NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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ANALYSIS OF STRESSES IN THE PLASTIC RANGE AROUND A CIRCULAR HOLE IN A PLATE SUBJECTED TO UNIAXIAL TENSION

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SUMMARY

An approximate theoretical solution is presented for the stresses in the plastic range around a circular hole in an infinite sheet subjected to uniaxial tension. The solution is based on the simple deformation theory of plasticity and is found by application of a variational principle in conjunction with the Rayleigh-Ritz procedure and the use of a high-speed computing machine (SEAC). Numerical results are obtained for four different materials, which are characterized by four distinct uniaxial stress-strain curves. The results for stress concentration factor in the plastic range are compared with those obtained from a formula due to Stowell.

INTRODUCTION

The stress distributions that occur around structural discontinuities such as holes and notches have been found theoretically for a wide variety of cases on the basis of the theory of elasticity. An important problem is the corresponding determination of such stress distributions for strain-hardening materials when the stresses exceed the elastic limit. A major obstacle to such an undertaking lies in the fact that basic stress-strain relations in the plastic range have not yet been definitely established for strain-hardening materials; in addition, even after the choice of a particular stress-strain relation is made, the concomitant nonlinear system of equations governing the stress distribution may generally be expected to defy exact analytical solution.

The present paper considers the problem of finding the stresses in the plastic range around a circular hole in an infinite plate subjected to uniaxial tension at infinity (fig. 1). The plate material is assumed to obey the stress-strain relations of the simple deformation theory of plasticity, and an approximate solution is effected by application of an appropriate variational principle in conjunction with the Rayleigh-Ritz procedure and the use of a high-speed computing machine, the Standards' Eastern Automatic Computer (SEAC) of the National Bureau of Standards. Numerical results are given for four different materials, each of which is characterized by a particular uniaxial stress-strain curve.

The results found for the stress concentration factor in the plastic range are compared with those predicted by a simple formula suggested by Stowell in 1950 (ref. 1).

SYMBOLS

σ_r, σ_θ, $\tau_{r\theta}$ — radial, circumferential, and shear stresses, respectively

σ_r^o, σ_θ^o, $\tau_{r\theta}^o$ — radial, circumferential, and shear stresses corresponding to elastic solution, respectively

$\bar{\sigma}_r$, $\bar{\sigma}_\theta$, $\bar{\tau}_{r\theta}$ — correction stresses (for example, $\bar{\sigma}_r = \sigma_r - \sigma_r^o$)

σ_e — effective stress, $\left(\sigma_r^2 + \sigma_\theta^2 - \sigma_r\sigma_\theta + 3\tau_{r\theta}^2\right)^{1/2}$

$$\sigma_e^o = \left[\left(\sigma_r^o\right)^2 + \left(\sigma_\theta^o\right)^2 - \sigma_r^o\sigma_\theta^o + 3\left(\tau_{r\theta}^o\right)^2\right]^{1/2}$$

$$\bar{\sigma}_e = \left(\bar{\sigma}_r^2 + \bar{\sigma}_\theta^2 - \bar{\sigma}_r\bar{\sigma}_\theta + 3\bar{\tau}_{r\theta}^2\right)^{1/2}$$

σ_∞ — uniaxial stress at infinity

σ_1 — nominal yield stress in Ramberg-Osgood equation (see eq. (3))

$\lambda = \sigma_\infty/\sigma_1$

K — stress concentration factor

ϵ_r, ϵ_θ, $\gamma_{r\theta}$ — radial, circumferential, and shear strains, respectively

u_r, u_θ — radial and circumferential displacements, respectively

u_r^o, u_θ^o	radial and circumferential displacements corresponding to elastic solution, respectively
$\bar{u}_r$, $\bar{u}_\theta$	correction displacements (for example, $\bar{u}_r = u_r - u_r^o$)
φ	stress function for $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, and $\bar{\tau}_{r\theta}$
ψ	nondimensional stress function $\left(\varphi = a^2\sigma_1\psi\right)$
a_{pq}	coefficient in expansion for ψ (see eq. (14))
a	hole radius (see fig. 1)
r	radial coordinate (see fig. 1)
θ	angular coordinate (see fig. 1)
ρ	nondimensional radial coordinate ($\rho = r/a$)
$\eta = 1/\rho$	
E	Young's modulus
E_s	secant modulus
G	shear modulus
ν	Poisson's ratio (elastic)
n	exponent in Ramberg-Osgood equation (eq. (3))
W	stress-energy density (defined by eq. (17))
F	complementary energy
F_{mod}	modified complementary energy
Φ	nondimensional modified complementary energy
i	integers 1, 2, . . . 10
j	integers 0, 1, 2, . . . 10
M, N	integers
p, q	integers 0, 1, 2, . . .

