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THE LINEARIZED EQUATIONS OF MOTION UNDERLYING THE DYNAMIC STABILITY OF AIRCRAFT, SPINNING PROJECTILES, AND SYMMETRICAL MISSILES

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SUMMARY

Linearized equations of motion are derived for both conventional aircraft having mirror symmetry and spinning projectiles or missiles having rotational and mirror symmetry. The aerodynamic coefficients are introduced as a formal series expansion in the customary variables, with additional terms being included to account for the aerodynamic effects of spin. The requirements of symmetry are used to reduce the system of aerodynamic coefficients and, in the projectile or missile case, to clarify markedly the geometry. A common mathematical approach and standard NACA nomenclature are used throughout.

The equations for aircraft are compared with those for missiles and shortcomings in the currently accepted theories are pointed out. The dynamic-stability requirements for spin-stabilized projectiles are discussed briefly.

The results are applied to the analysis of flight-test data from the aerodynamics range. Relations are derived between the aerodynamic coefficients and the constants of the equations of motion. A comparison is made with ballistic theory in current use and is found to be satisfactory.

INTRODUCTION

The dynamic stability of aircraft is a subject that has been explored at great length. The theory of the motion of spinning projectiles has also been studied extensively. No further development of either subject is needed per se, at least in so far as first-order effects are concerned. On the other hand, modern trends in aeronautics and ballistics have brought the flight performance and even the physical appearance of aircraft and projectiles closer and closer together. Hence, a need now exists for a

theory of motion that covers both cases simultaneously with a single nomenclature and a common mathematical development.

In the present analysis, aircraft are distinguished from projectiles and symmetrical missiles as follows:

1. Aircraft are assumed (a) to have a plane of mirror symmetry through the longitudinal axis and (b) to fly only slightly disturbed from a steady-state equilibrium attitude so that all components of the angular velocity of the aircraft are small.

2. Projectiles and symmetrical missiles are assumed (a) to have not only a plane of mirror symmetry but also 90° rotational symmetry (or its equivalent) and (b) to fly similarly to aircraft except that the axial component of the angular velocity, the spin, may be large (with the restriction that the change in the spin must be small).

Despite these differences, the flight of both aircraft and projectiles takes place under closely similar circumstances. The analysis of the motion in each case is the classical analysis of a rigid body moving under the action of external forces. The differential equations of motion are derived in both cases from the vector equations relating the rates of change of the linear and angular momenta to the external force and moment. Furthermore, the conditions of flight prescribed are the same, namely, both must fly in nearly a straight line at nearly constant velocity and the inclination to the flight path must be small. The aerodynamic-force system postulated is the same: The aerodynamic forces and moments are assumed to be linear functions of the velocity and the angular velocity. In both theories the equations of motion are linearized by the neglect of second-order terms. One might well suppose that the dynamic stability of aircraft and projectiles had been treated by a common development. Such is far from the case.

The dynamic stability of aircraft was first analyzed by Lanchester around 1900; the corresponding analysis of the dynamic stability of spin-stabilized projectiles was made by Fowler and his associates in 1920 (see refs. 1 and 2). Both men had very practical objectives in mind. Lanchester was interested in the flying qualities of airplanes. Fowler wished to find the design criteria for an artillery shell that would insure a true flight to the target and a strike head on. Both attacked the problem independently and their divergence of interest is strongly reflected in the two developments of the theory, a divergence which has continued for all practical purposes to the present time.

A casual observer reading the literature on dynamic stability would be left with the impression that ballisticians and aeronautical engineers were concerned with entirely different problems. The nomenclature is different; the geometry does not appear at first glance to be related; the mathematical treatments differ radically. The casual observer's judgment

would not be so superficial after all, for it is only fair to say that the theory developed to handle the dynamic stability of aircraft is not adequate in its present form to predict the dynamic stability of spinning projectiles, and vice versa. For example, the aircraft equations do not describe the gyroscopic nutation and precession of a spinning shell, and the projectile equations do not describe the phugoid oscillation of aircraft.

Recently, R. E. Bolz and J. D. Nicolaides have derived the dynamic stability of spinning projectiles and symmetrical missiles in terms familiar to the aerodynamicist (see refs. 3 and 4). Although their derivations go a long ways toward joining the theories of Lanchester and Fowler, both omit the force of gravity and contain certain other restrictions with the result that a gap as yet remains between the case of the aircraft, flying with its weight balanced by lift, and the case of the gyroscopically stabilized projectile, flying with varying velocity and spin. It is the purpose of this paper to bridge the gap, that is, to treat the two cases with strictly similar mathematical developments and with a common nomenclature. In addition, it is desired to apply the results of this theory to the analysis of experimental data obtained from a relatively new flight-test facility, the aerodynamics range.

The development of the theory presented herein follows closely the customary treatment of the dynamic stability of aircraft with the primary difference being the simultaneous treatment of longitudinal and lateral stability. The nomenclature conforms throughout to NACA standards. The other departures of importance from conventional aircraft theory are a rather formal development of the aerodynamic force system and the thorough use of the condition of symmetry assumed for the body, as is done in the ballistic theory. In fact, it might be said that the present development is a welding together of these aspects of the conventional aerodynamic and ballistic theories which, in the author's opinion, represent the most effective means of attacking the problem.

SYMBOLS

a_0, a_1, a_2, a_3 constants in the $x(t)$ equation

A any quantity
(This symbol is used in the development of general transformations.)

A_1, A_2 constants in the $\xi(t^+)$ equation

b_0, b_1, b_2, b_3 constants in the $\Delta(x^+)$ equation

c_0, c_1, c_2, c_3	constants in the $\varphi(x^+)$ equation
C	aerodynamic coefficients: $\frac{\text{force}}{(\rho/2)V^2S}, \frac{\text{moment}}{(\rho/2)V^2Sl}$
C_F	coefficient of aerodynamic asymmetry force
C_M	coefficient of aerodynamic asymmetry moment
d_0, d_1, d_2, d_3, d_4	constants in the $\xi(x^+)$ equation
D	operator, $DA = \frac{dA}{dt^+}$
$\mathbb{D}$	operator, $\mathbb{D}A = \frac{dA}{dx^+}$
f_0	$\frac{C_{F2} + C_{F3}}{2}$
f_0, f_1, f_2, f_3	constants in the $\eta(x^+)$ equation
F	force, external, acting on body
g	acceleration of gravity
H	angular momentum of body
i	unit vector along X axis
i	$\sqrt{-1}$
I	moment of inertia of body
j	unit vector along Y axis
J_{XZ}	product of inertia of body
k	unit vector along Z axis
K	$\frac{I_X}{I_Y}$
K_X^2	$\frac{I_X}{ml^2}$

K_Y^2	$\frac{I_Y}{ml^2}$
K_Z^2	$\frac{I_Z}{ml^2}$
K_1	$\frac{J_{XZ}}{I_X}$
K_2	$\frac{J_{XZ}}{I_Z}$
l	characteristic length
l_p	$\frac{C_{l_p}}{4K_X^2}$
l_r	$\frac{C_{l_r}}{4K_X^2}$
l_β	$\frac{\mu C_{l_\beta}}{2K_X^2}$
$l_{\dot{\beta}}$	$\frac{C_{l_{\dot{\beta}}}}{4K_X^2}$
l_o	$\frac{\mu C_{l_o}}{2K_X^2}$
m	mass of body
m_o	$\frac{\mu(C_{M3} - iC_{M2})}{2K_Z^2}$
m_o	$\frac{\mu C_{m_o}}{4K_Y^2}$
m_q	$\frac{C_{m_q}}{4K_Y^2}$

m_{α}	$\frac{\mu C_{m_{\alpha}}}{2K_Y^2}$
$m_{\dot{\alpha}}$	$\frac{C_{m_{\dot{\alpha}}}}{4K_Y^2}$
m_{η}	$m_q + x_o + i\nu(K - m_{rp})$
m_{ξ}	$m_{\alpha} + i\nu m_{\beta p}$
$m_{\dot{\xi}}$	$m_{\dot{\alpha}} + i\nu m_{\dot{\beta} p}$
m_{rp}	$\frac{C_{m_{rp}}}{8\mu K_Y^2}$
$m_{\beta p}$	$\frac{C_{m_{\beta p}}}{4K_Y^2}$
$m_{\dot{\beta} p}$	$\frac{C_{m_{\dot{\beta} p}}}{8\mu K_Y^2}$
M	linear momentum of body
M	moment; external acting on body
n_p	$\frac{C_{n_p}}{4K_Z^2}$
n_r	$\frac{C_{n_r}}{4K_Z^2}$
n_{β}	$\frac{\mu C_{n_{\beta}}}{2K_Z^2}$
$n_{\dot{\beta}}$	$\frac{C_{n_{\dot{\beta}}}}{4K_Z^2}$

n_{qp} $\dfrac{C_{n_{qp}}}{8\mu K_Z^2}$

$n_{\alpha p}$ $\dfrac{C_{n_{\alpha p}}}{4K_Z^2}$

$n_{\dot{\alpha} p}$ $\dfrac{C_{n_{\dot{\alpha} p}}}{8\mu K_Z^2}$

p X component of $\bar{\omega}$

p' variation in p over measured trajectory

p_o constant component of angular velocity about which the angular velocity in flight varies ($p = p_o + p'$)

$P(S)$ polynomial factor in constants d_1, d_3 of $\xi(x^+)$ equation

q Y component of $\bar{\omega}$

$Q(S)$ polynomial factor in constants f_2, f_3 of $\eta(x^+)$ equation

r Z component of $\bar{\omega}$

$R(S)$ stability quartic

s stability factor, $\dfrac{K^2\nu^2}{4m_\alpha}$

s arc length along trajectory

S characteristic area

S independent variable in stability quadratic

S_1, S_2 roots of stability quartic, $S_1 = d_2$, $S_2 = d_4$

t time

t^+	$\frac{t}{\tau}$
T_1, T_2	terms in S_{1R} and S_{2R}
u	variation in velocity along trajectory
V	velocity of center of gravity of body
V_o	constant component of velocity about which the flight velocity varies ($V = V_o + u$)
$\dot{\bar{V}}_T$	$\dot{\beta}\bar{j} + \dot{\alpha}\bar{k}$
w	angular velocity of body with respect to XYZ axes
x	distance along X_o axis
x^+	$\frac{x}{\mu l}$
x_G	$\frac{mg \sin \gamma_o}{\rho S V_o^2}$
x_o	$\frac{C_{X_o}}{2}$
x_α	$\frac{C_{X_\alpha}}{2}$
$x_{\dot{\alpha}}$	$\frac{C_{X_{\dot{\alpha}}}}{4\mu}$
x_q	$\frac{C_{X_q}}{4\mu}$
x_{oD}	$\frac{C_{D_o}}{2}$
$x_{\alpha D}$	$\frac{C_{D_\alpha}}{2}$

x_{pp}	$\frac{C_{X_{pp}}}{8\mu^2}$
X	coordinate axis
X_o	space-fixed coordinate axis
y	distance along Y_o axis
y^+	$\frac{y}{\mu l}$
y_β	$\frac{C_{Y_\beta}}{2}$
$y_{\dot{\beta}}$	$\frac{C_{Y_{\dot{\beta}}}}{4\mu}$
y_p	$\frac{C_{Y_p}}{4\mu}$
y_r	$\frac{C_{Y_r}}{4\mu}$
$y_{\beta L}$	$y_\beta + x_o$
$y_{\alpha p}$	$\frac{C_{Y_{\alpha p}}}{4\mu}$
$y_{\dot{\alpha} p}$	$\frac{C_{Y_{\dot{\alpha} p}}}{8\mu^2}$
y_{qp}	$\frac{C_{Y_{qp}}}{8\mu^2}$
Y	coordinate axis
Y_o	space-fixed coordinate axis
z	distance along Z_o axis
z^+	$\frac{z}{\mu l}$

z_o	$\frac{C_{Z_o}}{2}$
z_G	$\frac{mg \cos \gamma_o}{\rho S V_o^2}$
z_q	$\frac{C_{Z_q}}{4\mu}$
z_α	$\frac{C_{Z_\alpha}}{2}$
$z_{\dot{\alpha}}$	$\frac{C_{Z_{\dot{\alpha}}}}{4\mu}$
z_η	$1 - z_q + i\nu z_{rp}$
z_ξ	$Z_{\alpha L} + i\nu Z_{\beta p}$
$z_{\dot{\xi}}$	$1 + z_{\dot{\alpha}} + i\nu z_{\dot{\beta}p}$
z_{oL}	$\frac{C_{L_o}}{2}$
$z_{\alpha L}$	$\frac{C_{L_\alpha}}{2}$; also, $z_{\alpha L} = z_\alpha - x_o$
$z_{\beta p}$	$\frac{C_{Z_{\beta p}}}{4\mu}$
$z_{\dot{\beta}p}$	$\frac{C_{Z_{\dot{\beta}p}}}{8\mu^2}$
z_{rp}	$\frac{C_{Z_{rp}}}{8\mu^2}$
Z	coordinate axis
Z_o	space-fixed coordinate axis

α angle of attack, $90^\circ - \sphericalangle Z\bar{V}$

β angle of sideslip, $90^\circ - \sphericalangle Y\bar{V}$

γ_o inclination of flight path of steady-state glide to horizontal, $\sphericalangle \bar{g}Z_o$

Δ $y^+ + iZ^+$, nondimensional transverse displacement

ζ variable of integration, replaces x^+ in integrand

η $\psi - i\theta$

θ angle of pitch, angle between intersection of X_oY_o and XZ_o planes and X

Λ arbitrary angle of rotation about X

μ aircraft density factor, $\frac{m}{\rho S l}$

ν τp_o

ξ $\beta + i\alpha$

ρ density of the air

σ $\sqrt{1 - \frac{1}{s}}$

τ time factor, $\frac{m}{\rho S V_o}$

φ angle of roll:
aircraft: angle between intersection of YZ and X_oY_o planes and Y;
symmetrical missile: angle between Y and $\bar{2}$

ψ angle of yaw, angle between X_o and intersection of X_oY_o and XZ_o planes

ω	angular velocity of body with respect to space-fixed axes
Ω	angular velocity of the XYZ axes with respect to the $X_oY_oZ_o$ axes
$\bar{1}$, $\bar{2}$, $\bar{3}$	body-fixed coordinate axes, fixed in body for missiles and projectiles
$\sphericalangle$	angle, $\sphericalangle XY$ = angle between X and Y
$(\dot{})$	operator, $\dot{A} = \frac{dA}{dt}$
$(\ddot{})$	operator, $\ddot{A} = \frac{d^2A}{dt^2}$
$(\bar{})$	a vector quantity

Subscripts

The definitions of all subscripts apply to their usage with all symbols except where the complete symbol with its subscripts is defined as a unit.

A	aerodynamic components
D	component along trajectory (drag)
G	gravity force components
I	imaginary component
l	component along X axis
L	component normal to trajectory (lift)
m	component along Y axis
n	component along Z axis
p	operation, $A_p = \frac{\partial A}{\partial p}$
q	operation, $A_q = \frac{\partial A}{\partial q}$

r	operation, $A_r = \frac{\partial A}{\partial r}$
R	real component
X	component along X
Y	component along Y
Z	component along Z
α	operation, $A_\alpha = \frac{\partial A}{\partial \alpha}$
$\dot{\alpha}$	operation, $A_{\dot{\alpha}} = \frac{\partial A}{\partial \dot{\alpha}}$
β	operation, $A_\beta = \frac{\partial A}{\partial \beta}$
$\dot{\beta}$	operation, $A_{\dot{\beta}} = \frac{\partial A}{\partial \dot{\beta}}$
o	value of coefficient with $u = \beta = \alpha = \dot{\beta} = \dot{\alpha} = p = q = r = 0$
o	value of quantity at $z = 0$ or $x = 0$
1	component along 1 axis
2	component along 2 axis
3	component along 3 axis

EQUATIONS OF MOTION FOR AIRCRAFT WITH ONE PLANE OF MIRROR SYMMETRY

The essential features of Lanchester's work have formed the basis of all subsequent treatments of this subject, although many details such as nomenclature have been modified to meet better the changing demands of aircraft design. Dynamic stability is a standard subject in aeronautical textbooks (e.g., see refs. 1 and 5) and, since the methods and results of the theory are well known, the aircraft case will be sketched rapidly.

Geometry

The geometry is shown in figures 1(a), 1(b), and 1(c). Two sets of Cartesian coordinate axes are used in developing the equations of motion. One set is fixed in the aircraft and is known as "stability axes." The other set is fixed with respect to the earth and is known as "earth axes." The orientation of the two sets is shown in figure 1(a). Figure 1(b) shows the location of the gravity vector with respect to earth axes and figure 1(c) the location of the velocity vector with respect to stability axes. In the figures the directions of the coordinate axes are indicated by unit vectors and the angles by circular arcs, which represent great circles on a unit sphere centered at the center of gravity. Both sets of axes are right-handed, as defined in reference 1; that is, the positive sense of a component rotation or couple about any axis is "determined by reference to a right-handed screw, when facing the positive direction of the axis." It will be noted that the various features of this geometry conform throughout to NACA standards.

The stability axes are designated by XYZ. They are fixed in the aircraft and are oriented so that Y is perpendicular to the plane of symmetry and Z is perpendicular to the relative wind in steady flight. Their origin of coordinates is at the center of gravity. In other words, they are body-fixed axes with the XZ plane being the aircraft's plane of symmetry and with the X axis being coincident with the velocity vector in the steady-state glide on which the flight path is a perturbation. The positive directions along the axes are as follows: X is positive forward; Y is positive to starboard; Z is positive towards the bottom of the aircraft. It should be noted that, whereas Y is always a principal inertia axis, X and Z are not necessarily so, and products of inertia terms are introduced into the equations of motion in order to account for the alinement of X with the steady-state relative wind rather than with a principal inertial direction (body axes alined in this particular way are known as stability axes).

The earth axes are designated by $X_OY_OZ_O$. They are fixed with respect to the earth, but, since the earth's rotation is negligibly slow compared to the angular velocity of the aircraft, they are considered to be Galilean axes fixed in space. As with the stability axes, their orientation is determined by the position of the aircraft in its steady-state glide, which is subject to the restriction that in the steady-state glide the plane of symmetry is a vertical plane. Each member of the $X_OY_OZ_O$ axes is taken to be parallel to and to point in the same direction as its corresponding member of the XYZ axes. In other words, X_O points in the direction of the steady-state glide path, Y_O is horizontal and points to the right (of an observer facing forward), and Z_O points down (but not necessarily vertically down). Two locations are designated for the origin of coordinates of the earth axes, depending on the component of the motion being considered. For angular measurements, the

origin of coordinates is taken to be the center of gravity of the aircraft at some particular point along its flight path. For linear measurements, the origin of coordinates is located as required by the over-all scheme of measurement which would be set up to analyze the flight of the aircraft.

The orientation of the aircraft in space is defined by three angular coordinates Φ, θ, and ψ which give the alinement of the stability axes, XYZ, with respect to the earth axes, $X_oY_oZ_o$ (see fig. 1(a)). If these angles are all zero, the stability axes point in the directions of the earth axes. Any other orientation of the aircraft is reached by three consecutive rotations, starting with XYZ pointing in the directions of $X_oY_oZ_o$. In defining each of the rotations, the point of view is taken that XYZ are either in their starting position or in the position given by the preceding rotation, and not necessarily in their final position. The rotations are noncommutative and must be taken in the order specified.

ψ Rotation 1. Start with XYZ pointing in the directions of $X_oY_oZ_o$, rotate about Z through ψ.

θ Rotation 2. Rotate about Y through θ, thereby bringing X to its final position.

φ Rotation 3. Rotate about X through φ, thereby bringing Y and Z to final positions and the aircraft to its actual orientation in space.

The angles ψ, θ, and φ may also be defined as the angles between coordinate axes and the intersections of certain planes. For the purpose of these definitions it is considered that the origin of coordinates of the $X_oY_oZ_o$ axes is momentarily coincident with that of the XYZ axes. The point of view taken here is that the XYZ axes are in their final positions given by the orientation of the aircraft in space at the moment in question. The sign of the angle is specified by giving the axis about which the rotation is taken in going from the line named first to the line named second in the definition. The axis of rotation is listed in parentheses in each definition.

ψ angle between X_o and intersection of X_oY_o and XZ_o planes (rotation about Z_o)

θ angle between intersection of X_oY_o and XZ_o planes and X (rotation about $Y_o' = Y_o$ rotated about Z_o through ψ)

φ angle between intersection of YZ and X_oY_o planes and Y; also angle between intersection of YZ and XZ_o planes and Z (rotation about X)

The inclination of the earth axes $(X_oY_oZ_o)$ to the horizontal is given by the angle γ_o (see fig. 1(b)). The angle γ_o is positive when X_o points above the horizontal (as shown in fig. 1(b)). It should be noted that Y_o is horizontal by definition and hence the X_oZ_o plane is a vertical plane and contains the gravity vector $\overline{g}$.

The orientation of the velocity vector $\overline{V}$ with respect to the stability axes XYZ is given by the angle of attack α and the angle of sideslip β (see fig. 1(c)). These angles are defined so that the product of the velocity magnitude V and the sine of β or α gives the component of $\overline{V}$ along Y or Z; that is, by definition

$$\beta = 90^\circ - \sphericalangle\, \overline{V}Y$$

$$\alpha = 90^\circ - \sphericalangle\, \overline{V}Z$$

and resolving $\overline{V}$ along the transverse stability axes YZ results in the desired relations, as follows

$$V_Y = V \cos \sphericalangle\, \overline{V}Y = V \sin \beta$$

$$V_Z = V \cos \sphericalangle\, \overline{V}Z = V \sin \alpha$$

The above definitions of α and β were chosen because it is believed that the transverse components of the velocity are the quantities having the most physical significance insofar as the aerodynamics of the aircraft is concerned. However, it should be noted that these definitions differ somewhat from common wind-tunnel practice. In a wind tunnel the model is normally placed at an angle of sideslip by supporting it on a bent sting. The angle of attack is set at zero, and the sting is alined to lie in the plane containing the axis of rotation of the angle-of-attack sector and the tunnel axis (it is assumed that the wind vector is along the tunnel axis). The model is then placed on the sting with its plane of symmetry perpendicular to this plane. The angle of attack is now varied by rotating the angle-of-attack sector. Consequently, the Z axis of the model rotates in a plane perpendicular to this sector's axis of rotation. The angles of attack, α, and sideslip, β, are defined as follows: α is the angle through which the angle-of-attack sector is rotated; β is the angle at which the sting is bent. Now, referring to figure 1(c), it is clear that α defined in this report is the same as α defined in wind-tunnel practice, since the $Z\overline{V}$ plane shown in this figure is the plane in which Z rotates and α is measured in this plane in both cases. On the other hand, β in this report is measured in the $Y\overline{V}$ plane whereas β in the wind tunnel is measured in the XY plane, as shown. The difference is second-order for small values of β and hence is insignificant in so far as linearized theory is concerned. However, should the theory be extended to the case of aircraft maneuvering at large angles, as is done in reference 6, the precise definitions of α and β become important.

It will be convenient to write down expressions for the direction cosines between various axes of the earth axes and stability axes systems. The direction cosines are readily obtained from the equations derived in section (5.0), "Kinematics," of reference 6 and are listed in the table below; each entry in the table gives the direction cosine between the axis heading the row and the axis heading the column belonging to the space in which the entry is listed.

	X	Y	Z
X_O	$\cos\theta \cos\psi$	$\cos\psi \sin\varphi \sin\theta - \sin\psi \cos\varphi$	$\cos\psi \cos\varphi \sin\theta + \sin\psi \sin\varphi$
Y_O	$\cos\theta \sin\psi$	$\sin\psi \sin\varphi \sin\theta + \cos\psi \cos\varphi$	$\sin\psi \cos\varphi \sin\theta - \cos\psi \sin\varphi$
Z_O	$-\sin\theta$	$\sin\varphi \cos\theta$	$\cos\theta \cos\varphi$

Now, the conditions postulated for the flight, which are discussed in the next section, require that corresponding axes of the two coordinate systems lie close to one another during the flight covered by the analysis and hence that the angles φ, θ, and ψ be small. Only first-order terms are retained in the development of the equations of motion and, since φ, θ, and ψ are first-order, quadratic terms in the expressions for the direction cosines will be second-order and may be neglected. The equations giving the direction cosines correct to first order are summarized in the table below; again each entry in the table gives the direction cosine between the axis heading the row and the axis heading the column belonging to the space in which the entry is listed.

	X	Y	Z
X_O	1	$-\psi$	θ
Y_O	ψ	1	$-\varphi$
Z_O	$-\theta$	φ	1

(1)

Conditions of Flight

The following conditions are postulated for the aircraft and for its motion in flight:

(1) The aircraft has a plane of mirror symmetry, the XZ plane.

(2) The magnitude of the velocity of the aircraft may be large but the change in the velocity must be small. Accordingly, the velocity may be represented by a constant plus a small perturbation; that is, $V=V_O+u$.

(3) The angles orienting the two sets of coordinate axes, φ, θ, and ψ, and locating the velocity vector, β and α, are all small.

(4) All components of the angular velocity, p, q, and r, are small.

(5) The flight path of the aircraft is a perturbation on a steady-state glide.

Conditions (2), (3), and (4) may be summarized by saying that the following variables must be small (e.g., have a numerical magnitude of 0.1 or less) and hence are first-order quantities:

$$\left.\begin{array}{ccc} \varphi & \theta & \psi \\ \dfrac{u}{V_O} & \beta & \alpha \\ \dfrac{pl}{2V_O} & \dfrac{ql}{2V_O} & \dfrac{rl}{2V_O} \end{array}\right\} \quad \text{small, for example, 0.1 or less} \qquad (2)$$

where l is a characteristic length. It should be noted that if β, α, $\frac{ql}{2V_O}$, and $\frac{rl}{2V_O}$ are first-order, then $\frac{\dot{\beta}l}{2V_O}$ and $\frac{\dot{\alpha}l}{2V_O}$ will also be first-order, where $\dot{\beta} = \frac{d\beta}{dt}$ and $\dot{\alpha} = \frac{d\alpha}{dt}$.

The theory developed in this paper includes only first-order terms. Products of first-order terms (except for the Magnus terms of spinning projectiles and symmetrical missiles), for example, $\varphi\theta$, are considered to be negligibly small and are left out of the equations of motion. As a result, the equations of motion become linear in form and their solution may be given by well-known, explicit functions. Experience has shown that the solution of the linearized equations of motion is, in fact, a reasonable representation of the flight motion of many aircraft. However, there may be circumstances under which the linearized equations do not describe the aircraft's flight, despite the fact that all the quantities listed in equation (2) are small. The aerodynamic characteristics of

certain aircraft are such that some of the quadratic terms are as large or larger than certain of the first-order terms included in the linear theory. For aircraft of this type, the significant quadratic terms must be retained in the equations of motion and the theory revised accordingly. Consequently, the analysis of an actual flight should include a check of the magnitude of second-order terms to be sure that they are in fact second-order, that is, negligible. In other words, the a priori assumptions of the theory should be checked a posteriori by an analysis of the measurements.

