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VORTEX INTERFERENCE ON SLENDER AIRPLANES

By Alvin H. Sacks

Ames Aeronautical Laboratory
Moffett Field, Calif.

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SUMMARY

Formulas are developed for the forces and moments (except drag) due to vortex interference on slender wing-body-tail combinations of general cross section in terms of the positions and strengths of the shed vortices. The analysis is applicable to steady motion and to motions which can be considered to be made up of a succession of steady states (i.e., quasi-steady motions). In order to illustrate the application of the analysis, the interference lift of a plane wing-body-tail combination in steady straight flight is determined by utilizing vortex positions obtained by numerical methods.

It is found that the impulse of each shed vortex and its image vortex in a transformed circle plane enters into all the interference forces and moments on the airplane. A simple theorem is given for the interference forces in steady straight flight which are found to depend on this impulse evaluated only at the wing trailing edge and at the base of the configuration.

INTRODUCTION

It has been recognized for some time that interference among the various airplane components can be very important in determining the aerodynamic characteristics of slender configurations. The use of slender-body theory for treating entire wing-body combinations is not new (see, e.g., refs. 1, 2, and 3), and for cases falling into this category considerable simplification has resulted. The calculation of interference effects due to wing wakes (wing-tail or wing-afterbody interference), on the other hand, is not so clear-cut. Unlike the wing-body problem, this calculation becomes more difficult as the effects become more important, since the rolling-up and displacement of the wing vortex sheets cannot be ignored for long slender configurations.

Various authors have treated certain specific cases of interference due to wing wakes by assuming simplified wake shapes (e.g., refs. 4 to 7). The present paper, however, is concerned with developing formulas for calculating the forces and moments due to wing wakes for slender wing-body-tail combinations; the formulas developed are in terms of the strengths and positions of the shed vortices and the mapping functions of the airplane cross sections. Since the purpose of this report is the presentation of the derived formulas rather than the presentation of calculated results, only the simplest example will be calculated here. Specifically, the interference lift of a plane wing-body-tail combination will be determined by making use of a numerical analysis for the vortex strengths and positions at the tail.

ANALYSIS

It is generally recognized that in the vicinity of a slender body the wave equation in three space dimensions can be approximated by the two-dimensional Laplace equation in planes normal to the body axis, provided the frequency of the motion is small compared with the flight velocity divided by the length of the body. Hence, the classical methods of hydrodynamics can be applied to quasi-steady motions of slender bodies in a compressible fluid. The analysis of reference 8 was the counterpart of a method due to H. Blasius for obtaining the forces and moments on a two-dimensional body in an incompressible stream, and the body cross section was mapped onto a circle in order to express the slender-body stability derivatives in terms of the mapping functions of the cross-sectional shapes. The pressures on the body were expressed in terms of the two-dimensional complex potential and the total forces and moments were calculated by integrating those pressures round the body cross section and over the body length. The complex potential was expressed in a Laurent series of the form

$$F = B\, ln\, \zeta + \sum_{n=1}^{\infty} \frac{A_n}{\zeta^n} + D \qquad (1)$$

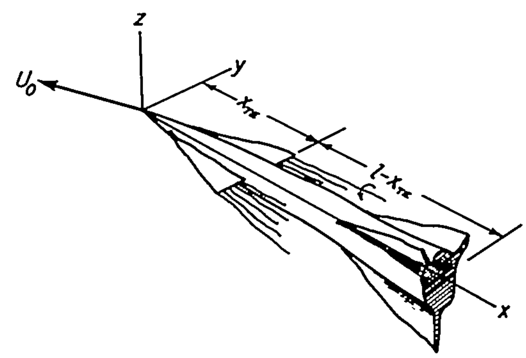
and the integrations were carried out by the method of residues. In the present analysis, it will be shown that by a simple extension of the resulting formulas of reference 8 for the forces and moments the effects of wing wakes can be included so that wing-body-tail combinations may be treated in the same fashion.

The extension to be made will consist of admitting any number of free vortices external to the body cross section along with their corresponding images inside the body. The boundary conditions are unchanged, then, and the only fundamental change lies in the expression for the complex potential which enters into some of the contour integrals to be taken round the body cross sections for the determination of the forces and moments.

It will first be observed that all the vortices in the flow field external to the body will be free vortices, representing wing wakes, and therefore can themselves sustain no forces. Hence the integrals for the forces, when evaluated round each external vortex, must vanish. In view of this fact, the resulting expressions for the forces and moments will be the same whether the integrals are evaluated at the body surface or on any other contour enclosing the body cross section. The integrals involving the potential can therefore be evaluated at large distances from the body so that, just as in reference 8, the complex potential will be expressed as a Laurent series valid at large distances, and the contour integrals will again be evaluated by the method of residues.