α, β, γ	integers 1, 2
$\delta_{\alpha\beta}$	Kronecker delta (equals 1 if $\alpha = \beta$, equals 0 if $\alpha \neq \beta$)

Superscripts:

E	elastic part
P	plastic part

STRESS-STRAIN RELATIONS

On the basis of the simple deformation theory of plasticity, the stress components σ_r, σ_θ, and $\tau_{r\theta}$ (see fig. 1) are related to the corresponding strain components ϵ_r, ϵ_θ, and $\gamma_{r\theta}$ as follows:

$$\left.\begin{aligned}
\epsilon_r &= \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right) + \left(\frac{1}{E_s} - \frac{1}{E}\right)\left(\sigma_r - \frac{1}{2}\sigma_\theta\right) \\
\epsilon_\theta &= \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right) + \left(\frac{1}{E_s} - \frac{1}{E}\right)\left(\sigma_\theta - \frac{1}{2}\sigma_r\right) \\
\gamma_{r\theta} &= \frac{1}{G}\tau_{r\theta} + 3\left(\frac{1}{E_s} - \frac{1}{E}\right)\tau_{r\theta}
\end{aligned}\right\} \tag{1}$$

In equations (1), the quantity E_s is defined as the secant modulus of the uniaxial stress-strain curve at an effective value of stress σ_e given by

$$\sigma_e = \left(\sigma_r^2 + \sigma_\theta^2 - \sigma_r\sigma_\theta + 3\tau_{r\theta}^2\right)^{1/2} \tag{2}$$

The first term in each of equations (1) constitutes the elastic part of the strain; the second term, in each case, is the plastic part. In keeping with the usual assumptions of plane stress, normal and shearing stresses in the direction of the plate thickness are considered to be negligibly small.

The simple stress-strain relations (1) should, strictly speaking, be used only if σ_e is continually increasing - that is, as long as no unloading takes place. However, it will be assumed in the ensuing calculations that equations (1) apply regardless of whether or not σ_e decreases; the extent to which the consequent solution actually exhibits unloading will be examined a posteriori.

In order to afford a complete analytical specification of the stress-strain law, an analytical formulation of the uniaxial stress-strain relation (which determines E_s) is desirable. One such formulation that appears to be useful for a variety of structural materials has been proposed by Ramberg and Osgood (ref. 2) and is given by

$$\epsilon = \frac{\sigma}{E}\left[1 + \frac{3}{7}\left(\frac{\sigma}{\sigma_1}\right)^{n-1}\right] \tag{3}$$

where E is the elastic modulus, σ_1 is the value of stress at which the secant modulus E_s is equal to 0.7E, and n is a parameter chosen to provide the best fit to the stress-strain curve of the actual material under consideration. Alternatively, σ_1 may also be considered as an arbitrary parameter that may be adjusted to provide a good overall fit to the actual stress-strain curve, and the requirement that it specify the actual stress at which $E_s/E = 0.7$ may accordingly be dropped. Equation (3) may be recast into the form

$$\left(\frac{E\epsilon}{\sigma_1}\right) = \frac{\sigma}{\sigma_1} + \frac{3}{7}\left(\frac{\sigma}{\sigma_1}\right)^{n} \tag{3a}$$

Figure 2 shows plots of σ/σ_1 against $E\epsilon/\sigma_1$ for values of $n = 3$, 5, 9, 19, and ∞. The gently sloping stress-strain curve for $n = 3$ is typical of some stainless steels; the sharply breaking curve for $n = 19$ is similar to the stress-strain curves of some aluminum alloys.