Kinematic Relations

The equations of motion are derived from the basic vector equations

$$\frac{d\overline{M}}{dt} = \overline{F} \tag{3}$$

$$\frac{d\overline{H}}{dt} = \overline{M} \tag{4}$$

The first step in their derivation is to obtain relations between the physical quantities, $\overline{M}$ and $\overline{H}$, and the kinematic variables (of the motion), β, α, φ, θ, and ψ. The components of the linear momentum, $\overline{M}$ along the XYZ axes in terms of the components of the velocity vector, $\overline{V}$, and the components of $\overline{V}$ in terms of β and α are listed in the following table.

	$\overline{M}$	$\overline{V}$
i	mV_X	$V_X = V\sqrt{1 - \sin^2\alpha - \sin^2\beta}$
j	mV_Y	$V_Y = V \sin \beta$
k	mV_Z	$V_Z = V \sin \alpha$

(5)

where the component along X, Y, or Z is designated by i, j, or k, the unit vector along X,Y,Z, respectively.

Similarly, the components of the angular momentum, $\overline{H}$, are given in terms of the components of the angular velocity, $\overline{\omega}$, and, in turn, the components of $\overline{\omega}$ in terms of φ, θ, and ψ are given in the table below (see ref. 6).

	$\overline{H}$	$\overline{\omega}$
i	$I_X p - J_{XZ} r$	$p = \dot{\phi} - \dot{\psi} \sin \theta$
j	$I_Y q$	$q = \dot{\theta} \cos \varphi + \dot{\psi} \sin \varphi \cos \theta$
k	$I_Z r - J_{XZ} p$	$r = \dot{\psi} \cos \theta \cos \varphi - \dot{\theta} \sin \varphi$

(6)

For convenience in computing the relative magnitudes of terms in the equations of motion, the components of $\overline{\omega}$ (p, q, and r) may be written as follows, expanding sines and cosines and retaining only first- and second-order terms.

$$\left.\begin{aligned} p &= \dot{\phi} - \theta r \\ q &= \dot{\theta} + \varphi r \\ r &= \dot{\psi} - \varphi q \end{aligned}\right\} \tag{7}$$

The expressions for $\overline{V}$ and $\overline{\omega}$ given in equations (5) and (7) include second-order terms in order to clarify certain steps taken in linearizing the equations of motion. The components of $\overline{V}$ and $\overline{\omega}$ appearing in the inertial terms in the equations of motion are multiplied by large numerical factors. Consequently, it is desirable to compare the second-order inertial terms with the first-order aerodynamic terms in order to establish that the second-order inertial terms are truly negligible. The comparison is made in Appendix A, which covers the linearization procedure.

Physical quantities are defined with respect to the body-fixed axes, XYZ, as is customary in the treatment of the motion of a rigid body. On the other hand, the rates of change involved in the basic vector equations (3) and (4) are taken with respect to space-fixed axes. Hence, equations (3) and (4) must be transformed in order to account for the movement of the XYZ axes. The well-known transformation for the rate of change of any vector $\overline{A}$ from fixed to moving axes is given by

$$\frac{d\overline{A}}{dt} = \dot{\overline{A}} + \overline{\omega} \times \overline{A} \tag{8}$$

where $\dot{\overline{A}}$ is the rate of change of $\overline{A}$ measured with respect to XYZ axes. The components of the transformation are

$$\left.\begin{aligned} \frac{dA_X}{dt} &= \dot{A}_X + qA_Z - rA_Y \\ \frac{dA_Y}{dt} &= \dot{A}_Y + rA_X - pA_Z \\ \frac{dA_Z}{dt} &= \dot{A}_Z + pA_Y - qA_X \end{aligned}\right\} \tag{9}$$

Aerodynamic and Gravitational Force Systems

The resultant external force, $\overline{F}$, of equation (3) is the sum of the aerodynamic forces and the gravitational force; that is,

$$\overline{F} = \overline{F}_A + \overline{F}_G \tag{10}$$

The resultant external moment, $\overline{M}$, of equation (4) is due solely to aerodynamic reactions.

Concerning the aerodynamic forces and moments, it is assumed that the components of the resultant force and moment, $\overline{F}_A$ and $\overline{M}$, are given by

$$F_{A_{X,Y,Z}} = C_{X,Y,Z} \frac{\rho}{2} V^2 S \tag{11}$$

and

$$M_{X,Y,Z} = C_{l,m,n} \frac{\rho}{2} V^2 S l \tag{12}$$

It is further assumed that the coefficients, the C's, in equations (11) and (12) are functions of the variables β, α, $\dot{\beta}$, $\dot{\alpha}$, p, q, and r and that these functions may be expanded as a series in the variables named. The general formula for any coefficient, C_a, where a stands for X, Y, Z, l, m, or n, is

$$C_a = C_{a_o} + C_{a_\beta}\beta + C_{a_\alpha}\alpha + C_{a_{\dot{\beta}}} \frac{\dot{\beta} l}{2V} + C_{a_{\dot{\alpha}}} \frac{\dot{\alpha} l}{2V} + C_{a_p} \frac{pl}{2V} + C_{a_q} \frac{ql}{2V} +$$

$$C_{a_r} \frac{rl}{2V} + (\text{higher-order terms}) \tag{13}$$

The coefficients of this series are assumed to be independent of the variables β, α, $\dot{\beta}$, $\dot{\alpha}$, p, q, and r and to be functions only of the external shape of the aircraft and of fundamental aerodynamic parameters, such as the Reynolds and Mach numbers. They will be known herein as the "aerodynamic coefficients" and correspond to the conventional stability derivatives.

Examples of the higher-order terms are $C_{a_{\alpha\alpha}}\alpha^2$ and $C_{a_{\alpha\beta}}\alpha\beta$, two of the quadratic terms of the series. In the aircraft case all of the variables β, α, p, q, and r are small (first-order). Hence, their products will be second-order and the quadratic and higher-order terms of the series may be neglected.

If the aircraft had no plane of symmetry, all 48 coefficients would be required to account for the complete force and moment system. However, the majority of aircraft have one plane of mirror symmetry as postulated

here and, as a result, certain of the coefficients must be zero. The requirement imposed by the existence of a plane of mirror symmetry can be stated succinctly as follows.

Let the aircraft execute two motions, the first being the mirror image of the second about the XZ plane, the plane of symmetry; the aerodynamic force due to the first motion must be the mirror image of that due to the second. The mirror image of a motion (or force) is defined as the motion (or the force) that will look the same to a second observer stationed on the opposite side of the aircraft from the first; that is, one observer being to port, the other to starboard.

To illustrate, consider the terms $C_{l_p}(pl/2V)$, the rolling moment due to rolling velocity, and $C_{m_p}(pl/2V)$, the pitching moment due to rolling velocity. The mirror image of the rolling velocity p is always -p. The mirror image of the rolling moment M_X is $-M_X$; hence, the mirror image of $C_{l_p}(pl/2V)$ should be $-C_{l_p}(pl/2V)$. It can be seen that this term reverses sign as it should, since p mirrors to -p, and the coefficient C_{l_p} is therefore an allowable one. On the other hand, the mirror image of the pitching moment M_Y is $+M_Y$; hence, the mirror image of $C_{m_p}(pl/2V)$ should be $+C_{m_p}(pl/2V)$. It can be seen that the term reverses sign as it should not, since p mirrors to -p, and therefore the coefficient C_{m_p} is not an allowable one. Consequently the coefficient C_{m_p} must vanish; that is, $C_{m_p} = 0$.

Applying the criterion of symmetry to equation (13), one can see that the following coefficients must vanish:

$$\left.\begin{aligned} C_{X_\beta} = C_{X_{\dot\beta}} = C_{X_p} = C_{X_r} = 0 \\ C_{Y_o} = C_{Y_\alpha} = C_{Y_{\dot\alpha}} = C_{Y_q} = 0 \\ C_{Z_\beta} = C_{Z_{\dot\beta}} = C_{Z_p} = C_{Z_r} = 0 \\ C_{l_o} = C_{l_\alpha} = C_{l_{\dot\alpha}} = C_{l_q} = 0 \\ C_{m_\beta} = C_{m_{\dot\beta}} = C_{m_p} = C_{m_r} = 0 \\ C_{n_o} = C_{n_\alpha} = C_{n_{\dot\alpha}} = C_{n_q} = 0 \end{aligned}\right\} \qquad (14)$$

Substituting equations (14) in equation (13) and arbitrarily assigning a minus sign to the coefficients of C_X and C_Z in conformity with standard

practice (since the axial drag force usually acts in the -X direction and the normal force in the -Z direction), give the force and moment coefficients for the aircraft as follows:

$$\left.\begin{aligned}
C_X &= -C_{X_o} - C_{X_\alpha}\alpha - C_{X_{\dot\alpha}}\frac{\dot\alpha l}{2V} - C_{X_q}\frac{ql}{2V} \\
C_Y &= C_{Y_\beta}\beta + C_{Y_{\dot\beta}}\frac{\dot\beta l}{2V} + C_{Y_p}\frac{pl}{2V} + C_{Y_r}\frac{rl}{2V} \\
C_Z &= -C_{Z_o} - C_{Z_\alpha}\alpha - C_{Z_{\dot\alpha}}\frac{\dot\alpha l}{2V} - C_{Z_q}\frac{ql}{2V} \\
C_l &= C_{l_\beta}\beta + C_{l_{\dot\beta}}\frac{\dot\beta l}{2V} + C_{l_p}\frac{pl}{2V} + C_{l_r}\frac{rl}{2V} \\
C_m &= C_{m_o} + C_{m_\alpha}\alpha + C_{m_{\dot\alpha}}\frac{\dot\alpha l}{2V} + C_{m_q}\frac{ql}{2V} \\
C_n &= C_{n_\beta}\beta + C_{n_{\dot\beta}}\frac{\dot\beta l}{2V} + C_{n_p}\frac{pl}{2V} + C_{n_r}\frac{rl}{2V}
\end{aligned}\right\} \qquad (15)$$

Concerning the force of gravity, F_G, its components along the XYZ axes are readily computed by first resolving F_G along the $X_oY_oZ_o$ axes and then computing the XYZ components using the direction cosines given in equations (1). The force of gravity F_G lies in the X_oZ_o plane since Y_o is horizontal by definition and, hence, is given by

$$\overline{F}_G = \overline{i}_o(-mg \sin \gamma_o) + \overline{k}_o(mg \cos \gamma_o) \qquad (16)$$

The XYZ components of $\overline{F}_G$ are now obtained by resolving the X_oZ_o components of equation (16) along the XYZ axes using equations (1):

$$\left.\begin{aligned}
F_{G_X} &= mg\,(-\sin \gamma_o - \theta \cos \gamma_o) \\
F_{G_Y} &= mg\,(\psi \sin \gamma_o + \varphi \cos \gamma_o) \\
F_{G_Z} &= mg\,(-\theta \sin \gamma_o + \cos \gamma_o)
\end{aligned}\right\} \qquad (17)$$

Derivation of the Differential Equations of Motion

The differential equations of motion are derived from the basic vector equations (3) and (4) by resolving the vectors along coordinates axes and using the kinematic relations and the equations for the external forces and moments, thereby obtaining six scalar equations, three associated with the force and three with the moment. Both earth axes and

stability axes are employed as coordinate axes in resolving the basic vector equations. Earth axes are used to compute the motion of the center of gravity of the aircraft, since linear displacements are measured along these axes. Stability axes are used to compute the angular motions of the aircraft and of the velocity vector, since the inertial properties and aerodynamic coefficients of the aircraft are invariant with respect to these axes. The detailed steps taken in deriving the equations of motion are outlined as follows.

Both equations (3) and (4) are transformed from fixed to moving axes by equation (9), except for the X component of equation (3), as discussed below. Equation (17) for the gravity force and equations (11) and (15) for the aerodynamic forces are substituted into equation (10) for the external force. The kinematic relations for linear momentum and velocity are given by equations (5) and these are used to complete the development of equation (3). Equations (12) and (15) for the aerodynamic moments and equations (6) and (7) giving the kinematic relations for the angular momentum and angular velocity are substituted into equation (4) to complete its development.

The details of this particular step are given in Appendix A. It should be noted that the equation associated with the axial drag force is treated as a special case. This equation is derived by resolving the vector linear momentum equation (3) along the space-fixed X_o axis rather than the X axis, thereby avoiding the term $(qA_Z - rA_Y)$ which comes from the transformation from fixed to moving axes (eq. 9). The change in approach is desirable because in the linearization of the equation of motion the term $(qA_Z - rA_Y)$ gives rise to second-order terms whose magnitude compared to first-order terms is uncertain and whose neglect consequently may be questionable. In so far as the validity of the derivation is concerned, either axis system, stability axes or earth axes, may be used in deriving any of the equations of motion provided the vector quantities are resolved correctly along the axes in question and the angular velocity of the coordinate axes (if any) is accounted for.

The next step in the derivation is to separate each of the six equations into first- and second-order terms (and higher-order terms) and to neglect the second-order (and higher-order) terms. This step is described in detail in Appendix A. Hence, at this point, the equations of motion are linearized and the theory is restricted to a consideration of first-order effects.

The third step in the derivation is to introduce a nondimensional time as the independent variable and a concise notation for the factors of the dependent variables. This nondimensional time, t^+, is sometimes referred to as the reduced time and is based on the aircraft time factor, τ. The concise notation stems directly from the subscripts of the aerodynamic coefficients and at the same time includes any other multiplicative factors. For example,

$$m_\alpha = \frac{\mu C_{m_\alpha}}{2K_Y^2}$$

In the same spirit, the derivative notation is shortened by defining

$$DA = \frac{dA}{dt^+}$$

This step parallels a similar procedure followed in reference 7. The advantages accrued thereby speak for themselves.

The final step goes back to (5) of the condition postulated for the flight, namely, that the flight path is a perturbation on a steady-state glide. Hence, the equilibrium condition of flight with all the dependent variables $(u,\beta,\alpha,\dot{\beta},\dot{\alpha},p,q,r)$ equal to zero is one in which the lift, drag, speed, glide angle, and weight are all in balance. It is assumed here that the X_O axis and the X axis are both coincident with the trajectory along the steady-state glide path. The equations of motion for the steady-state glide are

$$\left.\begin{aligned} x_O &= -\frac{mg \sin \gamma_O}{\rho S V_O^2} = -x_G \\ z_O &= \frac{mg \cos \gamma_O}{\rho S V_O^2} = z_G \\ m_O &= 0 \end{aligned}\right\} \qquad (18)$$

Equations (18) are used to replace the gravity terms with corresponding aerodynamic terms in the final differential equations of motion.

At the same time, the factors $x_O, x_\alpha, z_O, z_\alpha$ are replaced by the factors $x_{OD}, x_{\alpha D}, z_{OL}, z_{\alpha L}$ which refer to the drag and lift of the aircraft as measured in a wind tunnel. Now, it is customary in deriving the equations of motion to use drag and lift coefficients rather than the X and Z coefficients. This is a perfectly reasonable choice since the static stability characteristics of many aircraft are measured in wind-tunnel tests. Wind-tunnel data are ordinarily presented as lift and drag polars and it is desirable to formulate the theory in terms of lift and drag so that these data can be used directly. A support system commonly employed in wind tunnels has already been described. In so far as the coordinate axes of this report are concerned, lift and drag are measured along and perpendicular to the steady-state relative wind, that is, along the $-Z_O$ and $-X_O$ axes, respectively. The aerodynamic lift and drag factors corresponding to the lift and drag coefficients are designated by the subscripts L and D, and the relations between these and the aerodynamic factors corresponding to the X and Z coefficients are given by

$$\left.\begin{aligned} x_o &= x_{oD} \\ x_\alpha &= x_{\alpha D} - z_{oL} \\ z_o &= z_{oL} \\ z_\alpha &= z_{\alpha L} + x_{oD} \end{aligned}\right\} \tag{19}$$

The six differential equations of motion which result from the preceding steps are given below in the following order: the three force equations, axial drag, side force, and normal lift force, and the three moment equations, roll, pitch, and yaw.

$$D\left(\frac{u}{V_o}\right) + 2x_{oD}\left(\frac{u}{V_o}\right) + x_{\dot{\alpha}}D\alpha + (x_{\alpha D} - z_{oL})\alpha + x_q D\theta + z_{oL}\theta = 0 \tag{20}$$

$$(1 - y_{\dot{\beta}})\, D\beta - y_\beta \beta + (1 - y_r)\, D\psi + x_{oD}\psi - y_p D\varphi - z_{oL}\varphi = 0 \tag{21}$$

$$(1 + z_{\dot{\alpha}})\, D\alpha + (z_{\alpha L} + x_{oD})\alpha - (1 - z_q)\, D\theta - x_{oD}\theta + 2z_{oL}\left(\frac{u}{V_o}\right) = 0 \tag{22}$$

$$D^2\varphi - l_p D\varphi - K_1 D^2\psi - l_r D\psi - l_{\dot{\beta}} D\beta - l_\beta \beta = 0 \tag{23}$$

$$D^2\theta - m_q D\theta - m_{\dot{\alpha}} D\alpha - m_\alpha \alpha = 0 \tag{24}$$

$$D^2\psi - n_r D\psi - K_2 D^2\varphi - n_p D\varphi - n_{\dot{\beta}} D\beta - n_\beta \beta = 0 \tag{25}$$

Discussion of the Differential Equations of Motion

An inspection of equations (20) through (25) shows that the six equations may be divided into two distinct groups of three equations each. One group consisting of equations (20), (22), and (24) involves only the variables u/V_o, α, and θ and describes the longitudinal motion of the aircraft. The other group consisting of equations (21), (23), and (25) involves only the variables β, φ, and ψ and describes the lateral motion of the aircraft. Hence, it may be concluded that the linearized equations of motion of aircraft may be divided into two separate parts and that the parts do not interact. This result is, of course, well known and the over-all stability of aircraft is customarily subdivided into longitudinal

stability involving u/V_0, α, and θ and into lateral stability involving β, φ, and ψ. In this connection, it should be noted that different representative lengths, l, may be used in the two groups of equations since the groups are independent of one another. In practice, it is common to use the mean aerodynamic chord of the wing ($l = c$) in the longitudinal stability equations and the wing span ($l = b$) in the lateral stability equations.

Equations (20) through (25) agree satisfactorily with corresponding equations presented in standard aeronautical texts (see, e.g., eqs. (10-89a, b, c) and (11-34a, b, c) of ref. 5). The equations here do contain certain extra terms not ordinarily found elsewhere but it is believed that all of the terms in these equations are required for a complete and consistent first-order aerodynamic force system and to account accurately for gravity forces. On the other hand, the additions are so small for representative aircraft that they do not affect the comparison in so far as practical application of the equations is concerned.

To summarize briefly, it may be stated that equations (20) through (25) are the standard equations underlying the longitudinal and lateral stability of aircraft. Hence, it has been demonstrated that the approach used in the present development leads to the commonly accepted result. The solution and application of the differential stability equations have been treated at length and reference is made to the extensive aeronautical literature for a discussion of this aspect of the subject (e.g., see refs. 1, 5, 6, 7, 8, and 9).

EQUATIONS OF MOTION FOR PROJECTILES AND MISSILES WITH 90° ROTATIONAL AND MIRROR SYMMETRY

In this paper the equations of motion describing the flights of spinning projectiles and symmetrical missiles are treated simultaneously, and the results may be applied to either case. Both projectiles and missiles are mentioned in the title because the means of stabilization customarily employed in the two cases depends on different physical principles, the majority of artillery shell being spin stabilized and missiles being fin stabilized. Despite this difference, the motions of both stem from a common set of equations and each represents an application of the general equations to a particular condition of flight. The terms, projectiles and missiles, will be used interchangeably for the sake of succinctness, but, although only one may be named, the other will always be implied.

Projectiles and missiles are considered to differ from aircraft in two respects:

(1) They are postulated to have 90° roll symmetry or its equivalent in addition to a plane of mirror symmetry.

(2) The component of their angular velocity along the axis of symmetry is allowed to be large; that is, they may roll continuously (or spin) around the axis of symmetry.

It is immediately evident that the freedom of the missile to spin will lead to difficulties if axes fixed in the missile are specified in this case as they were for aircraft. The angle of roll, φ, will become large and the components of the gravity force will vary with the sine and cosine of φ. Both of these consequences will violate the requirement that the dependent variables remain small during the flight, an essential feature of a linearized theory. It is clear that the geometry postulated in the aircraft case must be modified before it is suitable for the development of linearized equations of motion for projectiles and missiles.

The 90° rotational symmetry, the addition to the requirement of mirror symmetry, provides the key to the problem. It will be shown that the aerodynamic force system is modified for this case so that the resultant force and moment vectors are independent of the roll orientation (to first order). Consequently, only one coordinate axis need be fixed in the missile, that is, along the axis of rotational symmetry, the longitudinal axis of the missile. One of the transverse coordinate axes will be prescribed to lie in a certain space-fixed plane oriented so that the angle of roll of the coordinate axis system is always zero. As a result, the change in orientation of the coordinate axes remains small during flight, and the variation of all of the dependent variables except the roll angle of the missile is correspondingly small so that the basic requirement of a first-order theory can again be satisfied.

Consequences of 90° Rotational Symmetry

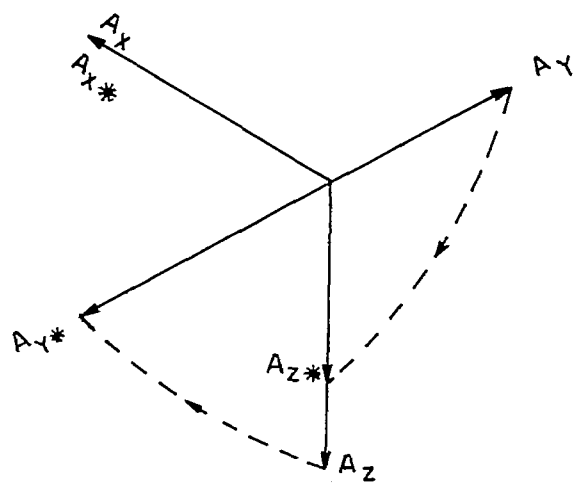
Ninety-degree rotational symmetry may be readily visualized by picturing the missile (or projectile) in two positions, one rotated by 90° about the axis of symmetry with respect to the other. If the missile has 90° rotational symmetry, the two pictures will look precisely the same. In mathematical terms, rotation through 90° transforms the missile into itself. It is assumed in this paper that the axis of rotational symmetry coincides with the longitudinal axis of the missile.

First, the consequences of rotational symmetry to the aerodynamic force system will be investigated. Let the XYZ axes remain axes fixed in the missile, but place the X axis along the axis of rotational symmetry. The X axis is now a principal axis of inertia so that J_{XZ} vanishes, furthermore $I_Y = I_Z$. It is clear that the aerodynamic coefficients must have values such that a rotation of the $\overline{V}$, $\dot{\overline{V}}_T$, and $\overline{\omega}$ vectors by 90° about X will produce a similar rotation of the $\overline{F}$ and $\overline{M}$ vectors of 90° about X, since $\overline{F}$ and $\overline{M}$ are functions of $\overline{V}$, $\dot{\overline{V}}_T$, and $\overline{\omega}$

and the aerodynamic configuration will be precisely the same after the 90° rotation, as required by symmetry. The components of $\dot{\overline{V}}_T$ are given to first order by $\dot{\overline{V}}_T = \dot{\beta}\overline{j} + \dot{\alpha}\overline{k}$. It will be noted that $\dot{\overline{V}}_T$ transforms as $\overline{V}$ does. Only $\overline{V}$ will be written down explicitly in the subsequent analysis but $\dot{\overline{V}}_T$ will be included implicitly in all derivations in which $\overline{V}$, $\dot{\overline{V}}_T$, and $\overline{\omega}$ are involved.

Now, if $\overline{A}$ is any vector and $\overline{A}_*$ is $\overline{A}$ rotated 90° around the axis of symmetry, and if the direction of rotation is taken to be from Y to Z, the components of $\overline{A}$ are related to the components of $\overline{A}_*$ as follows:

$$\left.\begin{aligned} A_{X*} &= A_X \\ A_{Y*} &= -A_Z \\ A_{Z*} &= A_Y \end{aligned}\right\} \qquad (26)$$



Sketch (a)

Applying equation (26) to the aerodynamic force and moment vectors gives the relations between the components of the force and moment before and after rotation of $\overline{V}$, $\dot{\overline{V}}_T$, and $\overline{\omega}$, which are

$$\left.\begin{aligned} F_{X*}(\overline{V}_*, \overline{\omega}_*) &= F_X(\overline{V}, \overline{\omega}) \\ F_{Y*}(\overline{V}_*, \overline{\omega}_*) &= -F_Z(\overline{V}, \overline{\omega}) \\ F_{Z*}(\overline{V}_*, \overline{\omega}_*) &= F_Y(\overline{V}, \overline{\omega}) \\ M_{X*}(\overline{V}_*, \overline{\omega}_*) &= M_X(\overline{V}, \overline{\omega}) \\ M_{Y*}(\overline{V}_*, \overline{\omega}_*) &= -M_Z(\overline{V}, \overline{\omega}) \\ M_{Z*}(\overline{V}_*, \overline{\omega}_*) &= M_Y(\overline{V}, \overline{\omega}) \end{aligned}\right\} \qquad (27)$$

It should be noted that the values of the components after rotation must be computed using the rotated values of velocity and angular velocity, $\overline{V}_*$ and $\overline{\omega}_*$, as indicated by the functional notation of equations (27).

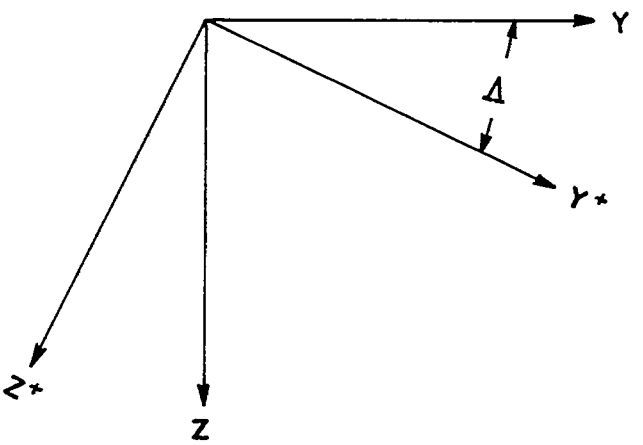
Equations (27) must hold for all values of $\overline{V}$ and $\overline{\omega}$. Hence, computing the components of $\overline{V}_*$ and $\overline{\omega}_*$ from equations (27) and substituting the values of the components of $\overline{V}$ and $\overline{\omega}$ from equations (5), (6), and (7) (neglecting second-order terms) and of $\overline{F}_A$ and $\overline{M}$ from equations (11), (12), and (15), the following equalities must hold between the aerodynamic coefficients:

$$\left.\begin{aligned}
&C_{Y_\beta} = -C_{Z_\alpha} \\
&C_{Y_{\dot\beta}} = -C_{Z_{\dot\alpha}} \\
&C_{Y_r} = C_{Z_q} \\
&C_{n_\beta} = -C_{m_\alpha} \\
&C_{n_{\dot\beta}} = -C_{m_{\dot\alpha}} \\
&C_{n_r} = C_{m_q} \\
&C_{X_\alpha} = C_{X_{\dot\alpha}} = C_{X_q} = C_{Y_p} = C_{Z_o} = C_{l_\beta} = C_{l_{\dot\beta}} = C_{l_r} = C_{m_o} = C_{n_p} = 0
\end{aligned}\right\} \quad (28)$$

Consequently, the equations for the components of $\overline{F}$ and $\overline{M}$ become

$$\left.\begin{aligned}
C_X &= -C_{X_o} \\
C_Y &= -C_{Z_\alpha}\beta - C_{Z_{\dot\alpha}}\frac{\dot\beta l}{2V} + C_{Z_q}\frac{rl}{2V} \\
C_Z &= -C_{Z_\alpha}\alpha - C_{Z_{\dot\alpha}}\frac{\dot\alpha l}{2V} - C_{Z_q}\frac{ql}{2V} \\
C_l &= +C_{l_p}\frac{pl}{2V} \\
C_m &= C_{m_\alpha}\alpha + C_{m_{\dot\alpha}}\frac{\dot\alpha l}{2V} + C_{m_q}\frac{ql}{2V} \\
C_n &= -C_{m_\alpha}\beta - C_{m_{\dot\alpha}}\frac{\dot\beta l}{2V} + C_{m_q}\frac{rl}{2V}
\end{aligned}\right\} \quad (29)$$

Second, since the aerodynamic coefficients for rotationally symmetric bodies have been established, the variation of the aerodynamic coefficients with rotation of the YZ axes around X will be investigated. Let the YZ axes be turned through an arbitrary angle Λ to a new position Y^+Z^+, as shown in the sketch.



