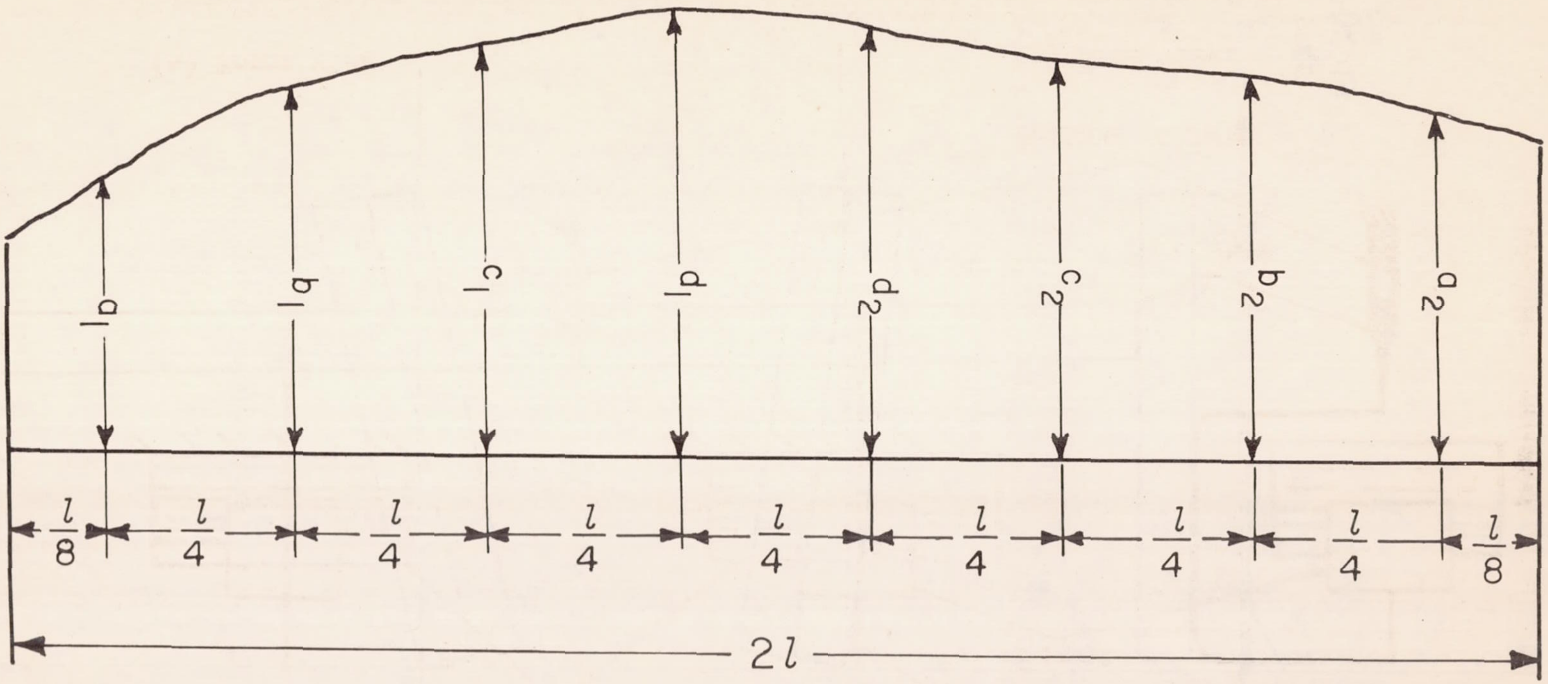


Figure 4.- Sketch showing points at which initial geometric deviations were measured.