With the use of equation (3), the secant modulus E_s needed in the stress-strain relations (eqs. (1)) may be written in terms of the effective stress σ_e as

$$E_s = \frac{E}{1 + \frac{3}{7}\left(\frac{\sigma_e}{\sigma_1}\right)^{n-1}} \tag{4}$$

ANALYSIS

Elastic Solution

Under the action of an applied stress σ_∞ at infinity, the elastic stress distribution in the plate shown in figure 1 is given by (ref. 3, p. 80)

$$\left.\begin{aligned}
\sigma_r{}^o &= \frac{\sigma_\infty}{2}\left[1 - \frac{1}{\rho^2} + \left(1 - \frac{4}{\rho^2} + \frac{3}{\rho^4}\right)\cos 2\theta\right] \\
\sigma_\theta{}^o &= \frac{\sigma_\infty}{2}\left[1 + \frac{1}{\rho^2} - \left(1 + \frac{3}{\rho^4}\right)\cos 2\theta\right] \\
\tau_{r\theta}{}^o &= -\frac{\sigma_\infty}{2}\left(1 + \frac{2}{\rho^2} - \frac{3}{\rho^4}\right)\sin 2\theta
\end{aligned}\right\} \tag{5}$$

where $\rho = r/a$.

At the hole ($\rho = 1$), the maximum value of σ_θ occurs at $\theta = \pm\frac{\pi}{2}$ and is equal to $3\sigma_\infty$. Thus, in the elastic range, the stress concentration factor is 3.

Assumed Form of Plastic Solution

The plastic stress distribution may be written in the form

$$\left.\begin{aligned}
\sigma_r &= \sigma_r{}^o + \bar{\sigma}_r \\
\sigma_\theta &= \sigma_\theta{}^o + \bar{\sigma}_\theta \\
\tau_{r\theta} &= \tau_{r\theta}{}^o + \bar{\tau}_{r\theta}
\end{aligned}\right\} \tag{6}$$

and attention may be directed to the determination of the "correction" stresses $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, and $\bar{\tau}_{r\theta}$ which, when added to the stresses corresponding to the elastic solution, yield the stress distribution in the plastic range. Since the correction stresses must be self-equilibrating, they may be expressed in terms of a stress function $\varphi(r,\theta)$ as

$$\left.\begin{aligned} \bar{\sigma}_r &= \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \\ \bar{\sigma}_\theta &= \frac{\partial^2\varphi}{\partial r^2} \\ \bar{\tau}_{r\theta} &= \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\varphi}{\partial r\,\partial\theta} \end{aligned}\right\} \tag{7}$$

By letting $\varphi = a^2\sigma_1\psi(\rho,\theta)$ the correction stresses, normalized with respect to the nominal yield stress σ_1, may be conveniently expressed in terms of ψ as

$$\left.\begin{aligned} \frac{\bar{\sigma}_r}{\sigma_1} &= \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} \\ \frac{\bar{\sigma}_\theta}{\sigma_1} &= \frac{\partial^2\psi}{\partial\rho^2} \\ \frac{\bar{\tau}_{r\theta}}{\sigma_1} &= \frac{1}{\rho^2}\frac{\partial\psi}{\partial\theta} - \frac{1}{\rho}\frac{\partial^2\psi}{\partial\rho\,\partial\theta} \end{aligned}\right\} \tag{8}$$

It will be assumed that, at each value of the radial coordinate ρ, the function ψ may be expressed as a Fourier series; thus,

$$\psi = \sum_{q=0}^{\infty} f_q(\rho) \cos 2q\theta \tag{9}$$

where the Fourier coefficients f_q are functions of ρ. Since the correction stress must leave the boundary of the circular hole stress-free, it is necessary that $\bar{\sigma}_r = \bar{\tau}_{r\theta} = 0$ at $\rho = 1$. Hence, it follows from equations (8) and (9) that

