# NATIONAL ADVISORY COMMITTEE
# FOR AERONAUTICS

## TECHNICAL NOTE 3533

THE PROPER COMBINATION OF LIFT LOADINGS FOR LEAST DRAG

ON A SUPERSONIC WING

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SUMMARY


Lagrange's method of undetermined multipliers is applied to the problem of properly combining lift loadings for the least drag at a given lift on supersonic wings. The method shows the interference drag between the optimum loading and any loading at the same lift coefficient to be constant. This is an integral form of the criterion established by Robert T. Jones for optimum loadings.

The best combination of four loadings on a delta wing with subsonic leading edges is calculated for several Mach numbers as a numerical example. The loadings considered have finite pressures everywhere on the plan form. At each Mach number the optimum combination of the four non-singular loadings has about the same drag coefficient as a flat plate with leading-edge thrust.

INTRODUCTION


The problem of minimizing the supersonic drag for a given lift on a fixed plan form has been approached in different ways. Jones, in references 1 and 2, makes use of reverse-flow theorems to derive several simple properties of the optimum load distribution and to present as well the optimum distribution for elliptic plan forms. Graham, in reference 3, shows how the proper use of orthogonal loadings can reduce the drag at fixed lift. Orthogonal loadings are loadings of zero interference drag. The interference drag between two loadings is the total drag of each in the downwash field of the other. In reference 4, theorems concerning orthogonality and reverse flow are developed, whereas in references 5 and 6 numerical examples of drag reduction by use of orthogonal loadings are given. For delta wings with conical camber the optimum shapes are derived by Ritz's method in reference 7.

In this paper Lagrange's method of undetermined multipliers is applied to the problem of properly combining loadings for the least drag

at a given lift.  By use of this method a simply expressed property of
the optimum loading is found which is an integral form of a property
established by Jones in reference 1.  Jones' property of the optimum
loading is that the downwash on the plan form is constant in the combined
forward- and reverse-flow fields.  The best combination of four types
of nonsingular loading on a delta wing is calculated as a numerical
example of the use of the method.

<div align="center">SYMBOLS</div>

| | |
|---|---|
| $A$ | loading strength parameter |
| $b$ | span |
| $c$ | local chord |
| $C_D$ | drag coefficient |
| $C_{D,ii}$ or $C_{D,i}$ | drag coefficient of ith loading |
| $C_{D,ij}$ | drag coefficient of interference between ith and jth component loadings |
| $C_L$ | lift coefficient |
| $C_{L,i}$ | lift coefficient of ith loading |
| $c_l(x)$ | lift coefficient on spanwise line |
| $c_l(y)$ | lift coefficient on local chord |
| $C_p$ | lifting pressure coefficient |
| $M$ | Mach number |
| $m$ | tangent of semiapex angle |
| $N$ | number of loadings |
| $n = \beta m$ | |
| $R$ | functions of $\theta$ and $n$ (see appendix) |
| $S$ | wing area |

s,t                integers

x,y,z              Cartesian coordinates of lifting surface (see fig. 2)

α                  local angle of attack of lifting surface

$\beta = \sqrt{M^2 - 1}$

ε                  small positive number

$\theta = \dfrac{y}{mx}$

λ                  Lagrange's multiplier

τ                  plan form

1 - μ              root chord of arrow wing

Subscripts:

i,j                ith, jth loading component

M                  minimum among all loadings

O                  minimum among  N  loadings

X                  arbitrary loading

<p align="center">ANALYSIS</p>

<p align="center">Theory</p>

Consider a superposition of  N  loadings of the form

$$C_p = A_1 C_{p,1} + A_2 C_{p,2} + A_3 C_{p,3} + \ldots + A_N C_{p,N} \tag{1}$$

where  A  is the strength parameter and  $C_p$  is the resultant lifting
pressure coefficient at a point on the plan form.  The corresponding
local angle of attack may be written as

$$\alpha = A_1 \alpha_1 + A_2 \alpha_2 + A_3 \alpha_3 + \ldots + A_N \alpha_N \tag{2}$$

The local drag coefficient $C_p\alpha$ is a quadratic in $A$ which may be integrated over the plan form $\tau$ to give the drag coefficient of the wing. Thrust-loaded singularities at the leading edge are therefore excluded from the drag. This exclusion is merely for convenience and is not necessary. The formula for the drag coefficient is

$$C_D = \frac{1}{S} \int_\tau C_p\alpha \, dS = \sum_{i=1}^{N} \sum_{j=1}^{N} C_{D,ij} A_i A_j \tag{3}$$

The average lifting pressure coefficient on the plan form is the lift coefficient, which is

$$C_L = \frac{1}{S} \int_\tau C_p \, dS = \sum_{i=1}^{N} C_{L,i} A_i \tag{4}$$

The problem is to find the set of $A$'s which yields the minimum value of $C_D$ subject to the condition that $C_L$ is constant. Because of the quadratic nature of $C_D$ and the linear form of $C_L$, Lagrange's method of undetermined multipliers is particularly suitable for the solution as it leads to a set of linear algebraic equations.

As shown in reference 8, a function of the $A$ coefficients $F = C_D + \lambda C_L$ is formed, where $\lambda$ is Lagrange's multiplier. The minimum value of $F$ as determined by the $N$ linear algebraic equations $\frac{\partial F}{\partial A_i} = 0$ plus condition (4) is Lagrange's solution. A schematic drawing of the coefficient matrix of the $N + 1$ linear equations is shown in figure 1.