Sketch (b)

Relations between the components of a vector $\overline{A}$ along YZ and along Y^+Z^+ are given by

$$\left.\begin{aligned} A_Y &= A_Y{}^+ \cos\Lambda - A_Z{}^+ \sin\Lambda \\ A_Z &= A_Z{}^+ \cos\Lambda + A_Y{}^+ \sin\Lambda \end{aligned}\right\} \tag{30a}$$

$$\left.\begin{aligned} A_Y{}^+ &= A_Y \cos\Lambda + A_Z \sin\Lambda \\ A_Z{}^+ &= A_Z \cos\Lambda - A_Y \sin\Lambda \end{aligned}\right\} \tag{30b}$$

Since the rotation is about X, the X component of $\overline{A}$ is unchanged.

Now, vector quantities themselves do not change with rotation of coordinate axes; only their components along the axes vary. Hence, $\overline{F}_A$ and $\overline{M}$ are not affected by rotation of Y and Z; that is,

$$\left.\begin{aligned} \overline{F}_A &= \overline{F}_A{}^+ \\ \overline{M} &= \overline{M}^+ \end{aligned}\right\} \tag{31a}$$

Resolving equation (31a) into components and using equations (11) and (12) which define the aerodynamic coefficients give

$$\left.\begin{aligned} C_X\overline{i} + C_Y\overline{j} + C_Z\overline{k} &= C_X{}^+\overline{i} + C_Y{}^+\overline{j}^+ + C_Z{}^+\overline{k}^+ \\ C_l\overline{i} + C_m\overline{j} + C_n\overline{k} &= C_l{}^+\overline{i} + C_m{}^+\overline{j}^+ + C_n{}^+\overline{k}^+ \end{aligned}\right\} \tag{31b}$$

If $\bar{j}^+$ and $\bar{k}^+$ are resolved along Y and Z (eq. (30b)) and the components are equated (eqs. (31b)), the following equations are obtained for the C's in terms of the C^+'s and Λ.

$$\left.\begin{aligned} C_X &= C_X{}^+ \\ C_Y &= C_Y{}^+ \cos\Lambda - C_Z{}^+ \sin\Lambda \\ C_Z &= C_Y{}^+ \cos\Lambda + C^+ \sin\Lambda \\ C_l &= C_l{}^+ \\ C_m &= C_m{}^+ \cos\Lambda - C_n{}^+ \sin\Lambda \\ C_n &= C_n{}^+ \cos\Lambda + C_n{}^+ \sin\Lambda \end{aligned}\right\} \tag{32}$$

The components of the aerodynamic force and moment along Y^+ and Z^+ are given by equations similar to equations (29) with all coefficients and all variables marked with the superscript $+$. Using this set of equations similar to equations (29) for the C^+'s and substituting into this set the values of β^+, α^+, $\dot{\beta}^+$, $\dot{\alpha}^+$, q^+, and r^+ given in terms of β, α, $\dot{\beta}$, $\dot{\alpha}$, q, r, and Λ by equations (30b), one obtains the following relations between the aerodynamic coefficients associated with the XYZ axes and with the XY^+Z^+ axes from equations (32)

$$\left.\begin{aligned} -C_{X_O} &= -C_{X_O}{}^+ \\ -C_{Z_\alpha}\beta - C_{Z_{\dot{\alpha}}}\frac{\dot{\beta}l}{2V} + C_{Z_q}\frac{rl}{2V} &= -C_{Z_\alpha}{}^+\beta - C_{Z_{\dot{\alpha}}}{}^+\frac{\dot{\beta}l}{2V} + C_{Z_q}{}^+\frac{rl}{2V} \\ -C_{Z_\alpha}\alpha - C_{Z_{\dot{\alpha}}}\frac{\dot{\alpha}l}{2V} - C_{Z_q}\frac{ql}{2V} &= -C_{Z_\alpha}{}^+\alpha - C_{Z_{\dot{\alpha}}}{}^+\frac{\dot{\alpha}l}{2V} - C_{Z_q}{}^+\frac{ql}{2V} \\ C_{l_p}\frac{pl}{2V} &= C_{l_p}{}^+\frac{pl}{2V} \\ C_{m_\alpha}\alpha + C_{m_{\dot{\alpha}}}\frac{\dot{\alpha}l}{2V} + C_{m_q}\frac{ql}{2V} &= C_{m_\alpha}{}^+\alpha + C_{m_{\dot{\alpha}}}{}^+\frac{\dot{\alpha}l}{2V} + C_{m_q}{}^+\frac{ql}{2V} \\ -C_{m_\alpha}\beta - C_{m_{\dot{\alpha}}}\frac{\dot{\beta}l}{2V} + C_{m_q}\frac{rl}{2V} &= -C_{m_\alpha}{}^+\beta - C_{m_{\dot{\alpha}}}{}^+\frac{\dot{\beta}l}{2V} + C_{m_q}{}^+\frac{rl}{2V} \end{aligned}\right\} \tag{33}$$

Since equations (33) must hold for all values of β, α, $\dot{\beta}$, $\dot{\alpha}$, p, q, and r, it is immediately evident that the aerodynamic coefficients associated with the XY^+Z^+ axes, the C^+'s, must be equal to the aerodynamic

coefficients associated with the XYZ axes, the C's. It may be concluded, therefore, that the aerodynamic coefficients are invariant with respect to rotation of coordinate axes about the axis of symmetry. Consequently, the Y and Z axes may be oriented at will around the X axis without regard to the orientation of the missile about X, since the aerodynamic coefficients are solely functions of the missile's external shape (and such nondimensional parameters as Reynolds and Mach numbers) and it has been shown that the aerodynamic coefficients do not change with orientation of the missile in roll. In fact, the missile may be allowed to spin about the X axis with respect to the YZ axes and the aerodynamic coefficients will be unaffected.[1]

Another consequence of the preceding analysis, which is immediately evident, is that the aerodynamic force and moment, $\overline{F}_A$ and $\overline{M}$, are also invariant with respect to the orientation of the body in roll. The values of $\overline{F}_A$ and $\overline{M}$ change, of course, with spin through the rolling moment, $C_{l_p} \frac{pl}{2V}$, and through the Magnus forces and moments, which will be discussed shortly.

A more adequate treatment of the consequences of rotational symmetry than has been presented here has been carried out in references 10, 11, 12, and 13. The general case of rotational symmetry about any submultiple of 360° is studied in these references in a most elegant and rigorous manner and it is shown that the full set of aerodynamic coefficients is reduced to the set given by equations (28) and (29) for rotational symmetry about all submultiples of 360° less than 180°, that is for an angle of 360°/n with n = 3, 4, 5 . . . (any integer greater than 2). Ninety-degree rotational symmetry, or cruciform symmetry as it is commonly called, is only a special case and was chosen primarily to aid in visualizing the physical aspects of the analysis. Hence the phrase "or its equivalent" means rotational symmetry about any submultiple of 360° less than 180°. The 180° case lies in between the aircraft and missile cases but will not be treated in this paper (for an analysis of the 180° case, see reference 14).

Geometry, Conditions of Flight, and Kinematic Relations

The geometry is shown in figure 2. Three sets of Cartesian coordinate axes are used in developing the equations of motion, instead of two sets as formerly in the aircraft case. One set is fixed with respect to the earth and is known as before as "earth axes." The second set is fixed

[1]The author is indebted to Mr. C. H. Murphy of the Ballistic Research Laboratories for pointing out to him the roll invariance of the aerodynamic coefficients of a roll symmetric projectile or missile.

partly in the missile and partly in space and is known as "pseudo-stability axes." The third set is fully fixed in the missile and is known as "body axes." The relative orientation of the three sets of axes is shown in figure 2. Other features of the geometry such as the locations of the gravity and velocity vectors and the designations of quantities in the figures are the same as in the aircraft case and reference is made to the previous section "Geometry" and to figures 1(b) and 1(c).

The earth axes are designated as before by $X_oY_oZ_o$. They are fixed with respect to the earth, but, since the earth's rotation is negligibly slow compared to the angular velocity of the missile, they are considered to be Galilean axes fixed in space. Their orientation in space is determined by two factors: First, the X_o axis is alined parallel to the X axis in the neutral position of the missile; that is, the position for which the angles α, β, ψ, θ, and φ are taken to be zero. In other words, the X_o axis is located by the initial conditions of the flight. Second, the Y_o axis is horizontal and points to the right (of an observer facing forward) and Z_o points down (but not necessarily vertically down). As formerly, two locations are designated for the origin of coordinates of the earth axes, depending on the component of the motion being considered. For angular measurements, the origin of coordinates is taken to be the center of gravity of the aircraft at some particular point along its flight path. For linear measurements, the origin of coordinates is located as required by the over-all scheme of measurement which would be set up to analyze the flight of the missile.

The pseudo-stability axes are designated by XYZ. This set of axes exploits the freedom brought by the symmetry of the missile to orient the axes at will about the axis of rotational symmetry. The X axis lies along the axis of rotational symmetry. The Y axis lies in the space-fixed X_oY_o plane. Their origin of coordinates is at the center of gravity. In other words, the X axis is fixed in the missile and moves with it while the Y axis slides about in the X_oY_o plane. The positive directions along the axes are as follows: X is positive forward; Z is positive down; Y is positive in accordance with the right-hand screw rule. Should the X axis happen to be vertical in the neutral position of the missile, the orientations of the Y and Z axes (and the Y_o and Z_o axes) become arbitrary and would be determined by the conditions of the particular flight under consideration. It should be noted that these axes do not roll, since Y remains in the X_oY_o plane, and consequently the φ and the derivatives of φ ($d\varphi/dt$, $d^2\varphi/dt^2$, etc.) of the XYZ axes are zero and remain zero during the flight. It should also be noted that all three axes, X, Y, and Z, are principal inertia axes, as a consequence of the rotational symmetry of the missile.

The body axes are designated by $\overline{1}$, $\overline{2}$, $\overline{3}$. They are firmly fixed in the missile throughout the flight and are oriented so that $\overline{1}$ lies along the axis of rotational symmetry and $\overline{2}$ and $\overline{3}$ are coincident with Y and Z

at that point on the flight path which determines the initial conditions of the flight ($\bar{1}$, of course, is coincident with X at all times). Their origin of coordinates is at the center of gravity.

The orientation of the missile in space is defined by three angular coordinates, two of which, ψ and θ, give the alinement of the pseudo-stability axes XYZ with respect to the earth axes $X_OY_OZ_O$; and one of which, φ, gives the alinement of the body axes $\bar{1}\ \bar{2}\ \bar{3}$ with respect to the pseudo-stability axes XYZ (see fig. 2). If these angles are all zero, both the pseudo-stability axes and the body axes point in the direction of the earth axes. Any other orientation of the missile is reached by two consecutive rotations of the XYZ axes, starting with XYZ coincident with $X_OY_OZ_O$, and one rotation of the $\bar{1}\ \bar{2}\ \bar{3}$ axes, starting with $\bar{1}\ \bar{2}\ \bar{3}$ coincident with XYZ. In defining the rotations, the point of view is taken that the axes in question start in the position specified and proceed in ordered sequence to their final position. In particular, the two rotations of the XYZ axes are noncommutative and must be taken in the order listed.

ψ first rotation of XYZ. Start with XYZ pointing in the directions of $X_OY_OZ_O$. Rotate about Z_O through ψ bringing Y to its final position.

θ second rotation of XYZ. Rotate about Y through θ, bringing X and Z to their final positions.

φ rotation of $\bar{1}\ \bar{2}\ \bar{3}$. Start with $\bar{1}\ \bar{2}\ \bar{3}$ coincident with XYZ. Rotate about X through φ.

The angles, ψ, θ, φ, may also be defined as the angles between coordinate axes and the intersections of certain planes. For the purpose of these definitions it is considered that the origin of coordinates of the $X_OY_OZ_O$ axes is momentarily coincident with that of the XYZ axes. The point of view taken here is that the XYZ and $\bar{1}\ \bar{2}\ \bar{3}$ axes are in their final positions given by the orientation of the missile in space at the moment in question. The sign of the angle is specified by giving the axis about which the rotation is taken in going from the line named first to the line named second in the definition. The axis of rotation is listed in parentheses in each definition.

ψ angle between X_O and intersection of X_OY_O and XZ_O planes; also angle between Y_O and Y (rotation about Z_O)

θ angle between intersection of X_OY_O and XZ_O planes and X; also angle between Z_O and Z (rotation about Y)

φ angle between Y and $\bar{2}$; also angle between Z and $\bar{3}$ (rotation about X)

It should be noted that the roll angle φ in the spinning-projectile and symmetrical-missile cases measures the rotation of the body axes $\bar{1}\ \bar{2}\ \bar{3}$ and not the rotation of the pseudo-stability axes XYZ. As pointed out previously the XYZ axes do not roll.

It may be of interest to note that the angles, ψ, θ, φ, are the Eulerian angles as defined in the classical treatment of the motion of a gyroscope (e.g., see section 43, "Heavy Symmetrical Top or Gyroscope," of ref. 15). The axis X corresponds to the "axis of spin" (axis of rotational symmetry) of the gyroscope, and the axis Y corresponds to the "line of nodes."

The following conditions are postulated for the missile and for its motion in flight:

(1) The missile has not only mirror symmetry, but also 90° rotational symmetry, or its equivalent, the axis of symmetry being the X axis.

(2) The magnitude of the velocity of the missile may be large but the change in the velocity must be small. Accordingly, the velocity may be represented by a constant plus a small perturbation; that is, $V = V_o + u$. (See eq. (2).)

(3) The angles orienting the pseudo-stability axes with respect to the earth axis, ψ, θ, and locating the velocity vector, β, α, are all small.

(4) The Y and Z components, q and r, of the angular velocity of the missile are small.

(5) The X component, p, of the angular velocity of the missile may be large but the change in p must be small. Accordingly, p may be represented by a constant plus a small perturbation; that is,

$$p = p_o + p' \tag{34}$$

(6) The flight path of the missile is a perturbation on a linear trajectory.

Conditions (2), (3), (4), and (5) may be summarized by saying that the following variables must be small (e.g., have a numerical magnitude of 0.1 or less) and hence are first-order quantities:

$$\left.\begin{array}{ccc} & \theta & \psi \\ \dfrac{u}{V_o} & \beta & \alpha \\ \dfrac{p'l}{2V_o} & \dfrac{ql}{2V_o} & \dfrac{rl}{2V_o} \end{array}\right\} \text{small, for example, 0.1 or less} \tag{35}$$

where l is a characteristic length. Again it should be noted that if β, α, $ql/2V_o$, and $rl/2V_o$ are first-order, then $\dot{\beta}l/2V_o$ and $\dot{\alpha}l/2V_o$ will also be first-order. Also, it should be noted that V_o, φ, p_o, and γ_o may all be large.

Again it should be emphasized that, although the linear theory gives an adequate description of the motion in flight of many projectiles and missiles, there may be cases in which the second-order terms neglected in the derivation of this theory are as large or larger than the first-order terms retained. Consequently, the analysis of an actual flight should include a check of the relative magnitudes of first- and second-order terms. The point here is that in both the aircraft and missile cases the a priori assumptions of the theory should be checked a posteriori by an analysis of the measurements.

As in the aircraft case, the equations of motion are derived from the basic vector equations

$$\frac{d\overline{M}}{dt} = \overline{F} \tag{3}$$

$$\frac{d\overline{H}}{dt} = \overline{M} \tag{4}$$

The components of the linear momentum, $\overline{M}$, and the velocity, $\overline{V}$, along the XYZ axes are given as formerly by the tabular listing of equations (5).

In deriving relations for the components of the angular momentum, $\overline{H}$, there are three angular velocities involved:

(1) $\overline{\omega}$ angular velocity of the missile with respect to the earth axes, $X_oY_oZ_o$

(2) $\overline{\Omega}$ angular velocity of the XYZ axes with respect to the $X_oY_oZ_o$ axes

(3) $\overline{w}$ angular velocity of the missile with respect to the XYZ axes

It can be shown that

$$\overline{\omega} = \overline{\Omega} + \overline{w} \tag{36}$$

The components of $\overline{\Omega}$ are given by equations (7b) with $\varphi = \dot{\varphi} = 0$, since the XYZ axes do not roll. For convenience in computing the relative magnitudes of terms in the equations of motion, these components,

p,q,r, may be written as follows, expanding sines and cosines and retaining only first- and second-order terms:

$$\left.\begin{aligned} \Omega_X &= -\theta r \\ \Omega_Y &= \frac{d\theta}{dt} \\ \Omega_Z &= \frac{d\psi}{dt} \end{aligned}\right\} \tag{37}$$

Since the $\overline{I}$ and X axes are coincident, the components of $\overline{w}$ are given by

$$\left.\begin{aligned} w_X &= \frac{d\varphi}{dt} \\ w_Y &= 0 \\ w_Z &= 0 \end{aligned}\right\} \tag{38}$$

Substituting equations (37) and (38) in equation (36), with $J_{XZ} = 0$, gives the components of $\overline{H}$ and $\overline{\omega}$ along XYZ in terms of φ, θ, and ψ by (correct to second order)

	$\overline{H}$	$\overline{\omega}$
i	$I_X p$	$p = \frac{d\varphi}{dt} - \theta r$
j	$I_Y q$	$q = \frac{d\theta}{dt}$
k	$I_Z r$	$r = \frac{d\psi}{dt}$

(39)

In equations (39) the transverse components of $\overline{H}$, H_j and H_k, are written as $I_Y q$ and $I_Z r$. Now, it can be shown that rotational symmetry requires that $I_Y = I_Z$. Consequently, a single value may be assigned for the transverse moment of inertia. This fact is used in deriving and solving the equations of motion for symmetrical missiles. In deriving equations for β, α, φ, θ, and ψ the rates of change of $\overline{M}$ and $\overline{H}$ will be computed

with respect to the XYZ axes and, hence, equations (3) and (4) must be transformed in order to account for the movement of the XYZ axes. The transformation for the rate of change of any vector $\overline{A}$ from fixed to moving axes is given in the missile case by

$$\frac{d\overline{A}}{dt} = \dot{\overline{A}} + \overline{\Omega} \times \overline{A} \tag{40a}$$

where $\dot{\overline{A}}$ is the rate of change of $\overline{A}$ measured with respect to XYZ axes and $\overline{\Omega}$ is the angular velocity of the XYZ axes with respect to fixed (earth) axes. The components of the transformation are given by

$$\left.\begin{aligned} \frac{dA_X}{dt} &= \dot{A}_X + qA_Z - rA_Y \\ \frac{dA_Y}{dt} &= \dot{A}_Y + rA_X \\ \frac{dA_Z}{dt} &= \dot{A}_Z - qA_X \end{aligned}\right\} \tag{40b}$$

Aerodynamic and Gravitational Force Systems

As in the aircraft case, the resultant external force $\overline{F}$ of equation (4) is the sum of the aerodynamic and gravitational forces, as given by equation (10), while the external moment $\overline{M}$ of equation (5) is solely an aerodynamic moment.

Concerning the aerodynamic forces and moments, it is assumed, as formerly, (a) that the components of the resultant force and moment, $\overline{F}_A$ and $\overline{M}$, are given by equations (11) and (12); (b) that the coefficients, the C's in equations (11) and (12), are functions of the variables β, α, $\dot{\beta}$, $\dot{\alpha}$, p, q, and r; and (c) that these functions may be expanded in a series in the variable named, where the general formula for any coefficient, C_a (a = X, Y, Z, l, m, n), is given by equation (13). Again, the coefficients of the series, the aerodynamic coefficients, are assumed to be independent of the dependent variables named and to be functions only of the body's external contour and of such fundamental aerodynamic parameters as Reynolds and Mach numbers.

In the aircraft case, all of the variables, β, α, $\dot{\beta}$, $\dot{\alpha}$, p, q, and r, are assumed to be small, and consequently, all quadratic terms in the series expansion of the aerodynamic coefficients (eq.(13)) are second-order and may be neglected. However, in the missile case one of these variables, p, may be large and the quadratic terms of the series which involve p may be first-order and must be included in the equations for the aerodynamic forces and moments, equations (28) and (29). There are also

higher-order terms in p which are first-order in the other variables and therefore should be included from a strictly logical standpoint. However, the available experimental data indicate that the forces and moments in question vary linearly with p to within the accuracy justified by a first-order theory. Accordingly, only the quadratic terms involving p will be included. The general formula for the quadratic terms of any coefficient C_a, which involve p, is given as follows, where a stands for X, Y, Z, l, m, or n:

$$C_a = C_{a_{\beta p}}\beta \frac{pl}{2V} + C_{a_{\alpha p}}\alpha \frac{pl}{2V} + C_{a_{\dot{\beta} p}} \frac{\dot{\beta} l}{2V} \frac{pl}{2V} +$$

$$C_{a_{\dot{\alpha} p}} \frac{\dot{\alpha} l}{2V} \frac{pl}{2V} + C_{a_{pp}} \frac{pl}{2V} \frac{pl}{2V} + C_{a_{qp}} \frac{ql}{2V} \frac{pl}{2V} +$$

$$C_{a_{rp}} \frac{rl}{2V} \frac{pl}{2V} \tag{41}$$

The quadratic terms involving p are known in the ballistic nomenclature as the Magnus forces and moments.

The conditions of symmetry require that many of the Magnus coefficients vanish. Furthermore, rotational symmetry establishes relations between coefficients associated with motions in the pitch plane (XZ) and in the yaw plane (XY). The Magnus terms remaining are listed below:

C_X	$- C_{X_{pp}} \left(\frac{pl}{2V}\right)^2$
C_Y	$C_{Y_{\alpha p}}\alpha \frac{pl}{2V} + C_{Y_{\dot{\alpha} p}} \frac{\dot{\alpha} l}{2V} \frac{pl}{2V} + C_{Y_{qp}} \frac{ql}{2V} \frac{pl}{2V}$
C_Z	$- C_{Z_{\beta p}}\beta \frac{pl}{2V} - C_{Z_{\dot{\beta} p}} \frac{\dot{\beta} l}{2V} \frac{pl}{2V} - C_{Z_{rp}} \frac{rl}{2V} \frac{pl}{2V}$
C_l	(There are no rolling-moment Magnus terms)
C_m	$C_{m_{\beta p}}\beta \frac{pl}{2V} + C_{m_{\dot{\beta} p}} \frac{\dot{\beta} l}{2V} \frac{pl}{2V} + C_{m_{rp}} \frac{rl}{2V} \frac{pl}{2V}$
C_n	$C_{n_{\alpha p}}\alpha \frac{pl}{2V} + C_{n_{\dot{\alpha} p}} \frac{\dot{\alpha} l}{2V} \frac{pl}{2V} + C_{n_{qp}} \frac{ql}{2V} \frac{pl}{2V}$

(42)

The equalities between coefficients are given by

$$\left.\begin{aligned} C_{Y_{\alpha p}} &= C_{Z_{\beta p}} & C_{n_{\alpha p}} &= C_{m_{\beta p}} \\ C_{Y_{\dot{\alpha} p}} &= C_{Z_{\dot{\beta} p}} & C_{n_{\dot{\alpha} p}} &= C_{m_{\dot{\beta} p}} \\ C_{Y_{qp}} &= -C_{Z_{rp}} & C_{n_{qp}} &= -C_{m_{rp}} \end{aligned}\right\} \quad (43)$$

The aerodynamic forces and moments specified up to this point (eqs. (28), (29), (42), and (43)) arise from the interaction of the air with the missile's principal aerodynamic surfaces, which surfaces are assumed to conform to the requirements of mirror and rotational symmetry. However, in practice there may be small asymmetries in the missile's contour or asymmetries due to control-surface deflections, and under certain circumstances it may be desirable to include aerodynamic forces and moments caused by the asymmetries. Forces and moments of this nature will be omitted from the main development of the equations of motion carried out in this section. However, they are described in Appendix B and their effect on the motion is discussed briefly. For a more complete discussion of the consequences of aerodynamic asymmetry see reference 4.

In summarization of the aerodynamic contributions to the external force and moment, the conventional aerodynamic coefficients are given by equations (28) and (29) and the Magnus coefficients by equations (42) and (43).

The remaining contribution to the external force, the force of gravity, F_G, may be resolved at once along the XYZ axes from equations (17), since $\varphi = 0$ for the XYZ axes:

$$\left.\begin{aligned} F_{GX} &= mg(-\sin\gamma_o - \theta\cos\gamma_o) \\ F_{GY} &= mg(\psi\sin\gamma_o) \\ F_{GZ} &= mg(-\theta\sin\gamma_o + \cos\gamma_o) \end{aligned}\right\} \quad (44)$$

Derivation of the Differential Equations of Motion

The differential equations of motion for the projectile and missile cases are derived in a similar manner to the aircraft case. The kinematic relations and the equations for the external force and moment are substituted into the basic vector equations (3) and (4), resulting in

six scalar equations, three associated with components of the force and three with components of the moment. Specifically, the substitutions involved are these:

Equations (3) and (4) are first transformed from fixed to moving axes by equations (40b). Equation (3) for the linear momentum is developed by equations (4) for the kinematic momentum relations and by equations (10), (11), (28), (29), (42), (43), and (44) for the external force. Equation (4) for the angular momentum is developed by equations (39) for the kinematic momentum relations and by equations (12), (28), (29), (42), and (43) for the external moment.

Again, the equation associated with the axial drag force (F_X) is treated as a special case and derived by resolving the vector equation (3) along the X_O axis rather than the X axis in order to avoid the (qA_Z - rA_Y) term. The equations associated with the side force (F_Y) and the normal lift force (F_Z) contain a term in $m\dot{V}$ which is given precisely by the axial drag equation. However, in the side-force and lift equations, this term is approximated by $-C_{X_O}\rho V^2 S/2$, since it is believed that the remaining terms in the drag equation are negligibly small under most circumstances. Full details of this step are given in Appendix A.

The next two steps in the derivation are the same as in the aircraft case. The equations are linearized by separating into first- and second-order (and higher in order) terms and retaining only the first-order terms. The details of this step are given in Appendix A. Nondimensional time and concise notation are introduced next.

