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INFLUENCE OF SHEAR DEFORMATION OF THE CROSS SECTION ON TORSIONAL FREQUENCIES OF BOX BEAMS

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SUMMARY

An exact analysis has been carried out on the torsional vibrations of a four-flange box beam with cross sections which can change shape because the stiffness of the bulkheads is finite. The effect of shear deformation of the cross section on the torsional frequencies is illustrated by numerical calculations. An approximate method for quickly estimating the effects of bulkhead shear stiffness on the torsional frequencies of box beams has been devised.

INTRODUCTION

In an experimental investigation described in reference 1, analyses based on the assumption that the changes in the shape of the cross section are negligible were found to be completely inadequate in predicting the experimental torsional frequencies of a thin-walled tube of rectangular cross section. One form of cross-sectional distortion - due to local deflections normal to the surface of the covers and webs - was investigated in reference 2. The magnitude of this effect on torsional frequencies, however, was shown to be practically negligible for the particular test beam and frequency range investigated in reference 1. Another type of cross-sectional distortion - due to overall changes of shape of a shearing nature - could be an important influence if the bulkheads were not rigid. Although the effect of bulkhead shear deformation has been considered in some analyses (see, for example, ref. 3), no investigations of the importance of this effect on torsional frequencies seem to exist.

The importance of the shear stiffness of bulkheads in torsional vibrations is assessed in the present paper by means of an analysis of a four-flange box beam containing bulkheads with finite shear stiffness. Frequency equations are derived for torsional vibrations of a uniform free-free beam, and numerical results obtained by use of these equations are shown. A set of curves based on an approximate solution, from which the effects of cross-sectional shear deformation can be quickly estimated, is also presented.

SYMBOLS

A_F	area of flange
A, B	parameters defined in equations (22)
C	inertial coupling constant, $\dfrac{I_z - I_y}{I_z + I_y}$
E	Young's modulus of elasticity
G	shear modulus of elasticity
G_e	effective shear modulus of bulkheads
I_y	mass moment of inertia per unit length about y-axis
I_z	mass moment of inertia per unit length about z-axis
I_p	mass polar moment of inertia per unit length, $I_y + I_z$
J	torsional-stiffness constant
K	restraint-of-warping parameter (see eqs. (22))
L	half-length of beam
M	parameter, $\dfrac{2}{1 - C^2}\left(1 - \dfrac{AC}{B}\right)$
P	frequency parameter for special case (see eq. (A9))
S	bulkhead stiffness parameter (see eqs. (22))
T	maximum kinetic energy
U	maximum strain energy
a_0,a_i,b_i,c_i	Fourier series coefficients
a	half-depth of beam
b	half-width of beam

i, m, n	integers
k_T	frequency coefficient, $\omega_T \sqrt{\frac{I_p L^2}{GJ}}$
k_{b_0}	frequency coefficient for uniform shear mode
t_C	thickness of cover sheet
t_W	thickness of web
u, v, w	displacement of flange in x-direction, y-direction, and z-direction
x, y, z	coordinates defined in figure 1
ϵ_F	longitudinal strain in flange
γ_C	shear strain in cover
γ_W	shear strain in web
γ_b	shear strain in bulkhead
θ	average angle of twist
ω_T	natural torsional frequency of four-flange box beam
ω_{b_0}	frequency of cross-sectional uniform shear mode
ρ	mass density of the actual beam
δ_{0i}	Kronecker delta (1 if $i = 0$; 0 if $i \neq 0$)

Subscripts:

i, n	integers
m	mode number
rig	rigid bulkheads

THEORETICAL ANALYSIS

Basic Equations

In order to investigate the influence of shear stiffness of bulkheads on the torsional frequencies of uniform box beams, it is convenient to make an idealization of the actual structure. In reference 4, the idealization of the webs and covers of a beam into flanges and a shear-carrying sheet was successfully used in an analysis of a beam with rigid bulkheads. In the present analysis, the same idealization is used for a beam which contains nonrigid bulkheads. For simplicity of calculation, the cross section of the four-flange box beam is assumed to be doubly symmetrical.

Assumptions.- In the present analysis the following assumptions are made:

(1) The flanges of the beam carry only normal stresses.

(2) The sheets connecting the flanges carry only shear.

(3) The beam contains continuously distributed, independently acting bulkheads that have finite shear stiffness in their planes but are entirely free to warp out of their planes.

(4) The influence of longitudinal inertia is negligible.

In accordance with the foregoing assumptions and the double symmetry of the cross section, the displacement of any point on the cross section can be defined in terms of u, v, and w, the displacements of one of the flanges in the x-, y-, and z-directions, respectively. (See fig. 1.)

Strain relations.- The longitudinal strain in the flanges ϵ_F and the shear strains in the covers, webs, and bulkheads $\left(\gamma_C,\ \gamma_W,\right.$ and γ_b, respectively$\left.\right)$ are given in terms of the displacements u, v, and w as

$$\epsilon_F = \frac{du}{dx} \tag{1}$$

$$\gamma_C = \frac{dv}{dx} + \frac{u}{b} \tag{2}$$

$$\gamma_W = \frac{dw}{dx} - \frac{u}{a} \tag{3}$$

$$\gamma_b = \frac{v}{a} - \frac{w}{b} \tag{4}$$

where a and b are the half-depth and the half-width of the beam, respectively.