The equations may be written more simply if first the interference drag between the optimum loading and the ith component of the loading is computed. From equations (1) and (2), the following expressions may be written:

$$\left. \begin{array}{l} C_{p,0}\alpha_i = A_1 C_{p,1}\alpha_i + A_2 C_{p,2}\alpha_i + A_3 C_{p,3}\alpha_i + \ldots + A_i C_{p,i}\alpha_i + \ldots + A_N C_{p,N}\alpha_i \\ \alpha_0 C_{p,i} = A_1 \alpha_1 C_{p,i} + A_2 \alpha_2 C_{p,i} + A_3 \alpha_3 C_{p,i} + \ldots + A_i \alpha_i C_{p,i} + \ldots + A_N \alpha_N C_{p,N} \end{array} \right\} \tag{5}$$

Adding equations (5) and integrating over the plan form gives

$$C_{D,0i} = \frac{1}{S} \int_\tau \left( C_{p,0}\alpha_i + \alpha_0 C_{p,i} \right) dS = A_1 C_{D,1i} + A_2 C_{D,2i} + A_3 C_{D,3i} + \ldots +$$

$$2A_i C_{D,ii} + \ldots + A_N C_{D,iN} \tag{6}$$

This expression for $C_{D,0i}$ is a part of the left-hand side of the ith equation of the linear set which is now written as

$$C_{D,0i} + \lambda C_{L,i} = 0 \qquad (7)$$

A simple property of the optimum load distribution may now be derived. First $C_{D,0}$ is written in terms of $C_{D,0i}$:

$$C_{D,0} = A_1 \left( C_{D,01} - A_1 C_{D,11} \right) + A_2 \left( C_{D,02} - A_1 C_{D,12} - A_2 C_{D,22} \right) + \cdots +$$

$$A_N \left[ C_{D,0N} - A_1 C_{D,1N} - A_2 C_{D,2N} - A_3 C_{D,3N} - \cdots - A_N C_{D,NN} \right] \qquad (8)$$

The sum of the terms not containing $C_{D,0i}$ is $-C_{D,0}$. Substituting for $C_{D,0i}$ from equation (7) gives

$$C_{D,0} = -C_{D,0} - \lambda \left( A_1 C_{L,1} + A_2 C_{L,2} + A_3 C_{L,3} + \cdots + A_N C_{L,N} \right) \qquad (9)$$

The term in parentheses is $C_L$; therefore $\lambda$ may be written as

$$\lambda = -2 \frac{C_{D,0}}{C_L} \qquad (10)$$

Substituting equation (10) into equation (7) gives

$$C_{D,0i} = 2 \frac{C_{D,0}}{C_L} C_{L,i} \qquad (11)$$

Since equation (11) holds for any number of loadings, let the number of components increase without limit to include all possible loadings. For an arbitrary loading X and the absolute minimum M, equation (11) may be written as

$$C_{D,MX} = 2 \frac{C_{D,M}'}{C_L} C_{L,X} \qquad (12)$$

The meaning of equation (12) may be simply expressed as follows: The interference drag between the optimum loading and any loading at the same lift coefficient is constant. If the reversibility theorem is applied, equation (12) is an integral equivalent of a condition established by Jones in reference 1. Jones' condition states that for the optimum loading the downwash on the plan form is constant in the combined forward- and reverse-flow fields. Barred variables will represent the reverse flow

which has the same lift loading on the plan form but, in general, a different surface shape. Then, by reversibility,

$$\int_\tau C_{p,M}\alpha_X \, dS = \int_\tau \overline{C_{p,M}}\alpha_X \, dS = \int_\tau \bar{\alpha}_M C_{p,X} \, dS \tag{13}$$

By definition, $C_{D,MX}$ is

$$C_{D,MX} = \frac{1}{S} \int_\tau \left( C_{p,M}\alpha_X + \alpha_M C_{p,X} \right) dS$$

Therefore, equation (12) may be written as

$$\int_\tau C_{p,X}\left(\alpha_M + \bar{\alpha}_M\right) dS = 2 \frac{C_{D,M}}{C_L} \int_\tau C_{p,X} \, dS \tag{14}$$

Since $C_{p,X}$ is arbitrary, $\alpha_M + \bar{\alpha}_M$ must be constant. Hence,

$$\alpha_M + \bar{\alpha}_M = 2 \frac{C_{D,M}}{C_L} \tag{15}$$

This is the condition derived by Jones in reference 1. Equation (12) is then an equivalent integral form of this condition.

Equation (12) shows the orthogonality of the optimum loading to, and only to, zero lift loadings. This result, which was stated by Graham in reference 3, is seen to be a special case of a more general interference drag property given by equation (12).

### Comparison With the Method of Orthogonal Loadings

If two loadings are to be combined, it may be shown that Graham's method of orthogonal loadings (ref. 3) and the present method are equivalent. If the resultant combination of two loadings is combined by the method of reference 3 with a third loading, the lift ratio of the first two loadings is unchanged in the best combination of the three. If $n > 2$ loadings are successively combined in the manner of reference 3, the first $n - 1$ loadings are not allowed to adjust their relative strengths upon addition of the nth. In the present Lagrangian method every loading has equal freedom to adjust. For this reason, the Lagrangian method should be more rapidly convergent.

## NUMERICAL EXAMPLE

Tucker in reference 9 presents formulas for the surface coordinates of delta and arrow wings which support four types of pressure distribution. The formulas are given for subsonic leading edges and supersonic trailing edges. In the notation of this paper (fig. 2) a combination of the four loadings may be written.

$$C_p = A_1 + A_2 x + A_3 \frac{|y|}{m} + A_4 \frac{y^2}{m^2} \tag{16}$$

Formulas for the $C_{D,ij}$ quantities may be derived from equation (16) and the surface formulas given in reference 9, by integrations over the plan form. Details are given in the appendix.

The optimum drag results and strength parameters are presented in figures 3 and 4. In figure 3 the corresponding drag values for a flat delta wing with and without leading-edge thrust (ref. 10) are given for comparison. The drag values for the four component loadings taken alone are also shown. In addition, the drag of the conically cambered optimum delta wing (ref. 7) and Jones' absolute minimum for narrow wings (ref. 1) are plotted.

Noteworthy in figure 3 is the closeness with which all the optimum drags agree with each other and with the drag of a flat delta wing which has a thrust-loaded leading edge. The close approach of the present optimum of four loadings to Jones' absolute minimum for narrow wings is also evident.