The final step in the derivation differs from that taken in the aircraft case in one respect and is similar in another respect. The neutral attitude of the missile is not the steady-state glide of the aircraft but is one in which the longitudinal axis of the missile points in the direction of motion ($\alpha = \beta = 0$). Since symmetry requires the C_{Z_O} coefficient to be zero, the main lift force due to the effective angle of attack of the aerodynamic surfaces is lost in the neutral attitude, and the lift, drag, speed, weight, and flight-path elevation will not be in balance with the missile in this position. Consequently, the gravity terms cannot be replaced with corresponding aerodynamic terms as in the case of aircraft. On the other hand, it is appropriate to replace the factor, z_α, by the lift and drag factors, $z_{\alpha L}$, x_{oD}, utilizing thereby the static characteristics of the missile measured in a wind tunnel. Symmetry suggests that it would also be appropriate to replace the factor y_β by the corresponding side-force and drag factors, $y_{\beta L}$, x_{oD}, which would be measured in a tunnel. The reasons for making these substitutions are the same as in the aircraft case and reference is made to the discussion leading up to equations (19). The corresponding equations for the missile are

$$\left.\begin{aligned} x_o &= x_{oD} \\ x_\alpha &= z_{\alpha L} + x_{oD} \\ y_\beta &= y_{\beta L} - x_{oD} \end{aligned}\right\} \tag{45a}$$

Substituting equation (28) into equation (45a) gives the relation between the side-force and lift factors of a symmetrical missile from wind-tunnel tests, namely

$$y_{\beta L} = -z_{\alpha L} \tag{45b}$$

The six differential equations are listed below in the same order as in the aircraft case: the three force equations, axial drag, side force, and normal lift force, and the three moment equations, roll, pitch, and yaw.

$$D\left(\frac{u}{V_o}\right) + 2x_o\left(\frac{u}{V_o}\right) + x_o + 2\nu x_{pp}D\varphi - \nu^2 x_{pp} + x_G = 0 \tag{46}$$

$$(1 - y_{\dot{\beta}})D\beta - y_{\beta L}\beta + (1 - y_r)D\psi - x_G\psi -$$

$$\nu y_{\dot{\alpha}p}D\alpha - \nu y_{\alpha p}\alpha - \nu y_{qp}D\theta = 0 \tag{47}$$

$$(1 + z_{\dot{\alpha}})D\alpha + z_{\alpha L}\alpha - (1 - z_q)D\theta + x_G\theta +$$

$$\nu z_{\dot{\beta}p}D\beta + \nu z_{\beta p}\beta + \nu z_{rp}D\psi - z_G = 0 \tag{48}$$

$$D^2\varphi - l_pD\varphi - \nu l_p\left(\frac{u}{V_o}\right) = 0 \tag{49}$$

$$D^2\theta - m_qD\theta - m_{\dot{\alpha}}D\alpha - m_\alpha\alpha - \nu m_{\dot{\beta}p}D\beta -$$

$$\nu m_{\beta p}\beta + \nu(K - m_{rp})D\psi = 0 \tag{50}$$

$$D^2\psi - n_rD\psi - n_{\dot{\beta}}D_\beta - n_\beta\beta - \nu n_{\dot{\alpha}p}D\alpha -$$

$$\nu n_{\alpha p}\alpha - \nu(K + n_{qp})D\theta = 0 \tag{51}$$

Discussion of the Differential Equations of Motion

If equations (46) through (51) for the projectile and missile cases are compared with the corresponding equations (20) through (25) for the aircraft case, it is evident that they can no longer be divided into two separate and distinct groups in the same way that was possible in the aircraft case. Equations that formerly involved only the set of variables u/V_o, α, and θ or the set φ, β, and ψ now contain members from both sets. Closer inspection discloses that the new members in the aircraft equations are all multiplied by the mean spin, ν, and hence the spin is shown to be responsible for the interaction between that phase of the motion associated with u/V_o, α, and θ and the phase associated with φ, β, and ψ. In a word, the spin couples the longitudinal and lateral motions.

Although the six equations can no longer be separated into two parts, one defining the longitudinal stability and the other the lateral stability, it is possible to divide them into two distinct groups. Equations (46) and (49) involve only the variables u/V_o and φ and constitute one group. Equations (47), (48), (50), and (51) involve only the variables β, α, θ, and ψ and constitute the other group. Furthermore, it will be shown that the later group can be reduced from four equations to two equations by introducing complex variables. Consequently, the differential equations of motion for projectiles and missiles can be reduced to two distinct pairs of equations with each pair involving two dependent variables. The differential equations will be reformulated in this manner and will be solved in the following section.

It may be instructive to return at this point to a statement made in the introduction that "the aircraft equations do not describe the gyroscopic nutation and precession of a spinning shell, and the projectile equations do not describe the phugoid oscillation of aircraft" and to discuss briefly the reasons for this statement. The phugoid oscillation of aircraft will be considered first. Equations (20), (22), and (24) may be simplified to give the essential features of the phugoid as follows (see eqs. (10-100) of ref. 5):

$$D\left(\frac{u}{V_o}\right) + z_o\theta = 0 \tag{52}$$

$$2z_o\left(\frac{u}{V_o}\right) - D\theta = 0 \tag{53}$$

Comparison of equation (52) with equation (46) and of equation (53) with equation (48) shows that the $z_o\theta$ and $2z_o(u/V_o)$ terms are missing from equations (46) and (48) for the missile and projectile case. Now, the

z_o factor is the constant lift coefficient that arises from the balance of weight by lift required by the equilibrium condition of a steady-state glide. Since the equilibrium condition of projectiles and missiles is with their lift zero (or what corresponds to an equilibrium condition, since they are never in a truly steady state in flight with all dependent variables zero, except in a terminal velocity vertical dive), the z_o terms vanish from the projectile and missile equations and they do not predict the phugoid oscillation. In other words, projectiles are hurled rather than flown through the air.

The gyroscopic nutation and precession of spin-stabilized projectiles will be considered next. The essentials of the gyroscopic motion are shown if the only external force or moment acting is taken to be the moment given by m_α, referred to in the ballistic nomenclature as the "overturning moment." In this case, the equations of motion reduce to

$$D^2\theta + \nu KD\psi - m_\alpha\alpha = 0 \tag{54}$$

$$D^2\psi - \nu KD\theta - n_\beta\beta = 0 \tag{55}$$

Comparing equation (54) with equation (24) and equation (55) with equation (25), shows that the $\nu KD\psi$ and $\nu KD\theta$ terms are missing from the aircraft equations. Since these particular terms are essential to the gyroscopic motion (as will be shown shortly), it is clear that the aircraft equations cannot predict the motions of rapidly spinning projectiles.

Dynamic stability requirements for spin-stabilized projectiles are not as widely known among aeronautical engineers as those for aircraft and it may be informative to discuss this aspect of the subject briefly. The essential features of the gyroscopic precession and nutation can be developed from equations (54) and (55). In this simplified case, the trajectory is a straight line, since there are no external forces; X, X_o, and $\overline{V}$ are all coincident; and the angles of pitch and yaw are related to the angles of attack and sideslip as follows:

$$\left.\begin{aligned} \beta &= -\psi \\ \alpha &= \theta \end{aligned}\right\} \tag{56}$$

Using equations (56) and the equality $n_\beta = -m_\alpha$ from equations (28), equations (54) and (55) become

$$D^2\alpha - \nu KD\beta - m_\alpha\alpha = 0 \tag{57}$$

$$D^2\beta + \nu KD\alpha - m_\alpha\beta = 0 \tag{58}$$

Equations (57) and (58) can be reduced to a single equation in the complex variable,

$$\xi = \beta + i\alpha$$

namely,

$$D^2\xi - i\nu KD\xi - m_\alpha\xi = 0 \tag{59}$$

where $i(=\sqrt{-1})$ has the significance of an accounting parameter permitting the simultaneous solution of the pitch and yaw equations.

The solution of equation (59) is given by

$$\xi = A_1 e^{S_1 t^+} + A_2 e^{S_2 t^+} \tag{60}$$

where A_1 and A_2 are constants which are functions of the initial conditions and of S_1 and S_2, and S_1 and S_2 are given by

$$S_1 = \frac{i\nu K + \sqrt{-K^2\nu^2 + 4m_\alpha}}{2} \qquad S_2 = \frac{i\nu K - \sqrt{-K^2\nu^2 + 4m_\alpha}}{2} \tag{61}$$

It should be noted that m_α is positive for spin-stabilized projectiles (i.e., the center of pressure is ahead of the center of gravity; if m_α were negative, the projectile would be arrow stable and spin would not be needed for stability) and the radical will be either real or imaginary depending on whether $K^2\nu^2$ is smaller or greater than $4m_\alpha$. The other factor in equation (61), $i\nu K$, is always imaginary and hence the motion is always an oscillation.

Now, if the radical in equation (61) is real, the S's will be complex and one of them will have a positive real part. Hence, the oscillatory motion will diverge. On the other hand, if the radical is imaginary, the S's will be purely imaginary, and the motion will be an oscillation at constant amplitude, which is the stable type of motion in this simplified case since damping terms have not been included. Consequently, the criterion for stability is

$$K^2\nu^2 > 4m_\alpha$$

or

$$\frac{K^2\nu^2}{4m_\alpha} > 1$$

In ballistic nomenclature, $K^2\nu^2/4m_\alpha$ is called the "stability factor." In other words, the analysis of the simplified equations of motion has led to the well-known requirement for stable spinning shell, namely, that the stability factor must be greater than unity.

The complete differential equations of motion are solved in the following section of this report. In the solution, the equations are transformed from time to distance as the independent variable. However, it can be shown that the nondimensional equations in t^+ are the same as those in x^+ except for minor differences in coefficients, which will be accounted for in the development that follows.

The integrated equation for the $\xi(t^+)$ history will have precisely the same form as equation (93) for the $\xi(x^+)$ history. It can be seen that the complete ξ equation has the same form as the simplified equation (60) except for an additional term, the constant, d_o. For the $\xi(x^+)$ equation (93) the relations between the exponents in the equation and the aerodynamic factors are given by equations (162) and (163). For the $\xi(t^+)$ equation (60) the corresponding relations are given by

$$\left.\begin{aligned} S_{1R} &= T_1 + T_2, \qquad S_{2R} = T_1 - T_2 \\ S_{1I} &= \frac{\nu K(1+\sigma)}{2}, \qquad S_{2I} = \frac{\nu K(1-\sigma)}{2} \end{aligned}\right\} \tag{62}$$

where

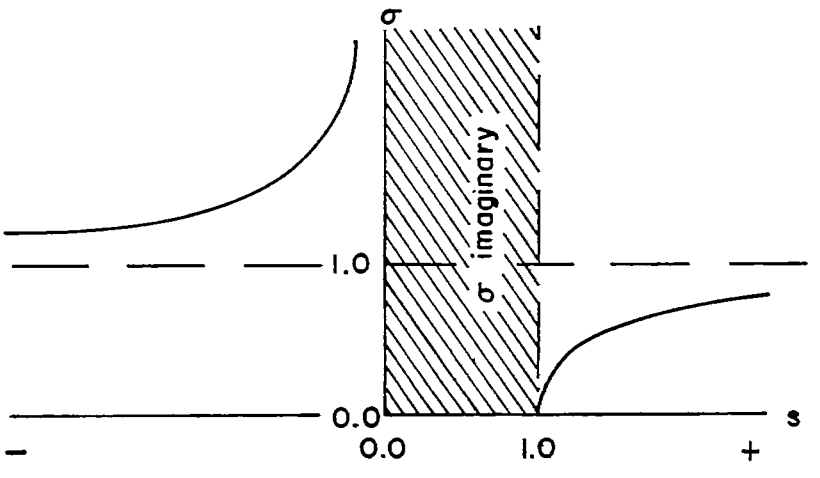
$$T_1 = -\frac{(z_{\alpha L} - m_q - m_{\dot{\alpha}})}{2}$$

$$T_2 = \frac{z_{\alpha L} + m_q + m_{\dot{\alpha}} + \frac{2m_{\beta p}}{K}}{2\sigma}$$

$$\sigma = \sqrt{1 - \frac{1}{s}}$$

$$s = \frac{K^2\nu^2}{4m_\alpha} = \text{stability factor}$$

The above solution is valid for $-\infty < s < 0$, that is, for missiles with arrow stability corresponding to m_α being negative, and for $1 < s < +\infty$, that is, for spin-stabilized projectiles with a stability greater than unity. The values of σ corresponding to values of s are shown in sketch (c). The solution is not valid,



Sketch (c)

however, in the region $0 < s < 1$ (shown crosshatched in the sketch) for which σ is imaginary.

Equations (62) are in the most suitable form for spin-stabilized projectiles ($s > 1$). However, they have a clearer physical significance for missiles with arrow stability ($m_\alpha < 0$) if they are rewritten as follows:

$$\left.\begin{aligned} S_{1R} &= T_1 + T_2 \qquad , \quad S_{2R} = T_1 - T_2 \\ S_{1I} &= \frac{1}{2}\left(\nu K + \sqrt{-4m_\alpha + \nu^2K^2}\right), \quad S_{2I} = \frac{1}{2}\left(\nu K - \sqrt{-4m_\alpha + \nu^2K^2}\right) \end{aligned}\right\} \tag{63}$$

where

$$T_1 = -\frac{(z_{\alpha L} - m_q - m_{\dot{\alpha}})}{2}$$

$$T_2 = \frac{\nu K}{2\sqrt{-4m_\alpha + \nu^2K^2}}\left(z_{\alpha L} + m_q + m_{\dot{\alpha}} + \frac{2m_{\beta p}}{K}\right)$$

Equations (62) show that the criteria for the dynamic stability of spin-stabilized projectiles, that is, the requirements for a convergent oscillation, are

$$\left.\begin{aligned} &s > 1 \ (\text{or } \sigma > 0) \\ &S_{1R} < 0 \\ &S_{2R} < 0 \end{aligned}\right\} \tag{64}$$

The term T_1 is normally negative. Hence, it can be seen that dynamic stability requires that $T_2 < T_1$ regardless of the sign of T_2. In practice, the magnitude of T_2 depends on a balance between the lift, $z_{\alpha L}$; the damping moment, $m_q + m_{\dot{\alpha}}$; and the Magnus moment, $m_{\beta p}$. That spin-stabilized artillery projectiles are in reality fully dynamically stable is a fact that has been thoroughly verified by experiment.

Lift and Magnus moment are both vital elements and must be included in the equations of motion in order to derive the correct criteria for dynamic stability. If they are omitted, equations (62) take the form

$$\left.\begin{aligned} S_{1R} &= \frac{(m_q + m_{\dot{\alpha}})}{2}\left(1 + \frac{1}{\sigma}\right) \\ S_{2R} &= \frac{(m_q + m_{\dot{\alpha}})}{2}\left(1 - \frac{1}{\sigma}\right) \end{aligned}\right\} \tag{65}$$

Now $0 < \sigma < 1$ for $1 < s < \infty$; hence,

$$\left(1 + \frac{1}{\sigma}\right) > 0$$

$$\left(1 - \frac{1}{\sigma}\right) < 0$$

Therefore, one S will be positive even though $(m_q + m_{\dot{\alpha}})$ is negative, and a simplified analysis with lift and Magnus moment omitted would lead to the erroneous conclusion that spin-stabilized artillery projectiles cannot have a convergent oscillatory motion (see section on "Application of Results: Missiles" of reference 14).

APPLICATION TO THE FREE-FLIGHT TEST FACILITY: THE AERODYNAMIC RANGE

During the past two decades a new facility, the aerodynamics range, has been developed for measuring the aerodynamic properties of bodies in free flight. The range is properly classed as a flight-test facility, since the aerodynamic measurements are made during a completely free flight of the model. Accurate records are taken of the model's movements along a certain length of its flight path and the aerodynamic characteristics are determined from these records.

The experiment consists of recording the positions, angular orientations, and times of the model at a series of stations placed along its flight path through the range. Photography is the primary medium used for recording, since it is a precise technique and one which does not interfere in any way with the model's flight. An electrical spark discharge generates the light for the photography and its duration can be made so short that the picture is nearly instantaneous in relation to the movement of the model during the time of exposure. Hence the photographic record gives the x, y, z, φ, θ, and ψ of the model at the particular instant of time that the spark produces the exposure. The remaining flight datum, t, the time at which the photograph is taken, is measured by a special high-precision chronograph. To enumerate specifically, the experimental data from the flight records are y, z, φ, θ, and ψ at a series of x

and t along the trajectory. In addition, measurements are made of the model's physical properties, that is, its dimensions, weight, center of gravity, and moments of inertia, before flying it in the range.

To the author's knowledge, there are seven aerodynamics ranges in operation at the present time: The Aerodynamics and Transonic Ranges and the Controlled Temperature-Pressure Chamber at the Ballistic Research Laboratories (U. S. Army), the Pressurized and Aerodynamics Ranges at the U. S. Naval Ordnance Laboratory (U. S. Navy), the Aeroballistics Field Laboratory at the U. S. Naval Ordnance Test Station (U. S. Navy), and the Supersonic Free-Flight Wind Tunnel[2] at the Ames Aeronautical Laboratory (NACA). The forerunner of all these facilities, the Aerodynamics Range at the National Physical Laboratory, Teddington, England, is dismantled at the moment of this writing, although it is understood that plans are in effect for its reconstruction at Fort Halstead (Kent, England).

The aerodynamics range is unique among flight-test facilities not only in its measurement techniques but also in the conditions under which its testing is carried out. The extent of the flight path is severely limited. The region of space under observation varies from a length of 15 feet with a cross section 1-foot square in the smallest range to a length of 750 feet with a cross section 25-feet square in the largest range. Testing is confined to flights for which the trajectory is nearly a straight line and the changes in velocity and angular inclination of the model over the length of the range are small. Most of the models themselves are either simple bodies of revolution or bodies with cruciform wings and fins having 90° rotational symmetry. Consequently, test conditions in the range agree with the flight conditions assumed for projectiles and missiles in the present analysis, and the equations derived herein describe correctly the linear and angular motions of models flown through the range.

Two steps are required to obtain the aerodynamic characteristics of the model from the flight-test data. First, the constants in the equations of motion are evaluated to give the best possible "fit" to the experimental measurements. Second, the aerodynamic coefficients are computed from certain of these constants. In this section of the report, the differential equations of motion will be solved for the particular test conditions prevailing in the aerodynamics range, the process of "fitting" the equations to the test data will be discussed briefly, and relations between the constants of the equations and the aerodynamic coefficients of the model will be derived.

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[2]At the Ames Laboratory, the test chamber of the range is the working section of a supersonic wind tunnel. Thus the Ames facility combines the wind stream of a supersonic wind tunnel with the measurement techniques of the range. The purpose is to carry out free-flight testing at much higher Mach numbers than would be possible in still air at normal room temperatures (see ref. 16).

Equations of Motion

The conditions of flight are modified slightly to fit the particular circumstances of testing in the range. The trajectory is so nearly horizontal that γ_o will be considered equal to zero in the x_G and z_G factors, thereby making

$$\left.\begin{aligned} x_G &= 0 \\ z_G &= \frac{mg}{\rho S V_o^2} \end{aligned}\right\} \tag{66}$$

The variation of drag with spin is so small according to the limited experimental evidence available (see ref. 17) that the x_{pp} terms may be neglected in the drag-force equation, (46), compared to the x_o term. However, should the conditions of any particular test be such that the x_G and x_{pp} terms are not truly negligible, they may be readily included in the general solution of the equations of motion since both are linear.

The independent variable in the equations of motion will be changed from time to distance and the equations transformed accordingly. It has been claimed that distance is the natural variable rather than time (see ref. 11). From the experimental standpoint, the distance is recorded in every photograph while time is recorded only at a few, widely spaced stations along the range. From the theoretical standpoint, the inertial forces vary with the square of the time and, in transformation to distance as independent variable, with the square of the velocity. Now, the aerodynamic forces also vary with the square of the velocity, so it might be expected that the velocity would disappear as a primary parameter from the transformed equations.

These advantages are realized in the case of the rolling motion (eq. (49) for φ), for which distance is the better variable on which to base the analysis than time. The velocity no longer appears explicitly in the $\varphi(x)$ equation and the speed range over which the equation correctly describes the rolling motion is limited only by the variation of the aerodynamic coefficients with Mach and Reynolds numbers. Furthermore, the basic differential equation for the rolling moment is linear in φ and contains no first-order term involving the other dependent variables. It is not necessary to limit the variation in the value of φ in order to linearize the equation and its solution is valid for large changes in φ as well as for small changes. Also the rolling motion is independent of the other motions and may be treated as a special case. Accordingly, a constant aerodynamic rolling moment, C_{l_o}, is introduced in order to include the class of missiles having aerodynamic surfaces with aileron deflection. Of course, aileron deflection violates the requirement of mirror symmetry but is permissible in the rolling-moment equation since

relaxing this requirement introduces no new terms in the rolling-moment equation. It is to be understood in the subsequent analysis that only the $\varphi(x)$ equation is valid for large changes in spin and possibly also in velocity; all other equations are still subject to the limitation of small changes in both spin and velocity.

Actually, the advantages of using distance rather than time are not as great as might be supposed. The experimental data are reduced in such a manner that time can be computed easily and accurately for all of the photographic records. The transformed equations, with the exception of the rolling-moment equation (49) for φ, still contain terms involving the velocity, and it is necessary to postulate that the change in velocity be small in order to linearize the equations of motion. In fact, the equations for the transverse displacement and for the pitching and yawing motions are practically identical for either the nondimensional time or the nondimensional distance as independent variable, and flight data from the range could be analysed on either basis. However, in conformity with accepted ballistic practice, distance will be selected as the variable used in developing the equations of motion for the range.

In developing the transformation from time to distance it will be recalled that the x distance is measured along the space-fixed axis, X_o. On the other hand, the velocity is strictly determined by the rate of change of distance along the trajectory, that is, letting s be the arc length along the trajectory

$$V = \frac{ds}{dt}$$

Hence, the relation between V and x is given by

$$V = \left(\frac{ds}{dx}\right)\left(\frac{dx}{dt}\right)$$

where

$$\frac{ds}{dx} = \sec \measuredangle \bar{V}X_o = 1 + \frac{(\measuredangle \bar{V}X_o)^2}{2} + \cdots$$

Now, the angle between $\bar{V}$ and X_o is normally very small, being less than 2° (0.03 radians). Therefore the term $(\measuredangle \bar{V}X_o)^2/2$ is of the order of magnitude of 0.001 or less and may be safely neglected compared to unity. Hence, the velocity may be approximated by

$$V = \frac{dx}{dt} \tag{67}$$

Using equation (67) for the velocity gives the transformation from time to distance for any dependent variable A

$$\left.\begin{aligned} \frac{dA}{dt} &= V\,\frac{dA}{dx} \\ \frac{d^2A}{dt^2} &= V^2\,\frac{d^2A}{dx^2} + \frac{dV}{dt}\,\frac{dA}{dx} \end{aligned}\right\} \tag{68}$$

Now dV/dt is given by equation (A18), (since the x_{pp} and x_G terms are neglected) as follows

$$\frac{dV}{dt} = -\frac{\rho S x_o}{m}\,V^2 \tag{69}$$

After equation (69) is substituted in equation (68) the transformation becomes

$$\left.\begin{aligned} \frac{dA}{dt} &= V\,\frac{dA}{dx} \\ \frac{d^2A}{dt^2} &= V^2\,\frac{d^2A}{dx^2} - \frac{\rho S x_o}{m}\,V^2\,\frac{dA}{dx} \end{aligned}\right\} \tag{70}$$

Distances are nondimensionalized in a manner similar to times by defining a base distance, μl, and denoting the nondimensional distance by a plus; that is,

$$\left.\begin{aligned} x^+ &= \frac{x}{\mu l} \\ y^+ &= \frac{y}{\mu l} \\ z^+ &= \frac{z}{\mu l} \end{aligned}\right\} \tag{71}$$

Differentiation with respect to x^+ is denoted by $\mathbb{D}$; that is, for any quantity, A

$$\mathbb{D}A = \frac{dA}{dx^+} \tag{72}$$

Hence, the nondimensional form of the transformation from time to distance is given by

$$\left.\begin{aligned} DA &= \left(\frac{V}{V_o}\right)\mathbb{D}A \\ D^2A &= \left(\frac{V}{V_o}\right)^2\mathbb{D}^2A - x_o\left(\frac{V}{V_o}\right)^2\mathbb{D}A \end{aligned}\right\} \tag{73}$$

One possible procedure at this point would be to substitute equation (73) into equations (46) through (51), thereby transforming directly from t^+ to x^+. However, it can be seen that neither (V/V_o) nor $(V/V_o)^2$ is a common factor in any of the equations of motion, (46) through (51), and that, if this procedure is followed, it will be necessary to make further approximations in order to linearize the equations. A better procedure is to return to the exact vector equations, (3) and (4), the kinematic relations, equations (39), and the moving-axes transformation, equations (40), to transform these from t to x using equations (70), and to derive the nondimensional linearized equations in x^+ as formerly in t^+. The detailed steps are carried out in Appendix A.

Two new equations in addition to the set corresponding to equations (46) through (51) are needed for the analysis of flight tests in the aerodynamics range. The trajectory is recorded and serves to determine the aerodynamic lift coefficient of the model. The position of the model transverse to the X_o axis is given by the y and z coordinates of its center of gravity measured along the Y_o and Z_o axes. In the derivation the force system is simplified by retaining only the lift, drag, and Magnus aerodynamic forces together with the gravity force, since experience in reducing experimental data from the range has shown that the contributions to the transverse motions by the remaining components of the aerodynamic force are less than the errors of measurement and, hence, may be neglected. The detailed derivation of the y and z equations is given in Appendix A.