If the bulkheads were assumed to be rigid in their own planes, v and w would each be proportional to the twist of the cross section in such a manner that $\gamma_b = 0$. The distortions of the structure could then be defined as in reference 4 in terms of u and the twist of the cross section. Since the bulkheads have a finite shear stiffness, the cross section is allowed to distort and v and w are considered separately. It is then convenient to define an average twist θ as

$$\theta = \frac{1}{2}\left(\frac{v}{a} + \frac{w}{b}\right) \tag{5}$$

From equations (4) and (5), the displacements v and w can now be written in terms of θ and γ_b as

$$v = a\left(\theta + \frac{\gamma_b}{2}\right) \tag{6}$$

$$w = b\left(\theta - \frac{\gamma_b}{2}\right) \tag{7}$$

Energy relations.- For the four-flange beam vibrating in a natural mode, the maximum strain energy U is

$$U = 2Gbt_C \int_{-L}^{L} \gamma_C{}^2 dx + 2Gat_W \int_{-L}^{L} \gamma_W{}^2 dx + 2EA_F \int_{-L}^{L} \epsilon_F{}^2 dx + 2G_e ab \int_{-L}^{L} \gamma_b{}^2 dx \tag{8}$$

where L is the half-length of the beam, t_C is the cover-sheet thickness, t_W is the web thickness, A_F is the area of a flange, and G_e is the effective shear modulus of the bulkheads.

The first two terms of equation (8) represent the energy due to the shear strains in the covers and webs of the beam; these are the only kinds of energy considered in elementary theories. The third term represents the contribution of the restraint of warping. The last term represents the contribution of the shear strain in the bulkheads.

The strain energy expressed by equation (8) can now be expressed in terms of u, γ_b, and θ as

$$U = 2Gbt_C \int_{-L}^{L}\left(a\,\frac{d\theta}{dx} + \frac{a}{2}\,\frac{d\gamma_b}{dx} + \frac{u}{b}\right)^2 dx + 2Gat_W \int_{-L}^{L}\left(b\,\frac{d\theta}{dx} - \frac{b}{2}\,\frac{d\gamma_b}{dx} - \frac{u}{a}\right)^2 dx +$$

$$2EA_F \int_{-L}^{L}\left(\frac{du}{dx}\right)^2 dx + 2G_e ab \int_{-L}^{L} \gamma_b{}^2 dx \tag{9}$$

In the calculation of the maximum kinetic energy, the inertial properties of the actual beam are used and the following assumption is made: At any point in the cross section, the displacement v in the y-direction is proportional to z, and the displacement w in the z-direction is proportional to y. (See fig. 1.) Thus the maximum kinetic energy can be written as

$$T = \frac{\omega_T{}^2}{2} \int_{-L}^{L} \iint_R \rho\left[\left(\frac{w}{b}\,y\right)^2 + \left(\frac{v}{a}\,z\right)^2\right] dz\,dy\,dx \tag{10}$$

where ω_T is a natural torsional frequency, ρ is the mass density of any point in the actual beam, and R is the region in the plane of the cross section containing all of the mass elements of the beam.

When equations (6) and (7) are substituted into equation (10), the expression for the maximum kinetic energy can be written in terms of γ_b and θ as follows:

$$T \doteq \frac{\omega_T{}^2}{2}\, I_p \int_{-L}^{L}\left(\theta^2 + \frac{1}{4}\,\gamma_b{}^2 - C\theta\gamma_b\right) dx \tag{11}$$

where

$$C = \frac{I_z - I_y}{I_z + I_y} \tag{12}$$

The quantity I_p is the mass polar moment of inertia per unit length, and I_y and I_z are the mass moments of inertia per unit length about the y- and z-axes; these quantities are, of course, based on the assumption of uniform spanwise mass distribution in the actual structure.

The constant C can be looked upon as an inertial coupling coefficient between the rotational and cross-sectional shear motions.

Method of Analysis

The following analysis is based on the principle that a natural mode of vibration must satisfy the variational equation

$$\delta(U-T) = 0 \tag{13}$$

where the variation is taken with respect to the distortions u, θ, and γ_b and where these displacements satisfy the geometric boundary conditions. Application of the techniques of the calculus of variations to equation (13), with u, θ, and γ_b as the independent variables, yields the differential equations and natural boundary conditions given in the appendix. It is more convenient, however, to carry out the exact solution of equation (13) by means of the Rayleigh-Ritz procedure.

Symmetrical vibrations of a free-free beam.- Appropriate assumptions for the distortions u, θ, and γ_b of a free-free beam in a symmetrical mode of vibration are

$$u = a_0 x + \sum_{n=1,2,3}^{\infty} a_n \sin \frac{n\pi x}{L} \tag{14}$$

$$\theta = \sum_{n=0,1,2}^{\infty} b_n \cos \frac{n\pi x}{L} \tag{15}$$

$$\gamma_b = \sum_{n=0,1,2}^{\infty} c_n \cos \frac{n\pi x}{L} \tag{16}$$

The choice of the particular trigonometric functions used in equations (14), (15), and (16) was guided by consideration of the orthogonality required for the simplification of the strain-energy expression. The linear term $a_0 x$ is included in the expression for u in order to allow the deflection of the tip of the beam to be unrestricted.

Substitution of equations (14), (15), and (16) into equations (9) and (11) yields, after subtraction,

$$U - T = 2Gbt_C \int_{-L}^{L}\left[-a \sum_{n=0,1,2}^{\infty} b_n\left(\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L} - \frac{a}{2}\sum_{n=0,1,2}^{\infty} c_n\left(\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L} + \right.$$

$$\left.\frac{1}{b}\sum_{n=1,2,3}^{\infty} a_n \sin\frac{n\pi x}{L} + \frac{a_0}{b} x\right]^2 dx +$$

$$2Gat_W \int_{-L}^{L}\left[-b \sum_{n=0,1,2}^{\infty} b_n\left(\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L} + \frac{b}{2}\sum_{n=0,1,2}^{\infty} c_n\left(\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L} - \right.$$

$$\left.\frac{1}{a}\sum_{n=1,2,3}^{\infty} a_n \sin\frac{n\pi x}{L} - \frac{a_0}{a} x\right]^2 dx + 2EA_F \int_{-L}^{L}\left[a_0 + \right.$$

$$\left.\sum_{n=1,2,3}^{\infty} a_n\left(\frac{n\pi}{L}\right)\cos\frac{n\pi x}{L}\right]^2 dx + 2G_e ab \int_{-L}^{L}\left(\sum_{n=0,1,2}^{\infty} c_n \cos\frac{n\pi x}{L}\right)^2 dx -$$

$$\frac{1}{2} I_p\omega_T^2 \int_{-L}^{L}\left(\sum_{n=0,1,2}^{\infty} b_n \cos\frac{n\pi x}{L}\right)^2 dx -$$

$$\frac{1}{8} I_p\omega_T^2 \int_{-L}^{L}\left(\sum_{n=0,1,2}^{\infty} c_n \cos\frac{n\pi x}{L}\right)^2 dx +$$

$$\frac{1}{2} I_p\omega_T^2 C \int_{-L}^{L}\left(\sum_{n=0,1,2}^{\infty} b_n \cos\frac{n\pi x}{L}\right)\left(\sum_{n=0,1,2}^{\infty} c_n \cos\frac{n\pi x}{L}\right) dx \qquad (17)$$