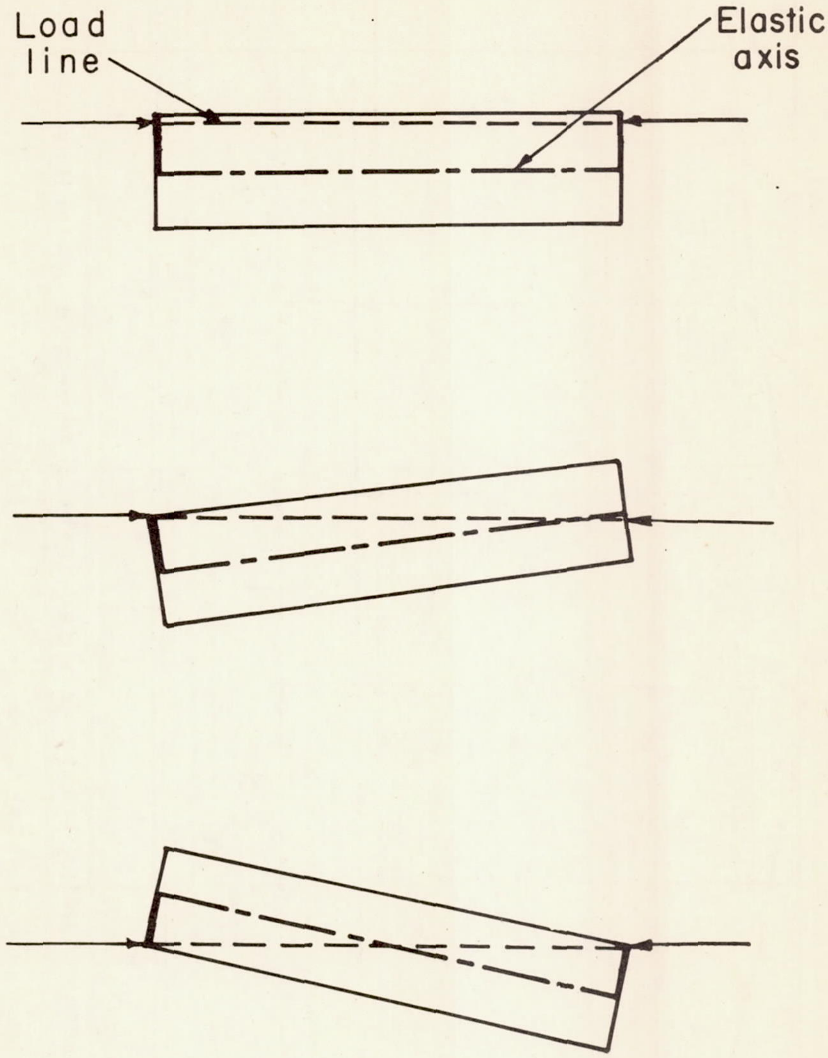
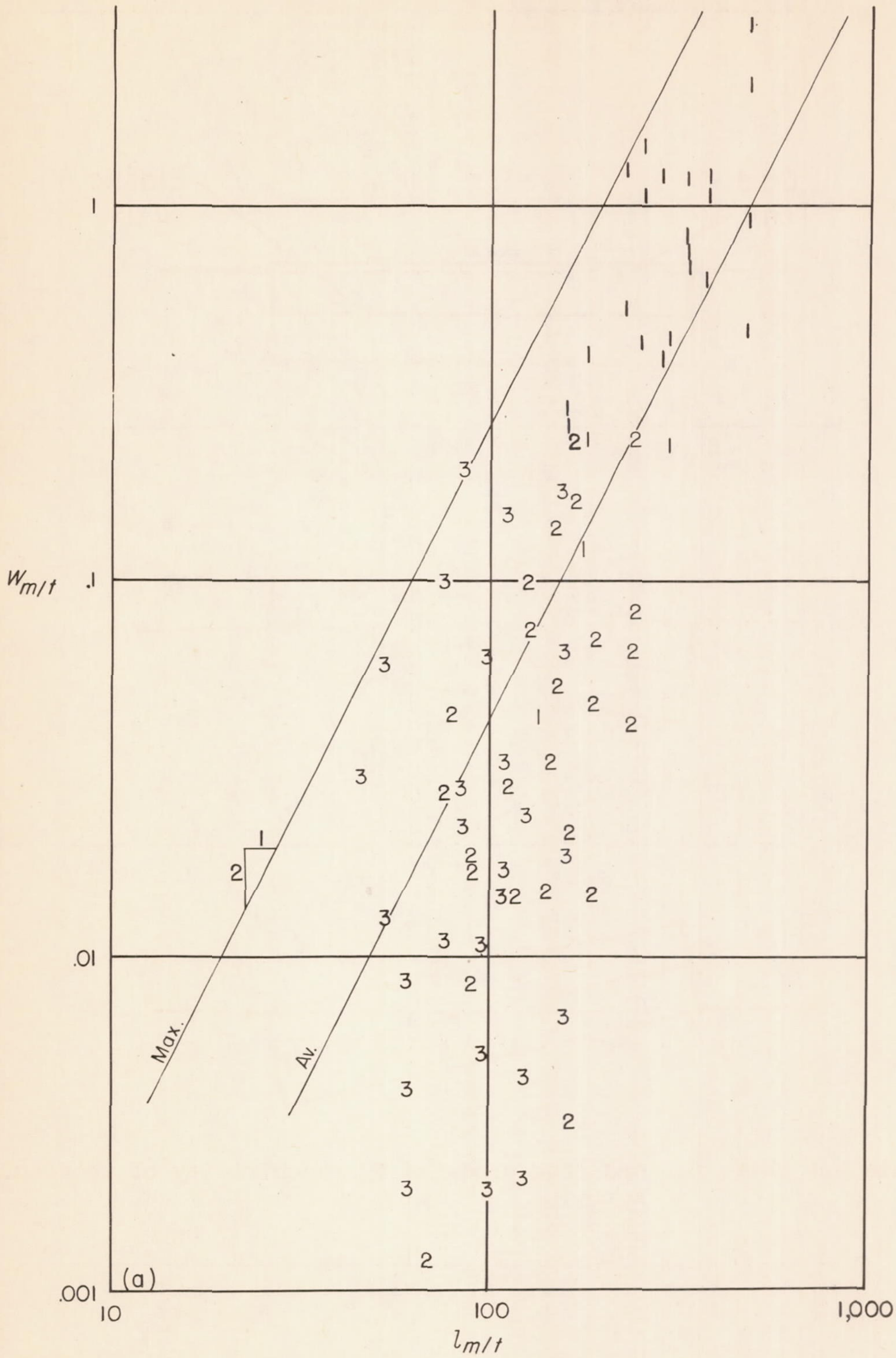
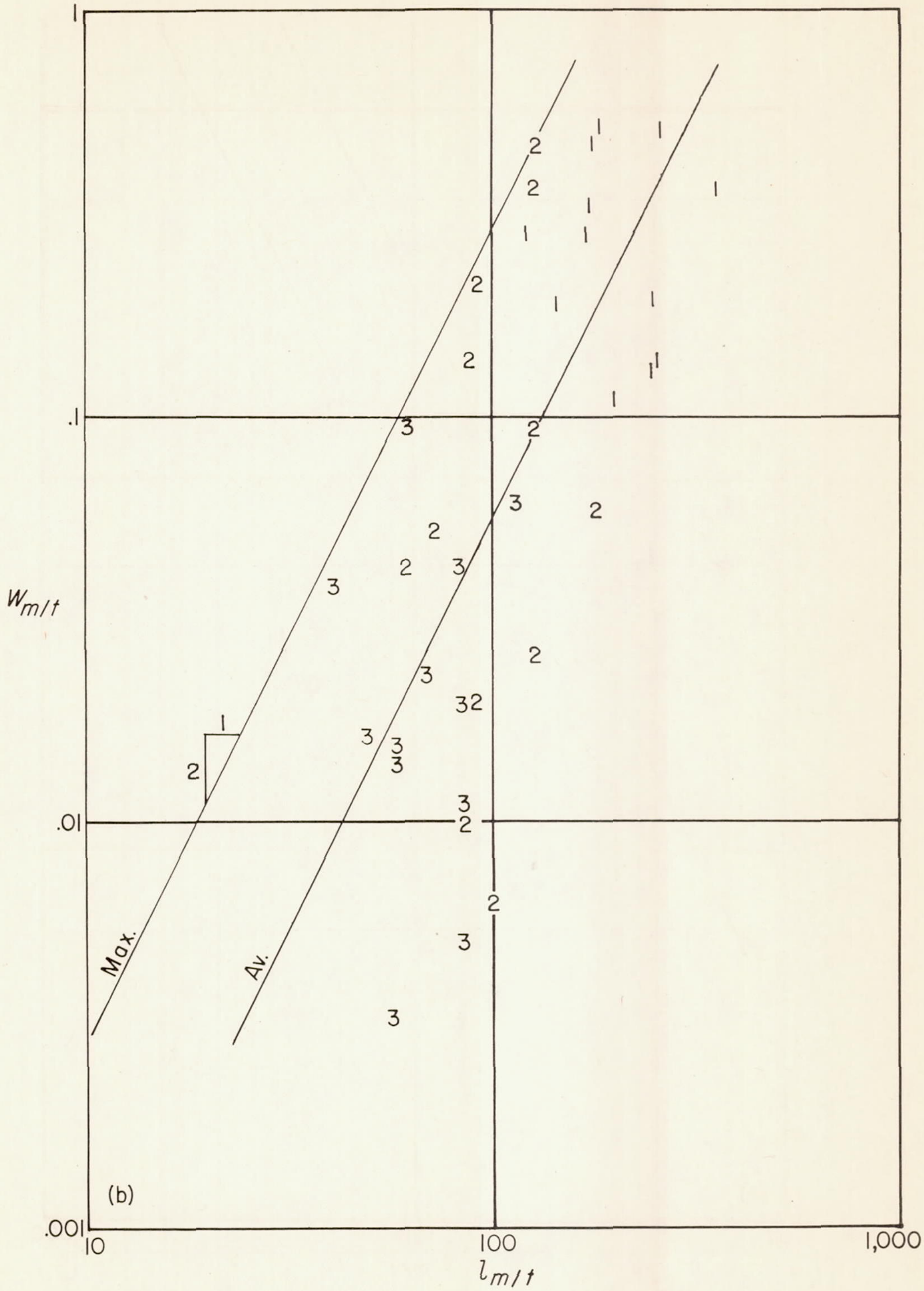