The eight differential equations of motion are listed below with the first three being the equations for the trajectory (x, y, and z) and the remaining five for the transverse velocity (β and α) and for the angular motion (φ, θ, and ψ). It will be recalled that rotational symmetry established certain equalities between aerodynamic coefficients associated with motions in the pitch and yaw planes and, taking advantage of this fact, the notation will be simplified by using only the aerodynamic coefficients associated with the pitch plane. The selection is arbitrary, of course, and the yaw plane coefficients could have been chosen just as well.

$$D^2x^+ + x_o\,(1 + x_o t^+)^{-2} = 0 \tag{74}$$

$$\mathbb{D}^2 y^+ + z_{\alpha L}\beta - \nu z_{\beta p}\alpha = 0 \tag{75}$$

$$\mathbb{D}^2 z^+ + z_{\alpha L}\alpha + \nu z_{\beta p}\beta = z_G \tag{76}$$

$$(1 + z_{\dot{\alpha}})\mathbb{D}\beta + z_{\alpha L}\beta + (1 - z_q)\mathbb{D}\psi - \nu z_{\dot{\beta}p}\mathbb{D}\alpha - \nu z_{\beta p}\alpha + \nu z_{rp}\mathbb{D}\theta = 0 \tag{77}$$

$$(1 + z_{\dot{\alpha}})\mathbb{D}\alpha + z_{\alpha_L}\alpha - (1 - z_q)\mathbb{D}\theta + \nu z_{\dot{\beta}_p}\mathbb{D}\beta + \nu z_{\beta_p}\beta + \nu z_{r_p}\mathbb{D}\psi = z_G \tag{78}$$

$$\mathbb{D}^2\varphi - (l_p + x_o)\mathbb{D}\varphi = l_o \tag{79}$$

$$\mathbb{D}^2\theta - (m_q + x_o)\mathbb{D}\theta - m_{\dot{\alpha}}\mathbb{D}\alpha - m_\alpha\alpha - \nu m_{\dot{\beta}_p}\mathbb{D}\beta - \nu m_{\beta_p}\beta + \nu(K - m_{r_p})\mathbb{D}\psi = 0 \tag{80}$$

$$\mathbb{D}^2\psi - (m_q + x_o)\mathbb{D}\psi + m_{\dot{\alpha}}\mathbb{D}\beta + m_\alpha\beta - \nu m_{\dot{\beta}_p}\mathbb{D}\alpha - \nu m_{\beta_p}\alpha - \nu(K - m_{r_p})\mathbb{D}\theta = 0 \tag{81}$$

Inspection of equations (75) and (76) shows that the coefficients, 1, z_{α_L}, and νz_{β_p} of $\mathbb{D}^2y^+$, β, and α, respectively, in equation (75) reappear as the coefficients of $\mathbb{D}^2z^+$, α, and β in equation (76). The same similarity of coefficients occurs between equations (77) and (78) and between equations (80) and (81) and suggests that the three pairs of equations may be reduced to three single equations by a proper choice of dependent variables. This, in fact, is the case; by defining three new dependent variables

$$\left.\begin{aligned} \Delta &= y^+ + iz^+ \\ \xi &= \beta + i\alpha \\ \eta &= \psi - i\theta \quad (i\eta = \theta + i\psi) \end{aligned}\right\} \tag{82}$$

and including equations (74) and (79) in order to complete the set of differential equations of motion, equations (74) through (81) reduce to

$$D^2x^+ + x_o\,(1 + x_o t^+)^{-2} = 0 \tag{83}$$

$$\mathbb{D}^2\Delta + z_\xi\xi = iz_G \tag{84}$$

$$z_\eta\mathbb{D}\eta + z_{\dot{\xi}}\mathbb{D}\xi + z_\xi\xi = iz_G \tag{85}$$

$$\mathbb{D}^2\varphi - (l_p + x_o)\mathbb{D}\varphi = l_o \tag{86}$$

$$\mathbb{D}^2\eta - m_\eta\mathbb{D}\eta + m_{\dot{\xi}}\mathbb{D}\xi + m_\xi\xi = 0 \tag{87}$$

where

$$\left.\begin{aligned} z_\xi &= z_{\alpha L} + i\nu z_{\beta p} \\ z_{\dot\xi} &= 1 + z_{\dot\alpha} + i\nu z_{\dot\beta p} \\ z_\eta &= 1 - z_q + i\nu z_{rp} \\ m_\xi &= m_\alpha + i\nu m_{\beta p} \\ m_{\dot\xi} &= m_{\dot\alpha} + i\nu m_{\dot\beta p} \\ m_\eta &= m_q + x_o + i\nu(K - m_{rp}) \end{aligned}\right\} \tag{88}$$

The physical significance of the complex dependent variables, Δ, ξ, and η, is evident if they are considered to be vector quantities. The quantity Δ is the vector displacement of the trajectory transverse to the X_o axis. The quantity ξ is the component of the vector velocity, $\bar{V}$, transverse to the model's axis, X; in ballistic nomenclature ξ is known as the "vector yaw." The quantity $i\eta$ is the component of a unit vector along the model's X axis transverse to the space-fixed X_o axis, thereby defining the orientation of X axis in space. Because of its rotational symmetry, the model's motion in flight can be described by these three vector quantities together with the roll angle, φ, and the time-distance history, $t^+(x^+)$.

The solution of the differential equations of motion is quite straightforward. The coefficients in all equations are constants. Equation (83) contains only the dependent variable x^+ and, although not linear, may be integrated in closed form. Equation (84) is linear in the dependent variables Δ and ξ and may be integrated once a solution has been obtained for ξ. Equation (86) contains only the dependent variable, φ, is linear in φ, and may be integrated directly. Equations (85) and (87) form a pair of simultaneous, linear differential equations for ξ and η and may be solved by a variety of methods; for example, by the use of the Laplace transform (see ref. 18).

The exact solution to the $x^+(t^+)$ equation (83) is given by

$$x^+ = x_o^{-1} \ln (1+x_o t^+) \tag{89}$$

The logarithm in equation (89) is expanded in a series, since $x_o t^+$ is ordinarily a small quantity, and x is expressed as a function of t rather than x^+ as a function of t^+ so that V_o can be determined directly from the time-distance records as is required for the computation of τ. Since $x^+ = 0$, $V = V_o$, $\varphi = \varphi_o$, $\mathbb{D}\varphi = \nu$, $\Delta = \Delta_o$, $\mathbb{D}\Delta = (\mathbb{D}\Delta)_o$, $\xi = \xi_o$, $\eta = \eta_o$, and $\mathbb{D}\eta = (\mathbb{D}\eta)_o$ at $t^+ = 0$, the solutions to the differential equations of motion (eqs. 83 through 87) are given as follows:

$$x = a_o + a_1 t + a_2 t^2 + a_3 t^3 \tag{90}$$

$$\Delta = b_o + b_1 x^+ + b_2 (x^+)^2 + b_3 \left(\frac{d_1 e^{d_2 x^+}}{d_2^2} + \frac{d_3 e^{d_4 x^+}}{d_4^2} \right) \tag{91}$$

$$\varphi = c_o + c_1 x^+ + c_2 e^{c_3 x^+} \tag{92}$$

$$\xi = d_o + d_1 e^{d_2 x^+} + d_3 e^{d_4 x^+} \tag{93}$$

$$\eta = f_o + f_1 x^+ + f_2 e^{d_2 x^+} + f_3 e^{d_4 x^+} \tag{94}$$

where

a_o = constant required to adjust the zero of the distance scale to coincide with the zero of the time scale,

$$x = 0 \quad \text{at} \quad t = 0 \tag{95}$$

$$a_1 = V_o = \frac{\mu l}{\tau} \tag{96}$$

$$a_2 = - \frac{x_o V_o}{2\tau} \tag{97}$$

$$a_3 = \frac{x_o^2 V_o}{3\tau^2} \text{ (assuming } x_o \text{ constant)} \tag{98}$$

$$b_o = \Delta_o + z_\xi \left(\frac{d_1}{d_2^2} + \frac{d_3}{d_4^2} \right) \tag{99}$$

$$b_1 = (\mathbb{D}\Delta)_o + z_\xi \left(\frac{d_1}{d_2} + \frac{d_3}{d_4} \right) \tag{100}$$

$$b_2 = \frac{i z_G - z_\xi d_o}{2} \tag{101}$$

$$b_3 = - z_\xi \tag{102}$$

$$c_o = \varphi_o - \left[\frac{\nu(l_p + x_o) + l_o}{(l_p + x_o)^2} \right] \tag{103}$$

$$c_1 = \frac{-l_o}{(l_p + x_o)} \tag{104}$$

$$c_2 = \frac{\nu(l_p + x_o) + l_o}{(l_p + x_o)^2} \tag{105}$$

$$c_3 = l_p + x_o \tag{106}$$

$$d_o = -\frac{iz_G m_\eta}{z_\xi d_2 d_4} \tag{107}$$

$$d_1 = \frac{P(d_2)}{d_2(d_2 - d_4)} \tag{108}$$

$$d_3 = \frac{P(d_4)}{d_4(d_4 - d_2)} \tag{109}$$

$$P(S) = \xi_o S^2 + \left\{ \frac{iz_G}{z_\xi} - m_\eta \xi_o - \frac{z_\eta \left[(\mathbb{D}\eta)_o + m_\xi \xi_o \right]}{z_\xi} \right\} S - \frac{iz_G m_\eta}{z_\xi} \tag{110}$$

$$S = d_2, d_4 \tag{111}$$

$$d_2, d_4 = \text{roots of } R(S) = 0 \tag{112}$$

$$R(S) = (S - d_2)(S - d_4) = 0 \tag{113}$$

$$R(S) = S^2 + \left(\frac{z_\xi - m_\eta z_\xi - z_\eta m_\xi}{z_\xi} \right) S - \left(\frac{m_\eta z_\xi + z_\eta m_\xi}{z_\xi} \right) \tag{114}$$

$$f_o = \eta_o - \frac{Q(d_2)}{d_2^2 (d_2 - d_4)} - \frac{Q(d_4)}{d_4^2 (d_4 - d_2)} \tag{115}$$

$$f_1 = -\frac{iz_G m_\xi}{z_\xi d_2 d_4} \tag{116}$$

$$f_2 = \frac{Q(d_2)}{d_2^2 (d_2 - d_4)} \tag{117}$$

$$f_3 = \frac{Q(d_4)}{d_4^2 (d_4 - d_2)} \tag{118}$$

$$Q(S) = (\mathbb{D}\eta)_o S^2 + \left[\frac{(\mathbb{D}\eta)_o z_\xi + m_\xi z_\xi \xi_o - m_\xi z_\xi \xi_o - i z_G m_\xi}{z_\xi}\right] S - \frac{i z_G m_\xi}{z_\xi} \quad (119)$$

Equations (90) through (94) with the constants defined by equations (95) through (119) describe completely the motion of a projectile or missile flying through the range under the conditions postulated. They form the basis for determining the aerodynamic properties of the model from an analysis of the flight records.

Reduction of Flight Data

The first step in the reduction of data is the computation of the constants appearing in the equations of motion. It is assumed that the equations represent correctly the actual motion of the model and that any differences between the theoretical and experimental values are due to errors of measurement. The problem is to determine the values of the constants, the a's, b's, c's, d's, and f's, that give the best fit of theory and experiment. Specifically, the best fit normally means that the sum of the squares of the residuals between the computed and measured values is a minimum.

In principle the procedure is to compute an initial set of values of the constants by some approximate analysis of the data and to correct this initial set through a series of iterations. The sum of the squares of the residuals is computed at each step of the iteration, and the variation of the residuals with time or distance is studied in order to detect any systematic trends. If the process is convergent, it is carried out until the residuals are random in their distribution and until the sum of their squares approaches a steady, minimum value. Then, if the average value of the residuals after the final iteration is of the same order of magnitude as the estimated experimental error, the fit is considered to be satisfactory.

It should not be supposed that any one set of procedures has been drawn up to reduce the data from aerodynamic ranges. The computations are detailed and lengthy, and each range has its own unique methods adapted to the peculiarities of the apparatus and to the uses for which the data are obtained. Reference is best made to the facility of interest for a description of the methods in use there.

The final step in the reduction of data is the computation of the model's aerodynamic coefficients from the constants of its motions. The relations between the two are contained in equations (95) through (119). However, the equations named are formulated from the standpoint of computing the motions after the aerodynamic properties and the initial conditions have been given. It is desirable to transform these equations and to simplify certain of them in order to facilitate the computations involved in this step.

With regard to the constants in the $x(t)$ equation (90), the a's, the velocity and time factor are given by:

$$V_o = a_1 \tag{120}$$

$$\tau = \frac{\mu l}{a_1} \tag{121}$$

and the drag coefficient factor by:

$$x_o = -\frac{2\tau a_2}{a_1} = -\frac{2\mu l a_2}{a_1^2} \tag{122}$$

With regard to the constants in the $\Delta(x^+)$ equation (91), the b's, the lift coefficient factor, and Magnus force coefficient factor are given by the real and imaginary parts of b_3 as follows:

$$z_{\alpha L} = -b_{3R} \tag{123}$$

$$\nu z_{\beta p} = -b_{3I} \tag{124}$$

although it should be noted that the determination of $z_{\beta p}$ from b_{3I} may be marginal under certain circumstances.

With regard to the constants in the $\varphi(x^+)$ equation (92), the c's, the static and damping rolling-moment coefficient factors are given by

$$l_p + x_o = c_3 \qquad \text{(eq. (106))} \tag{125}$$

$$l_o = -c_1 c_3 \tag{126}$$

and the initial value of the nondimensional spin by

$$\nu = c_1 + c_2 c_3 \tag{127}$$

With regard to the constants in the $\xi(x^+)$ equation (93), the d's, the relations between these and the aerodynamic coefficient factors are derived by equating the sum of the roots of the quadratic in S, equation (113), to minus the coefficient of the linear term, equation (114), and the product of the roots to the constant term; that is,

$$\frac{z_\xi - m_\eta z_{\dot\xi} - z_\eta m_{\dot\xi}}{z_{\dot\xi}} = -(d_2 + d_4) \tag{128}$$

$$-\frac{m_\eta z_\xi + z_\eta m_\xi}{z_{\dot\xi}} = d_2 d_4 \tag{129}$$

Substituting equations (88) into equations (128) and (129), regrouping into real and imaginary parts, and replacing the aerodynamic factors by their definitions in terms of the aerodynamic coefficients gives for equations (128) and (129)

$$\left\{\left[\frac{C_{L_\alpha}}{2} - \frac{C_{X_o}}{2} - \frac{C_{m_q}}{4K_Y^2} - \frac{C_{m_{\dot\alpha}}}{4K_Y^2} - \frac{C_{Z_{\dot\alpha}}}{4\mu}\left(\frac{C_{X_o}}{2} + \frac{C_{m_q}}{4K_Y^2}\right) + \frac{C_{Z_q}}{4\mu}\frac{C_{m_{\dot\alpha}}}{4K_Y^2} + \right.\right.$$

$$\left.\nu^2 \frac{C_{Z_{\dot\beta p}}}{8\mu^2}\left(K - \frac{C_{m_{rp}}}{8\mu K_Y^2}\right) + \nu^2 \frac{C_{Z_{rp}}}{8\mu^2}\frac{C_{m_{\dot\beta p}}}{8\mu K_Y^2}\right] + i\nu\left[-K + \frac{C_{Z_{\beta p}}}{4\mu} - \right.$$

$$-\frac{C_{Z_{\dot\alpha}}}{4\mu}\left(K - \frac{C_{m_{rp}}}{8\mu K_Y^2}\right) + \frac{C_{m_{rp}}}{8\mu K_Y^2} - \frac{C_{Z_{\dot\beta p}}}{8\mu^2}\left(\frac{C_{X_o}}{2} + \frac{C_{m_q}}{4K_Y^2}\right) - \frac{C_{Z_{rp}}}{8\mu^2}\frac{C_{m_{\dot\alpha}}}{4K_Y^2} -$$

$$\left.\left.\frac{C_{m_{\dot\beta p}}}{8\mu K_Y^2}\left(1 - \frac{C_{Z_q}}{4\mu}\right)\right]\right\}\left\{\left[1 + \frac{C_{Z_{\dot\alpha}}}{4\mu}\right] + i\nu\frac{C_{Z_{\dot\beta p}}}{8\mu^2}\right\}^{-1}$$

$$= -(d_{2R} + d_{4R}) - i(d_{2I} + d_{4I}) \tag{130}$$

$$\left\{\left[\frac{\mu C_m}{2K_Y^2} - \frac{C_{Z_q}}{4}\ \frac{C_{m_\alpha}}{2K_Y^2} + \frac{C_{L_\alpha}}{2}\left(\frac{C_{m_q}}{4K_Y^2} + \frac{C_{X_o}}{2}\right) - \nu^2\ \frac{C_{Z_{rp}}}{8\mu^2}\ \frac{C_{m_{\beta p}}}{4K_Y^2} -\right.\right.$$

$$\left.\nu^2\ \frac{C_{Z_{\beta p}}}{4\mu}\left(K - \frac{C_{m_{rp}}}{8\mu K_Y^2}\right)\right] + i\nu\left[K\ \frac{C_{L_\alpha}}{2} + \frac{C_{m_{\beta p}}}{4K_Y^2} - \frac{C_{m_{rp}}}{8\mu K_Y^2}\ \frac{C_{L_\alpha}}{2} +\right.$$

$$\left.\left.\frac{C_{Z_{\beta p}}}{4\mu}\left(\frac{C_{m_q}}{4K_Y^2} + \frac{C_{X_o}}{2}\right) - \frac{C_{Z_q}}{4\mu}\ \frac{C_{m_{\beta p}}}{4K_Y^2} + \frac{C_{Z_{rp}}}{8\mu}\ \frac{C_{m_\alpha}}{2K_Y^2}\right]\right\}\left\{\left[1 + \frac{C_{Z_{\dot\alpha}}}{4\mu}\right] +\right.$$

$$\left.i\nu\ \frac{C_{Z_{\dot\beta p}}}{8\mu^2}\right\}^{-1} = \left(d_{2R}\ d_{4R} - d_{2I}\right) + i\left(d_{2I}\ d_{4R} + d_{2R}\ d_{4I}\right) \qquad (131)$$

It can be seen that many of the terms in equations (130) and (131) contain the projectile density factor, μ. Now, μ is a large number in all practical cases. It varies from 100 to 1,000 for full-scale projectiles and missiles and from 1,000 to 10,000 for models tested in the range. Consequently, certain terms will be much smaller than other terms and may be properly neglected consistent with the other approximations of the linearized equations.

Separating equation (130) into real and imaginary component equations and neglecting all terms of magnitude $1/\mu$ or smaller gives

$$(z_{\alpha L} - x_o - m_q - m_{\dot\alpha}) = -(d_{2R} + d_{4R}) \qquad (132)$$

$$\nu K = (d_{2I} + d_{4I}) \qquad (133)$$

Separating equation (131) into real and imaginary component equations, retaining only the terms of magnitude μ in the real part, neglecting all terms of magnitude $1/\mu$ or smaller in the imaginary part, and neglecting $d_{2R}\ d_{4R}$ compared to $d_{2I}\ d_{4I}$ gives

$$m_{\alpha} = d_{2I}\, d_{4I} \tag{134}$$

$$-\nu(Kz_{\alpha L} + m_{\beta p}) = (d_{2I}\, d_{4R} + d_{2R}\, d_{4I}) \tag{135}$$

The sum of the damping-moment factors, $m_q + m_{\dot{\alpha}}$, can be computed from equation (132) using equations (122) and (123) for x_o and $z_{\alpha L}$. The mean nondimensional spin ν can be computed from equation (133) and checked against the value from equation (127). The static-moment factor, m_{α}, is given by equation (134). The Magnus moment factor, $m_{\beta p}$, can be computed from equation (135).

From equations (74) through (81), it can be seen that all of the aerodynamic coefficient factors have been accounted for thus far in the reduction of data except $z_{\dot{\alpha}}$, z_q, $z_{\dot{\beta}p}$, z_{rp}, $m_{\dot{\beta}p}$, m_{rp}, and possibly, $z_{\beta p}$ Also, only the sum, $m_q + m_{\dot{\alpha}}$, can be computed, but not m_q and $m_{\dot{\alpha}}$ individually. These exceptions represent the limits of the analysis of test data presented up to this point.

One further possibility may be exploited in flight testing in the aerodynamics range. Instead of only one model of a given configuration being tested, two models are tested, both having identical external contours but one having a center-of-gravity position different from the other. In this way the sum of the damping-force factors, $z_{\dot{\alpha}} + z_q$, may be determined from measurements of the sum of the corresponding damping-moment factors, $m_{\dot{\alpha}} + m_q$, at the two center-of-gravity positions; similarly, the Magnus force factor, $z_{\beta p}$, may be determined from measurements of the Magnus moment factors, $m_{\beta p}$, at the two center-of-gravity positions. Also, the normal-force factor, z_{α}, may be determined in this way from the m_{α}'s, should measurements of the trajectory be unsuitable for some reason.

On the other hand, it does not appear possible to separate $(z_{\dot{\alpha}} + z_q)$ or $(m_{\dot{\alpha}} + m_q)$ or to determine $z_{\dot{\beta}p}$, z_{rp}, $m_{\dot{\beta}p}$, and m_{rp} from range tests. However, it can be seen from the development leading from equation (128) through to equation (135) that only the sum $(z_{\dot{\alpha}} + z_q)$ or the sum $(m_{\dot{\alpha}} + m_q)$ is retained in the final equations as a quantity having a significant magnitude and that all of the factors, $z_{\dot{\beta}p}$, z_{rp}, $m_{\dot{\beta}p}$, and m_{rp} are negligible. Consequently, it may be concluded that all of the aerodynamic coefficients of practical significance may be determined from free-flight testing in the aerodynamics range on projectiles and missiles with rotational and mirror symmetry flying under the conditions of small angles of sideslip, attack, pitch, and yaw, and small changes in velocity and spin, as specified.

Comparison With Ballistic Theory

Differences in nomenclature and derivation complicate a comparison of the projectile equations developed in this paper with those of ballistic theory. The differential equations giving the $x(t)$, $\varphi(x^+)$, and $\Delta(x^+)$ histories can be compared directly with the corresponding ballistic differential equations, as in the aircraft case. On the other hand, the equation giving the $\xi(x^+)$ history is best compared in its integrated form. It is first necessary to derive the relations between the nomenclature of this report and that used in ballistics. The comparison will be made by transforming the equations of this report into the ballistic terminology. The ballistic nomenclature will be taken from reference 10, except for that used in the $\varphi(x^+)$ equation, which will be taken from reference 19. The ballistic equations will be taken from the following sources: the $x(t)$ equation from reference 20; the $\varphi(x^+)$ equation from reference 19; the $\Delta(x^+)$ and $\xi(x^+)$ equations from reference 21. These references were selected because it is understood that they constitute the basis for the analysis of data from the Aerodynamics and Transonic Ranges at the Ballistic Research Laboratories and hence present a section of ballistic theory which is in use at the present time.

In ballistics[3] the aerodynamic coefficients are denoted by the capital letter K and the aerodynamic factors by the capital letter J, the two being related in all cases by

$$J = \frac{\rho d^3 K}{m} \tag{136}$$

where d is the diameter of the projectile. A study of the geometry and nomenclature of reference 10 shows that the following relations hold for the transformation of quantities defined in this report into ballistic terminology:

$$\left.\begin{array}{lll} F_{AX} = F_1 & F_{AY} = F_2 & F_{AZ} = F_3 \\ M_X = G_1 & M_Y = G_2 & M_Z = G_3 \\ V = u & \beta = u_2/u & \alpha = u_3/u \\ p = \omega_1 & q = \omega_2 & r = \omega_3 \\ I_X = A & I_Y = B & K_Y = k(d/l) \end{array}\right\} \tag{137}$$

$$\nu = \frac{\mu l}{d}\left(\nu_{\text{ballistic}}\right)_{t=0}, \quad \text{where } \nu_{\text{ballistic}} = \frac{\omega_1 d}{u}$$

$$s = \left(s_{\text{ballistic}}\right)_{t=0} \qquad \sigma = \left(\sigma_{\text{ballistic}}\right)_{t=0}$$

[3]If a ballistic symbol is not specifically defined in this section, it has the same definition in both systems of nomenclature and is given in the list of symbols at the beginning of this report.

The transformation of the aerodynamic factors can be derived from the definitions of the ballistic factors, the J's, given in reference 10, namely

$$\left.\begin{aligned} F_1 &= -\frac{mu^2}{d} J_{DA} \\ F_2 + iF_3 &= \frac{mu}{d}(-J_N + i\nu J_F)(u_2 + iu_3) + mu(\nu J_{XF} + iJ_S)(\omega_2 + i\omega_3) \\ G &= -mdu\,\omega_1 J_A \\ G_2 + iG_3 &= \frac{Bu}{k^2d^2}(-\nu J_T - iJ_M)(u_2 + iu_3) + \\ &\quad \frac{Bu}{k^2d}(-J_H + i\nu J_{XT})(\omega_2 + i\omega_3) \end{aligned}\right\} \quad (138)$$

If equations (138) for $F_{1,2,3}$ and $G_{1,2,3}$ and equations (11), (12), (28), (29), and (42) for $F_{AX,AY,AZ}$ and $M_{X,Y,Z}$ are substituted into the relations between the force and moment components given by equations (137) and the coefficients of the dependent variables are equated, since equations (137) must hold for all values of the dependent variables, the following relations are obtained:

$$\left.\begin{aligned} x_{oD} &= \mu\frac{l}{d} J_{DA} & l_p &= -\frac{\mu}{K_X^2}\left(\frac{d}{l}\right) J_A \\ z_\alpha &= \mu\frac{l}{d} J_N & m_\alpha &= \frac{\mu^2}{K_Y^2} J_M \\ z_q \boxed{+ z_{\dot\alpha}} &= -J_S & m_q \boxed{+ m_{\dot\alpha}} &= -\frac{\mu}{K_Y^2}\frac{d}{l} J_H \\ z_{\beta p} &= -J_F & m_{\beta p} &= -\frac{\mu}{K_Y^2}\frac{d}{l} J_T \\ z_{rp} &= -\frac{l}{\mu}\left(\frac{d}{l}\right) J_{XF} & m_{rp} &= -\frac{l}{K_Y^2}\left(\frac{d}{l}\right)^2 J_{XT} \end{aligned}\right\} \quad (139)$$

It will be noted that the factor $z_{\dot\alpha}$ has been added to z_q in the equation for J_S and similarly the factor $m_{\dot\alpha}$ has been added to m_q in the equation for J_T. In the strictest sense the added terms should have been omitted since the ballistic nomenclature does not include the $\dot\alpha$

components of force and moment. However, the relation between the aerodynamic factors and the constants of the $\xi(x^+)$ equation given by equation (132) shows clearly that $m_{\dot{\alpha}}$ should be added to m_q in the integrated equations of motion as they apply to flight in the aerodynamics range. Furthermore, since $(z_{\dot{\alpha}} + z_q)$ is the force corresponding to the moment $(m_{\dot{\alpha}} + m_q)$ just as J_S is the force corresponding to the moment J_H, $z_{\dot{\alpha}}$ should be added to z_q in order to have the equation for J_S consistent with the equation for J_H.

The lift and drag factors are introduced in the same manner in the ballistic development as in this report (see eq. (19)), that is

$$J_L = J_N - J_{DA}$$

hence

$$z_{\alpha L} = \mu \frac{l}{d} J_L \tag{140}$$

Since they may be of interest in reading the ballistic literature, the equations between the aerodynamic and ballistic coefficients are listed below:

$$\left.\begin{aligned}
K_D &= \frac{S}{2d^2} C_D & K_A &= \frac{1}{4} \frac{S}{d^2} \left(\frac{l}{d}\right)^2 C_{l_p} \\
K_N &= \frac{S}{2d^2} C_{Z_\alpha} & K_M &= \frac{S}{2d^2} \frac{l}{d} C_{m_\alpha} \\
K_S &= -\frac{1}{4} \frac{S}{d^2} \frac{l}{d} \left(C_{Z_q} + C_{Z_{\dot{\alpha}}}\right) & K_H &= -\frac{1}{4} \frac{S}{d^2} \left(\frac{l}{d}\right)^2 \left(C_{m_q} + C_{m_{\dot{\alpha}}}\right) \\
K_F &= -\frac{1}{4} \frac{S}{d^2} \frac{l}{d} C_{Z_{\beta p}} & K_{XT} &= -\frac{1}{4} \frac{S}{d^2} \left(\frac{l}{d}\right)^2 C_{m_{\beta_p}} \\
K_{XF} &= -\frac{1}{8} \frac{S}{d^2} \left(\frac{l}{d}\right)^2 C_{Z_{rp}} & K_{XT} &= -\frac{1}{8} \frac{S}{d^2} \left(\frac{l}{d}\right)^3 C_{m_{r_p}}
\end{aligned}\right\} \tag{141}$$

and

$$K_L = \frac{S}{2d^2} C_{L_\alpha} \tag{142}$$

The $x^+(t^+)$ history is given by equation (74); however, a more suitable form for the purposes of comparison is given during the development of equation (74) in Appendix A, as follows:

$$m \frac{dV}{dt} = -C_{X_o} \frac{\rho}{2} SV^2 \tag{143}$$

Equation (143) becomes, after transforming from time to distance by equation (68),

$$m \frac{dV}{dx} = - C_{X_o} \frac{\rho}{2} SV \tag{144}$$

The comparable equation from reference 20 is given on page 8 of the reference as follows:

$$\frac{dV}{dz} = - \frac{V}{F_1} K_D \tag{145}$$

where

V = velocity of projectile ($= V$)

z = distance along range ($= x$)

$$F_1 = \frac{m}{\rho d^2}$$

If equation (144) is transformed into ballistic notation, it becomes, using $C_D = C_{X_o}$ as specified by equation (19), which is correct to first order since $C_{X_\alpha} = 0$ for projectiles and missiles,

$$\frac{dV}{dz} = - \frac{V}{F_1} K_D \tag{146}$$

Comparison of equation (146) with equation (145) shows that the results of this paper agree precisely with those of reference 20.