In figure 5 the chordwise and spanwise loadings of the optimum combination are shown for the extreme Mach numbers calculated. Comparison with the elliptical loading shows good agreement spanwise and poor agreement chordwise.

The data indicate that the relatively low drag of the flat delta wing with leading-edge thrust can be equalled by properly combining a few loadings having finite pressures everywhere on the plan form.

A plausible speculation suggested by the data is that it is possible to come very close to the minimum drag on a delta wing with but a few steps in a series approximation. Perhaps, too, a restricted minimum, such as the one for conical camber, gives a close approximation to the absolute minimum drag if the restriction is not too unnatural.

## CONCLUDING REMARKS

Lagrange's method of undetermined multipliers is applied to the problem of properly combining lift loadings for the least drag at a given lift on supersonic wings.

The method shows the interference drag between the optimum loading and any loading at the same lift coefficient to be constant. This is an integral form of the criterion established by Robert T. Jones for optimum loadings.

The best combination of four loadings on a delta wing with subsonic leading edges is calculated for several Mach numbers as a numerical example. The loadings considered have finite pressures everywhere on the plan form. At each Mach number the optimum combination of these four nonsingular loadings has very nearly the same drag coefficient as a flat plate with leading-edge thrust.

Langley Aeronautical Laboratory,
       National Advisory Committee for Aeronautics,
              Langley Field, Va., July 27, 1955.

APPENDIX

DETAILS OF NUMERICAL EXAMPLE

Inasmuch as the pressure coefficient and corresponding $\alpha$ are given by

$$\left.\begin{array}{l} C_p = A_1 + A_2 x + A_3 \dfrac{|y|}{m} + A_4 \dfrac{y^2}{m^2} \\[2em] \alpha = A_1 \alpha_1 + A_2 \alpha_2 + A_3 \alpha_3 + A_4 \alpha_4 \end{array}\right\} \quad (A1)$$

then the local drag coefficient may be written as

$$\begin{aligned}
C_p \alpha = {}& A_1{}^2 (\alpha_1) + A_1 A_2 (x\alpha_1 + \alpha_2) + A_1 A_3 \left(\frac{|y|}{m}\alpha_1 + \alpha_3\right) + \\[1em]
& A_1 A_4 \left(\frac{y^2}{m^2}\alpha_1 + \alpha_4\right) + A_2{}^2 (x\alpha_2) + A_2 A_3 \left(\frac{|y|}{m}\alpha_2 + x\alpha_3\right) + \\[1em]
& A_2 A_4 \left(\frac{y^2}{m^2}\alpha_2 + x\alpha_4\right) + A_3{}^2 \left(\frac{|y|}{m}\alpha_3\right) + A_3 A_4 \left(\frac{y^2}{m^2}\alpha_3 + \frac{|y|}{m}\alpha_4\right) + \\[1em]
& A_4{}^2 \left(\frac{y^2}{m^2}\alpha_4\right)
\end{aligned} \quad (A2)$$

The required $C_{D,ij}$ functions are the averages over the plan form (fig. 2) of the quantities in parentheses in equation (A2). Rather than $\alpha_i$ itself, reference 9 gives the surface ordinate $z_i$ which is the chordwise integrated value of $\alpha_i$:

$$z_i = -\int \alpha_i \, dx \quad (A3)$$

The values given for $z_i$ are

$$\left.\begin{array}{l} z_1 = \dfrac{x}{m} R_1 \\[1.5em] z_2 = \dfrac{x^2}{m} R_2 \\[1.5em] z_3 = \dfrac{x^2}{m} R_3 \\[1.5em] z_4 = \dfrac{x^3}{m} R_4 \end{array}\right\} \quad (A4)$$

The values of $R_i$ are functions of $\theta = \dfrac{y}{mx}$ tabulated in reference 9 for different values of $n$. The equations for $R_i$ are

$$R_1 = \frac{1}{4\pi}\left[2\sqrt{1 - n^2\theta^2} - 2\cosh^{-1}\left|\frac{1}{n\theta}\right| + \sqrt{1 - n^2}\,(1 + \theta)\cosh^{-1}\left|\frac{1 + n^2\theta}{n(1 + \theta)}\right| + \right.$$

$$\left. \sqrt{1 - n^2}\,(1 - \theta)\cosh^{-1}\left|\frac{1 - n^2\theta}{n(1 - \theta)}\right|\right] \tag{A5a}$$

$$R_2 = -\frac{1}{4\pi}\left\{\sqrt{1 - n^2\theta^2} - 2\theta^2\cosh^{-1}\left|\frac{1}{n\theta}\right| + \right.$$

$$\frac{1}{\sqrt{1 - n^2}}\left[\frac{n^2(1 - \theta^2)}{2} + \theta + \theta^2\right]\cosh^{-1}\left|\frac{1 + n^2\theta}{n(1 + \theta)}\right| +$$

$$\left. \frac{1}{\sqrt{1 - n^2}}\left[\frac{n^2(1 - \theta^2)}{2} - \theta + \theta^2\right]\cosh^{-1}\left|\frac{1 - n^2\theta}{n(1 - \theta)}\right|\right\} \tag{A5b}$$

$$R_3 = -\frac{1}{4\pi}\left[\frac{5}{2}\sqrt{1 - n^2\theta^2} - \left(1 + 3\theta^2 - \frac{1}{2}n^2\theta^2\right)\cosh^{-1}\left|\frac{1}{n\theta}\right| + \right.$$

$$\frac{(1 + \theta)^2 + 2(1 - n^2)(\theta + \theta^2)}{2\sqrt{1 - n^2}}\cosh^{-1}\left|\frac{1 + n^2\theta}{n(1 + \theta)}\right| +$$