The form of the complex potential will be exactly the same as equation (1) since the same types of singularities are introduced by the addition of external vortices and their images. Therefore, any contour integrals which were identically zero before the addition of the vortices will remain zero. Furthermore, it was found in reference 8 that all the forces and moments (except drag) are linear in the potential. Hence it is possible to calculate the forces and moments due to vortex interference alone (by using only the additional potential due to the vortices and their images) and then to add these directly to those calculated without interference. The present analysis will therefore be concerned with the determination of the additional potential due to the shed vortices and their images and with the associated interference forces and moments.

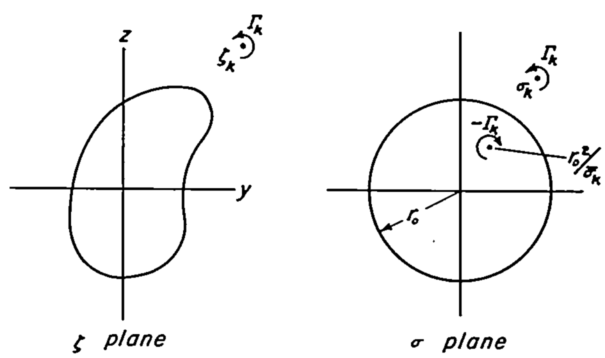
If one considers a slender wing-body-tail combination as shown in sketch (a), it is noted that the net circulation in planes perpendicular to the x axis must be zero for all values of x. Hence if the airplane cross section in the physical plane $\zeta = y + iz$ is transformed to a circle of radius r_o (center at the origin) in the σ plane by the transformation



Sketch (a)

$$\zeta = f(\sigma) = \sigma + \sum_{n=o}^{\infty} \frac{b_n}{\sigma^n} \qquad (2)$$

then the additional complex potential F' due to the shed vortices and their images can be expressed in the plane of the circle simply as (see sketch (b))



Sketch (b)

$$F'(\zeta) = F_1'(\sigma) = -\frac{i}{2\pi}\sum_{k=1}^{m}\Gamma_k \ln(\sigma - \sigma_k) + \frac{i}{2\pi}\sum_{k=1}^{m}\Gamma_k \ln\left(\sigma - \frac{r_o^2}{\bar{\sigma}_k}\right) \tag{3}$$

where m is the number of external (free) vortices and may be a function of x. All other symbols of the present analysis are defined in Appendix A. Now, since the transformation of equation (2) leaves the flow field at infinity unchanged, the residue of the complex potential is the same in either plane. Thus, expanding equation (3) for large σ, one finds

$$F'(\zeta) = F_1'(\sigma) = -\frac{i}{2\pi}\sum_{k=1}^{m}\Gamma_k\left(\ln \sigma - \frac{\sigma_k}{\sigma} - \frac{1}{2}\frac{\sigma_k^2}{\sigma^2} - \ldots\right) + \frac{i}{2\pi}\sum_{k=1}^{m}\Gamma_k\left(\ln \sigma - \frac{r_o^2}{\sigma\bar{\sigma}_k} - \frac{1}{2}\frac{r_o^4}{\sigma^2\bar{\sigma}_k^2} - \ldots\right) \tag{4}$$

so that the residue A_1' is seen to be

$$A_1' = \frac{i}{2\pi}\sum_{k=1}^{m}\Gamma_k\left(\sigma_k - \frac{r_o^2}{\bar{\sigma}_k}\right) \equiv \frac{i}{2\pi}\sum_{k=1}^{m}\Gamma_k\sigma_{k_r} \tag{5}$$

where σ_{k_r} represents the distance between the kth free vortex and its image in the complex σ plane. It is interesting to note that the quantity $\Gamma_k\sigma_{k_r}$ is proportional to the impulse of the vortex pair made up of the kth external vortex and its image in the σ plane.

According to reference 8 the total forces and moments on a slender configuration whose trailing edges all lie in the base plane $x = l$ are given by

$$Y - iL = 2\pi\rho U_o \bar{A}_{1_{x=l}} + \rho U_o\left[S(\bar{R} + 2ip\bar{\zeta}_c) + U_o\frac{d}{dx}(S\bar{\zeta}_c)\right]_{x=l} + 2\pi\rho\int_0^l \frac{\partial \bar{A}_1}{\partial t}dx + \rho\int_0^l S\frac{\partial \bar{R}}{\partial t}dx + 2\pi i\rho p\int_0^l \bar{A}_1 dx + i\rho p\int_0^l S(\bar{R} + ip\bar{\zeta}_c)dx \tag{6}$$