Figure 5.- Shift in load line produced by eccentricity of loading.



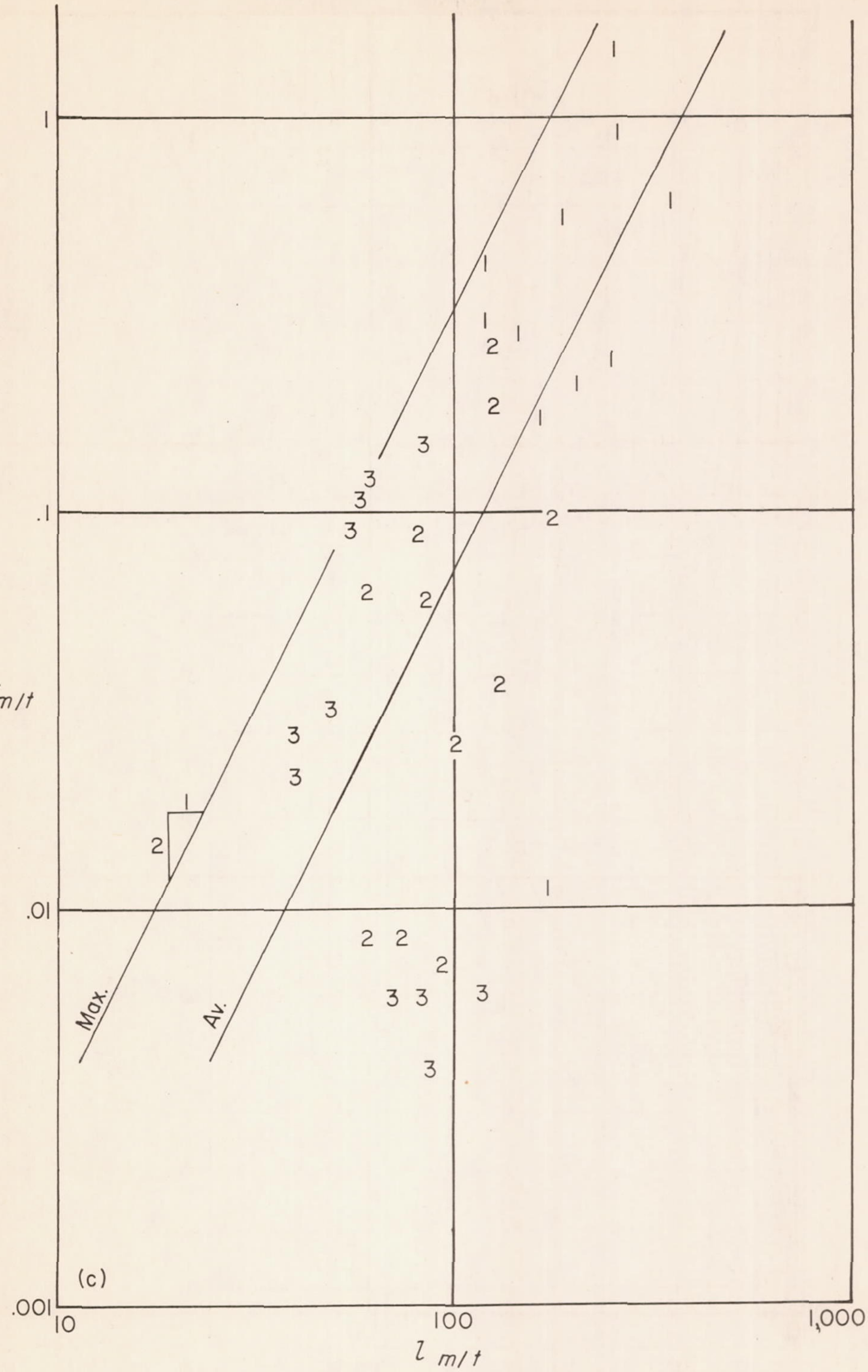
(a) Cold-rolled mild-steel bar stock.