$$\left. \frac{(1 - \theta)^2 - 2(1 - n^2)(\theta - \theta^2)}{2\sqrt{1 - n^2}}\cosh^{-1}\left|\frac{1 - n^2\theta}{n(1 - \theta)}\right|\right] \tag{A5c}$$

$$R_4 = \frac{1}{4\pi}\left\{\frac{\left(1 - n^2\theta^2\right)^{3/2}}{3\left(1 - n^2\right)} + \frac{12 - 10n^2}{3n^2\left(1 - n^2\right)}\,n^2\theta^2\sqrt{1 - n^2\theta^2} - 6\theta^2\cosh^{-1}\left|\frac{1}{n\theta}\right| + \right.$$

$$\frac{1}{\left(1 - n^2\right)^{3/2}}\left[\frac{6 - 9n^2 + 2n^4}{2}\left(\theta^2 + \theta^3\right) + \frac{2 - 3n^2}{2}\left(\theta - \theta^3\right) - \right.$$

$$\left.\frac{n^2}{6}\left(1 + \theta^3\right)\right]\cosh^{-1}\left|\frac{1 + n^2\theta}{n(1 + \theta)}\right| + \frac{1}{\left(1 - n^2\right)^{3/2}}\left[\frac{6 - 9n^2 + 2n^4}{2}\left(\theta^2 - \theta^3\right) - \right.$$

$$\left.\left.\frac{2 - 3n^2}{2}\left(\theta - \theta^3\right) - \frac{n^2}{6}\left(1 - \theta^3\right)\right]\cosh^{-1}\left|\frac{1 - n^2\theta}{n(1 - \theta)}\right|\right\} \qquad \text{(A5d)}$$

For terms in equation (A2) of the type $(y/m)^s\alpha_1$, a spanwise integration of $z_i$ gives the following average on the plan form:

$$\frac{1}{S}\int_\tau\left(\frac{y}{m}\right)^s\alpha_i\,dS = \frac{2}{1 - \mu}\,\frac{1}{m}\left\{\frac{R_i(1)}{s + t + 1} - (1 - \mu)^{s+t+1}\int_0^1\frac{\theta^s R_i(\theta)}{(1 - \mu\theta)^{s+t+2}}\,d\theta\right. \qquad \text{(A6)}$$

For terms of the type $x\alpha_i$ an additional integration by parts in the $x$ direction is required to maintain the $R_i$ functions intact under the integral signs. The result for this case is

$$\frac{1}{S}\int_\tau x\alpha_i\,dS = \frac{2}{1 - \mu}\,\frac{1}{m}\left[\frac{R_i(1)}{t + 2} - (1 - \mu)^{t+2}\int_0^1\frac{R_i(\theta)}{(1 - \mu\theta)^{t+3}}\,d\theta + \right.$$

$$\left.\frac{(1 - \mu)^{t+2}}{t + 2}\int_0^1\frac{R_i(\theta)}{(1 - \mu\theta)^{t+2}}\,d\theta\right] \qquad \text{(A7)}$$

In formulas (A6) and (A7) the value of $t$ for each $i$ is as follows:

| i | t |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 2 |
| 4 | 3 |

By applying formulas (A6) and (A7) to the integration of (A2), the following equations for $C_{D,ij}$ are derived:

$$2mC_{D,11} = \frac{2}{1 - \mu} R_1(1) - 4(1 - \mu) \int_0^1 \frac{R_1(\theta)}{(1 - \mu\theta)^3} \, d\theta \qquad (A8a)$$

$$mC_{D,12} = \frac{2}{3(1 - \mu)}\left[R_1(1) + R_2(1)\right] + \frac{2(1 - \mu)^2}{3} \int_0^1 \frac{R_1(\theta)}{(1 - \mu\theta)^3} \, d\theta -$$

$$2(1 - \mu)^2 \int_0^1 \frac{R_1(\theta)}{(1 - \mu\theta)^4} \, d\theta - 2(1 - \mu)^2 \int_0^1 \frac{R_2(\theta)}{(1 - \mu\theta)^4} \, d\theta$$

$$(A8b)$$

$$mC_{D,13} = \frac{2}{3(1 - \mu)}\left[R_1(1) + R_3(1)\right] - 2(1 - \mu)^2 \int_0^1 \frac{\theta R_1(\theta)}{(1 - \mu\theta)^4} \, d\theta -$$

$$2(1 - \mu)^2 \int_0^1 \frac{R_3(\theta)}{(1 - \mu\theta)^4} \, d\theta \qquad (A8c)$$

$$mC_{D,14} = \frac{1}{2(1 - \mu)}\left[R_1(1) + R_4(1)\right] - 2(1 - \mu)^3 \int_0^1 \frac{\theta^2 R_1(\theta)}{(1 - \mu\theta)^5} \, d\theta -$$

$$2(1 - \mu)^3 \int_0^1 \frac{R_4(\theta)}{(1 - \mu\theta)^5} \, d\theta \qquad (A8d)$$

$$2mC_{D,22} = \frac{R_2(1)}{1 - \mu} - 4(1 - \mu)^3 \int_0^1 \frac{R_2(\theta)}{(1 - \mu\theta)^5} \, d\theta + (1 - \mu)^3 \int_0^1 \frac{R_2(\theta)}{(1 - \mu\theta)^4} \, d\theta$$

$$(A8e)$$

$$mC_{D,23} = \frac{1}{2(1-\mu)}\left[R_2(1) + R_3(1)\right] - 2(1-\mu)^3 \int_0^1 \frac{\theta R_2(\theta)}{(1-\mu\theta)^5}\,d\theta -$$

$$2(1-\mu)^3 \int_0^1 \frac{R_3(\theta)}{(1-\mu\theta)^5}\,d\theta + \frac{1}{2}(1-\mu)^3 \int_0^1 \frac{R_3(\theta)}{(1-\mu\theta)^4}\,d\theta$$