$$N - iM = -2\pi\rho U_0 \int_0^l (x - c_1) \frac{\partial \overline{A}_1}{\partial x} dx -$$

$$\rho U_0 \int_0^l (x - c_1) \frac{\partial}{\partial x} \left[S(\overline{R} + 2ip\overline{\zeta}_c) + U_0 \frac{d}{dx} (S\overline{\zeta}_c) \right] dx -$$

$$2\pi\rho \int_0^l (x - c_1) \frac{\partial \overline{A}_1}{\partial t} dx - \rho \int_0^l (x - c_1) S \frac{\partial \overline{R}}{\partial t} dx -$$

$$2\pi i\rho p \int_0^l (x - c_1)\overline{A}_1 dx - i\rho p \int_0^l (x - c_1)(\overline{R} + ip\overline{\zeta}_c) S\, dx \qquad (7)$$

$$L' = \frac{1}{2} \rho U_0 \mathbf{R} \oint_{x=l} F d(\zeta\overline{\zeta}) + \frac{1}{2} \rho \mathbf{R} \int_0^l dx \frac{\partial}{\partial t} \oint F d(\zeta\overline{\zeta}) - 2\pi\rho \mathbf{I} \int_0^l \overline{R} A_1 dx -$$

$$\rho p \mathbf{R} \int_0^l S(R\overline{\zeta}_c - 2\overline{R}\zeta_c) dx - \rho U_0 \mathbf{I} \int_0^l \overline{R} \frac{d}{dx} (S\zeta_c) dx \qquad (8)$$

where $\mathbf{R}$ denotes the real part, $\mathbf{I}$ denotes the imaginary part, and p is the rate of roll about the body axis. Now, in order to write the corresponding expressions for the airplane of sketch (a), it will be convenient to divide the airplane length into the two segments shown. It should be mentioned that the present analysis will be restricted to wings with trailing edges normal to the x axis. However, if the wing trailing edges do not all lie in the same plane $x = \text{constant}$, further division will of course be necessary, but the procedure will be the same. Equations (6), (7), and (8) can be applied directly to each segment of the airplane of sketch (a) to give the total forces and moments including vortex interference if we replace F by $(F + F')$ and A_1 by $(A_1 + A_1')$, where F' and A_1' are the additional complex potential due to the shed vortices (and their images) and the residue of that additional complex potential. Thus, making use of equation (5) for the residue A_1', and stipulating that the interference forces act only on portions of the airplane behind the wing trailing edge, we find that the additional forces and moments due to vortex interference (and denoted by the subscript I) are given by

$$Y_I - iL_I = -i\rho U_o\left[\left(\sum_{k=1}^{m}\Gamma_k\bar{\sigma}_{k_r}\right)_{x=l} - \left(\sum_{k=1}^{m}\Gamma_k\bar{\sigma}_{k_r}\right)_{x=x_{TE+}}\right] - i\rho\int_{TE}^{l}\frac{\partial}{\partial t}\left(\sum_{k=1}^{m}\Gamma_k\bar{\sigma}_{k_r}\right)dx + \rho p\int_{TE}^{l}\sum_{k=1}^{m}\Gamma_k\bar{\sigma}_{k_r}dx \tag{9}$$

$$N_I - iM_I = i\rho U_o\int_{TE}^{l}(x - c_1)\frac{\partial}{\partial x}\left(\sum_{k=1}^{m}\Gamma_k\bar{\sigma}_{k_r}\right)dx + i\rho\int_{TE}^{l}(x - c_1)\frac{\partial}{\partial t}\left(\sum_{k=1}^{m}\Gamma_k\bar{\sigma}_{k_r}\right)dx - \rho p\int_{TE}^{l}(x - c_1)\sum_{k=1}^{m}\Gamma_k\bar{\sigma}_{k_r}dx \tag{10}$$

$$L_I' = \frac{1}{2}\rho U_o\mathbf{R}\oint_{x=l}F'd(\zeta\bar{\zeta}) - \frac{1}{2}\rho U_o\mathbf{R}\oint_{x=x_{TE+}}F'd(\zeta\bar{\zeta}) + \frac{1}{2}\rho\mathbf{R}\int_{TE}^{l}dx\frac{\partial}{\partial t}\oint F'd(\zeta\bar{\zeta}) - \rho\mathbf{R}\int_{TE}^{l}\bar{R}\sum_{k=1}^{m}\Gamma_k\sigma_{k_r}dx \tag{11}$$

where x_{TE+} refers to a station immediately behind the wing trailing edge. Inasmuch as F' has been expressed in the transformed circle plane (eq. (3)) it will, in general, be convenient to carry out the contour integrations of equation (11) in the σ plane, particularly since the integrals become analytic in that plane due to the relation $\sigma\bar{\sigma} = r_o^2$ on the circle boundary.