Differentiating $U - T$ with respect to a_0, the a_i's, the b_i's, and the c_i's and setting the respective results equal to zero yields

$$\sum_{n=1,2,3}^{\infty}\left[\frac{L}{a} B \frac{(-1)^n}{n\pi}\frac{a_n}{b} + A(-1)^n b_n - \frac{1}{2} B(-1)^n c_n\right] - \frac{L^2}{ab}\left(\frac{B}{3} + K^2\right) a_0 = 0 \qquad (18)$$

$$\left[K^2(i\pi)^2 + B\right]\frac{L}{a}\,\frac{a_i}{b} + A(i\pi)b_i - \frac{1}{2}B(i\pi)c_i - \frac{2L^2}{ab}B\,\frac{(-1)^i}{i\pi}a_0 = 0$$

$$(i = 1,2,3, \ldots) \qquad (19)$$

$$\frac{L}{a}A(i\pi)\,\frac{a_i}{b} - \left[k_T^2\left(1 + \delta_{0i}\right) - B(i\pi)^2\right]b_i + \left[\frac{1}{2}Ck_T^2\left(1 + \delta_{0i}\right) - \frac{1}{2}A(i\pi)^2\right]c_i - \frac{2L^2}{ab}A\left(1 - \delta_{0i}\right)(-1)^i a_0 = 0 \qquad (i = 0,1,2, \ldots) \qquad (20)$$

$$\frac{1}{2}\,\frac{L}{a}B(i\pi)\,\frac{a_i}{b} - \left[\frac{1}{2}Ck_T^2\left(1 + \delta_{0i}\right) - \frac{1}{2}A(i\pi)^2\right]b_i + \frac{1}{4}\left[k_T^2\left(1 + \delta_{0i}\right) - B(i\pi)^2 - S\left(1 + \delta_{0i}\right)\right]c_i - \frac{L^2}{ab}B(-1)^i\left(1 - \delta_{0i}\right)a_0 = 0$$

$$(i = 0,1,2, \ldots) \qquad (21)$$

where

$$\left.\begin{aligned}
k_T &= \omega_T\sqrt{\frac{I_p L^2}{GJ}} \qquad & J &= \frac{16a^2b^2}{\frac{a}{t_W} + \frac{b}{t_C}} \\
K &= \sqrt{\frac{EA_F}{4GL^2}\left(\frac{a}{t_W} + \frac{b}{t_C}\right)} \qquad & S &= \frac{G_e\left(\frac{a}{t_W} + \frac{b}{t_C}\right)L^2}{Gab} \\
A &= \frac{t_W t_C\left(\frac{b^2}{t_C^2} - \frac{a^2}{t_W^2}\right)}{4ab} & & \\
B &= \frac{t_W t_C\left(\frac{a}{t_W} + \frac{b}{t_C}\right)^2}{4ab} = \frac{1}{2}\left(1 + \sqrt{1 + 4A^2}\right) & & \\
\delta_{0i} &= 0 \qquad (\text{for} \quad i \neq 0) & & \\
\delta_{00} &= 1 & &
\end{aligned}\right\} \qquad (22)$$

The unknown natural frequency is contained in the parameter k_T which is in common use in torsional vibration analyses. The quantity J, which appears in the expression for k_T, is the well-known torsional stiffness constant. The parameter K is associated with the effects of restraint of warping, whereas the parameter S is associated with the effects of bulkhead stiffness. The parameters A and B are simply geometrical properties of the four-flange box beam.

Examination of equations (18) to (21) shows that the coefficients b_0 and c_0 appear only in equations (20) and (21), and then only for $i = 0$. When $i = 0$, equations (20) and (21) reduce to

$$\left.\begin{aligned} 2k_T{}^2 b_0 - Ck_T{}^2 c_0 &= 0 \\ 2Ck_T{}^2 b_0 - \left(k_T{}^2 - S\right)c_0 &= 0 \end{aligned}\right\} \tag{23}$$

The condition for a nontrivial solution of b_0 and c_0 gives the frequency equation

$$k_T{}^2\left[k_T{}^2\left(1 - C^2\right) - S\right] = 0 \tag{24}$$

From equations (15) and (16), it can be seen that a given value of b_0 corresponds to a rigid-body rotation of the beam, whereas a given value of c_0 corresponds to a bulkhead shear distortion which is uniform along the length of the beam. Equation (24), therefore, gives the frequency coefficient for a uniform cross-sectional shear mode,

$$k_{b_0} = \sqrt{\frac{S}{1 - C^2}} \tag{25}$$

in addition to the frequency coefficient $k_T = 0$ for the rigid-body torsional mode. The frequency coefficient k_{b_0} for the uniform cross-sectional shear mode is shown to be a pertinent parameter in the evaluation of the influence of the shear stiffness of bulkheads on torsional frequencies.