Figure 6.- Total initial deviations of columns.



(b) 2024-T3 aluminum alloy.

Figure 6.- Continued.



(c) 7075-T6 aluminum alloy.

Figure 6.- Concluded.

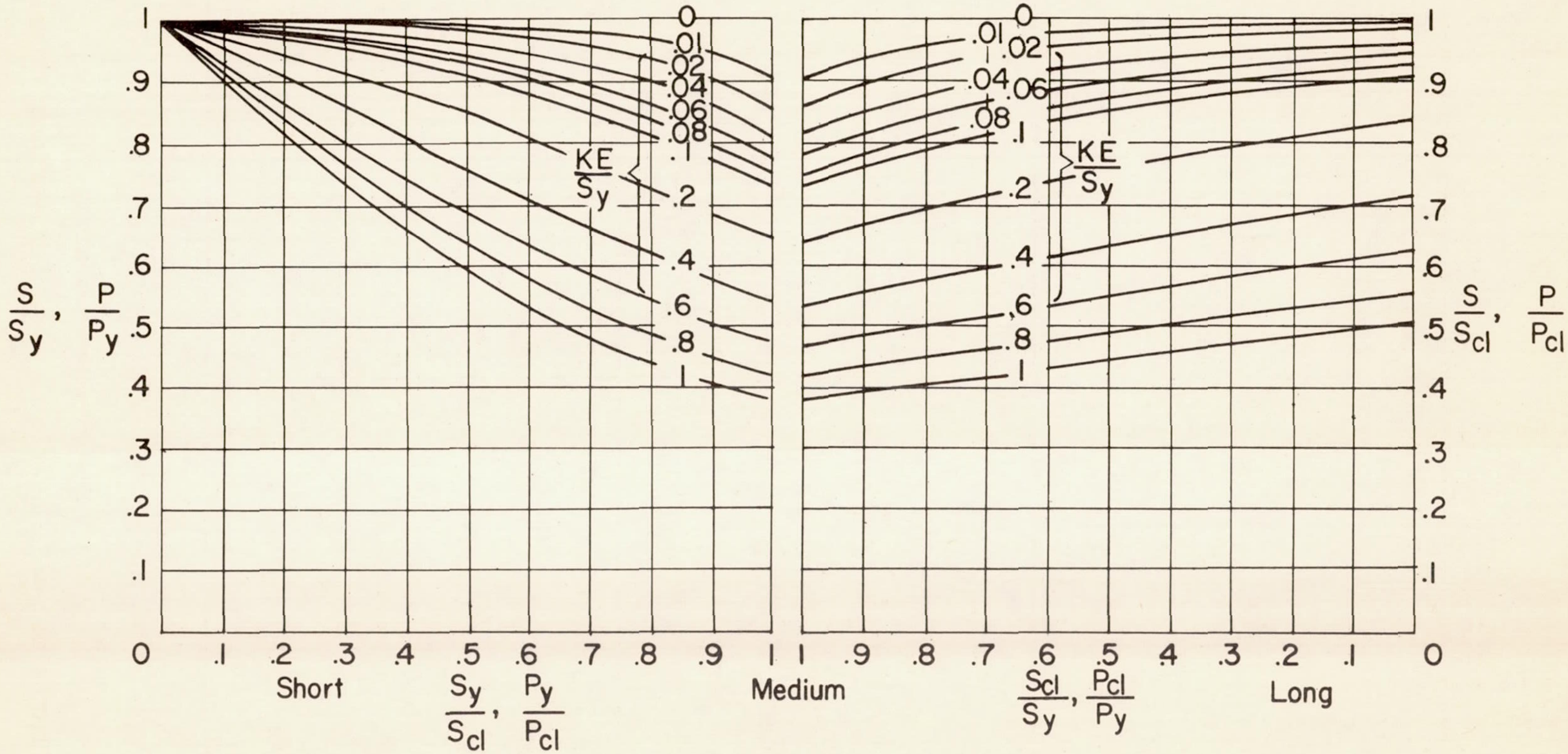


Figure 7.- Universal column chart of $\frac{S}{S_y} = \frac{P}{P_y} = q - \sqrt{q^2 - \frac{S_{cl}}{S_y}}$ for case $n = 2$.