For the special case of steady straight flight, equations (9), (10), and (11) reduce to

$$Y_I - iL_I = -i\rho U_o\left[\left(\sum_{k=1}^{m}\Gamma_k\bar{\sigma}_{k_r}\right)_{x=l} - \left(\sum_{k=1}^{m}\Gamma_k\bar{\sigma}_{k_r}\right)_{x=x_{TE+}}\right] \tag{12}$$

$$N_I - iM_I = i\rho U_o \int_{TE}^{l} (x - c_1) \frac{\partial}{\partial x}\left(\sum_{k=1}^{m} \Gamma_k \bar{\sigma}_{k_r}\right) dx \tag{13}$$

$$L_I' = \frac{1}{2} \rho U_o \mathbf{R} \oint_{x=l} F' d(\zeta\bar{\zeta}) - \frac{1}{2} \rho U_o \mathbf{R} \oint_{x=x_{TE+}} F' d(\zeta\bar{\zeta}) - \rho \mathbf{R} \int_{TE}^{l} \bar{R} \sum_{k=1}^{m} \Gamma_k \sigma_{k_r} dx \tag{14}$$

and it is interesting to note that the quantity $\sum_{k=1}^{m} \Gamma_k \sigma_{k_r}$ enters the calculations in much the same manner as the apparent mass enters into the stability derivatives of wing-body combinations (see ref. 8). In fact, equation (12) can be stated in the form of a simple theorem for the determination of the interference forces.

> <u>THEOREM</u>: The lateral force $Y_I + iL_I$ due to each vortex of strength Γ shed from a forward wing of a slender wing-body-tail combination in steady straight flight is equal to the change, from wing trailing edge to base of the airplane, of the quantity $i\rho U_o \Gamma \sigma_r$ where σ_r is the (complex) distance between the vortex and its image in the plane in which the body cross section is mapped onto a circle while leaving the flow field at infinity unchanged.

An important point to notice in regard to equation (12) and the above theorem is that the quantity $\rho U_o \mathbf{R} \sum_{k=1}^{m} \Gamma_k \sigma_{k_r}$ evaluated at the wing trailing edge is equal to the lift of the airplane segment lying ahead of the wing trailing edge. Hence the interference lift in steady straight flight is equal to the difference between the quantity $\rho U_o \mathbf{R} \left(\sum_{k=1}^{m} \Gamma_k \sigma_{k_r}\right)_{x=l}$ and the lift of the airplane segment ahead of the wing trailing edge. Thus, if we write for the total lift of the airplane of sketch (a)

$$L = L_a + L_b + L_I$$

where L_a and L_b are the lifts of the isolated segments of the airplane lying ahead and behind the wing trailing edge (i.e., with no interference), it is clear that

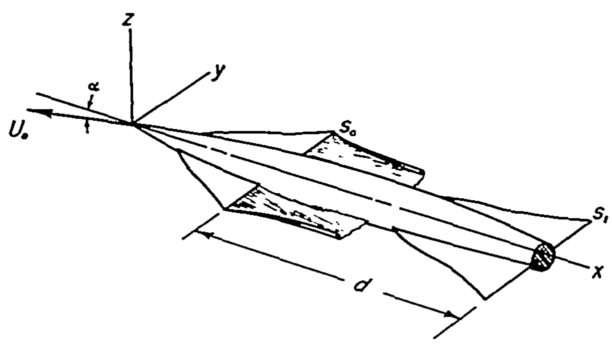
$$L = L_a + L_b + \rho U_o \mathbf{R}\left(\sum_{k=1}^{m} \Gamma_k \sigma_{k_r}\right)_{x=l} - L_a = L_b + \rho U_o \mathbf{R}\left(\sum_{k=1}^{m} \Gamma_k \sigma_{k_r}\right)_{x=l}$$

The total lift of the airplane is therefore equal to the lift of the isolated airplane segment behind the wing trailing edge plus the quantity $\rho U_o \mathbf{R}\left(\sum_{k=1}^{m} \Gamma_k \sigma_{k_r}\right)_{x=l}$. The lift L_b can often be calculated by ordinary slender-body theory and for many interesting configurations this part of the lift is already known. The above statements are independent of the form in which the vortex sheet leaves the wing, but this will of course determine the positions and strengths of the vortices at the base of the configuration.