$$\text{(A8f)}$$

$$mC_{D,24} = \frac{2}{5(1-\mu)}\left[R_2(1) + R_4(1)\right] - 2(1-\mu)^4 \int_0^1 \frac{\theta^2 R_2(\theta)}{(1-\mu\theta)^6}\,d\theta -$$

$$2(1-\mu)^4 \int_0^1 \frac{R_4(\theta)}{(1-\mu\theta)^6}\,d\theta + \frac{2}{5}(1-\mu)^4 \int_0^1 \frac{R_4(\theta)}{(1-\mu\theta)^5}\,d\theta$$

$$\text{(A8g)}$$

$$2mC_{D,33} = \frac{R_3(1)}{1-\mu} - 4(1-\mu)^3 \int_0^1 \frac{\theta R_3(\theta)}{(1-\mu\theta)^5}\,d\theta \qquad \text{(A8h)}$$

$$mC_{D,34} = \frac{2}{5(1-\mu)}\left[R_3(1) + R_4(1)\right] - 2(1-\mu)^4 \int_0^1 \frac{\theta^2 R_3(\theta)}{(1-\mu\theta)^6}\,d\theta -$$

$$2(1-\mu)^4 \int_0^1 \frac{\theta R_4(\theta)}{(1-\mu\theta)^6}\,d\theta \qquad \text{(A8i)}$$

$$2mC_{D,44} = \frac{2}{3(1-\mu)}R_4(1) - 4(1-\mu)^5 \int_0^1 \frac{\theta^2 R_4(\theta)}{(1-\mu\theta)^7}\,d\theta \qquad \text{(A8j)}$$

The required $C_{L,i}$ functions are simple integrals over the plan form which yield

$$\left.\begin{array}{l} C_{L,1} = 1 \\[2mm] C_{L,2} = \dfrac{2 - \mu}{3} \\[2mm] C_{L,3} = \dfrac{1}{3} \\[2mm] C_{L,4} = \dfrac{1}{6} \end{array}\right\} \tag{A9}$$

The integrals in equations (A8) were, in general, evaluated numerically. However, several of the integrands in equations (A8) have the form $\dfrac{R_1(\theta)}{(1 - \mu\theta)^t}$ and $\dfrac{R_3(\theta)}{(1 - \mu\theta)^t}$. These functions have an infinite discontinuity at $\theta = 0$. For such a discontinuity, numerical methods break down. Near zero the following approximation is integrated analytically:

$$\left.\begin{array}{l} R_1(\theta) \approx R_1(\epsilon) + \dfrac{1}{2\pi} \cosh^{-1} \dfrac{1}{n\epsilon} - \dfrac{1}{2\pi} \cosh^{-1} \dfrac{1}{n\theta} \\[3mm] R_3(\theta) \approx R_3(\epsilon) - \dfrac{1}{4\pi} \cosh^{-1} \dfrac{1}{n\epsilon} + \dfrac{1}{4\pi} \cosh^{-1} \dfrac{1}{n\theta} \end{array}\right\} 0 < \theta \leqq \epsilon \ll 1 \tag{A10}$$

The integrals for the region $0 \leqq \theta \leqq \epsilon$ can then be approximated:

$$\left.\begin{array}{l} \displaystyle\int_0^\epsilon \dfrac{R_1(\theta)}{(1 - \mu\theta)^t}\, d\theta \approx f(\epsilon)\left[R_1(\epsilon) + \dfrac{1}{2\pi} \cosh^{-1} \dfrac{1}{n\epsilon}\right] - \dfrac{I(\epsilon)}{2\pi} \\[4mm] \displaystyle\int_0^\epsilon \dfrac{R_3(\theta)}{(1 - \mu\theta)^t}\, d\theta \approx f(\epsilon)\left[R_3(\epsilon) - \dfrac{1}{4\pi} \cosh^{-1} \dfrac{1}{n\epsilon}\right] + \dfrac{I(\epsilon)}{4\pi} \end{array}\right\} \tag{A11}$$

where

$$f(\epsilon) = \int_0^\epsilon \dfrac{d\theta}{(1 - \mu\theta)^t} = \epsilon\left[1 + \dfrac{t}{1}\mu\dfrac{\epsilon}{2} + \dfrac{t(t + 1)}{2!}\dfrac{\overline{\mu\epsilon}^2}{3} + \dfrac{t(t + 1)(t + 2)}{3!}\dfrac{\overline{\mu\epsilon}^3}{4} + \ldots\right] \tag{A12}$$

and

$$I(\epsilon) = \int_0^\epsilon \frac{\cosh^{-1} \frac{1}{n\theta}}{(1 - \mu\theta)^t} \, d\theta \tag{A13}$$

The integral in equation (A13) may be evaluated by expanding the denominator by the binomial theorem and writing $I(\epsilon)$ as an infinite series

$$I(\epsilon) = a_0 i_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 + \cdots \tag{A14}$$

where

$$
\left.
\begin{aligned}
a_0 &= 1 & i_0 &= \int_0^\epsilon \cosh^{-1} \frac{1}{n\theta} \, d\theta \\[2mm]
a_1 &= \frac{t}{1} \mu & i_1 &= \int_0^\epsilon \theta \cosh^{-1} \frac{1}{n\theta} \, d\theta \\[2mm]
a_2 &= \frac{t(t + 1)}{2!} \mu^2 & i_2 &= \int_0^\epsilon \theta^2 \cosh^{-1} \frac{1}{n\theta} \, d\theta \\[2mm]
a_3 &= \frac{t(t + 1)(t + 2)}{3!} \mu^3 & & \\[2mm]
a_s &= \frac{t(t + 1) \, \cdots \, (t + s - 1)}{s!} \mu^s & i_s &= \int_0^\epsilon \theta^s \cosh^{-1} \frac{1}{n\theta} \, d\theta
\end{aligned}
\right\} \tag{A15}
$$