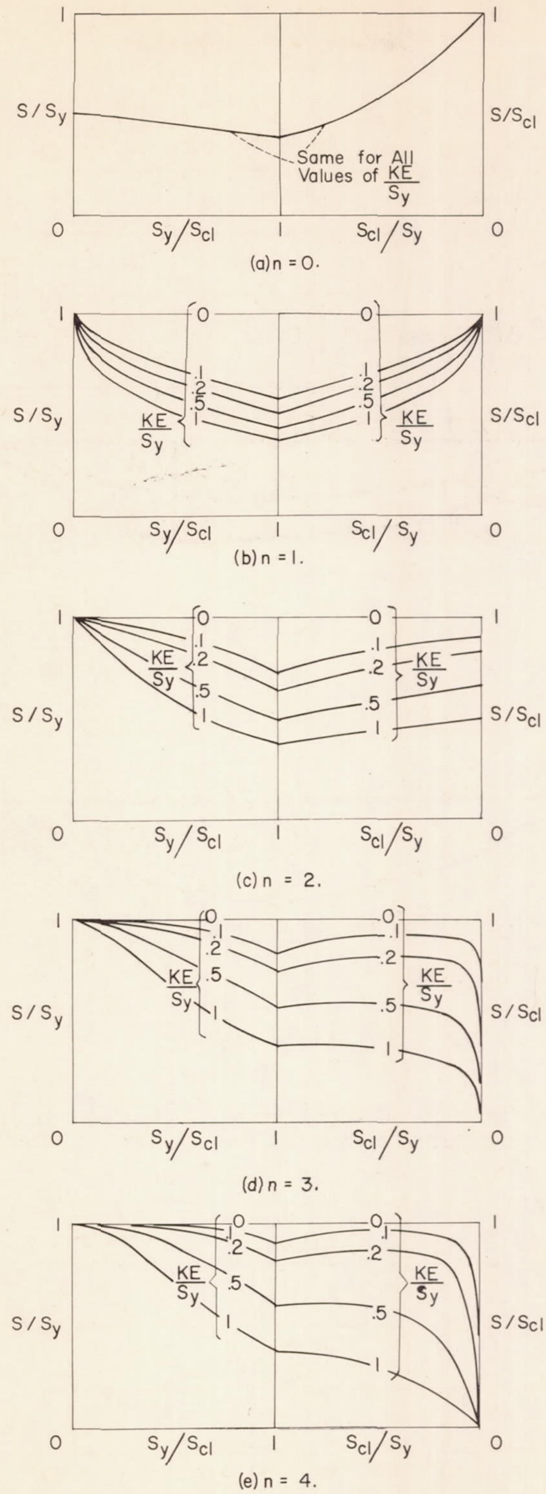
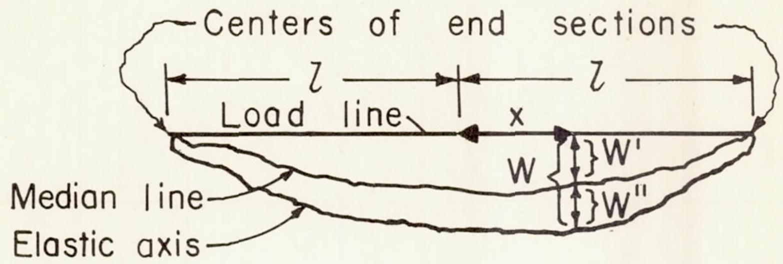
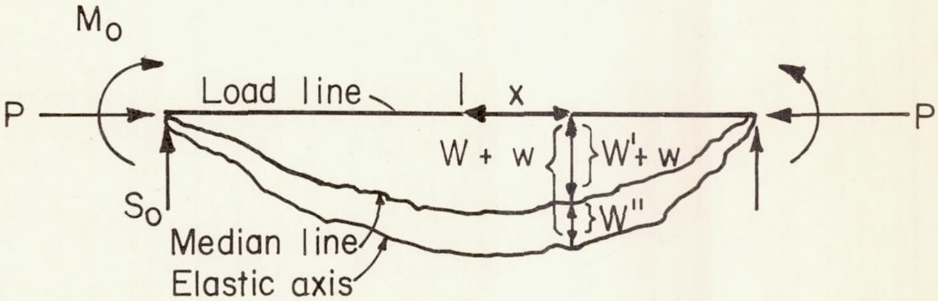


Figure 8.- Charts of $\frac{S}{S_y} = q - \sqrt{q^2 - \frac{S_{cl}}{S_y}}$ for various values of n.



(a) No-load condition.



(b) Loaded condition.

Figure 9.- Diagram illustrating factors involved in total deviation and movement for a fixed-end strut.