The $\varphi(x^+)$ history is given by equation (86). The comparable equation from reference 19 is equation (14) of that report:

$$\varphi'' + C_1 \varphi' - C_2 = 0 \tag{147}$$

where (in the notation of ref. 19)

$$\left.\begin{aligned} C_1 &= \frac{K_{l_p}}{I} - \frac{K_R}{m} \\ C_2 &= \frac{K_{l_\delta}}{I} \end{aligned}\right\} \tag{148}$$

A study of the nomenclature of reference 19 discloses the following relations between its nomenclature and that of the present report:

$$\left.\begin{aligned}
\varphi &= \varphi \\
\varphi' &= \frac{d\varphi}{dx} = \frac{1}{\mu l}\,\mathbb{D}\varphi \\
\varphi'' &= \frac{d^2\varphi}{dx^2} = \frac{1}{(\mu l)^2}\,\mathbb{D}^2\varphi \\
I &= I_X \\
K_{l_p} &= -\frac{I_X l_p}{\mu l} \\
K_{l_\delta} &= \frac{I_X l_o}{(\mu l)^2} \\
K_R &= \frac{m x_o}{\mu l}
\end{aligned}\right\} \qquad (149)$$

After it is transformed into the nomenclature of reference 19 by the use of equations (148) and (149), equation (86) becomes

$$\varphi'' + C_1\varphi' - C_2 = 0 \qquad (150)$$

Comparison of equation (150) with equation (147) shows that the results of this paper agree precisely with those of reference 19.

The differential equation for the $\Delta(x^+)$ history is equation (93), whose constants are given by equations (99) through (102). The comparable equations from reference 21 are equations (40) and (5) of that report:

$$S'' = (J_L - i\nu J_F)\frac{\zeta}{d} + i(\nu J_{XF} + iJ_S)\zeta' - \frac{ig}{u^2} \qquad (151)$$

$$\zeta = \zeta_H + i\zeta_V = k_1 e^{i\varphi_1} + k\, e^{i\varphi_2} - \frac{gdA\nu}{Bk^{-2}u^2}\,J_M \qquad (152)$$

where k_1, k_2, φ_1, and φ_2 are given by equations (6) and (7) of reference 21:

$$k_1 = k_{1o} \exp \frac{1}{2d} \int_o^l \left\{ \left[-J_L + J_D - \frac{J_H}{k^2} - \frac{(J_D - md^2J_A/A)}{\sigma^2} \right] + \frac{1}{\sigma}\left[J_L - \frac{J_H}{k^2} - (2J_T - J_A)\frac{md^2}{A} \right] \right\} dz \qquad (153a)$$

$$k_2 = k_{2o} \exp \frac{1}{2d} \int_o^l \left\{ \left[- J_L + J_D - \frac{J_H}{k^2} - \frac{\left(J_D - \frac{md^2 J_A}{A} \right)}{\sigma^2} \right] - \frac{1}{\sigma} \left[J_L - \frac{J_H}{k^2} - (2J_T - J_A) \frac{md^2}{A} \right] \right\} dz \tag{153b}$$

$$\left. \begin{aligned} \varphi_1 &= \varphi_{1o} + \frac{1}{2d} \int_o^z \left[\frac{\nu A(1+\sigma)}{B} \right] dz \\ \varphi_2 &= \varphi_{2o} + \frac{1}{2d} \int_o^z \left[\frac{\nu A(1-\sigma)}{B} \right] dz \end{aligned} \right\} \tag{154}$$

Experience indicates that the term $i(\nu J_{XF} + iJ_S)\zeta'$ in equation (151) is negligible compared to the remaining terms, and consequently the ballistic equation for the transverse displacement reduces to

$$S'' = (J_L - i\nu J_F) \frac{\zeta}{d} - \frac{ig}{u^2} \tag{155}$$

A study of reference 21 shows that its nomenclature is transformed into the nomenclature of this report by equations (137), (139), and (140) together with the following:

$$\left. \begin{aligned} S &= - \mu l \Delta \\ \zeta &= \xi \\ z &= x \end{aligned} \right\} \tag{156}$$

If equation (84) is transformed into the ballistic notation, it becomes

$$S'' = (J_L - i\nu J_F) \frac{\zeta}{d} - \frac{ig}{u^2} - \frac{2ig(u - u_o)}{u^3} \tag{157}$$

where $u_o = u$ at $t = 0$.

Comparison of equation (157) with equation (155) shows that the transformed $\Delta(x^+)$ equation agrees exactly with the ballistic equation except for the term $2ig(u - u_o)/u^3$. For representative tests in the aerodynamics range, this term is negligible compared to the remaining terms in equation (157). Hence, it may be concluded that the agreement between equations (157) and (155) is satisfactory.

From the comparison of equation (93) for the $\xi(x^+)$ history with the corresponding ballistic equation (152), it can be seen that both equations have the same form. Hence, the question of agreement between the two is concerned with a comparison of the constants of the equations. The aerodynamic factors are determined solely from the constants d_2 and d_4 (see eqs. (93) and (132) through (135)) and it is believed sufficient for the purposes of this report to confine the comparison to these two constants.

Comparison of equation (93) with equation (152) shows that the real and imaginary parts of the exponents in equation (93) are related to the exponents in equation (152) as follows:

$$d_{2R}\, x^+ = \frac{1}{2d} \int_o^z \left\{ \left[- J_L + J_D - \frac{J_H}{k^2} - \frac{1}{\sigma^2} \left(J_D - \frac{md^2 J_A}{A} \right) \right] + \frac{1}{\sigma} \left[J_L - \frac{J_H}{k^2} - (2J_T - J_A) \frac{md^2}{A} \right] \right\} dz \tag{158a}$$

$$d_{4R}\, x^+ = \frac{1}{2d} \int_o^z \left\{ \left[- J_L + J_D - \frac{J_H}{k^2} - \frac{1}{\sigma^2} \left(J_D - \frac{md^2 J_A}{A} \right) \right] - \frac{1}{\sigma} \left[J_L - \frac{J_H}{k^2} - (2J_T - J_A) \frac{md^2}{A} \right] \right\} dz \tag{158b}$$

$$\left.\begin{aligned} d_{2I}\, x^+ &= \varphi_{1o} + \frac{1}{2d} \int_o^z \frac{\nu A(1+\sigma)}{B} dz \\ d_{4I}\, x^+ &= \varphi_{2o} + \frac{1}{2d} \int_o^z \frac{\nu A(1-\sigma)}{B} dz \end{aligned}\right\} \tag{159}$$

Differentiating equations (158) and (159) with respect to z, noting from equations (156) that $x = z$, and limiting the comparison to $t = 0$ gives

$$\frac{2d}{\mu l} d_{2R} = - \left[J_L - J_D + \frac{J_H}{k^2} + \frac{1}{\sigma^2} \left(J_D - \frac{md^2 J_A}{A} \right) \right] + \frac{1}{\sigma} \left[J_L + J_D - \frac{J_H}{k^2} - 2 \frac{md^2}{A} J_T - \left(J_D - \frac{md^2 J_A}{A} \right) \right] \tag{160a}$$

$$\frac{2d}{\mu l} d_{4R} = -\left[J_L - J_D + \frac{J_H}{k^2} + \frac{1}{\sigma^2}\left(J_D - \frac{md^2 J_A}{A} \right) \right] -$$

$$\frac{1}{\sigma}\left[J_L + J_D - \frac{J_H}{k^2} - 2\frac{md^2}{A} J_T - \left(J_D - \frac{md^2 J_A}{A} \right) \right] \qquad (160b)$$

$$\left.\begin{aligned} \frac{2d}{\mu l} d_{2I} &= \frac{\nu A}{B}(1+\sigma) \\ \frac{2d}{\mu l} d_{4I} &= \frac{\nu A}{B}(1-\sigma) \end{aligned}\right\} \qquad (161)$$

Up to this point explicit expressions have not been derived for the real and imaginary parts of d_2 and d_4, since such expressions are not required for the analysis of data from the range. However, they may be readily obtained from equations (132) through (135). The solution given below is valid for all values of s or σ except the region for which $0 \leq s \leq +1$ for which σ is imaginary. If m_α is negative, corresponding to $s < 0$ or $\sigma > 1$, the model will have arrow stability. If m_α is positive but $s > 1$, corresponding to $0 < \sigma < 1$, the model will be gyroscopically stable. Hence it is believed that the following solution will cover all cases of practical importance

$$\left.\begin{aligned} d_{2R} &= -\frac{(z_{\alpha L} - m_q - m_{\dot\alpha} - x_o)}{2} + \frac{\left(z_{\alpha L} + m_q + m_{\dot\alpha} + x_o + \frac{2m_{\beta p}}{K}\right)}{2\sigma} \\ d_{4R} &= -\frac{(z_{\alpha L} - m_q - m_{\dot\alpha} - x_o)}{2} - \frac{\left(z_{\alpha L} + m_q + m_{\dot\alpha} + x_o + \frac{2m_{\beta p}}{K}\right)}{2\sigma} \end{aligned}\right\} \qquad (162)$$

$$\left.\begin{aligned} d_{2I} &= \frac{\nu K(1+\sigma)}{2} \\ d_{4I} &= \frac{\nu K(1-\sigma)}{2} \end{aligned}\right\} \qquad (163)$$

Transforming equations (162) and (163) into ballistic notation and multiplying by $2d/\mu l$, as required for comparison with equations (160) and (161), gives

$$\frac{2d}{\mu l} d_{2R} = -(J_L - J_D + k^{-2}J_H) + \frac{1}{\sigma}\left(J_L + J_D - k^{-2}J_H - 2\frac{md^2}{A}J_T\right) \quad (164a)$$

$$\frac{2d}{\mu l} d_{4R} = -(J_L - J_D + k^{-2}J_H) - \frac{1}{\sigma}\left(J_L + J_D - k^{-2}J_H - 2\frac{md^2}{A}J_T\right) \quad (164b)$$

$$\left.\begin{aligned} \frac{2d}{\mu l} d_{2I} &= \frac{\nu A}{B}(1+\sigma) \\ \frac{2d}{\mu l} d_{4I} &= \frac{\nu A}{B}(1-\sigma) \end{aligned}\right\} \quad (165)$$

Comparison of equations (164) with equations (160) and equations (165) with equations (161) shows that corresponding equations agree except for the term $\left(J_D - \frac{md^2J_A}{A}\right)$ which is missing entirely from equations (164). Differences in the derivation of the equations of motion make it difficult to ascertain the reason for the discrepancy. A careful examination of the terms neglected in this paper in linearizing the equations of motion suggests that the term in question is actually a second-order quantity. In practice, the extra term is so small for the majority of tests carried out in the range that equations (164) can be said to agree satisfactorily with equations (160).

To summarize, the results of this paper are in good agreement with the results of ballistic theory. The equations of motion are the same except for an additional term in the equations giving the transverse displacement and for two additional terms containing a common factor in the relations between the aerodynamic coefficients and the damping rates of the oscillatory components of the angular motion. It is believed that these additional terms are negligibly small under the majority of representative test conditions in the aerodynamics range.

CONCLUSIONS

Equations of motion have been developed for conventional aircraft on the one hand, and for rotationally symmetric missiles and projectiles on the other. Similar mathematical derivations and standard NACA nomenclature were used for both developments. The essential difference between the two cases lies in allowing the axial component of the angular velocity

to be large in the projectile and missile case. Otherwise, the conditions of flight are limited to small changes in velocity and orientation, as is customary in treating the first-order dynamic stability of aircraft.

Two novel features appear in the derivation of the equations of motion. First, the aerodynamic coefficients are introducted as a formal series expansion in the components of the linear and angular velocities. This approach allows the introduction of coefficients of quadratic terms involving the axial component of angular velocity, which are second-order for the aircraft case but first-order for the missile and projectile case. Second, it is shown that the aerodynamic forces and moments are independent of the orientation in roll for rotationally symmetric missiles and projectiles. Advantage is taken of the independence to place the coordinate system used in the missile case with one axis along the missile's axis of rotational symmetry and another axis in a plane fixed in space. Hence, the position of the coordinate system is independent of the roll orientation of the missile.

Criteria are derived for the dynamic stability of missiles with arrow stability and projectiles with gyroscopic stability. It is shown that spin-stabilized projectiles may be completely dynamically stable, a result in agreement with experiment. On the other hand, it is clear that the dynamic stability of spin-stabilized projectiles is a delicate balance between lift, damping, and Magnus forces and that all three elements are required to predict adequately the character of the motion.

The equations of motion are applied to the flight testing of rotationally symmetric missiles and projectiles in an aerodynamics range. It is shown that the flight records obtained in this facility can be analyzed to give all the aerodynamic coefficients required to predict the first-order dynamic stability of the missile or projectile.

The results of this paper are compared with the accepted aeronautical theory for aircraft and the corresponding ballistic theory for missiles and projectiles. The agreement is considered to be satisfactory throughout.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., Oct. 1, 1954

APPENDIX A

LINEARIZATION OF THE EQUATIONS OF MOTION

In the author's opinion, the proper procedure to be followed in linearizing each of the equations of motion is to separate the exact equation into first- and second-order terms and to place all of the first-order terms that are to be retained on the left-hand side and all of the second-order terms that are to be neglected on the right-hand side. This procedure allows a term-by-term comparison of the second-order terms neglected with the first-order terms retained at one step in the derivation. As a result, it places the neglect of second-order terms on a firm basis that can be re-examined readily at any time should there be some doubt as to the relative magnitude of any particular term.

The procedure just recommended is carried through only in part. All of the second-order and some of the third-order inertia terms are included. However, only part of the second-order aerodynamic force and moment terms are written down. Although it would be desirable for the sake of rigor to include them, their omission is justified in order to make the derivation more concise since there is rarely any doubt as to their magnitude relative to the first-order aerodynamic terms. It is the comparison of the inertia terms with the aerodynamic terms that is ordinarily subject to scrutiny.

No explicit limit has been set so far on the values of the aerodynamic coefficients themselves; yet, there is the implicit assumption that their magnitude ranges in value from unity to ten. This restriction can usually be met by the appropriate selection of representative length and area. Also, there is the implicit assumption in the missile and projectile case that the axial component of the angular velocity (the spin) be bounded. Again, this restriction is normally satisfied by the values of the angular velocity which actually occur in practice.

It will be noted that an anomaly exists in setting limits on both φ,θ,ψ and p,q,r for a given aircraft. Once the physical and aerodynamic properties of the aircraft are specified together with the initial conditions of flight, the values of p, q, and r are determined. Physically, the consequence of assigning limits to both φ,θ,ψ and p,q,r is to assign a certain range of values to the physical properties of the aircraft, that is, to its moments of inertia, weight, and center of gravity. As a result, it is desirable to examine the values of p, q, and r occurring during the flight of any particular missile or projectile in order to make certain that their magnitudes fall within the limits of the theory.

In the following sections, the equations of motion will be developed up to the point of listing the first-order terms retained on the left-hand

side and the second-order terms neglected on the right-hand side, but a numerical comparison of their relative magnitudes will be omitted. In the aircraft case, the terms neglected are those customarily neglected. In the missile and projectile case, numerical examples were worked out for representative missiles and projectiles and it is believed that the terms neglected are truly second-order for the majority of conditions that will occur in actual practice.

Aircraft Equations With Time as Independent Variable

X component of force.- This equation is derived by resolving the vector force equation (3) along the space-fixed X_O axis.

Since

$$\overline{M} = m\overline{V} \tag{A1}$$

the component of equation (3) along X_O becomes

$$m \frac{d(V \cos \measuredangle \overline{V}X_O)}{dt} = F_{AX} \cos \measuredangle (XX_O) + F_{AY} \cos \measuredangle (YX_O) + F_{AZ} \cos \measuredangle (ZX_O) - mg \sin \gamma_O \tag{A2}$$

Substituting equations (11) and (15) for $F_{AX,Y,Z}$ and equations (1) for $\cos \measuredangle(XX_O)$, $\cos \measuredangle(YX_O)$, and $\cos \measuredangle(ZX_O)$ in equation (A2), and at the same time neglecting all second-order terms on the right-hand side of equation (A2), gives

$$m\left(\cos \measuredangle \overline{V}X_O \frac{dV}{dt} - V \sin \measuredangle \overline{V}X_O \frac{d \measuredangle VX_O}{dt}\right)$$

$$= \frac{\rho V_O^2 S}{2}\left(\frac{V}{V_O}\right)^2\left(- C_{X_O} - C_{X_\alpha}\alpha - C_{X_{\dot{\alpha}}} \frac{\dot{\alpha} l}{2V} - C_{X_q} \frac{ql}{2V} - C_{Z_O}\theta\right) - mg \sin \gamma_O \tag{A3}$$

$\cos \measuredangle \overline{V}X_O$ being approximated by

$$\cos \measuredangle \overline{V}X_O = 1 - \frac{\sin^2 \measuredangle \overline{V}X_O}{2}$$

and the velocity, V, is given by

$$\left.\begin{aligned} V &= V_o + u \\ \frac{V}{V_o} &= 1 + \frac{u}{V_o} \end{aligned}\right\} \tag{A4}$$

If equations (A4) are expanded in a series, equation (A3) may be separated into first- and second-order terms with the first-order listed on the left-hand side and the second-order on the right-hand side, as follows:

$$m\frac{du}{dt} + \frac{\rho V_o^2 S}{2}\left(C_{X_o} + C_{X_\alpha}\alpha + C_{X_{\dot\alpha}}\frac{\dot\alpha l}{2V_o} + C_{X_q}\frac{\dot\theta l}{2V_o} + C_{Z_o}\theta + 2\frac{u}{V_o}C_{X_o}\right) +$$

$$mg\sin\gamma_o = \frac{m}{2}\frac{du}{dt}\sin^2\measuredangle\bar{V}X_o + mV\sin\measuredangle\bar{V}X_o\frac{d\measuredangle VX_o}{dt} - \frac{\rho V_o^2 S}{2}\left(C_{X_\alpha}\alpha\frac{2u}{V_o} +\right.$$

$$\left. C_{X_{\dot\alpha}}\frac{\dot\alpha l}{2V_o}\frac{u}{V_o} + C_{X_q}\frac{ql}{2V_o}\frac{u}{V_o} + C_{Z_o}\theta\frac{2u}{V_o}\right) - \frac{\rho V^2 S}{2}C_{X_q}\varphi\frac{rl}{2V} \tag{A5}$$

If the right-hand side is neglected, the left-hand side is divided by $(\rho S V_o^2)$, and the equilibrium conditions (eq. (18)) and lift and drag coefficients (eq. (19)) are introduced, equation (A5) becomes equation (20).

A slight variation of equation (A5), which will be useful in deriving other equations of motion, comes from equation (A3) by introducing the equilibrium condition of equation (18)

$$x_o = -x_G$$

namely,

$$m\frac{dV}{dt} = \frac{\rho S V^2}{2}\left(-C_{X_\alpha}\alpha - C_{X_{\dot\alpha}}\frac{\dot\alpha l}{2V} - C_{X_q}\frac{ql}{2V} - C_{Z_o}\theta\right) \tag{A6}$$

Y component of force.- This equation is derived by resolving the vector force equation (3) along the Y axis.

The components of $\bar{M}$ and $\bar{V}$ are given by equation (6), the components of $\bar{F}_A$ are given by equations (11) and (15), the components of $\bar{F}_G$ are given by equation (17), and the transformation of moving axes is given by equation (9). Equation (3), modified by the foregoing, may be resolved along the Y axis as follows:

$$m\frac{dV}{dt}\sin\beta + V\frac{d\beta}{dt}\cos\beta + rV\sqrt{1-\sin^2\alpha-\sin^2\beta} - pV\sin\alpha$$

$$= \frac{\rho V^2 S}{2}\left(C_{Y_\beta}\beta + C_{Y_{\dot\beta}}\frac{\dot\beta l}{2V} + C_{Y_p}\frac{pl}{2V} + C_{Y_r}\frac{rl}{2V}\right) + mg(\psi\sin\gamma_o + \varphi\cos\gamma_o) \quad (A7)$$

The following quantities are approximated by

$$\cos\beta = 1 - \frac{\beta^2}{2}$$

$$\sin\beta = \beta$$

$$\sqrt{1-\sin^2\alpha-\sin^2\beta} = 1 - \frac{(\alpha^2+\beta^2)}{2}$$

The values of r and p are given by equation (7)

$$r = \frac{d\psi}{dt} - \varphi q$$

$$p = \frac{d\varphi}{dt} - \theta r$$

After division by V and the use of equations (A4) and (A6) for V and dV/dt, equation (A7) may be separated into first- and second-order terms with the first-order listed on the left-hand side and the second-order on the right-hand side, as follows:

$$m\left(\frac{d\beta}{dt}+\frac{d\psi}{dt}\right) - \frac{\rho V_o S}{2}\left(C_{Y_\beta}\beta + C_{Y_{\dot\beta}}\frac{\dot\beta l}{2V_o} + C_{Y_p}\frac{\dot\varphi l}{2V_o} + C_{Y_r}\frac{\dot\psi l}{2V_o}\right) - \frac{mg}{V_o}(\psi\sin\gamma_o +$$

$$\varphi\cos\gamma_o) = m\left(\frac{\beta^2}{2}\dot\beta + \varphi q + \frac{r(\alpha^2+\beta^2)}{2} + p\alpha\right) + \frac{\rho SV}{2}\beta\left(C_{X_\alpha}\alpha + C_{X_{\dot\alpha}}\frac{\dot\alpha l}{2V} +\right.$$

$$\left. C_{X_q}\frac{ql}{2V} + C_{Z_o}\theta\right) - \frac{\rho SV}{2}\left(C_{Y_p}\theta\frac{rl}{2V} + C_{Y_r}\varphi\frac{ql}{2V}\right) + \frac{u}{V_o}\left[\frac{\rho SV_o}{2}C_{Y_\beta}\beta -\right.$$

$$\left.\frac{mg}{V_o}(\psi\sin\gamma_o + \varphi\cos\gamma_o)\right] \quad (A8)$$

If the right-hand side is neglected, the left-hand side is divided by $(\rho V_o S)$, and the lift and drag coefficients (eq. 19) are introduced, equation (A8) becomes equation (21).

Z component of force.- This equation is derived by resolving the vector force equation (3) along the Z axis.

The components of $\overline{M}$, $\overline{V}$, $\overline{F}_A$, and $\overline{F}_G$ are given by equations (6), (11), (15), and (17), respectively. After transformation to moving axes by equation (9) and division by V, equation (3) resolved along the Z axis becomes

$$m\left(\frac{d\alpha}{dt}\cos\alpha + \frac{dV}{dt}\frac{\sin\alpha}{V} + p\sin\beta - q\sqrt{1 - \sin^2\alpha - \sin^2\beta}\right)$$

$$= \frac{\rho SV}{2}\left(-C_{Z_o} - C_{Z_\alpha}\alpha - C_{Z_{\dot{\alpha}}}\frac{\dot{\alpha}l}{2V} - C_{Z_q}\frac{ql}{2V}\right) + \frac{mg}{V}\left(-\theta\sin\gamma_o + \cos\gamma_o\right) \quad (A9)$$

The following quantities are approximated by

$$\sin\alpha = \alpha$$

$$\sqrt{1 - \sin^2\alpha - \sin^2\beta} = 1 - \frac{\alpha^2 + \beta^2}{2}$$

$$\cos\alpha = 1 - \frac{\alpha^2}{2}$$

The value of q is given by equation (7b), namely

$$q = \frac{d\theta}{dt} + \varphi r$$

By the use of equations (A4) and (A6) for V and dV/dt, equation (A9) may be separated into first- and second-order terms as follows:

$$m\left(\frac{d\alpha}{dt} - \frac{d\theta}{dt}\right) + \frac{\rho S V_o}{2}\left(C_{Z_o} + C_{Z_\alpha} + C_{Z_{\dot{\alpha}}}\frac{l}{2V_o}\frac{d\alpha}{dt} + C_{Z_q}\frac{l}{2V_o}\frac{d\theta}{dt} + \frac{u}{V_o}C_{Z_o}\right) -$$

$$\frac{mg}{V_o}\left[-\theta \sin\gamma_o + \left(1 - \frac{u}{V_o}\right)\cos\gamma_o\right]$$

$$= m\left[\frac{\dot{\alpha}\alpha^2}{2} + \varphi r - p\beta - \frac{q(\alpha^2 + \beta^2)}{2}\right] + \frac{\rho V_o S\alpha}{2}\left(C_{X_\alpha}\alpha + C_{X_{\dot{\alpha}}}\frac{\dot{\alpha}l}{2V} + C_{X_q}\frac{ql}{2V} + C_{Z_o}\theta\right) +$$

$$\frac{u}{V_o}\left(-\frac{C_{Z_\alpha}\rho S V_o}{2}\alpha + \frac{mg}{V_o}\theta \sin\gamma_o\right) - \frac{\rho S V}{2}\left(C_{Z_q}\varphi\frac{rl}{2V}\right) \qquad \text{(A10)}$$

If the right-hand side is neglected, the left-hand side divided by $(\rho V_o S)$, and the equilibrium conditions (eq. (18)) and lift and drag coefficients (eq. (19)) are introduced, equation (A10) becomes equation (22).

X component of moment.- This equation is derived by resolving the vector moment equation (4) along the X axis.