The $i_s$ integrals of equation (A15) are evaluated by use of the relation

$$\int \theta^s \cosh^{-1} \frac{1}{n\theta} \, d\theta = \frac{\theta^{s+1}}{s + 1} \cosh^{-1} \frac{1}{n\theta} + \frac{1}{s + 1} \int \frac{\theta^s}{\sqrt{1 - n^2\theta^2}} \, d\theta \tag{A16}$$

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**VARIABLES**

| $A_1$ | $A_2$ | $A_3$ | . | . | . | $A_N$ | $\lambda$ | |
|---|---|---|---|---|---|---|---|---|
| $2C_{D,11}$ | $C_{D,12}$ | $C_{D,13}$ | . | . | . | $C_{D,1N}$ | $C_{L,1}$ | $O$ |
| $C_{D,12}$ | $2C_{D,22}$ | $C_{D,23}$ | . | . | . | $C_{D,2N}$ | $C_{L,2}$ | $O$ |
| $C_{D,13}$ | $C_{D,23}$ | $2C_{D,33}$ | . | . | . | $C_{D,3N}$ | $C_{L,3}$ | $O$ |
| . | . | . | . | . | . | . | . | . |
| . | . | . | . | . | . | . | . | . |
| . | . | . | . | . | . | . | . | . |
| $C_{D,1N}$ | $C_{D,2N}$ | $C_{D,3N}$ | . | . | . | $2C_{D,NN}$ | $C_{L,N}$ | $O$ |
| $C_{L,1}$ | $C_{L,2}$ | $C_{L,3}$ | . | . | . | $C_{L,N}$ | $0$ | $C_L$ |

Figure 1.- Coefficient matrix.
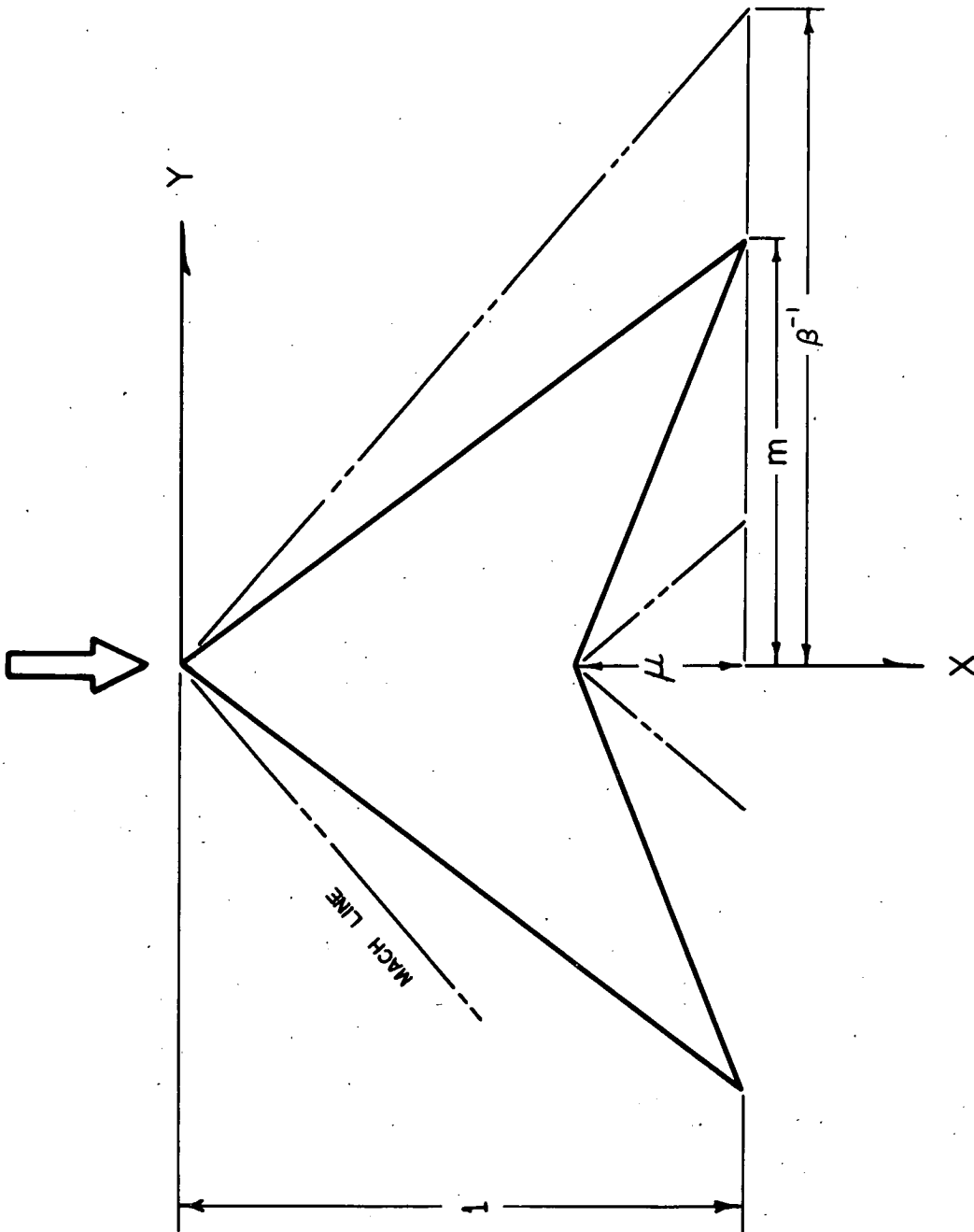
Figure 2.- Arrow plan form.
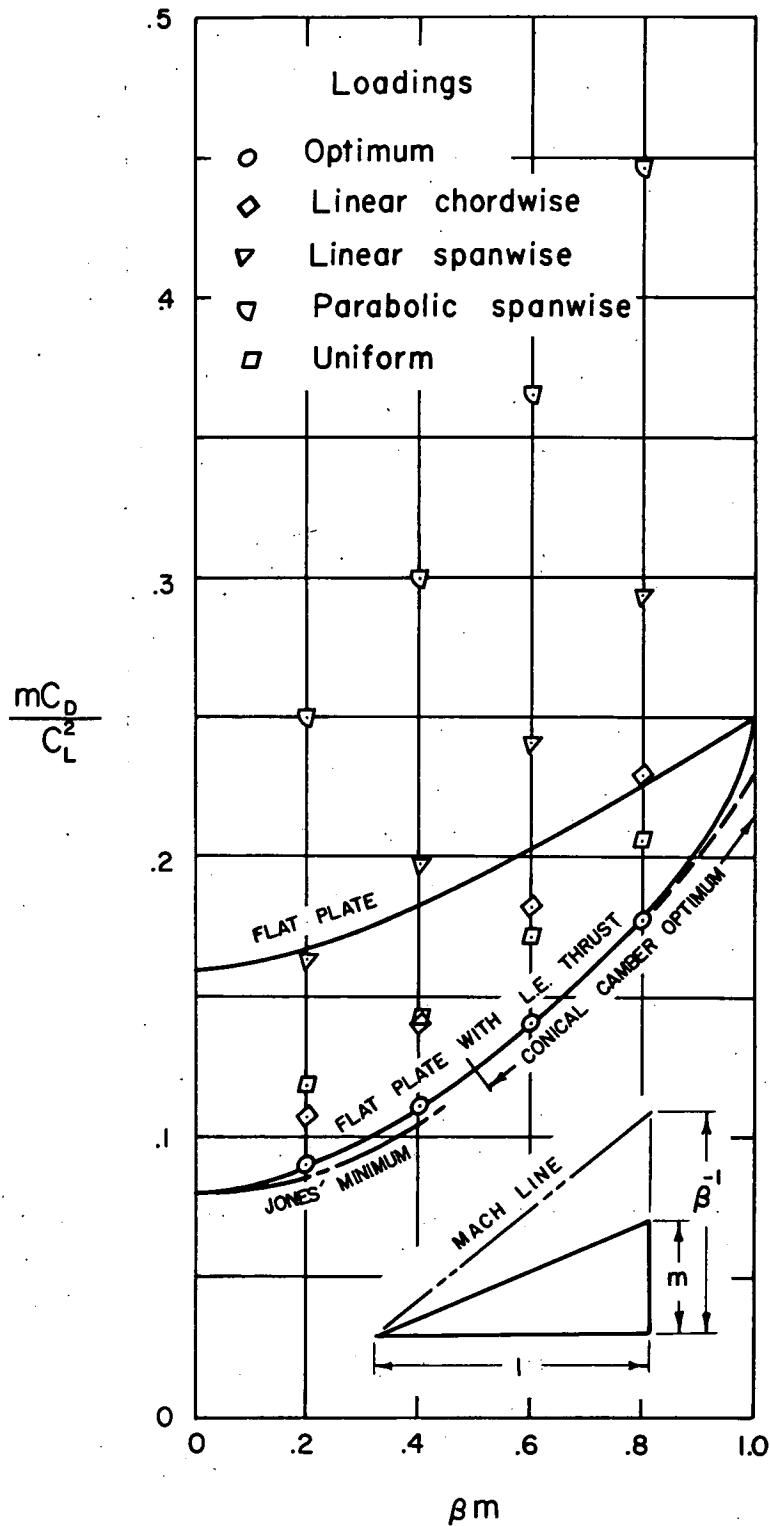
Figure 3.- Comparative drags on a delta plan form.