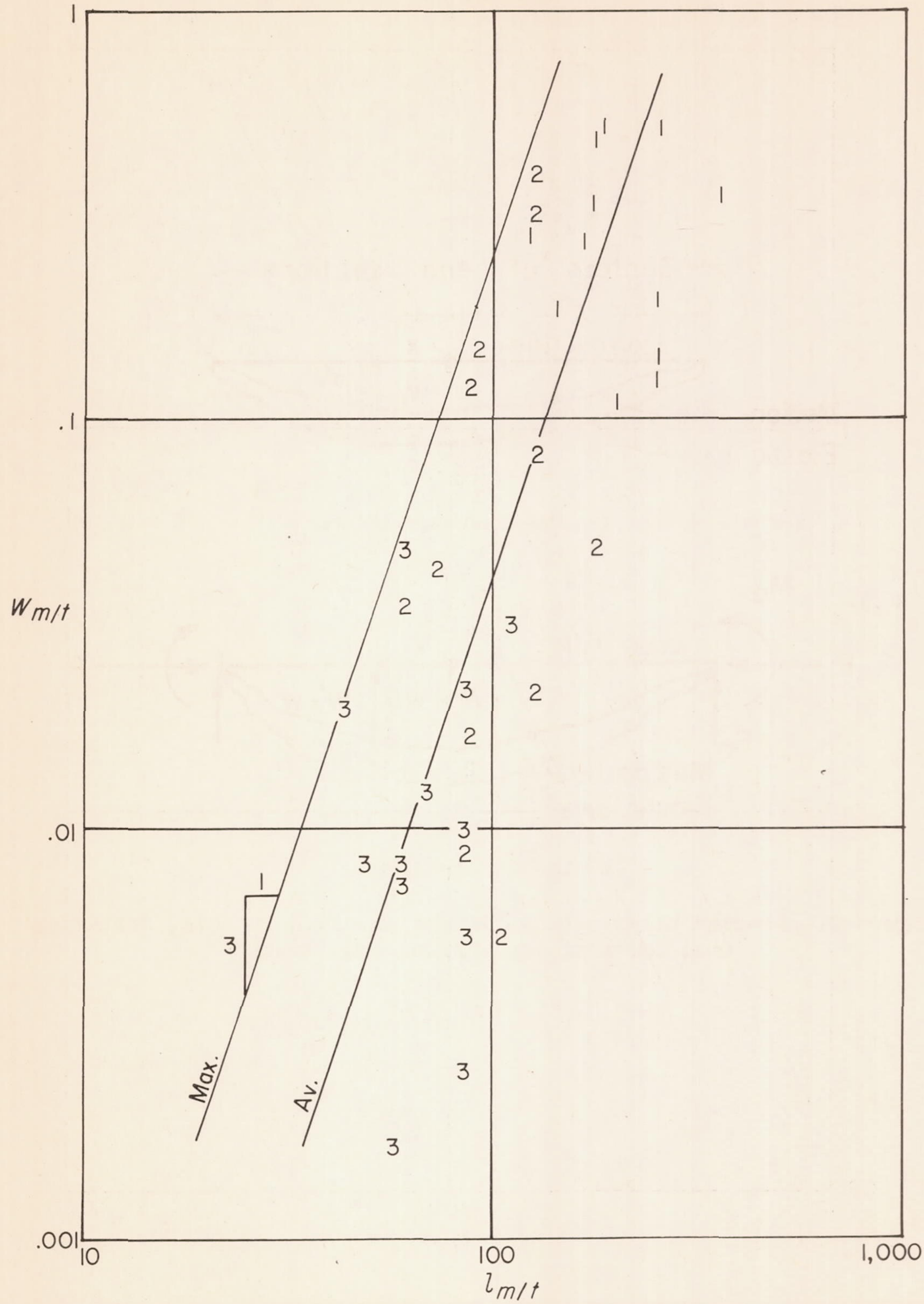
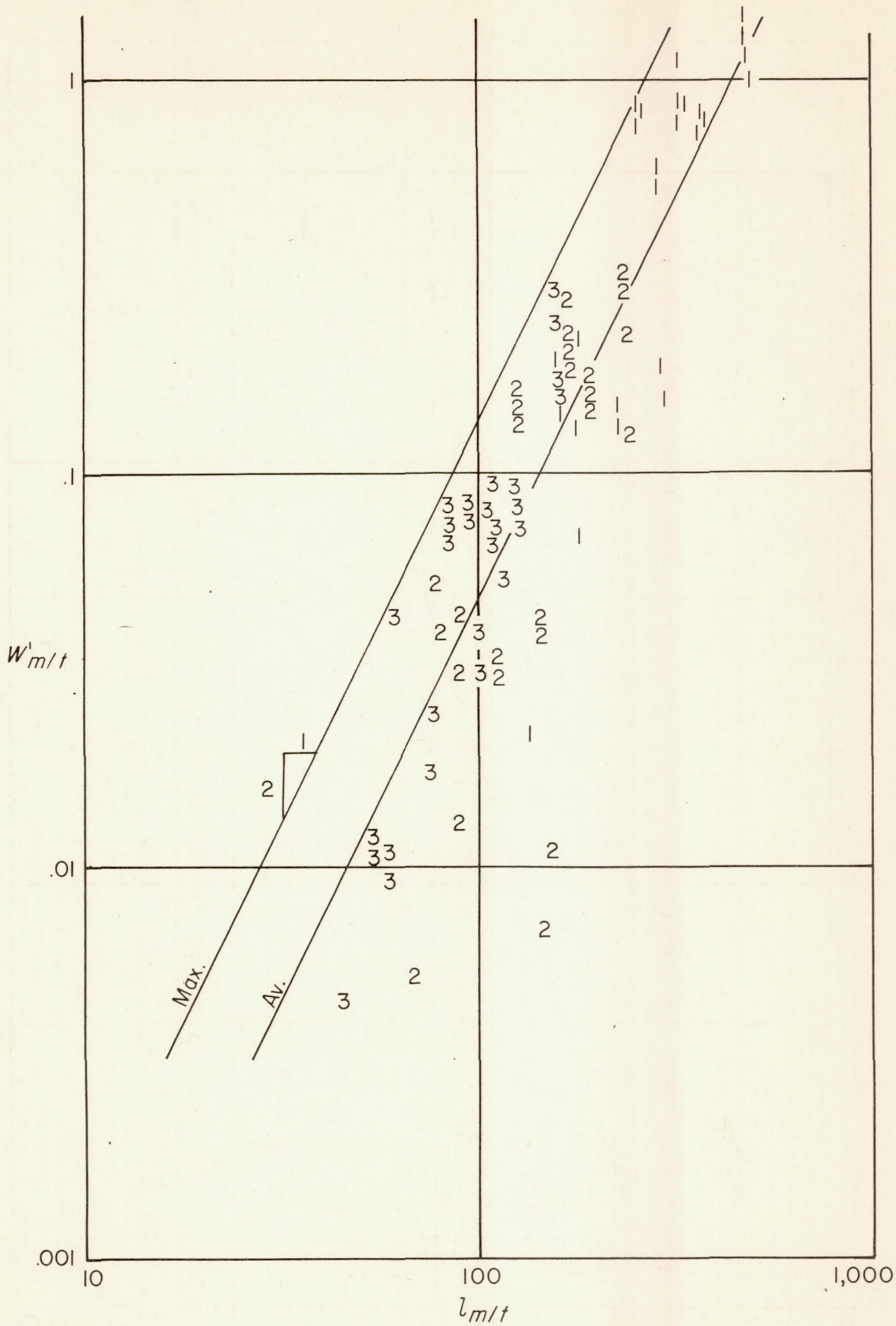
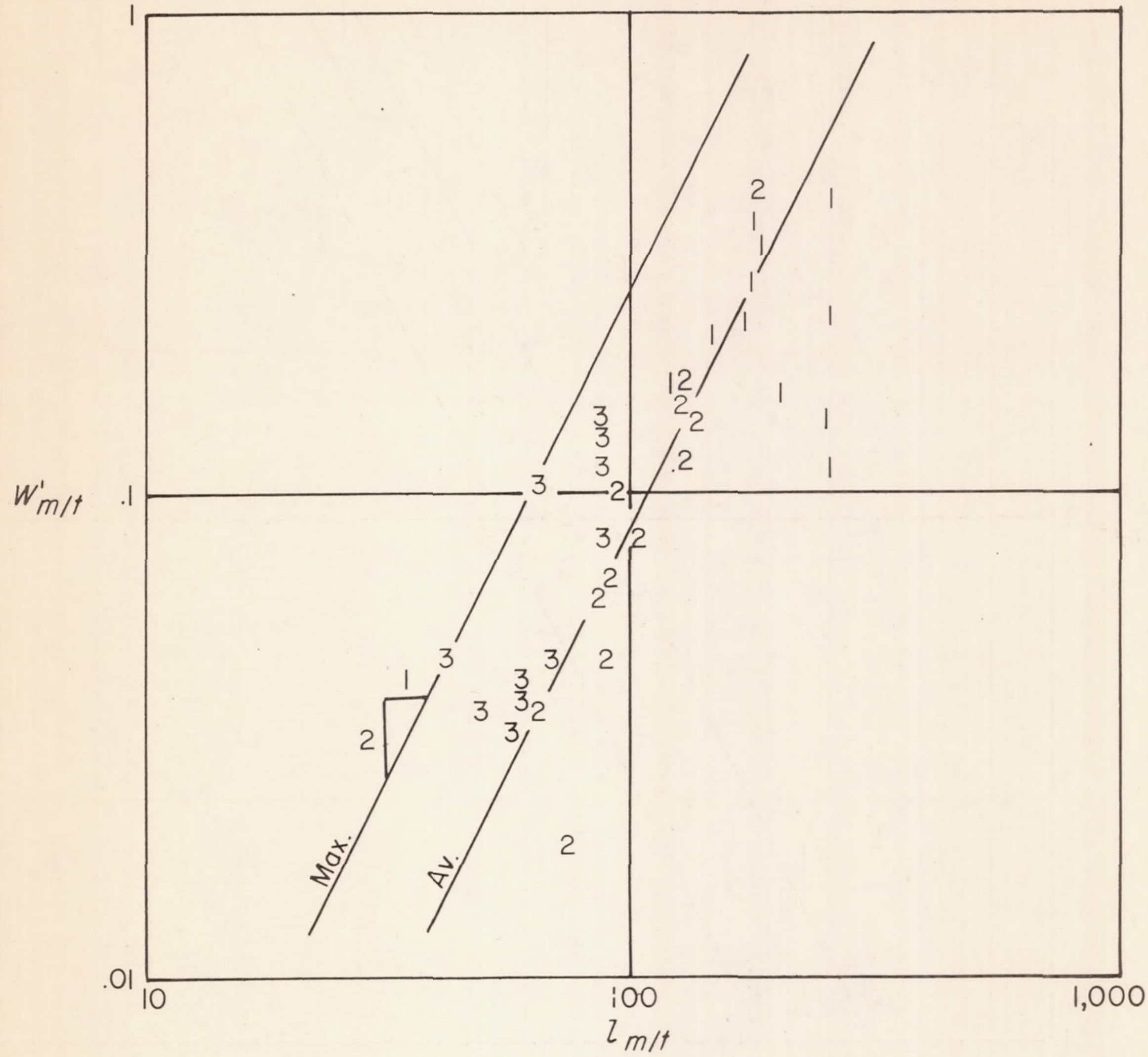