It can be seen from equation (12) and the associated theorem that in steady straight flight the interference side force and interference lift depend only on the shapes of the cross sections at the wing trailing edge and at the base of the configuration and on the strengths and positions of the shed vortices at these two stations. This statement closely parallels the corresponding statement for slender wing-body combinations that the side force and lift depend only on the shape of the base cross section (see ref. 1) and on the angle it makes with the flight direction. The forces and moments including interference effects will, however, be nonlinear with respect to the angles of attack and sideslip.

It is evident from equations (9), (10) and (11) that the calculation of the interference forces and moments requires, in all cases, a knowledge of the vortex strengths and positions as well as the mapping functions of the airplane cross sections. The calculation of the forces and moments without vortex interference requires only the latter.

APPLICATION TO A PLANE WING-BODY-TAIL COMBINATION

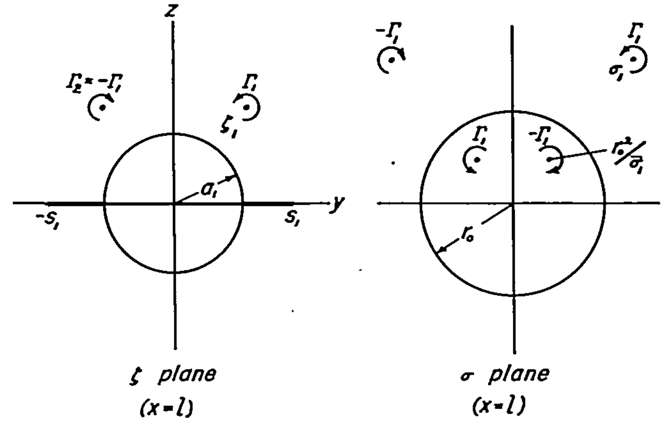


Sketch (c)

In order to illustrate the application of the foregoing analysis, a simple example will be treated briefly. In this example the interference lift will be determined by means of equation (12). Consider a plane wing-body-tail combination in steady straight flight at an angle of attack α as shown in sketch (c). The cross section is taken as a circle with symmetrical flat-plate wings mounted along a diameter. The complex variable is then given by

$$\sigma = \frac{1}{2}\left[\zeta + \frac{a^2}{\zeta} + \sqrt{\left(\zeta + \frac{a^2}{\zeta}\right)^2 - 4r_o^2}\right]$$

and $r_o = \frac{1}{2}\left(s + \frac{a^2}{s}\right)$ where a is the body radius and s is the local semispan of the wing or tail. It will be assumed that the trailing vortex sheet is fully rolled up into two vortices somewhere ahead of the tail trailing edge which lies in the plane $x = l$. Thus, denoting the station $x = l$ by the subscript 1, we have at $x = l$ (see sketch (d))



Sketch (d)

$$\bar{\sigma}_{1r} = \bar{\sigma}_1 - \frac{r_o^2}{\sigma_1} = \frac{1}{2}\left[\bar{\zeta}_1 + \frac{a_1^2}{\bar{\zeta}_1} - \zeta_1 - \frac{a_1^2}{\zeta_1} + \sqrt{\left(\bar{\zeta}_1 + \frac{a_1^2}{\bar{\zeta}_1}\right)^2 - \left(s_1 + \frac{a_1^2}{s_1}\right)^2} + \sqrt{\left(\zeta_1 + \frac{a_1^2}{\zeta_1}\right)^2 - \left(s_1 + \frac{a_1^2}{s_1}\right)^2}\right]$$

Furthermore, since here by symmetry

$$\Gamma_2 = -\Gamma_1 = -\Gamma$$

and

$$\zeta_2 = -\bar{\zeta}_1$$

it follows that

$$\left(\sum_{k=1}^{2}\Gamma_k\bar{\sigma}_{k_r}\right)_{x=l} = \Gamma(\bar{\sigma}_{1_r} - \bar{\sigma}_{2_r}) = 2\Gamma\mathbf{R}\sqrt{\left(\zeta_1 + \frac{a_1^2}{\zeta_1}\right)^2 - \left(s_1 + \frac{a_1^2}{s_1}\right)^2}$$

Now, from equation (12),

$$L_I = \rho U_o \mathbf{R}\left(\sum_{k=1}^{m}\Gamma_k\sigma_{k_r}\right)_{x=l} - \rho U_o \mathbf{R}\left(\sum_{k=1}^{m}\Gamma_k\sigma_{k_r}\right)_{x=x_{TE+}}$$

and the second term has already been identified as the lift of the airplane segment ahead of the wing trailing edge. For the present problem, if the vortex sheet leaves the wing trailing edge as a flat sheet from each wing panel, this lift is known to be (see, e.g., ref. 1)

$$\rho U_o \mathbf{R}\left(\sum_{k=1}^{\infty}\Gamma_k\sigma_{k_r}\right)_{x=x_{TE+}} = \pi\rho U_o^2\alpha s_o^2\left(1 - \frac{a_o^2}{s_o^2} + \frac{a_o^4}{s_o^4}\right)$$