The components of $\overline{H}$, $\overline{\omega}$, and $\overline{M}$ are given by equations (6), (7), (12), and (15). After transformation to moving axes by equation (9), equation (4) resolved along the X axis becomes

$$I_X\frac{dp}{dt} - J_{XZ}\frac{dr}{dt} + qr(I_Z - I_Y) - qp\,J_{XZ}$$

$$= \frac{\rho S l V^2}{2}\left(C_{l_\beta}\beta + C_{l_{\dot{\beta}}}\frac{\dot{\beta}l}{2V} + C_{l_p}\frac{pl}{2V} + C_{l_r}\frac{rl}{2V}\right) \qquad \text{(A11)}$$

The values of p and r are given by equation (7), namely,

$$p = \frac{d\varphi}{dt} - \theta r$$

$$r = \frac{d\psi}{dt} - \varphi q$$

The value of V is given by equation (A4), namely,

$$\frac{V}{V_o} = 1 + \frac{u}{V_o}$$

Equation (A11) may be separated into first- and second-order terms as follows,

$$I_X \frac{d^2\varphi}{dt^2} - J_{XZ} \frac{d^2\psi}{dt^2} - \frac{\rho S l V_o^2}{2} \left(C_{l_\beta} \beta + C_{l_{\dot\beta}} \frac{l}{2V_o} \frac{d\beta}{dt} + C_{l_p} \frac{l}{2V_o} \frac{d\varphi}{dt} + C_{l_r} \frac{l}{2V_o} \frac{d\psi}{dt} \right)$$

$$= I_X (r\dot\theta + \theta\dot r) - J_{XZ} (q\dot\varphi + \varphi\dot q) - qr(I_Z - I_Y) + qp\, J_{XZ} + \frac{\rho S V_o^2 l}{2} \frac{u}{V_o} \left(2C_{l_\beta} \beta + C_{l_{\dot\beta}} \frac{\dot\beta l}{2V_o} + C_{l_p} \frac{pl}{2V_o} + C_{l_r} \frac{rl}{2V_o} \right) - \frac{\rho S l V^2}{2} \left(C_{l_p\theta} \frac{rl}{2V} + C_{l_r\theta} \frac{ql}{2V} \right) \quad \text{(A12)}$$

If the terms on the right-hand side are neglected and the left-hand side is multiplied by $\mu/K_X^2\, \rho S l V_o^2$, equation (A12) becomes equation (23).

Y component of moment.- This equation is derived by resolving the vector moment equation (4) along the Y axis.

The components of $\bar H$, $\bar\omega$, and $\bar M$ are given by equations (6), (7), (12), and (15). The value of q is given by equation (7) to obtain $\dot q$, namely

$$\dot q = \frac{d^2\theta}{dt^2} + \frac{d(\varphi r)}{dt}$$

After transformation to moving axes by equation (9), equation (4) resolved along the Y axis becomes

$$I_Y \left[\frac{d^2\theta}{dt^2} + \frac{d(\varphi r)}{dt} \right] + rp(I_X - I_Z) + J_{XZ} (p^2 - r^2) = \frac{\rho S l V^2}{2} \left(C_{m_o} + C_{m_\alpha} \alpha + C_{m_{\dot\alpha}} \frac{\dot\alpha l}{2V} + C_{m_q} \frac{ql}{2V} \right) \quad \text{(A13)}$$

The value of V is given by equation (A4), namely,

$$\frac{V}{V_o} = 1 + \frac{u}{V_o}$$

Equation (A13) may be separated into first- and second-order terms as follows:

$$I_Y \frac{d^2\theta}{dt^2} - \frac{\rho S l V_o^2}{2}\left(C_{m_o} + C_{m_\alpha}\alpha + C_{m_{\dot\alpha}} \frac{\dot\alpha l}{2V_o} + C_{m_q} \frac{\dot\theta l}{2V_o}\right)$$

$$= -I_Y \frac{d(\varphi r)}{dt} + rp(I_Z - I_X) + J_{XZ}(r^2 - p^2) + \frac{\rho S l V_o^2}{2} \frac{u}{V_o}\left(2C_{m_o} + 2C_{m_\alpha}\alpha + C_{m_{\dot\alpha}} \frac{\dot\alpha l}{2V_o} + C_{m_q} \frac{q l}{2V_o}\right) + \frac{\rho S l V^2}{2}\left(C_{m_q\theta} \frac{r l}{2V}\right) \quad (A14)$$

If the right-hand side is neglected, the left-hand side multiplied by $\mu/\rho S l V_o^2 K_Y^2$, and C_{m_o} set equal to zero in accordance with equilibrium conditions specified by equation (18), equation (A14) becomes equation (24).

Z component of moment.- This equation is derived by resolving the vector moment equation (5) along the Z axis.

The components of $\overline{H}$, $\overline{\omega}$, and $\overline{M}$ are given by equations (6), (7), (12), and (15). The values of p and r are given by equation (7) to obtain $\dot p$ and $\dot r$, namely,

$$\dot p = \frac{d^2\varphi}{dt^2} - \frac{d(\theta r)}{dt}$$

$$\dot r = \frac{d^2\psi}{dt^2} - \frac{d(\varphi q)}{dt}$$

After transformation to moving axes by equation (9), equation (5) resolved along the Z axis becomes

$$I_Z\left[\frac{d^2\psi}{dt^2} - \frac{d(\varphi q)}{dt}\right] - J_{XZ}\left[\frac{d^2\varphi}{dt^2} - \frac{d(\theta r)}{dt}\right] + pqI_Y - q(pI_X - rJ_{XZ})$$

$$= \frac{\rho S l V^2}{2}\left(C_{n_\beta}\beta + C_{n_{\dot\beta}} \frac{\dot\beta l}{2V} + C_{n_p} \frac{p l}{2V} + C_{n_r} \frac{r l}{2V}\right) \quad (A15)$$

The value of V is given by equation (A4), namely,

$$\frac{V}{V_o} = 1 + \frac{u}{V_o}$$

equation (A15) may be separated into first- and second-order terms as follows,

$$I_Z \frac{d^2\psi}{dt^2} - J_{XZ} \frac{d^2\varphi}{dt^2} - \frac{\rho S l V_o^2}{2} \left(C_{n_\beta}\beta + C_{n_{\dot\beta}} \frac{l}{2V_o} \frac{d\beta}{dt} + C_{n_p} \frac{l}{2V_o} \frac{d\varphi}{dt} + C_{n_r} \frac{l}{2V_o} \frac{d\psi}{dt} \right)$$

$$= I_Z \frac{d(\varphi q)}{dt} - J_{XZ} \frac{d(\theta r)}{dt} + pq(I_X - I_Y) - J_{XZ}qr +$$

$$\frac{\rho S l V_o^2}{2} \frac{u}{V_o} \left(2C_{n_\beta}\beta + C_{n_{\dot\beta}} \frac{\dot\beta l}{2V_o} + C_{n_p} \frac{pl}{2V_o} + \right.$$

$$\left. C_{n_r} \frac{rl}{2V_o} \right) - \frac{\rho S l V_o^2}{2} \left(C_{n_p}\theta \frac{rl}{2V_o} + C_{n_r}\varphi \frac{ql}{2V_o} \right) \qquad \text{(A16)}$$

If the right-hand side is neglected and the left-hand side is multiplied by $\mu/\rho S l V_o^2 K_Z^2$, equation (A16) becomes equation (25).

Symmetrical Missile and Projectile Equations With Time as Independent Variable

The missile and projectile equations are derived by resolving the vector force and moment equations (3) and (4) along the XYZ axes, except for the X component of the vector force equation, which will be resolved along the space-fixed axis X_o instead of the body axis X. The derivation will proceed as follows: First, the component of the exact vector equation will be written down transformed to the moving XYZ axes, except, of course, for the X_o component equation; second, the kinematic relations will be substituted for certain of the dependent variables, various functions will be expanded in series, and the resulting equation will be written down separated into first- and second-order terms with the first-order terms on the left-hand side and the second-order terms on the right-hand side.

The following relations will be used: The transformation from space-fixed to moving axes is given by equation (40b); the direction cosines of the angles between X_o and X,Y,Z are given by equation (1); the components of $\overline{M}$ and $\overline{V}$ are given by equation (6) and of $\overline{H}$ and $\overline{\omega}$ by equation (39); the components of $\overline{F}_A$ are given by equations (11), (29), and (42), of $\overline{F}_G$ by equation (44), and of $\overline{M}$ by equations (12), (29), and (42); the magnitude of the velocity, V, is given by condition (2) preceding equation (2) and the axial component of the angular velocity, p, by equation (34).

The following series expansions will be utilized:

$$\left.\begin{aligned} \sin\alpha &= \alpha & \sin\beta &= \beta \\ \cos\alpha &= 1 - \frac{\alpha^2}{2} & \cos\beta &= 1 - \frac{\beta^2}{2} \\ \sqrt{1 - \sin^2\alpha - \sin^2\beta} &= 1 - \frac{\alpha^2 + \beta^2}{2} \\ \left(\frac{V}{V_o}\right)^n &= 1 + n\frac{u}{V_o} \end{aligned}\right\} \quad (A17)$$

In deriving the Y and Z components of the force equation, the following approximation will be used

$$\frac{1}{V}\frac{dV}{dt} = -C_{X_o}\frac{\rho SV}{2m} \quad (A18)$$

As will be shown in the derivation of the X component of the force equation, the above equation neglects the terms $C_{X_{pp}}\rho SV/2m$ and $g \sin\gamma_o/V$. From the limited experimental evidence available, it is believed that the $C_{X_{pp}}$ term is truly negligible under all practical circumstances. On the other hand, the gravity term is strictly negligible only for horizontal trajectories or in those cases for which the drag force greatly exceeds the weight. In other cases, for example, a bomb falling along a steep trajectory, the gravity term should be included. The gravity term will be omitted from the present treatment since the application covered in this paper is to the aerodynamics range in which the trajectories are nearly horizontal and the drag force is many times the weight in the great majority of tests.

X component of force.-

$$m\frac{dV_{X_o}}{dt} = F_{AX}\cos\measuredangle X_oX + F_{AY}\cos\measuredangle X_oY + F_{AZ}\cos\measuredangle X_oZ - mg\sin\gamma_o \quad (A19)$$

where V_{X_o} is the component of $\bar{V}$ along X_o and is given by

$$V_{X_o} = V\cos\measuredangle\bar{V}X_o$$

$$m\left[(\cos\measuredangle\bar{V}X_o)\left(\frac{dV}{dt}\right) - (V\sin\measuredangle\bar{V}X_o)\left(\frac{d\measuredangle\bar{V}X_o}{dt}\right)\right] = \frac{\rho SV^2}{2}\left[-C_{X_o} - C_{X_{pp}}\left(\frac{pl}{2V}\right)^2\right] - mg\sin\gamma_o \quad (A20)$$

$$m\frac{du}{dt} + \frac{\rho SV_o^2}{2}\left[C_{X_o} + 2C_{X_o}\frac{u}{V_o} + 2C_{X_{pp}}\left(\frac{p_o l}{2V_o}\right)\left(\frac{\dot{\phi} l}{2V_o}\right) - C_{X_{pp}}\left(\frac{p_o l}{2V_o}\right)^2\right] + mg\sin\gamma_o$$

$$= m\left[\left(\frac{\sin^2 \measuredangle \bar{V}X_o}{2}\right)\left(\frac{dV}{dt}\right) + (V\sin \measuredangle \bar{V}X_o)\left(\frac{d \measuredangle \bar{V}X_o}{2}\right)\right] -$$

$$\frac{\rho V_o^2 S}{2}\left[C_{X_o}\left(\frac{u}{V_o}\right)^2 + C_{X_{pp}}\left(\frac{p' l}{2V_o}\right)^2 - 2C_{X_{pp}}\theta\,\frac{r l}{2V_o}\right] \quad (A21)$$

Dividing equation (A21) by (ρSV_o^2) and neglecting the right-hand side gives equation (46).

A variation of equation (A21), which will be needed in the derivation of the Y and Z force component equations, derives from equation (A20). In this variation the $C_{X_{pp}}$ and g terms are neglected; hence equation (A20) may be written

$$m\frac{dV}{dt} + C_{X_o}\frac{\rho SV^2}{2} = m\left[\left(\frac{\sin^2 \measuredangle \bar{V}X_o}{dt}\right)\left(\frac{dV}{dt}\right) + (V\sin \measuredangle \bar{V}X_o)\left(\frac{d \measuredangle \bar{V}X_o}{dt}\right)\right] \quad (A22)$$

If the right-hand side is neglected, equation (A22) becomes equation (A18). after the terms are rearranged and the equation is divided by mV.

Y component of force.-

$$\frac{dM_Y}{dt} + rM_X = F_{AY} + F_{GY} \quad (A23)$$

$$m\left(\frac{d\beta}{dt} + \frac{d\psi}{dt}\right) - \frac{\rho SV_o}{2}\left[\left(C_{Y_\beta} + C_{X_o}\right)\beta + C_{Y_{\dot{\beta}}}\frac{l}{2V_o}\frac{d\beta}{dt} + C_{Y_r}\frac{l}{2V_o}\frac{d\psi}{dt} + C_{Y_{\alpha p}}\frac{p_o l}{2V_o}\alpha +\right.$$

$$\left. C_{Y_{\dot{\alpha}p}}\frac{p_o l}{2V_o}\frac{l}{2V_o}\frac{d\alpha}{dt} + C_{Y_{qp}}\frac{p_o l}{2V_o}\frac{l}{2V_o}\frac{d\theta}{dt}\right] - \left(\frac{mg\sin\gamma_o}{V_o}\right)\psi$$

$$= m\left[\frac{\beta^2}{2}\left(\frac{d\beta}{dt} + r\right) + \frac{r\alpha^2}{2}\right] + \frac{\rho SV_o}{2}\frac{u}{V_o}\left[\left(C_{Y_\beta} + C_{X_o}\right)\beta -\right.$$

$$\left. C_{Y_{\dot{\alpha}p}}\frac{pl}{2V_o}\frac{l}{2V_o}\frac{d\alpha}{dt} - C_{Y_{qp}}\frac{pl}{2V_o}\frac{ql}{2V_o}\right] + \frac{\rho SV_o}{2}\frac{p' l}{2V_o}\left(C_{Y_{\alpha p}}\alpha +\right.$$

$$\left. C_{Y_{\dot{\alpha}p}}\frac{l}{2V_o}\frac{d\alpha}{dt} + C_{Y_{qp}}\frac{ql}{2V_o}\right) - \left(\frac{mg\sin\gamma_o}{V_o}\right)\psi\,\frac{u}{V_o} \quad (A24)$$

After division by (ρSV_o) and neglect of the right-hand side, equation (A24) becomes equation (47).

Z component of force.-

$$\frac{dM_Z}{dt} - qM_X = F_{AZ} + F_{GZ} \tag{A25}$$

$$m\left(\frac{d\alpha}{dt} - \frac{d\theta}{dt}\right) + \frac{\rho SV_o}{2}\left[\left(C_{Z_\alpha} - C_{X_o}\right)\alpha + C_{Z_{\dot{\alpha}}}\frac{l}{2V_o}\frac{d\alpha}{dt} + C_{Z_q}\frac{l}{2V_o}\frac{d\theta}{dt} + C_{Z_{\beta p}}\frac{p_o l}{2V_o}\beta + C_{Z_{\dot{\beta}p}}\frac{p_o l}{2V_o}\frac{l}{2V_o}\frac{d\beta}{dt} + C_{Z_{rp}}\frac{p_o l}{2V_o}\frac{l}{2V_o}\frac{d\psi}{dt}\right] - \frac{mg}{V_o}(\cos\gamma_o - \theta\sin\gamma_o)$$

$$= m\left[\frac{\alpha^2}{2}\left(\frac{d\alpha}{dt} - q\right) - \frac{q\beta^2}{2}\right] + \frac{\rho SV_o}{2}\frac{u}{V_o}\left[\left(-C_Z + C_{X_o}\right)\alpha + C_{Z_{\dot{\beta}p}}\frac{pl}{2V_o}\frac{l}{2V_o}\frac{d\beta}{dt} + C_{Z_{rp}}\frac{pl}{2V_o}\frac{rl}{2V_o}\right] - \frac{\rho SV_o}{2}\frac{p'l}{2V_o}\left(C_{Z_{\beta p}}\beta + C_{Z_{\dot{\beta}p}}\frac{l}{2V_o}\frac{d\beta}{dt} + C_{Z_{rp}}\frac{rl}{2V_o}\right) + \frac{mg}{V_o}\frac{u}{V_o}(\theta\sin\gamma_o - \cos\gamma_o) \tag{A26}$$

If it is divided by (ρSV_o) and the right-hand side is neglected, equation (A26) becomes equation (48).

X component of moment.-

$$\frac{dH_X}{dt} + qH_Z - rH_Y = M_X \tag{A27}$$

Since rotational symmetry requires that $I_Z = I_Y$, equation (A27) becomes

$$I_X\frac{d^2\varphi}{dt^2} - \frac{\rho Sl{V_o}^2}{2}\left(C_{l_p}\frac{l}{2V_o}\frac{d\varphi}{dt} + C_{l_p}\frac{p_o l}{2V_o}\frac{u}{V_o}\right) = \frac{\rho Sl{V_o}^2}{2}\left(C_{l_p}\frac{u}{V_o}\frac{p'l}{2V_o} - C_{l_p\theta}\frac{rl}{2V_o}\right) + I_X\frac{d(\theta r)}{dt} \tag{A28}$$

If it is multiplied by $\mu/\rho Sl{V_o}^2{K_X}^2$ and the right-hand side is neglected, equation (A28) becomes equation (49).

Y component of moment.-

$$\frac{dH_Y}{dt} + rH_X = M_Y \tag{A29}$$

$$I_Y \frac{d^2\theta}{dt^2} + I_{XP_o} \frac{d\psi}{dt} - \frac{\rho S l V_o^2}{2} \left(C_{m_\alpha} \alpha + C_{m_{\dot{\alpha}}} \frac{l}{2V_o} \frac{d\alpha}{dt} + C_{m_q} \frac{l}{2V_o} \frac{d\theta}{dt} + \right.$$

$$\left. C_{m_{\beta p}} \frac{p_o l}{2V_o} \beta + C_{m_{\dot{\beta} p}} \frac{p_o l}{2V_o} \frac{l}{2V_o} \frac{d\beta}{dt} + C_{m_{rp}} \frac{p_o l}{2V_o} \frac{l}{2V_o} \frac{d\psi}{dt} \right)$$

$$= -I_{XrP'} + \frac{\rho S l V_o^2}{2} \frac{u}{V_o} \left(2C_{m_\alpha} \alpha + C_{m_{\dot{\alpha}}} \frac{\dot{\alpha} l}{2V_o} + C_{m_q} \frac{q l}{2V_o} + \right.$$

$$\left. C_{m_{\beta p}} \frac{p l}{2V_o} \beta \right) + \frac{\rho S l V_o^2}{2} \frac{p' l}{2V_o} \left(C_{m_{\beta p}} \beta + C_{m_{\dot{\beta} p}} \frac{l}{2V_o} \frac{d\beta}{dt} + C_{m_{rp}} \frac{r l}{2V_o} \right) \tag{A30}$$

Through multiplication by $\mu/\rho S l V_o^2 K_Y^2$ and neglect of the right-hand side, equation (A30) becomes equation (50).

Z component of moment.-

$$\frac{dH_Z}{dt} - qH_X = M_Z \tag{A31}$$

$$I_Z \frac{d^2\psi}{dt^2} - I_{XP_o} \frac{d\theta}{dt} - \frac{\rho S l V_o^2}{2} \left(C_{n_\beta} \beta + C_{n_{\dot{\beta}}} \frac{l}{2V_o} \frac{d\beta}{dt} + C_{n_r} \frac{l}{2V_o} \frac{d\psi}{dt} + \right.$$

$$\left. C_{n_{\alpha p}} \frac{p_o l}{2V_o} \alpha + C_{n_{\dot{\alpha} p}} \frac{p_o l}{2V_o} \frac{l}{2V_o} \frac{d\alpha}{dt} + C_{n_{qp}} \frac{p_o l}{2V_o} \frac{l}{2V_o} \frac{d\theta}{dt} \right)$$

$$= I_{XqP'} + \frac{\rho S l V_o^2}{2} \frac{u}{V_o} \left(2C_{n_\beta} \beta + C_{n_{\dot{\beta}}} \frac{\dot{\beta} l}{2V_o} + C_{n_r} \frac{r l}{2V_o} + C_{n_{\alpha p}} \frac{p l}{2V_o} \alpha \right) +$$

$$\frac{\rho S l V_o^2}{2} \frac{p' l}{2V_o} \left(C_{n_{\alpha p}} \alpha + C_{n_{\dot{\alpha} p}} \frac{l}{2V_o} \frac{d\alpha}{dt} + C_{n_{qp}} \frac{q l}{2V_o} \right) \tag{A32}$$

Through multiplication by $\mu/\rho S l V_o^2 K_Z^2$ and neglect of the right-hand side, equation (A32) becomes equation (51).

Symmetrical Missile and Projectile Equations With Distance as Independent Variable

The missile and projectile equations with distance as independent variable are derived in a similar manner to those with time as independent variable. The procedure followed closely parallels the preceding section. First, the component of the exact vector equation will be written down and for those components which are resolved along the XYZ axes it will be written transformed to moving axes. Second, the kinematic relations will be substituted for certain of the dependent variables, various functions will be expanded in series, differential quantities will be transformed from time to distance as independent variable, and the resulting equation will be written down separated into first- and second-order terms with the first-order terms on the left-hand side and the second-order terms on the right-hand side.

The same relations will be used as in the previous section. In addition, the transformation from time to distance is given by equation (70). Also, since the principal application of the equations with distance as independent variable is to the aerodynamics range, the earth axes will be oriented as they normally are in the range with Z_o vertical and X_o horizontal; consequently, $\gamma_o = 0$.

It is evident from the relation, $V = dx/dt$ (eq. (67)) and the development of equation (A22) that the X-component force equation gives the relationship between time and distance. Consequently, it is proper that time should remain the independent variable in this equation. However, the form of the $x^+(t^+)$ equation (74) may not be familiar and the steps leading from equation (A22) to equation (74) will be presented in this section. It should be noted that in this development the $C_{X_{pp}}$ term is neglected.

Application of this theory to the aerodynamics range requires equations for the transverse displacement of the trajectory, y and z. The equations for y and z are derived by resolving the vector force equation (3) along the space-fixed Y_o and Z_o axes. Their development will be presented in this section. It should be noted that the only forces retained in deriving the y,z equations are lift, drag, and Magnus forces $(C_{L_\alpha}, C_{D_o}, C_{Z_{\beta_p}})$.

Time-distance equation.- If equation (A22) is multiplied by dt/mV^2 and the right-hand side is neglected, it becomes

$$\frac{dV}{V^2} = -\frac{C_{X_o}\rho S}{2m}\,dt \tag{A33}$$

Since $V = V_o$ at $t = 0$, the integral of equation (A33) may be written

$$V = V_o \left(1 + \frac{C_{X_o} \rho S V_o t}{2m}\right)^{-1} \tag{A34}$$

If equation (A34) is substituted into equation (A33) and the relation $V = dx/dt$ is used, the resulting equation may be written

$$\frac{d^2x}{dt^2} + C_{X_o} \frac{\rho S V_o^2}{2m} \left(1 + C_{X_o} \frac{\rho S V_o}{2m} t\right)^{-2} = 0 \tag{A35}$$

After multiplication by $m/\rho S V_o^2$, equation (A35) becomes equation (74).

Transverse displacement of trajectory (y, z equations).-

$$\left.\begin{aligned} m \frac{d^2y}{dt^2} &= F_X \cos \measuredangle Y_o X + F_Y \cos \measuredangle Y_o Y + F_Z \cos \measuredangle Y_o Z \\ m \frac{d^2z}{dt^2} &= F_X \cos \measuredangle Z_o X + F_Y \cos \measuredangle Z_o Y + F_Z \cos \measuredangle Z_o Z \end{aligned}\right\} \tag{A36}$$

$$\left.\begin{aligned} m \frac{d^2y}{dt^2} &= \frac{\rho S V^2}{2} \left(- C_{X_o} \psi - C_{Z_\alpha} \beta + C_{Z_{\beta p}} \frac{pl}{2V} \alpha\right) \\ m \frac{d^2z}{dt^2} &= \frac{\rho S V^2}{2} \left(C_{X_o} \theta - C_{Z_\alpha} \alpha - C_{Z_{\beta p}} \frac{pl}{2V} \beta\right) + mg \end{aligned}\right\} \tag{A37}$$

Transforming equations (A37) from time to distance, dividing by V^2, and separating into first- and second-order terms gives

$$\left.\begin{aligned} m \frac{d^2y}{dx^2} - \frac{C_{X_o} \rho S}{2} \frac{dy}{dx} + \frac{\rho S}{2} \left(C_{Z_\alpha} \beta - C_{Z_{\beta p}} \frac{p_o l}{2V_o} \alpha + C_{X_o} \psi\right) \\ = \frac{\rho S C_{Z_{\beta p}}}{2} \alpha \left(\frac{p' l}{2V_o} - \frac{pl}{2V_o} \frac{u}{V_o}\right) \\ m \frac{d^2z}{dx^2} - \frac{C_{X_o} \rho S}{2} \frac{dz}{dx} + \frac{\rho S}{2} \left(C_{Z_\alpha} \alpha + C_{Z_{\beta p}} \frac{p_o l}{2V_o} \beta - C_{X_o} \theta\right) - \frac{mg}{V_o^2} \\ = - \frac{\rho S C_{Z_{\beta p}}}{2} \beta \left(\frac{p' l}{2V_o} - \frac{pl}{2V_o} \frac{u}{V_o}\right) - 2 \frac{mg}{V_o^2} \frac{u}{V_o} \end{aligned}\right\} \tag{A38}$$

For small angles

$$\frac{dy}{dx} = \psi + \beta$$

$$\frac{dz}{dx} = \alpha - \theta$$

if lift and drag coefficients are introduced and the right-hand sides are neglected, equations (A38) become

$$\left.\begin{aligned} m\frac{d^2y}{dx^2} + \frac{\rho S}{2}\left(C_{L_\alpha}\beta - C_{Z_{\beta p}}\frac{p_o l}{2V_o}\alpha\right) &= 0 \\ m\frac{d^2z}{dx^2} + \frac{\rho S}{2}\left(C_{L_\alpha}\alpha + C_{Z_{\beta p}}\frac{p_o l}{2V_o}\beta\right) &= \frac{mg}{V_o^2} \end{aligned}\right\} \tag{A39}$$

After multiplication by $(\rho S)^{-1}$, equations (A39) become equations (75) and (76).

Y component of force.-

$$\frac{dM_Y}{dt} + rM_X = F_{AY} + F_{GY} \tag{A23}$$

$$m\left(\frac{d\beta}{dx} + \frac{d\psi}{dx}\right) - \frac{\rho S}{2}\left[\left(C_{Y_\beta} + C_{X_o}\right)\beta + C_{Y_{\dot\beta}}\frac{l}{2}\frac{d\beta}{dx} + C_{Y_r}\frac{l}{2}\frac{d\psi}{dx} + C_{Y_{\alpha p}}\frac{p_o l}{2V_o}\alpha + C_{Y_{\dot\alpha p}}\frac{p_o l}{2V_o}\frac{l}{2}\frac{d\alpha}{dx} + C_{Y_{qp}}\frac{p_o l}{2V_o}\frac{l}{2}\frac{d\theta}{dx}\right]$$

$$= m\left[\frac{\beta^2}{2}\left(\frac{d\beta}{dx} + \frac{d\psi}{dx}\right) + \frac{\alpha^2}{2}\frac{d\psi}{dx}\right] - \frac{\rho S}{2}\frac{u}{V_o}\left(C_{Y_{\alpha p}}\frac{pl}{2V_o}\alpha + C_{Y_{\dot\alpha p}}\frac{pl}{2V_o}\frac{l}{2V}\frac{d\alpha}{dt} + C_{Y_{qp}}\frac{pl}{2V_o}\frac{ql}{2V}\right) + \frac{\rho S}{2}\frac{p'l}{2V_o}\left(C_{Y_{\alpha p}}\alpha + C_{Y_{\dot\alpha p}}\frac{l}{2V}\frac{d\alpha}{dt} + C_{Y_{qp}}\frac{ql}{2V}\right) \tag{A40}$$

Dividing equation (A40) by (ρS) and neglecting the right-hand side gives equation (77).