Figure 10.- Harmonic components of deviation. Specimen of 2024-T3 aluminum alloy.



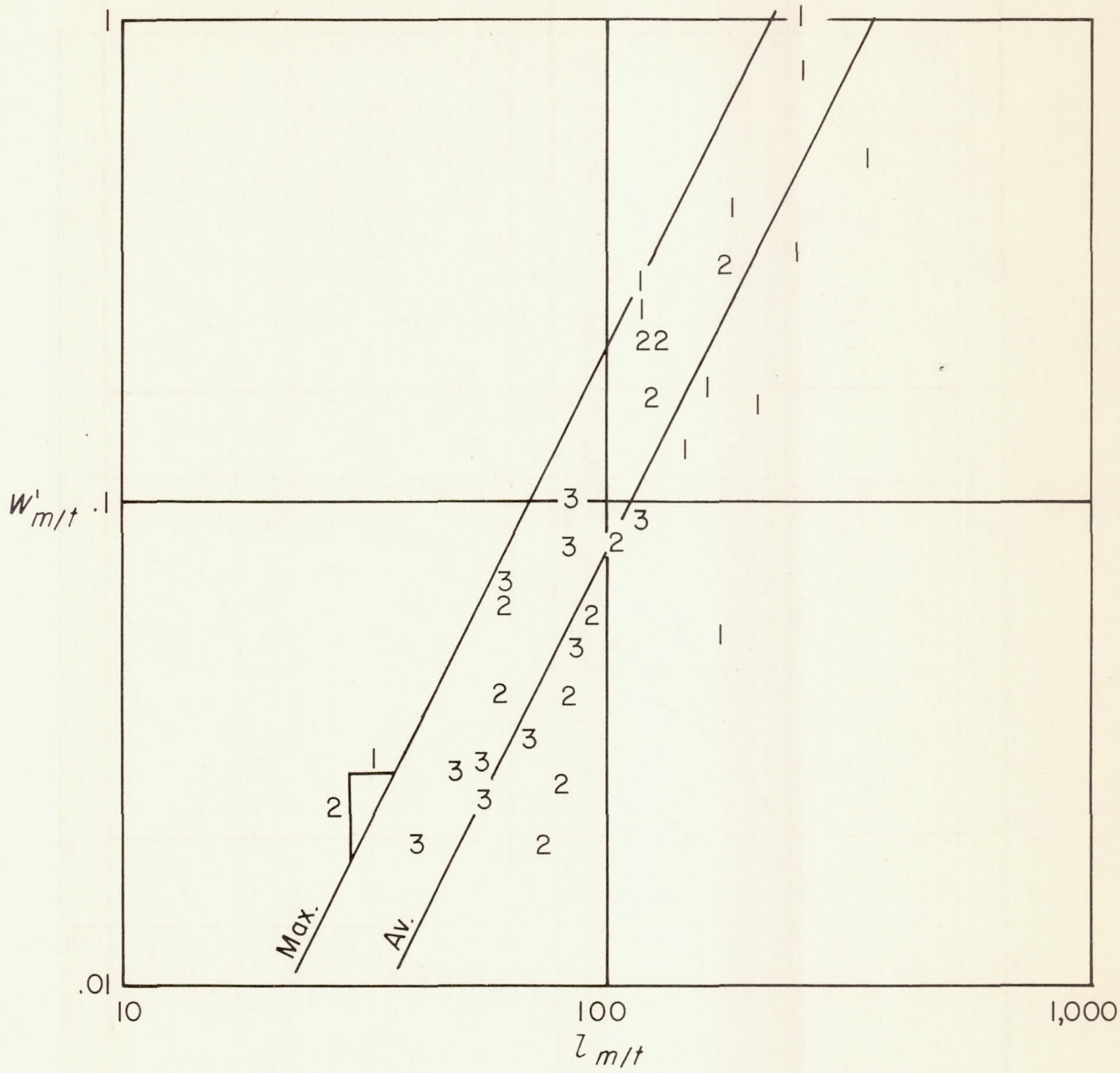
(a) Cold-rolled mild-steel bar stock.