Now consider the remaining equations (18) to (21). Substitution of equation (19) into equation (18) gives

$$a_0 + \frac{1}{L} \sum_{n=1,2,3}^{\infty} (-1)^n (n\pi) a_n = 0 \qquad (K^2 \neq 0) \qquad (26)$$

By solving equations (19), (20), and (21) simultaneously for a_i in terms of a_0 and then substituting the results into equation (26), the following equation results if it is required that a_0 is not equal to zero:

$$1 + 2 \sum_{n=1,2,3}^{\infty} \frac{\left(k_{b_0}{}^2 - \frac{k_T{}^2}{1 - C^2}\right)(n\pi)^2 - k_T{}^2\left(k_{b_0}{}^2 - k_T{}^2\right)}{\frac{K^2}{1 - C^2}(n\pi)^6 + K^2\left(k_{b_0}{}^2 - k_T{}^2 M\right)(n\pi)^4 + \left[k_{b_0}{}^2 - \frac{k_T{}^2}{1 - C^2} - \frac{K^2 k_T{}^2}{B}\left(k_{b_0}{}^2 - k_T{}^2\right)\right](n\pi)^2 - k_T{}^2\left(k_{b_0}{}^2 - k_T{}^2\right)} = 0$$

$$(K^2 \neq 0) \qquad (27)$$

where

$$M = \frac{2}{1 - C^2}\left(1 - \frac{AC}{B}\right) \qquad (28)$$

Equation (27) is the frequency equation for the symmetrical torsional vibration of a four-flange box beam when the influence of bulkhead shear deformation is included.

Antisymmetrical vibrations of a free-free beam.- For a free-free beam vibrating in an antisymmetrical torsional mode, appropriate assumptions for the distortions are

$$u = a_0 + \sum_{n=1,3,5}^{\infty} a_n \cos \frac{n\pi x}{2L} \qquad (29)$$

$$\theta = \sum_{n=1,3,5}^{\infty} b_n \sin \frac{n\pi x}{2L} \qquad (30)$$

$$\gamma_b = \sum_{n=1,3,5}^{\infty} c_n \sin \frac{n\pi x}{2L} \qquad (31)$$

As in the case of the symmetrical modes of vibration, the choice of the particular trigonometric functions was guided by considerations of the orthogonality required for the simplification of the strain-energy expression. The constant term a_0 in equation (29) is necessary to allow sufficient freedom for the beam to warp.

By substituting the expressions for u, θ, and γ_b (eqs. (29) to (31)) into equations (9) and (11) and by following a procedure similar to that described for the symmetrical modes of vibration, the following frequency equation for the antisymmetrical mode of vibration is derived:

$$\sum_{n=1,3,5}^{\infty} \frac{\left(k_{b_0}{}^2 - \frac{k_T{}^2}{1 - C^2}\right)\left(\frac{n\pi}{2}\right)^2 - k_T{}^2\left(k_{b_0}{}^2 - k_T{}^2\right)}{\frac{K^2}{1 - C^2}\left(\frac{n\pi}{2}\right)^6 + K^2\left[k_{b_0}{}^2 - k_T{}^2 M\right]\left(\frac{n\pi}{2}\right)^4 + \left[k_{b_0}{}^2 - \frac{k_T{}^2}{1 - C^2} - \frac{K^2 k_T{}^2}{B}\left(k_{b_0}{}^2 - k_T{}^2\right)\right]\left(\frac{n\pi}{2}\right)^2 - k_T{}^2\left(k_{b_0}{}^2 - k_T{}^2\right)} = 0$$

$$(K^2 \neq 0) \qquad (32)$$

Discussion of Limiting Cases

Before proceeding with a numerical evaluation of the influence of the shear stiffness of bulkheads on the torsional vibrations of box beams, a discussion of some limiting cases of the frequency equations is desirable.

The influence of cross-sectional shear distortions on torsional frequencies depends on the frequency coefficient k_{b_0} of the uniform cross-sectional shear mode. The stiffer the cross sections of a beam, the higher the frequency of this uniform shear mode. When the cross sections become rigid (that is, when G_e and, consequently, k_{b_0} are infinite), the frequency equations can be put into closed forms identical to those in reference 4 (where the influence of cross-sectional distortions is considered negligible).

The parameter K is associated with the influence of restraint of warping and, if this effect is to be neglected, it is sufficient to set K equal to zero. The frequency equations (eqs. (27) and (32)), however, become indeterminate for $K = 0$. A solution for this special case is presented in the appendix and has a particular importance which will be discussed later in the section entitled "Method for Estimating Effects."

Another special limiting case which is of interest is the box beam for which both C and A are equal to zero as, for example, a square beam with equal wall and cover thickness. For this case, equation (20), for the condition $b_1 \neq 0$, yields the elementary torsional frequency equation, whereas solution of the remaining equations (eqs. (18), (19), and (21)), which no longer contain b_1, results in a frequency equation for cross-sectional shear modes (that is, modes which contain only γ_b and u displacements). The frequency equations for these shear modes can be obtained from equations (27) and (32) by setting both A and C equal to zero.

NUMERICAL RESULTS

In order to evaluate the influence of shear stiffness of bulkheads on torsional vibrations, the frequencies of the rectangular tube shown in figure 2 were calculated from equations (27) and (32) for various values of k_{b_0}. The tube was assumed to have a width-depth ratio b/a

of 3.6, a plan-form aspect ratio L/b of 13.3, and a thickness ratio t_W/t_C of 1.0. A value for E/G of 2.65 was also assumed. (These proportions correspond to those of the tube for which experimental results were reported in ref. 1.) The cross-sectional area of the flanges of the four-flange box beam was taken as equal to one-sixth of the sectional area of the walls adjacent to each corner. The cover and web thicknesses of the four-flange box beam were set equal to the wall thickness of the tube. The results of these calculations for the first five free-free modes (three antisymmetrical and two symmetrical) are shown in figures 2 and 3.

In figure 2, the frequency coefficient k_{T_m} of the four-flange box beam is plotted as a function of the frequency coefficient k_{b_0} for the uniform cross-sectional shear mode. Two sets of curves are shown; the solid curves represent the frequency coefficients obtained from equations (27) and (32) when the influence of cross-sectional distortions is included, and the dashed curves represent the frequency coefficients obtained from a solution of the four-flange box beam when the cross-sectional distortions are assumed to be negligible. The solid curves representing the first five free-free modes are given for values of k_{b_0} from 2.73 to 25.