where a_o and s_o are the body radius and wing semispan at $x = x_{TE}$. Therefore, the interference lift for the plane wing-body-tail combination of sketch (c) is

$$L_I = 2\rho U_o\Gamma\mathbf{R}\sqrt{\left(\zeta_1 + \frac{a_1^2}{\zeta_1}\right)^2 - \left(s_1 + \frac{a_1^2}{s_1}\right)^2} - \pi\rho U_o^2\alpha s_o^2\left(1 - \frac{a_o^2}{s_o^2} + \frac{a_o^4}{s_o^4}\right) \tag{15}$$

and it is clear that Γ and ζ_1 (the strength and position of rolled-up vortex 1 at $x = l$) must be specified if the interference lift is to be calculated.

Before any calculations are performed, it is interesting to note that if the real part of the square root in equation (15) vanishes, then the interference lift is equal and opposite to the lift of the airplane segment ahead of the wing trailing edge. This condition is satisfied if $\left(\zeta_1 + \frac{a_1^2}{\zeta_1}\right)$ is real and less than $\left(s_1 + \frac{a_1^2}{s_1}\right)$ and it can easily be shown that this requires that ζ_1 be real and less than s_1. In other words, if the rolled-up vortices intersect the tail anywhere along its trailing edge, then the interference lift just cancels the lift forward of the wing trailing edge. It can therefore be concluded that, for a wing-body-tail combination having cross sections of the type shown in sketch (d), if the location and span of the tail are such that the rolled-up vortices pass through the tail trailing edge, the total lift of the combination is unchanged by removing the portion of the airplane forward of the wing trailing edge. This result has required no assumptions regarding the influence of the tail on the vortex paths but has required the assumption of inviscid vortices (no cores). While the above result would be modified by the presence of viscous cores, equation (15) would still apply at angles of attack at which the vortex cores do not touch the body or the tail.

In reference 9, numerical calculations were carried out to determine the positions of the vortices at various distances behind the wings of several plane wing-body combinations. In order to make use of those calculations, we shall consider a specific slender wing-body combination and a specific tail length. Thus let

$$\frac{a_o}{s_o} = 0.6 \quad \text{and} \quad \frac{d}{a_o} = 10$$

Also, to insure the validity of the slenderness assumption, we choose the case treated in reference 9 for $A = 2/3$, $M = 2$ where A is the wing aspect ratio and M is the free-stream Mach number. The body is cylindrical behind the wing trailing edge so that $a_1 = a_o$, and it will be assumed that the vortex positions are not influenced by the tail. Thus from figure 10 of reference 9

$$\Gamma = 1.27\, U_o \alpha s_o$$

and the position ζ_1 of the vortex at the base plane $x = l$ is given by $\zeta_1 = y_1 + iz_1$ where y_1 and z_1 are obtained from figure 7(a) of reference 9 and z_1 is transferred to body axes by adding αd to the values in the figure.

The above information can be put directly into equation (15) to determine the interference lift as a function of the ratio of tail span to wing span s_1/s_o for all angles of attack for which reference 9 has supplied the vortex positions. The resulting curves for the interference lift coefficient, based on the gross wing area, are shown in figure 1.[1] It can be seen that the interference lift is nonlinear with respect to the angle of attack and becomes more negative as the ratio of tail span to wing span increases. At $s_1/s_o = a/s_o = 0.6$, there is no tail and the interference lift is the lift on the cylindrical afterbody.

CONCLUDING REMARKS

The analysis reported in reference 8 has been extended to permit the calculation of the total forces and moments (except drag) for slender

[1]This calculation requires taking the real part of the square root indicated in equation (15). Since this process is not entirely straightforward if ambiguities are to be avoided, the procedure is given in Appendix B.

wing-body-tail combinations of general cross section performing quasi-stationary maneuvers in a compressible fluid. Expressions have been developed in the present paper for the forces and moments due to the influence of wing wakes on the other airplane components, and these equations were used to calculate the interference lift of a plane wing-body-tail combination.