Figure 11.- Geometric deviations of columns.



(b) 2024-T3 aluminum alloy.

Figure 11.- Continued.



(c) 7075-T6 aluminum alloy.

Figure 11.- Concluded.

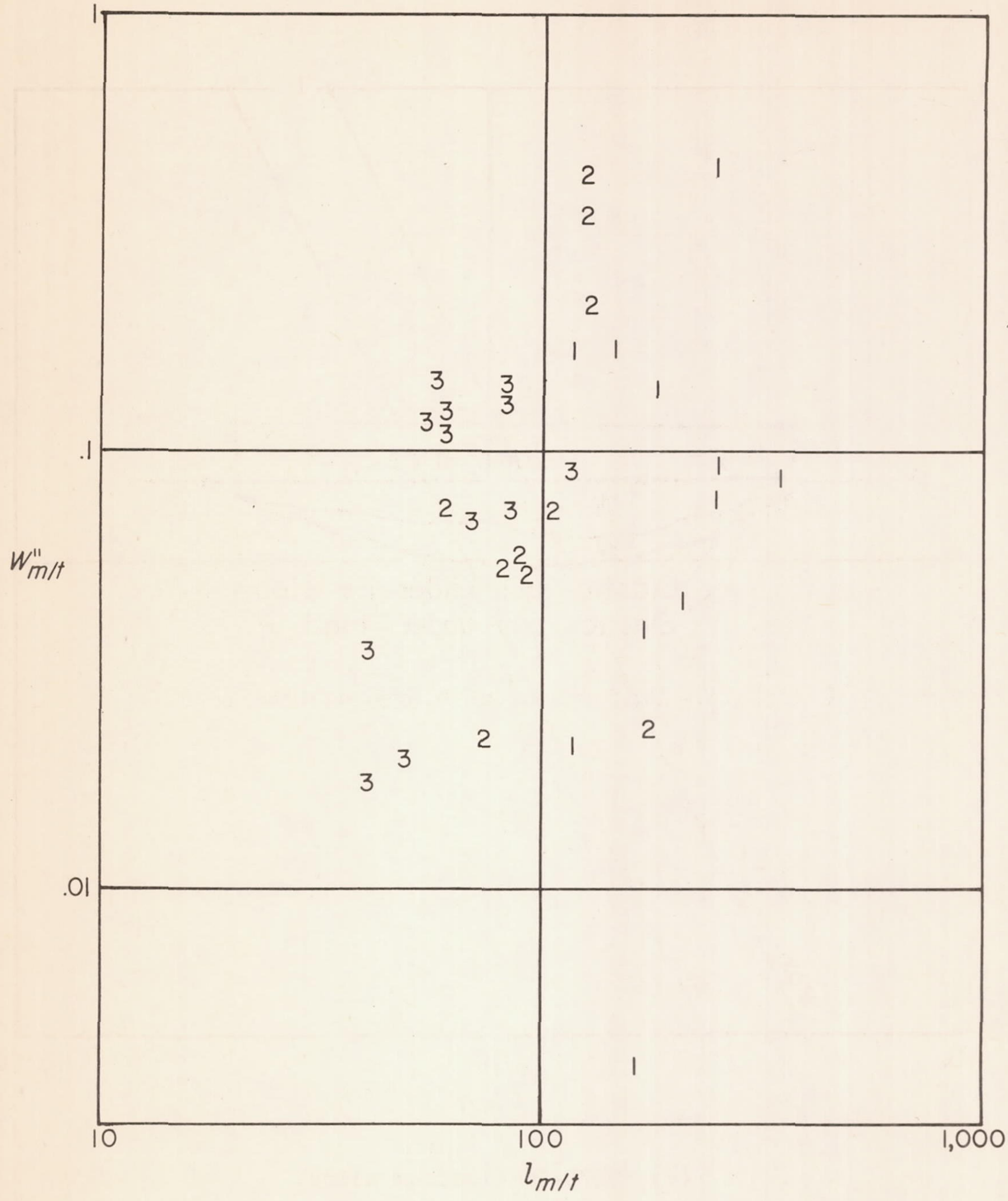


Figure 12.- Nongeometric deviation. Specimen of 7075-T6 aluminum alloy.

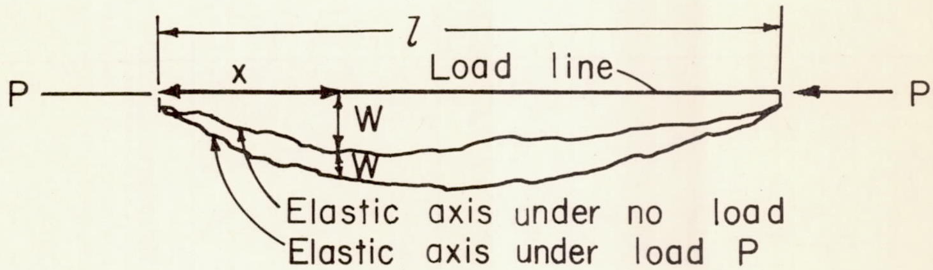


Figure 13.- Uniform column hinged at both ends.