The differences between the solid and dashed curves in figure 2, show that the reductions in torsional frequencies due to bulkhead shear flexibilities can be of considerable importance, especially for small values of k_{b_0}. For a tube with $k_{b_0} = 3.08$, for example, the reduction in torsional frequencies due to cross-sectional flexibilities, as obtained from figure 2 and verified experimentally in reference 1, ranges from 16 percent for the first mode to 63 percent for the fifth mode. This value of k_{b_0} is for the test beam used in reference 1. The beam contained no discrete bulkheads of any kind and therefore depended on the bent action of its own walls for cross-sectional stiffness. If, however, weightless bulkheads of the same thickness and shear modulus as the walls of the beam were spaced at a distance of twice the chord, the average value of k_{b_0} would be approximately 22.0, and the reductions in torsional frequencies would range from 1 percent for the first mode to 8 percent for the fifth mode.

In figure 3, the results of figure 2 are replotted in terms of the frequency ratios $\omega_{T_m}\Big/\left(\omega_{T_m}\right)_{rig}$ and $\omega_{b_0}\Big/\left(\omega_{T_m}\right)_{rig}$ where $\left(\omega_{T_m}\right)_{rig}$ is the frequency of the mth mode of the four-flange box beam with rigid cross sections. It should be noted that the ratios of the frequencies $\omega_{T_m}\Big/\left(\omega_{T_m}\right)_{rig}$ and $\omega_{b_0}\Big/\left(\omega_{T_m}\right)_{rig}$ are identical to the ratios of the corresponding frequency coefficients $k_{T_m}\Big/\left(k_{T_m}\right)_{rig}$ and $k_{b_0}\Big/\left(k_{T_m}\right)_{rig}$.

The curves for each mode in figure 3 fall so close together that it should be possible to draw one curve which would be representative of all modes considered. It would seem probable, therefore, that, with some simplification of the analysis, a relationship between $\omega_{T_m}/\left(\omega_{T_m}\right)_{rig}$ and $\omega_{b_0}/\left(\omega_{T_m}\right)_{rig}$ could be obtained that does not depend on the mode. Such a relationship would be useful for estimating the effects of cross-sectional shearing.

METHOD FOR ESTIMATING EFFECTS

It is shown in the appendix that, when the restraint of warping is neglected (that is, when EA_F and, consequently, K are set equal to zero), the following frequency equation will result:

$$\left[\frac{\omega_{T_m}}{\left(\omega_{T_m}\right)_{rig}}\right]^4 - \left\{\left[\frac{\omega_{b_0}}{\left(\omega_{T_m}\right)_{rig}}\right]^2 + \frac{1}{1 - c^2}\right\}\left[\frac{\omega_{T_m}}{\left(\omega_{T_m}\right)_{rig}}\right]^2 + \left[\frac{\omega_{b_0}}{\left(\omega_{T_m}\right)_{rig}}\right]^2 = 0 \qquad (33)$$

Equation (33) is similar to the equations obtained in reference 2 in which the influence of coupling between overall torsion and certain cross-sectional or panel vibrations was investigated. The particular type of cross-sectional vibration considered in reference 2, however, is one in which the corners of the cross sections do not move with respect to each other; thus, no cross-sectional shear distortions are allowed.

Equation (33) is a quadratic equation in $\left[\frac{\omega_{T_m}}{\left(\omega_{T_m}\right)_{rig}}\right]^2$ and will yield two real and positive values of $\frac{\omega_{T_m}}{\left(\omega_{T_m}\right)_{rig}}$. Only the smaller of the two roots, however, is of interest. Results of this approximate solution are shown by the dashed curve in figure 3 for the particular beam considered in figures 2 and 3 and are in good agreement with the results of the exact solution, even for the lower values of $\frac{\omega_{b_0}}{\left(\omega_{T_m}\right)_{rig}}$.

This frequency equation (eq. (33)), therefore, can be used to estimate quickly the effects of cross-sectional shear stiffness on the torsional frequencies of box beams. Solution of the equation for a range of values of I_y/I_z (which determines the value of C) from 1.0 to 0.05 is shown in figure 4. A value for I_y/I_z of 1.0 corresponds to a uniform-walled tube with a width-depth ratio b/a of 1.0, whereas a value for I_y/I_z of 0.05 corresponds to a tube with b/a approximately equal to 7.0. The curves in figure 4 show that the effect of cross-sectional flexibility increases not only with a decrease in uniform cross-sectional shear frequency but also with an increase in width-depth ratio.

Before these curves can be used, the ratio I_y/I_z must be known and the frequency ratios $\omega_{b_0}\Big/\left(\omega_{T_m}\right)_{rig}$ or $k_{b_0}\Big/\left(k_{T_m}\right)_{rig}$ must be evaluated. A value for the frequency coefficient $\left(k_{T_m}\right)_{rig}$ may be obtained from any torsional analysis in which the cross sections are assumed to be rigid. A value for the frequency coefficient k_{b_0} may be obtained from equation (25), provided an appropriate value of G_e for a representative beam cross section is known. In determining G_e for an actual structure where the bulkheads contribute most of the cross-sectional shear stiffness, the defining relation is

$$\frac{G_e}{G} = \frac{N\left(\overline{G}t_b\right)_e}{2LG} \tag{34}$$

where N is the number of bulkheads and $\left(\overline{G}t_b\right)_e$ is the effective shear stiffness of a bulkhead. If the bulkheads are solid sheets, the shear modulus of the bulkhead material should be used for $\overline{G}$ and the bulkhead thickness should be used for t_b. For other forms of bulkheads, estimates of the effective shear stiffness must be made. For beams that contain no bulkheads, special consideration, such as that given in appendix B of reference 1, is necessary.