The essential quantity required for the calculations is shown to be proportional to the impulse of each shed vortex and its image vortex in a transformed circle plane, and it is demonstrated that the calculation of the interference lift in steady straight flight requires the determination of this quantity only at the trailing edge of the wing and at the base of the configuration.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
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APPENDIX A

LIST OF IMPORTANT SYMBOLS

A_n	coefficient of $\frac{1}{\zeta^n}$ term in Laurent expansion of the complex potential $F(\zeta)$ with no vortices in the field
A_n'	coefficient of $\frac{1}{\zeta^n}$ term in expansion of the additional complex potential $F'(\zeta)$ due to shed vortices and their images
a	body radius
B	coefficient of $l n\ \zeta$ in expansion of $F(\zeta)$; $B = \frac{U_o}{2\pi}\frac{dS}{dx}$
c_1	distance from airplane nose to pivot point
D	a function of x and t which contributes nothing to the forces and moments considered herein
d	tail length, $l - x_{TE}$
F	complex potential $\varphi + i\psi$ with no vortices in the field
F'	additional complex potential due to shed vortices and their images
l	length of airplane
L	force in the z direction (approximately lift)
L'	rolling moment about the x axis
M	pitching moment about pivot point $x = c_1$
m	number of external (free) vortices
N	yawing moment about pivot point $x = c_1$
p	angular rolling velocity about the x axis
q	angular pitching velocity about the y axis
R	$V + iW$
r	angular yawing velocity about the z axis

r_o radius of transformed circle corresponding to airplane cross section

S cross-sectional area of the airplane

s local semispan of wing or tail

s_o maximum semispan of wing (at $x = x_{TE}$)

s_1 maximum semispan of tail (at $x = l$)

t time

U_o component of flight velocity along negative x axis

V_o component of flight velocity along positive y axis ($V_o = U_o\beta$ if $p = 0$)

V $V_o - r(x - c_1)$

W_o component of flight velocity along positive z axis ($W_o = -U_o\alpha$ if $p = 0$)

W $W_o - q(x - c_1)$

Y force in the y direction (side force)

x,y,z Cartesian coordinates fixed in the body (x rearward, y to starboard, z upward, origin at the body nose)

x_{TE} distance from airplane nose to wing trailing edge

α angle of attack (angle between arbitrarily chosen xy plane and flight direction)

β angle of sideslip (angle between xz plane and flight direction)

Γ_k circulation strength of kth external (free) vortex, positive counterclockwise

ρ fluid mass density

ζ $y + iz$

ζ_c complex coordinate of centroid of cross-sectional area

ζ_k position of kth external (free) vortex

σ complex coordinate in transformed circle plane

σ_k position of kth external vortex in transformed circle plane

σ_{k_r} position of kth external vortex relative to its image in the transformed circle plane, $\sigma_k - \frac{r_o^2}{\bar{\sigma}_k}$

φ velocity potential

ψ stream function

Special Notations

I subscript indicating vortex interference

$\oint$ contour integral taken once round the body cross section in the positive (counterclockwise) sense

I imaginary part

R real part

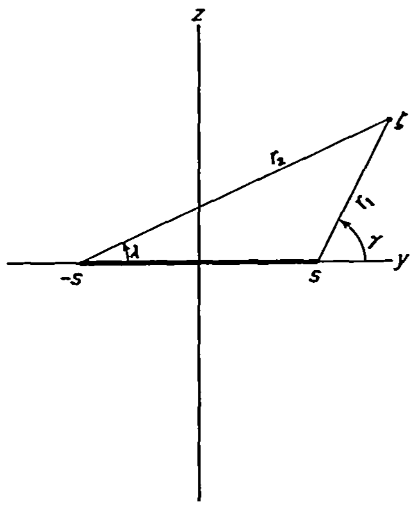
$(\bar{\ })$ complex conjugate of ( )

APPENDIX B

EVALUATION OF THE REAL PART OF A COMPLEX SQUARE ROOT

A complex square root of the type $\sqrt{\zeta^2 - s^2}$ where $\zeta = y + iz$ and s is real has branch points at $\zeta = \pm s$ and can be written in its factored form

$$\sqrt{\zeta^2 - s^2} = \sqrt{(\zeta - s)(\zeta + s)}$$



Sketch (e)

Now each factor can be written in polar coordinates referred to one of the branch points. Thus (see sketch (e)) let

$$\zeta - s = r_1 e^{i\gamma}; \qquad \zeta + s = r_2 e^{i\lambda}$$

where γ and λ are both limited to a range of, say, 0 to 2π (to give the proper sign changes through the line segment shown). This is commonly called a bipolar coordinate system and enables one to write

$$\sqrt{\zeta^2 - s^2} = \sqrt{r_1 r_2}\, e^{i\left(\frac{\gamma+\lambda}{2}\right)} = \sqrt{r_1 r_2}\left[\cos\left(\frac{\gamma + \lambda}{2}\right) + i \sin\left(\frac{\gamma + \lambda}{2}\right)\right]$$

so that the real part is

$$\mathbf{R}\sqrt{\zeta^2 - s^2} = \sqrt{r_1 r_2}\cos\left(\frac{\gamma + \lambda}{2}\right)$$

where

$$r_1 = \left|\sqrt{(y - s)^2 + z^2}\right|$$

$$r_2 = \left|\sqrt{(y + s)^2 + z^2}\right|$$

$$\gamma = \tan^{-1}\left(\frac{z}{y - s}\right)$$

$$\lambda = \tan^{-1}\left(\frac{z}{y + s}\right)$$

The sign of the real part of the square root is therefore determined by the quadrants of γ and λ which depend on the position of the point ζ relative to the branch points $\zeta = \pm s$ and also on the range of γ and λ specified above.