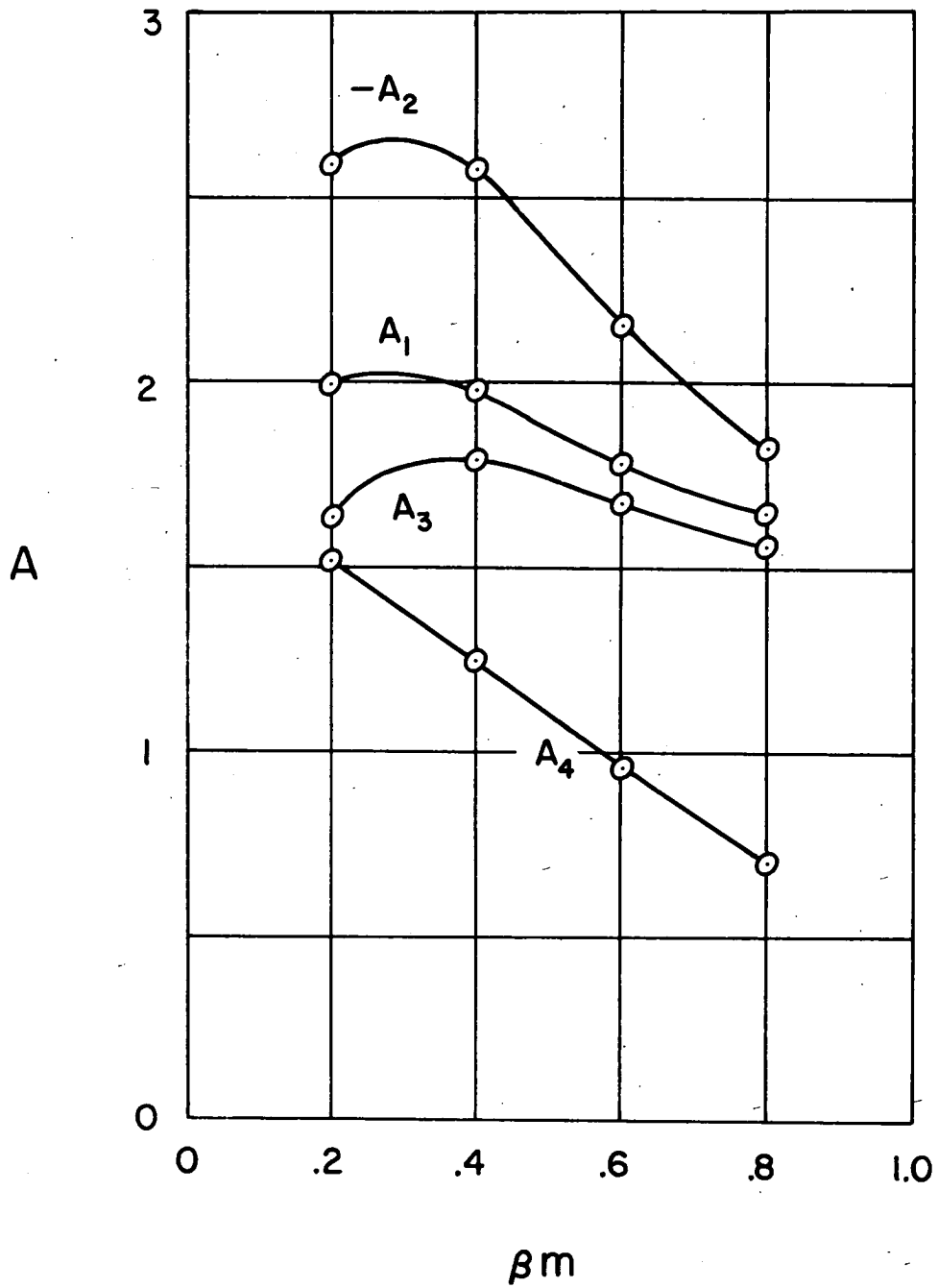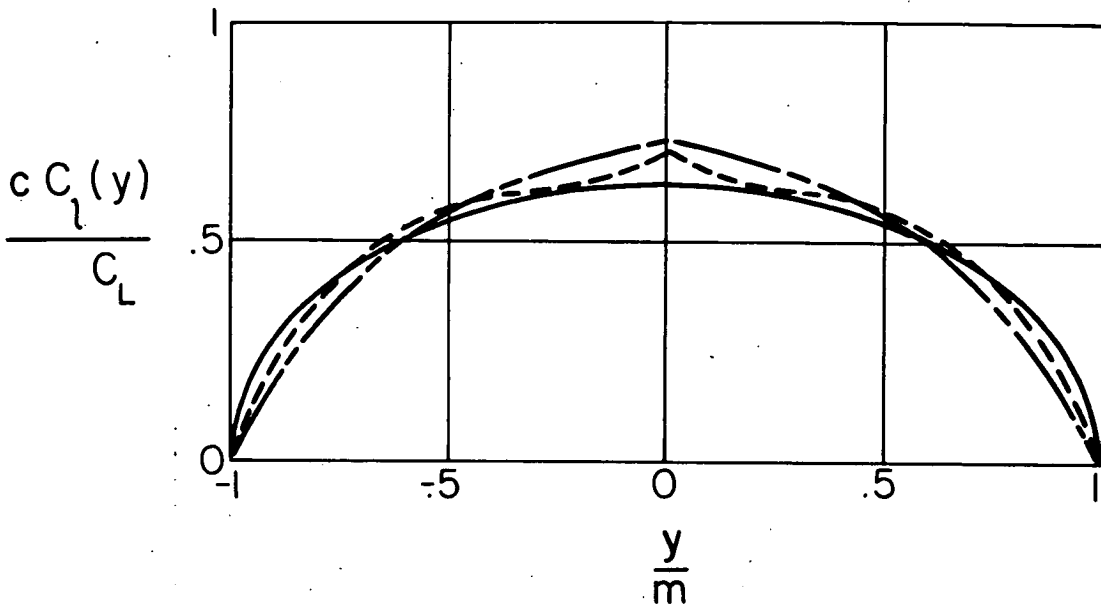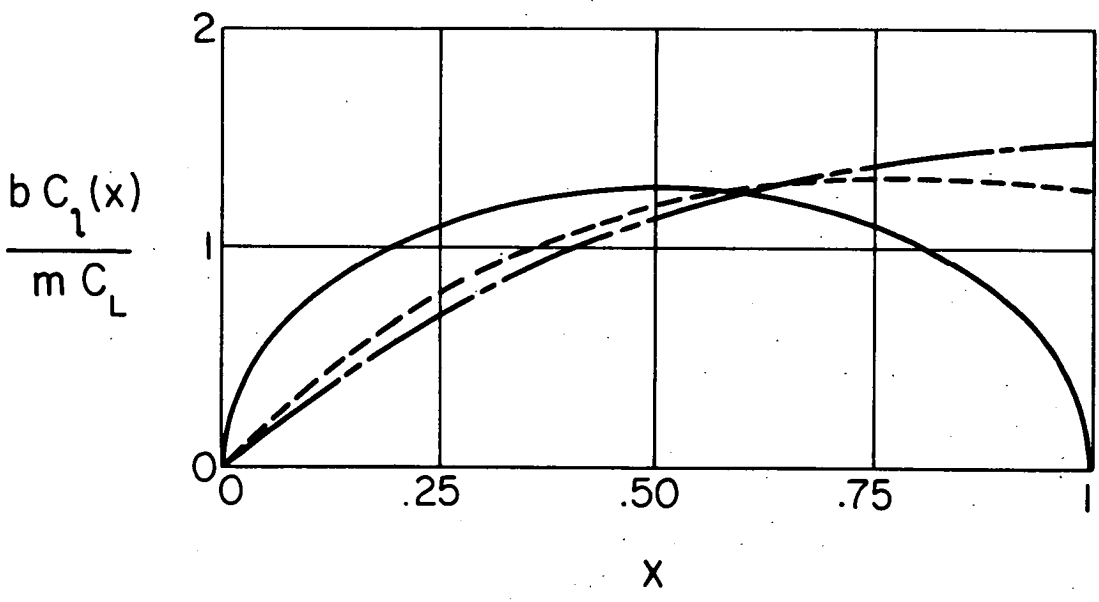Figure 4.- Optimum values of A parameters on a delta plan form.

Figure 5.- Optimum chordwise and spanwise loadings on a delta plan form.