CONCLUDING REMARKS

The influence of cross-sectional or bulkhead shear deformations on the torsional frequencies of box beams has been obtained by means of an analysis of a four-flange box beam. For conventional constructions where bulkheads are spaced approximately 1 chord apart, the influence of cross-sectional shear deformations generally is negligible. For beams that depend on the bent action of their own walls for most of their cross-sectional stiffness, the effect of cross-sectional shear deformations is considerable. Since the trend in wing design is toward structures with fewer bulkheads and higher width-depth ratios, the influence of cross-sectional flexibilities on torsional frequencies could become important. Curves based on an approximate solution are presented to allow quick approximation of the reduction in torsional frequencies of box beams having flexible rather than rigid bulkheads.

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National Advisory Committee for Aeronautics,
Langley Field, Va., April 27, 1955.

APPENDIX

APPROXIMATE SOLUTION

Although a solution for $K = 0$ can be obtained directly from equations (18) to (21), a solution based on the differential equations might be of more interest.

From equations (9), (11), and (13), where the variation is taken independently with respect to u, θ, and γ_b, the following differential equations and the natural boundary conditions associated with the vibration problem may be obtained:

$$B \frac{d^2\theta}{dx^2} - \frac{1}{2} A \frac{d^2\gamma_b}{dx^2} - \frac{A}{ab} \frac{du}{dx} + \frac{k_T^2}{L^2} \theta - \frac{1}{2} \frac{k_T^2 C}{L^2} \gamma_b = 0 \tag{A1}$$

$$-A \frac{d^2\theta}{dx^2} + \frac{1}{2} B \frac{d^2\gamma_b}{dx^2} + \frac{B}{ab} \frac{du}{dx} - \frac{k_T^2 C}{L^2} \theta + \frac{1}{2}\left(k_T^2 - S\right)\frac{1}{L^2} \gamma_b = 0 \tag{A2}$$

$$-\frac{L^2}{ab} K^2 \frac{d^2u}{dx^2} - A \frac{d\theta}{dx} + \frac{1}{2} B \frac{d\gamma_b}{dx} + \frac{B}{ab} u = 0 \tag{A3}$$

$$K^2 \frac{du}{dx} \delta u \Big|_{-L}^{L} = 0 \tag{A4}$$

$$\left(B \frac{d\theta}{dx} - \frac{1}{2} A \frac{d\gamma_b}{dx} - \frac{A}{ab} u\right) \delta\theta \Big|_{-L}^{L} = 0 \tag{A5}$$

$$\left(A \frac{d\theta}{dx} - \frac{1}{2} B \frac{d\gamma_b}{dx} - \frac{B}{ab} u\right) \delta\gamma_b \Big|_{-L}^{L} = 0 \tag{A6}$$

The parameters used in these equations are defined in equations (22).

In order to neglect the influence of restraint due to warping in equations (A1) to (A6), it is sufficient to set K^2 equal to 0. These equations may then be reduced to the following differential equation and boundary conditions:

$$\frac{d^2\theta}{dx^2} + P^2\theta = 0 \tag{A7}$$

$$\frac{d\theta}{dx}\,\delta\theta \Big|_{-L}^{L} = 0 \tag{A8}$$

where

$$P^2 = \frac{k_T^2}{L^2}\,\frac{k_T^2 - k_{b_0}^2}{\dfrac{k_T^2}{1 - C^2} - k_{b_0}^2} \tag{A9}$$

For a free-free beam, equation (A8) will yield

$$\frac{d\theta}{dx}\Big|_{-L} = \frac{d\theta}{dx}\Big|_{L} = 0 \tag{A10}$$

The general solution for equation (A7) is given by

$$\theta = D_1 \sin Px + D_2 \cos Px \tag{A11}$$

where D_1 and D_2 are constants of integration.

Now, by use of equations (A10) and (A11) and the condition for a nontrivial solution for D_1 and D_2, the following frequency equation for a free-free beam may be obtained:

$$\sin PL \cos PL = 0 \tag{A12}$$

Equation (A12) is satisfied by the relation:

$$P^2L^2 = \left(\frac{m\pi}{2}\right)^2 \qquad (m = 0, 1, 2, \ldots) \qquad \text{(A13)}$$

By substitution of the definition of P^2 from equation (A9), equation (A13) may be written as

$$k_T^{\,2}\,\frac{k_T^{\,2} - k_{b_0}^{\,2}}{\dfrac{k_T^{\,2}}{1 - C^2} - k_{b_0}^{\,2}} = \left(\frac{m\pi}{2}\right)^2 \qquad (m = 0, 1, 2, \ldots) \qquad \text{(A14)}$$

For $m = 0$, equation (A14) yields the frequencies of the rigid-body torsion mode and the uniform cross-sectional shear mode (see eq. (25)). If the case in which $m = 0$ is neglected, equations (A14) may be written as

$$\left(k_{T_m}\right)_{rig}^2\left[\frac{\omega_{T_m}}{\left(\omega_{T_m}\right)_{rig}}\right]^2 \frac{\left[\dfrac{\omega_{T_m}}{\left(\omega_{T_m}\right)_{rig}}\right]^2 - \left[\dfrac{\omega_{b_0}}{\left(\omega_{T_m}\right)_{rig}}\right]^2}{\dfrac{1}{1 - C}\left[\dfrac{\omega_{T_m}}{\left(\omega_{T_m}\right)_{rig}}\right]^2 - \left[\dfrac{\omega_{b_0}}{\left(\omega_{T_m}\right)_{rig}}\right]^2} = \left(\frac{m\pi}{2}\right)^2$$

$$(m = 1, 2, 3, \ldots) \qquad \text{(A15)}$$

At this point, it should be noted that $\left(k_{T_m}\right)_{rig}$ reduces to the mth elementary frequency coefficient for the case in which $K^2 = 0$; that is, $\left(k_{T_m}\right)_{rig}^2 = \left(\frac{m\pi}{2}\right)^2$ where $m = 1, 2, 3, \ldots$