Z component of force.-

$$\frac{dM_Z}{dt} - qM_X = F_{AZ} + F_{GZ} \tag{A25}$$

$$m\left(\frac{d\alpha}{dx} - \frac{d\theta}{dx}\right) + \frac{\rho S}{2}\left[\left(C_{Z_\alpha} - C_{X_o}\right)\alpha + C_{Z_{\dot{\alpha}}}\frac{l}{2}\frac{d\alpha}{dx} + C_{Z_q}\frac{l}{2}\frac{d\theta}{dx} + C_{Z_{\beta p}}\frac{p_o l}{2V_o}\beta + C_{Z_{\dot{\beta} p}}\frac{p_o l}{2V_o}\frac{l}{2}\frac{d\beta}{dx} + C_{Z_{rp}}\frac{p_o l}{2V_o}\frac{l}{2}\frac{d\psi}{dx}\right] - \frac{mg}{V_o^2}$$

$$= \frac{m}{2}\left[\alpha^2\left(\frac{d\alpha}{dx} - \frac{d\theta}{dx}\right) - \beta^2\frac{d\theta}{dx}\right] + \frac{\rho S}{2}\frac{u}{V_o}\left(\frac{C_{Z_{\beta p}} p l\beta}{2V_o} + C_{Z_{\dot{\beta} p}}\frac{pl}{2V_o}\frac{l}{2V}\frac{d\beta}{dt} + C_{Z_{rp}}\frac{pl}{2V_o}\frac{rl}{2V}\right) - \frac{\rho S}{2}\frac{p'l}{2V_o}\left(C_{Z_{\beta p}}\beta + C_{Z_{\dot{\beta} p}}\frac{l}{2V}\frac{d\beta}{dt} + C_{Z_{rp}}\frac{rl}{2V}\right) - \frac{2mg}{V_o^2}\frac{u}{V_o} \tag{A41}$$

Dividing equation (A41) by (ρS) and neglecting the right-hand side gives equation (78).

X component of moment.-

$$\frac{dH_X}{dt} + qH_Z - rH_Y = M_X \tag{A27}$$

With the introduction of a constant rolling moment, C_{l_o}, equation (A27) becomes

$$I_X\left(\frac{d^2\varphi}{dx^2} - \frac{\rho S x_o}{m}\frac{d\varphi}{dx}\right) - \frac{\rho S l}{2}\left(C_{l_o} + C_{l_p}\frac{l}{2}\frac{d\varphi}{dx}\right) = \frac{I_X}{V^2}\frac{d(\theta r)}{dt} - \frac{\rho S l}{2}C_{l_p}\theta\frac{rl}{2V} \tag{A42}$$

Multiplying equation (A42) by $\mu/\rho S l K_X^2$ and neglecting the right-hand side gives equation (79).

Y component of moment.-

$$\frac{dH_Y}{dt} + rH_X = M_Y \tag{A29}$$

$$I_Y \frac{d^2\theta}{dx^2} + I_X \frac{p_o}{V_o} \frac{d\psi}{dx} - \frac{\rho S l}{2} \left[C_{m_\alpha} \alpha + C_{m_{\dot{\alpha}}} \frac{l}{2} \frac{d\alpha}{dx} + \left(C_{m_q} + \frac{2 I_Y C_{X_o}}{m l^2} \right) \frac{l}{2} \frac{d\theta}{dx} + C_{m_{\beta p}} \frac{p_o l}{2V_o} \beta + C_{m_{\dot{\beta} p}} \frac{p_o l}{2V_o} \frac{l}{2} \frac{d\beta}{dx} + C_{m_{rp}} \frac{p_o l}{2V_o} \frac{l}{2} \frac{d\psi}{dx} \right]$$

$$= \frac{I_X}{l} \frac{d\psi}{dx} \left(\frac{p l}{V_o} \frac{u}{V_o} - \frac{p' l}{V_o} \right) - \frac{\rho S l}{2} \frac{p l}{2V_o} \frac{u}{V_o} \left(C_{m_{\beta p}} \beta + C_{m_{\dot{\beta} p}} \frac{l}{2V} \frac{d\beta}{dt} + C_{m_{rp}} \frac{r l}{2V} \right) + \frac{\rho S l}{2} \frac{p' l}{2V_o} \left(C_{m_{\beta p}} + C_{m_{\dot{\beta} p}} \frac{l}{2V} \frac{d\beta}{dt} + C_{m_{rp}} \frac{r l}{2V} \right) \qquad (A43)$$

If equation (A43) is multiplied by $\mu/\rho S l K_Y^2$ and the right-hand side is neglected, it becomes equation (80).

Z component of moment.-

$$\frac{dH_Z}{dt} - q H_X = M_Z \qquad (A31)$$

$$I_Z \frac{d^2\psi}{dx^2} - \frac{I_Z C_{X_o} \rho S}{2m} \frac{d\psi}{dx} - I_X \frac{p_o}{V_o} \frac{d\theta}{dx} - \frac{\rho S l}{2} \left(C_{n_\beta} \beta + C_{n_{\dot{\beta}}} \frac{l}{2} \frac{d\beta}{dx} + C_{n_r} \frac{l}{2} \frac{d\psi}{dx} + C_{n_{\alpha p}} \frac{p_o l}{2V_o} \alpha + C_{n_{\dot{\alpha} p}} \frac{p_o l}{2V_o} \frac{l}{2} \frac{d\alpha}{dx} + C_{n_{qp}} \frac{p_o l}{2V_o} \frac{l}{2} \frac{d\theta}{dx} \right)$$

$$= -I_X \frac{p}{V_o} \frac{d\theta}{dx} \frac{u}{V_o} + \frac{I_X}{l^2} \frac{q l}{V} \frac{p' l}{V_o} - \frac{\rho S l}{2} \frac{p l}{2V_o} \frac{u}{V_o} \left(C_{n_{\alpha p}} \alpha + C_{n_{\dot{\alpha} p}} \frac{l}{2V} \frac{d\alpha}{dx} + C_{n_{qp}} \frac{q l}{2V} \right) + \frac{\rho S l}{2} \frac{p' l}{2V_o} \left(C_{n_{\alpha p}} \alpha + C_{n_{\dot{\alpha} p}} \frac{\dot{\alpha} l}{2V} + C_{n_{qp}} \frac{q l}{2V} \right) \qquad (A44)$$

If equation (A44) is multiplied by $\mu/\rho S l K_Z^2$ and the right-hand side is neglected, it becomes equation (81).

APPENDIX B

MODIFICATION OF THE SYMMETRICAL MISSILE AND PROJECTILE EQUATIONS TO INCLUDE THE EFFECTS OF SMALL AERODYNAMIC ASYMMETRIES

Forces and moments due to aerodynamic asymmetries are not independent of the roll angle as are the principal aerodynamic forces and moments. Hence, they will be defined as components along the $\bar{1}$, $\bar{2}$, $\bar{3}$ body-fixed axes as follows:

$$\left.\begin{aligned} F_{1,2,3} &= C_{F_{1,2,3}} \frac{\rho V^2 S}{2} \\ M_{1,2,3} &= C_{M_{1,2,3}} \frac{\rho V^2 S l}{2} \end{aligned}\right\} \tag{B1}$$

The components of C_F and C_M along the X,Y,Z axes are given by

$$\left.\begin{aligned} C_{X_o} \text{ (due to asymmetries)} &= C_{F_1} \\ C_{Y_o} &= C_{F_2}\cos\varphi - C_{F_3}\sin\varphi \\ C_{Z_o} &= C_{F_3}\cos\varphi + C_{F_2}\sin\varphi \\ C_{l_o} &= C_{M_1} \\ C_{m_o} &= C_{M_2}\cos\varphi - C_{M_3}\sin\varphi \\ C_{n_o} &= C_{M_3}\cos\varphi + C_{M_2}\sin\varphi \end{aligned}\right\} \tag{B2}$$

It is assumed that the coefficients, $C_{F_{1,2,3}}$ and $C_{M_{1,2,3}}$ are first-order quantities, in contrast to all other aerodynamic coefficients. This is evidently the case if the asymmetries are small. Also, if C_F or C_M are due to control-surface deflections, δ, they will have the form $C_{F_\delta}\delta$ and, hence, the requirement that they be small is equivalent to assuming that the control surface deflections be small. Consequently the variable velocity, V^2, may be replaced by the constant, V_o^2, since the term $2C\ u/V_o$ will be second-order and hence negligible insofar as the linearized theory is concerned.

Certain of the models tested in the aerodynamics range have small asymmetries of the type just described. It is desirable, therefore, to derive the modifications to the equations of motion which are required to account for the effects of the asymmetries. Two of the components have already been accounted for. The X force component, F_1, is a constant and its coefficient, C_{F_1}, may be considered to be included in the axial drag coefficient, C_{X_o}. The X moment component, M_1, has been included in the $\varphi(x^+)$ equation by the l_o term. It should be noted that in this particular equation the C_{M_1} coefficient is not required to be small

(first-order). The remaining components affect the $\Delta(x^+)$ and $\xi(x^+)$ motions[1] and the changes to the equations involved will be taken up in this order.

$\Delta(x^+)$ Equation

The modification to the equations involved in the development of the $\Delta(x^+)$ equation are listed as follows: Add to the right-hand side of the equation listed the term following.

(A37), first equation F_{Y_o}

(A37), second equation F_{Z_o}

(A39), first equation $\frac{\rho S}{2} C_{Y_o}$

(A39), second equation $\frac{\rho S}{2} C_{Z_o}$

(75) $\frac{C_{Y_o}}{2}$

(76) $\frac{C_{Z_o}}{2}$

(84) $f_o e^{i\varphi}$

where $f_o = \frac{C_{F_2} + iC_{F_3}}{2}$

(91) $f_o e^{i\varphi_o}\left[\frac{ix^+}{\nu} - \frac{(e^{i\nu x^+} - 1)}{\nu^2}\right]$

If νx^+ is small, the added term in equation (91) is approximated by

(91) $\frac{f_o e^{i\varphi_o}(x^+)^2}{2}\left(1 + \frac{i\nu x^+}{3} + \ldots\right)$

Hence, if the missile does not roll ($\nu = 0$), the force due to asymmetry will cause the transverse displacement, Δ, to increase with the square of the distance, x^+.

$\xi(x^+)$ Equation

The modifications to the equations involved in the development of the $\xi(x^+)$ equation are listed as follows: Add to the right-hand side of the equation listed the term following.

[1]The $\eta(x^+)$ motion is affected also but the changes required to the $\eta(x^+)$ equations will be omitted from this treatment since the $\xi(x^+)$ motion is the one ordinarily reduced in the analysis of flight data.

(A40) $\frac{\rho S C_{Y_o}}{2}$

(A41) $\frac{\rho S C_{Z_o}}{2}$

(A43) $\frac{\rho S l C_{m_o}}{2}$

(A44) $\frac{\rho S l C_{n_o}}{2}$

(77) $\frac{C_{Y_o}}{2}$

(78) $\frac{C_{Z_o}}{2}$

(80) $\frac{\mu C_{m_o}}{2K_Y^2}$

(81) $\frac{\mu C_{n_o}}{2K_Z^2}$

(85) $f_o e^{i\varphi}$

where $f_o = \frac{C_{F_2} + iC_{F_3}}{2}$

(87) $m_o e^{i\varphi}$

where $m_o = \frac{\mu(C_{M_3} - iC_{M_2})}{2K_Z^2}$

The aerodynamic asymmetries will change the values of the coefficients d_1 and d_3 of the integrated $\xi(x^+)$ equation (93); the following term should be added to $P(S)$, defined by equation (110):

$$z_\xi^{-1} S(Sf_o - m_\eta f_o - z_\eta m_o) \int_o^{x^+} e^{(i\varphi - S\zeta)} d\zeta$$

where ζ is the variable of integration replacing x^+ in functions of x^+ appearing in the integrand and φ is considered to be a function of ζ. The added term may be readily derived in the integration of the

simultaneous differential equations (85) and (87) if the integration is carried out by the Laplace transform and use is made of the convolution of the "natural" frequency and the "asymmetrical forcing function." A clear explanation of the operational mathematics involved is given in chapters I and II of reference (16) (particularly section 14 on the convolution; Churchill states that the convolution is also known as the Faltung integral).

It may be of interest to note that if the roll rate with respect to distance is constant, that is, if

$$\varphi = \nu x^{+} + \varphi_{o}$$

then the term added to P(S) becomes

$$\frac{S(Sf_{o} - m_{\eta}f_{o} - z_{\eta}m_{o})e^{i\varphi_{o}}\left[e^{(i\nu-S)x^{+}} - 1\right]}{z_{\xi}(i\nu - S)}$$

Hence if $\nu = S_{1I}$ or S_{2I} then $(i\nu - S) = S_{1R}$ or S_{2R}, ordinarily a small value. Consequently, the value of the corresponding term will become very large. In other words, the oscillation experiences a divergent resonance as the roll rate approaches the pitch or yaw rate.[2]

[2]This result has been noted elsewhere in the literature; for example, see reference 4.

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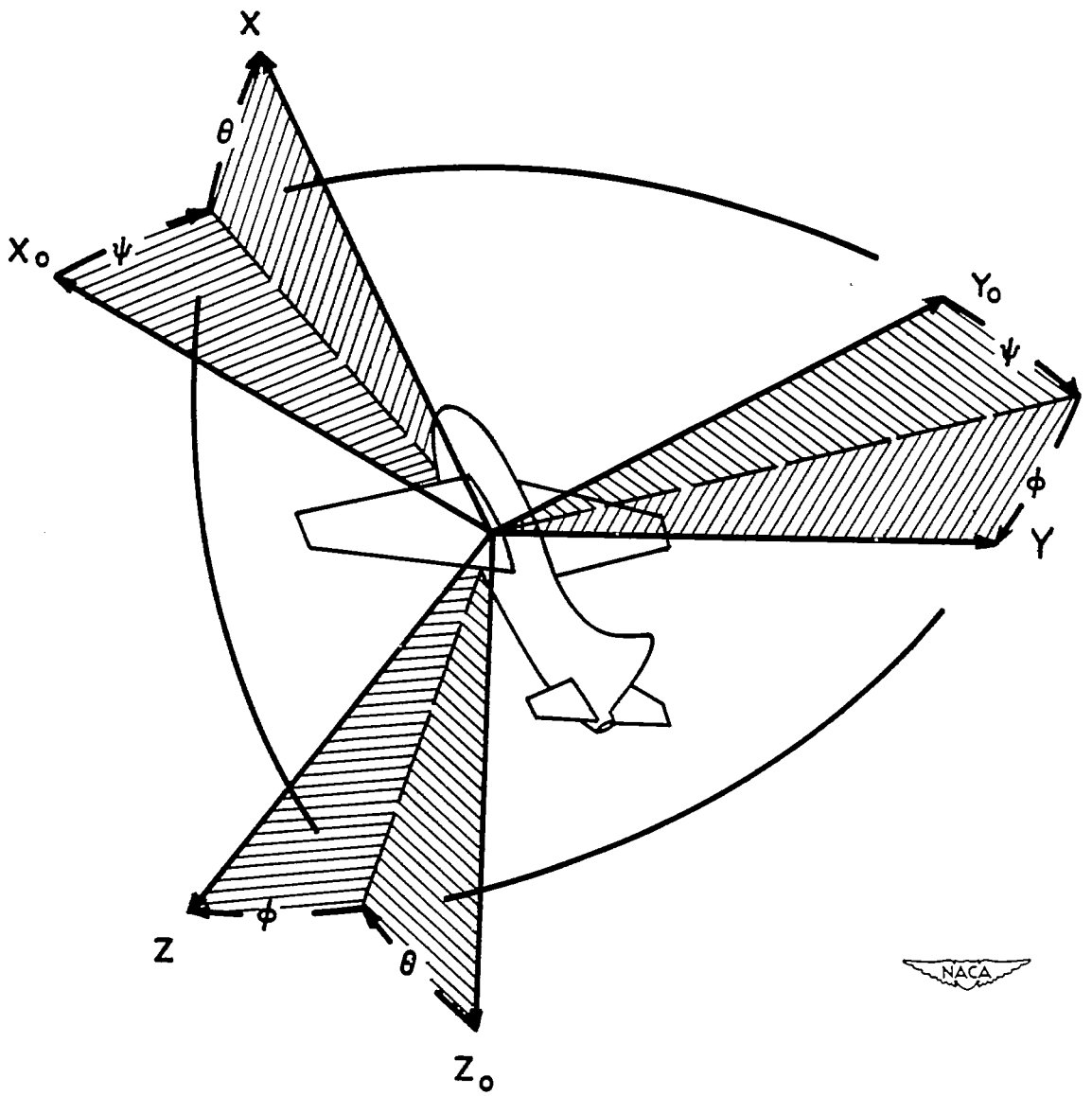
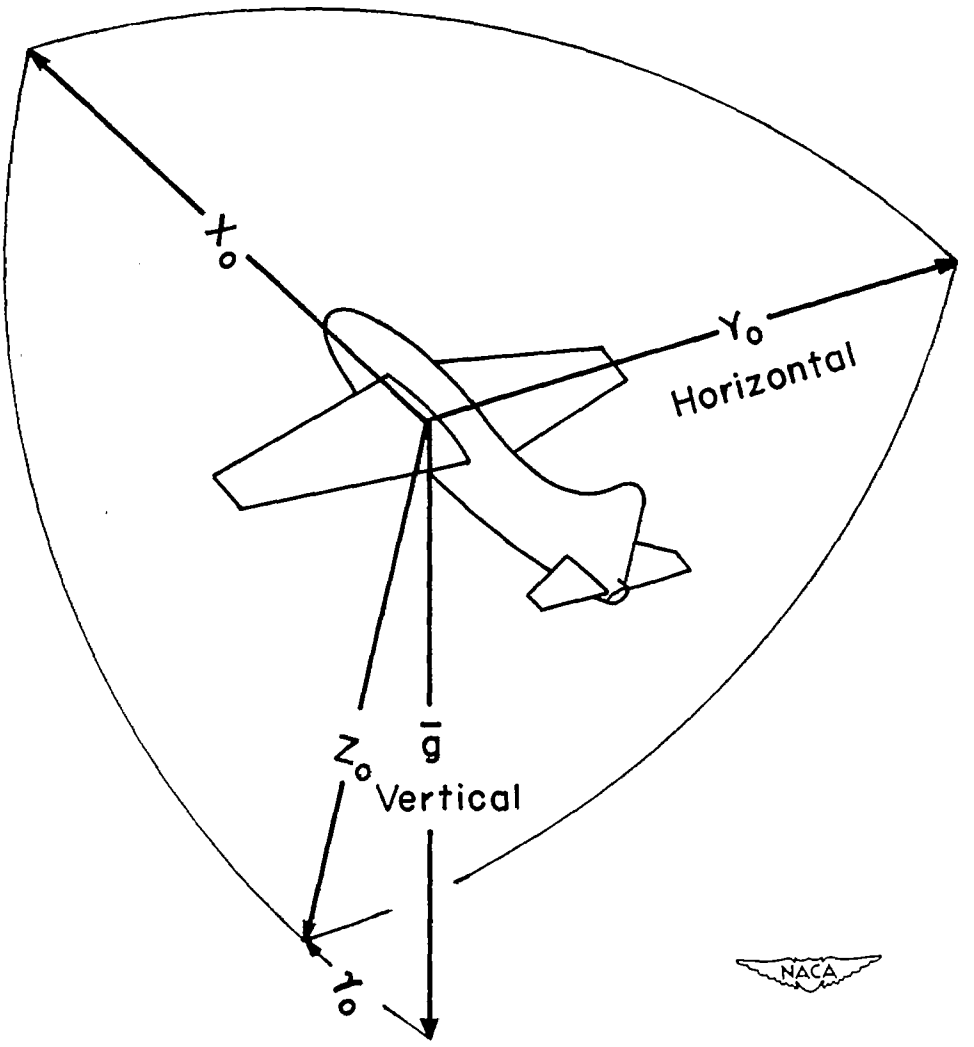


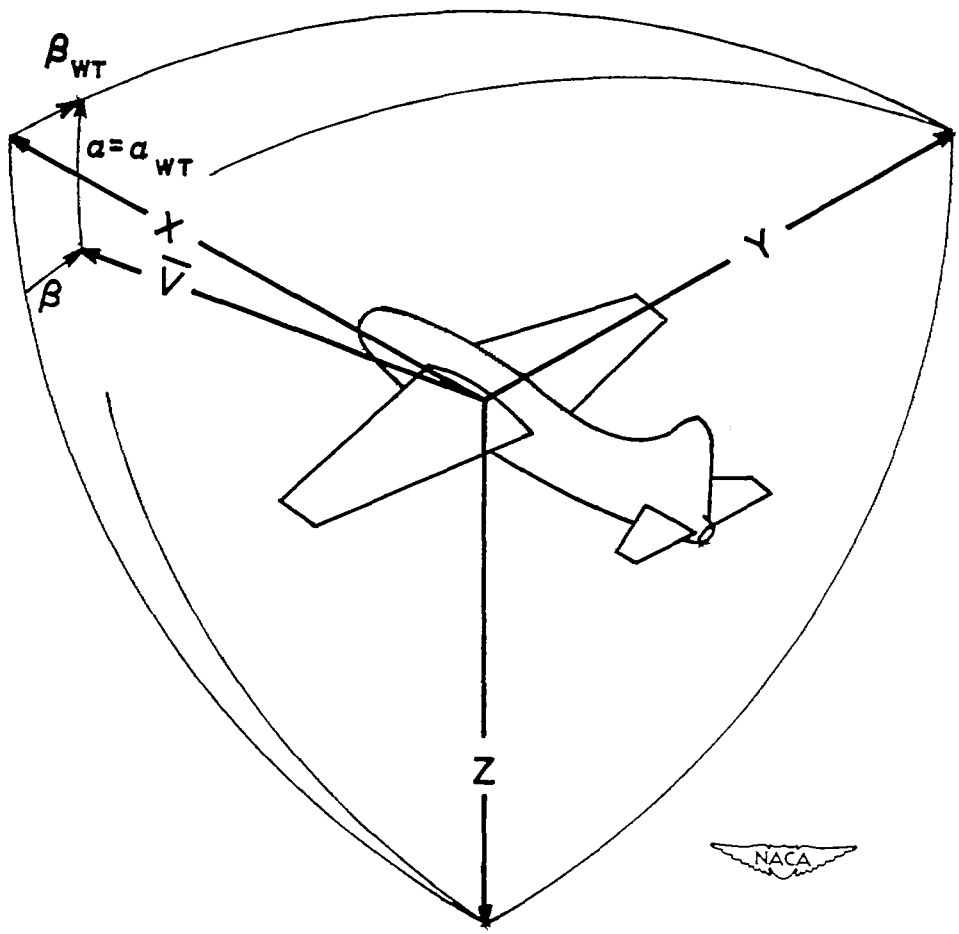
(a) Orientation of stability axes with respect to earth axes.

Figure 1.- Space geometry for aircraft.



(b) Orientation of gravity vector with respect to earth axes.

Figure 1.- Continued.



(c) Orientation of velocity vector with respect to stability axes.

Figure 1.- Continued.

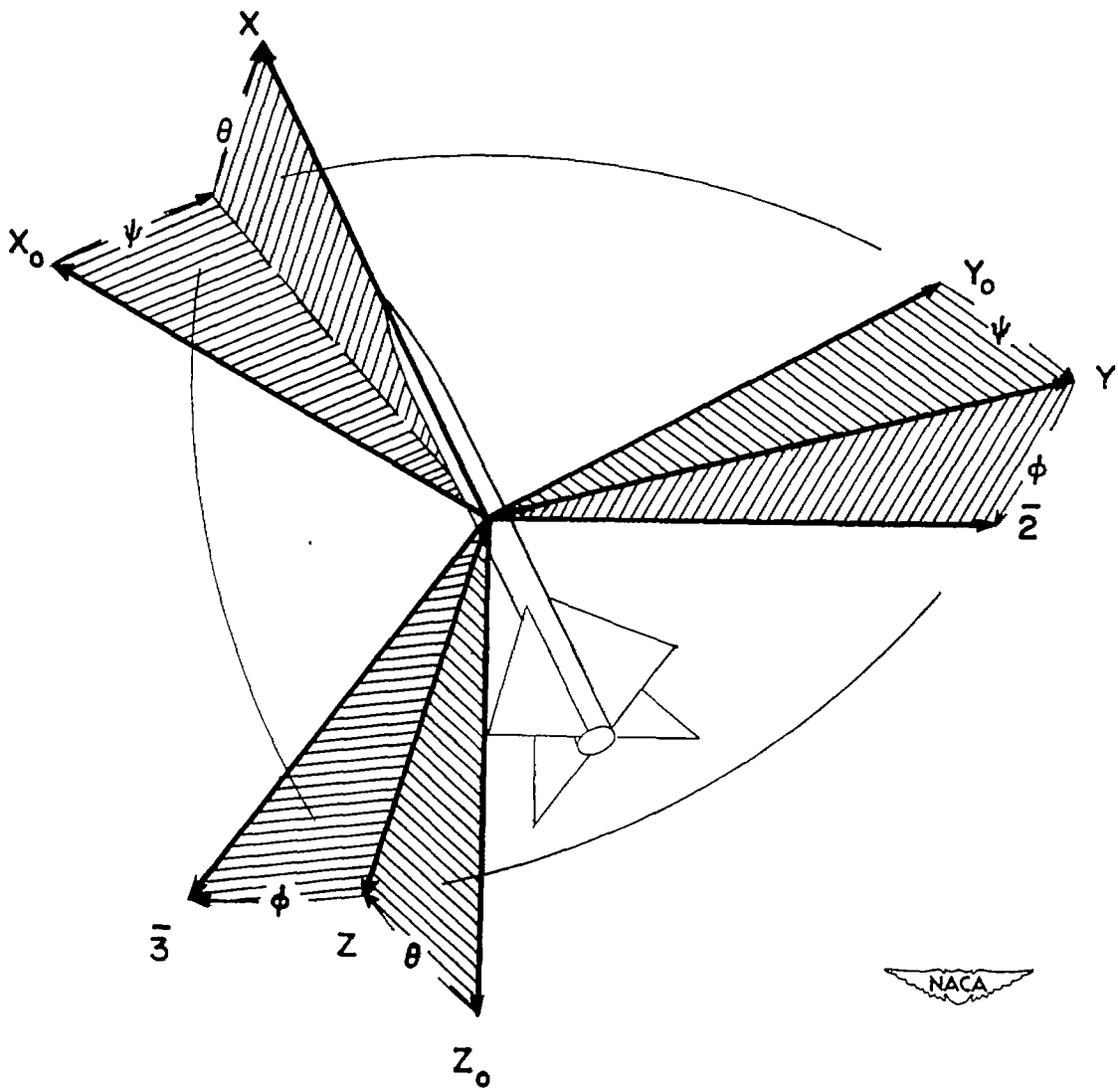


Figure 2.- Space geometry for projectiles and missiles with 90° rotational and mirror symmetry. Orientation of body axes and pseudo-stability axes with respect to earth axes.