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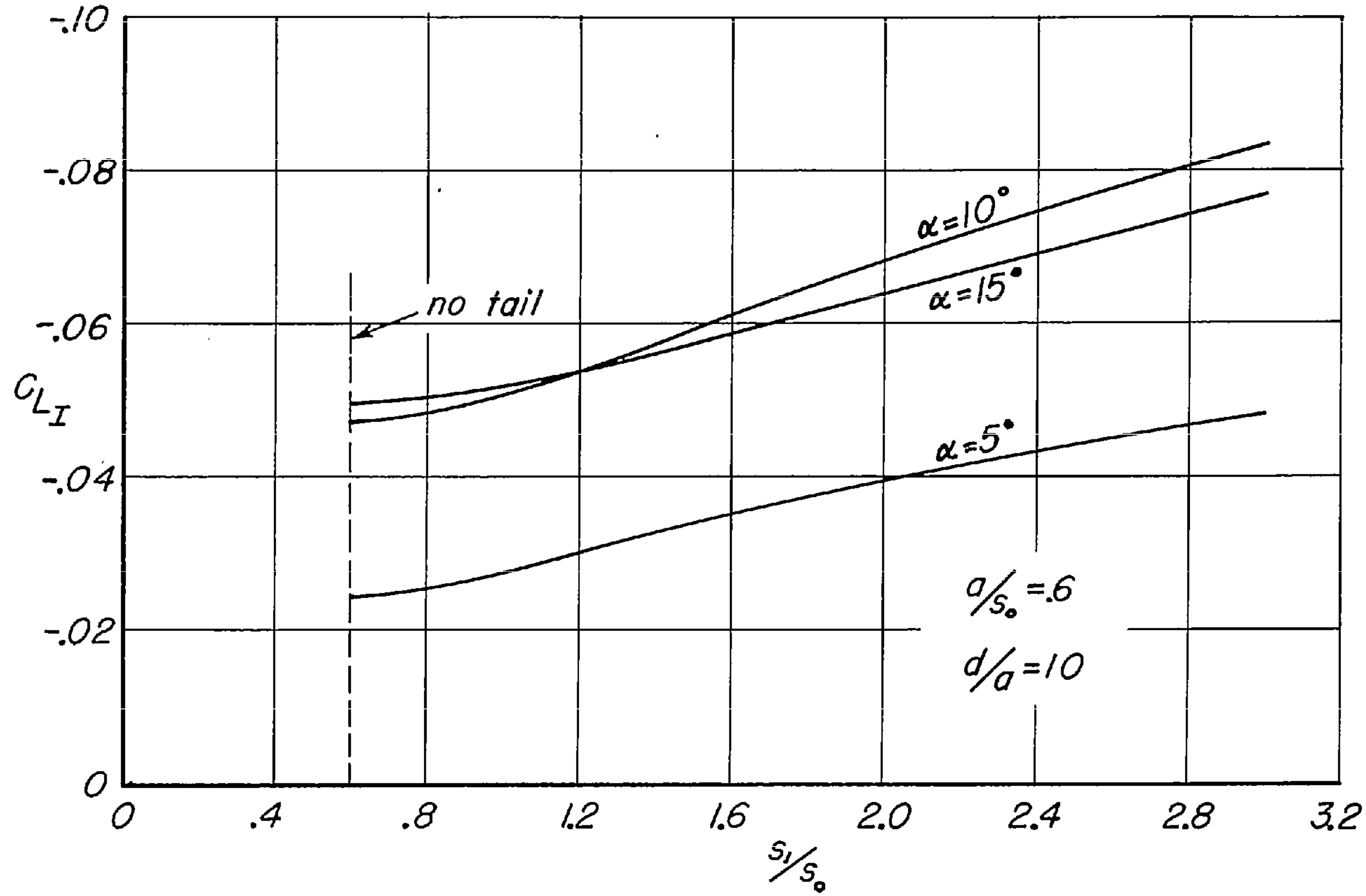

Figure 1.- Interference lift coefficient (based on gross wing area) for a plane wing-body-tail combination having a cylindrical body behind the wing.

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