Thus, equation (A15) may be written as

$$\left[\frac{\omega_{T_m}}{\left(\omega_{T_m}\right)_{rig}}\right]^4 - \left\{\left[\frac{\omega_{b_0}}{\left(\omega_{T_m}\right)_{rig}}\right]^2 + \frac{1}{1 - C^2}\right\}\left[\frac{\omega_{T_m}}{\left(\omega_{T_m}\right)_{rig}}\right]^2 + \left[\frac{\omega_{b_0}}{\left(\omega_{T_m}\right)_{rig}}\right]^2 = 0 \qquad \text{(A16)}$$

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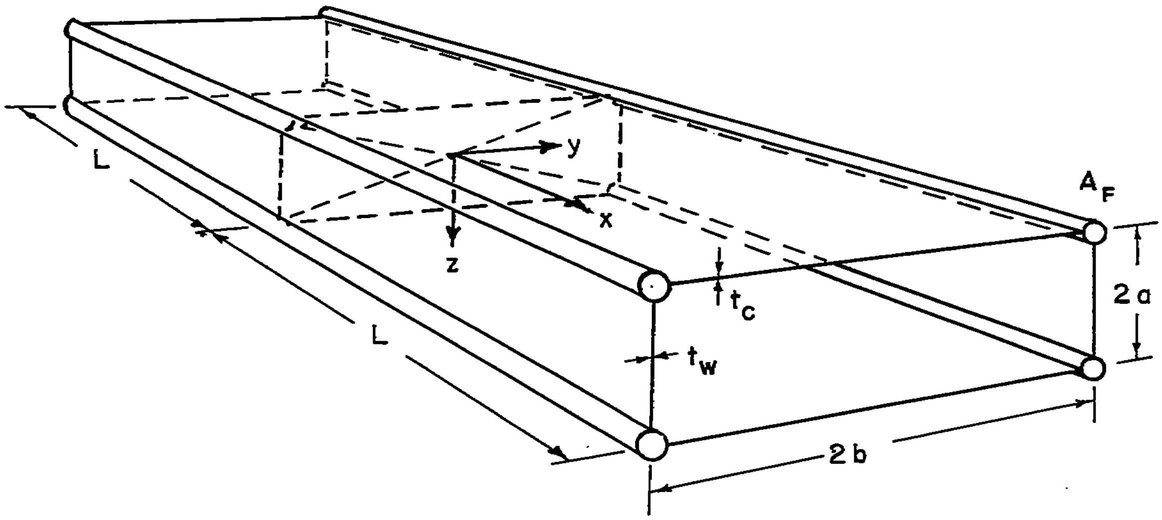

Figure 1.- Four-flange box beam and the distortions of a cross section.

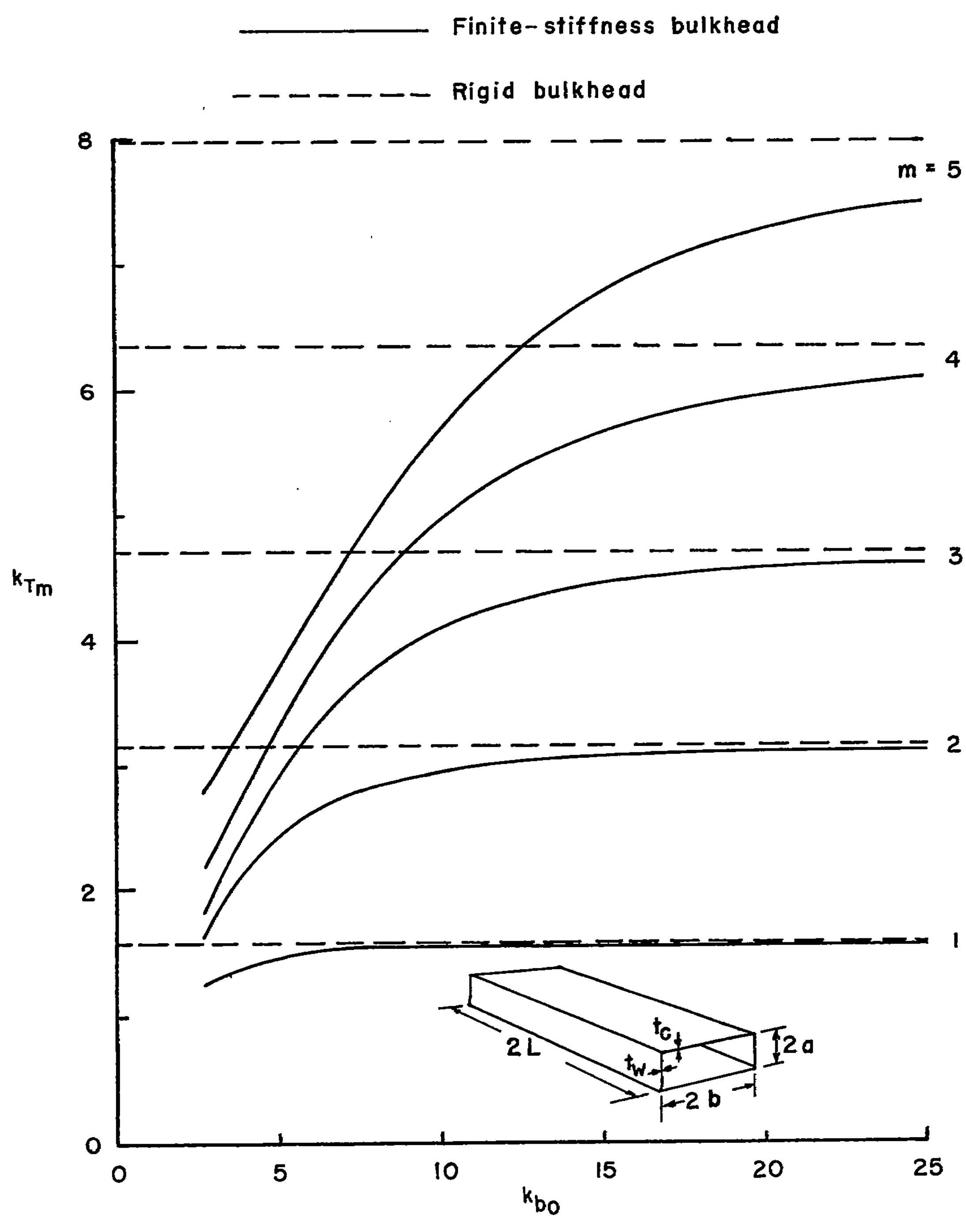

Figure 2.- Influence of bulkhead shear stiffness on the torsional frequencies of a box beam where $\frac{L}{b} = 13.3$, $\frac{b}{a} = 3.6$, and $\frac{t_w}{t_c} = 1.0$.

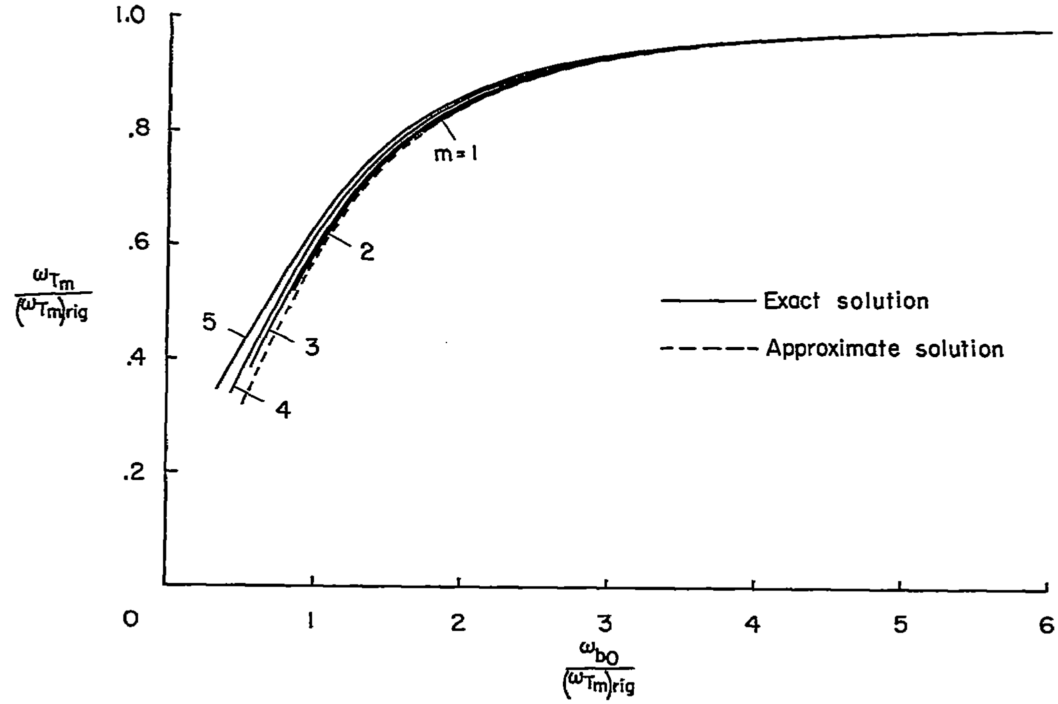

Figure 3.- Comparison of exact and approximate solutions which include the influence of bulkhead shear flexibility for a beam with $\frac{L}{b} = 13.3$, $\frac{b}{a} = 3.6$ and $\frac{t_w}{t_c} = 1.0$.

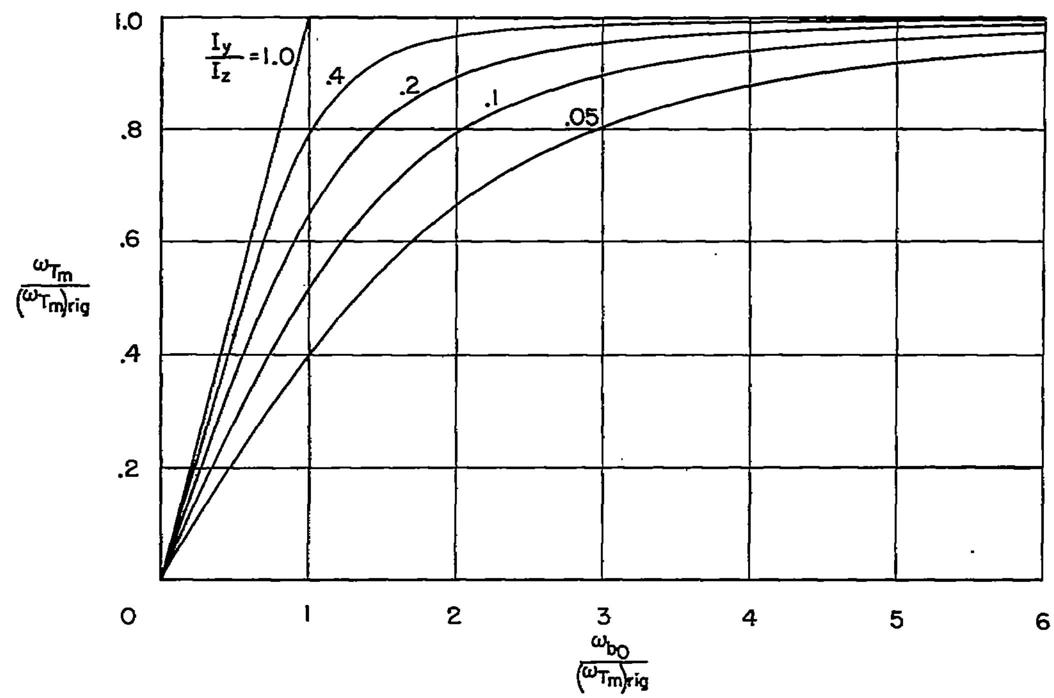

Figure 4.- Influence of bulkhead shear flexibility given by an approximate solution for various values of $\frac{I_y}{I_z}$.