$$f_0'(1) = 0 \tag{10}$$

$$f_q(1) = f_q'(1) = 0 \qquad (q = 1,2,3,\ldots) \tag{11}$$

Furthermore, the correction stresses must all vanish at infinity, and, indeed, they will be assumed to vanish as $1/\rho^2$. The plausibility of this assumption stems, in part, from an examination of the elastic solution (see eqs. (5)), which shows that the effect of the hole in perturbing the uniform stress field drops off as $1/\rho^2$ in the elastic case. It seems reasonable to expect that plasticity will not introduce stronger perturbation of the uniform stress state. Combining this assumption with equations (10) and (11) leads, then, to the following assumed expressions as approximations for the functions $f_q(\rho)$:

$$f_0'(\rho) = -\left(1 - \frac{1}{\rho}\right) \sum_{p=0}^{M} a_{p0}\rho^{-p-1} \tag{12}$$

$$f_q(\rho) = \left(1 - \frac{1}{\rho}\right)^2 \sum_{p=0}^{M} a_{pq}\rho^{-p} \qquad (q = 1,2,3,\ldots) \tag{13}$$

where the coefficients a_{pq} are, of course, as yet unknown; the stress function may thus be approximated by

$$\psi = -\int^{\rho}\left(1 - \frac{1}{\rho}\right)\sum_{p=0}^{M} a_{p0}\rho^{-p-1}d\rho + \left(1 - \frac{1}{\rho}\right)^2 \sum_{p=0}^{M}\sum_{q=1}^{N} a_{pq}\rho^{-p}\cos 2q\theta \tag{14}$$

where M and N are integers to be chosen as large as is practicable. If the substitution $\eta = 1/\rho$ is made, equation (14) becomes

$$\psi = \int^{\eta}(1 - \eta)\sum_{p=0}^{M} a_{p0}\eta^{p-1}d\eta + (1 - \eta)^2 \sum_{p=0}^{M}\sum_{q=1}^{N} a_{pq}\eta^{p}\cos 2q\theta \tag{15}$$

In terms of derivatives with respect to η, the relationship between stress and stress function is

$$\left.\begin{aligned}
\frac{\bar{\sigma}_r}{\sigma_1} &= -\eta^3 \frac{\partial\psi}{\partial\eta} + \eta^2 \frac{\partial^2\psi}{\partial\theta^2} \\
\frac{\bar{\sigma}_\theta}{\sigma_1} &= \eta^4 \frac{\partial^2\psi}{\partial\eta^2} + 2\eta^3 \frac{\partial\psi}{\partial\eta} \\
\frac{\bar{\tau}_{r\theta}}{\sigma_1} &= \eta^2 \frac{\partial\psi}{\partial\theta} + \eta^3 \frac{\partial^2\psi}{\partial\theta\,\partial\eta}
\end{aligned}\right\} \tag{16}$$

Solution of the problem (for a given material stress-strain curve) now depends upon the determination of the coefficients a_{pq} for given values of the applied stress at infinity. This determination is made possible through the use of the variational principle to be discussed in the next section.

Variational Principle

Variational principles governing the solutions to boundary-value problems associated with a variety of stress-strain laws of plasticity have been reviewed comprehensively in reference 4. These variational principles, however, are specifically limited in applicability to problems involving finite domains; for this reason, it will be necessary to introduce a new modified variational principle for the determination of stresses in the infinite two-dimensional region presently considered. A similar situation in fluid dynamics was encountered by Chi-Teh Wang (refs. 5 and 6) who found it necessary to modify Bateman's variational principle in order to render it applicable to compressible-flow problems in infinite domains.

For deformation-type theories, the variational principles for the stresses involve the so-called stress-energy density, which in the plane stress case can be written as

$$w(\sigma_r,\sigma_\theta,\tau_{r\theta}) = \int_0^{\sigma_r,\sigma_\theta,\tau_{r\theta}} (\epsilon_r \, d\sigma_r + \epsilon_\theta \, d\sigma_\theta + \gamma_{r\theta} \, d\tau_{r\theta}) \tag{17}$$

where the strains are considered as functions of the stresses; these functions are assumed to be such that the line integral in equation (17) is path independent. (This condition is satisfied by all deformation theories that have been seriously considered, including the simple deformation theory.) For a finite two-dimensional domain A having stresses prescribed over its boundaries, the complementary energy is defined as

$$F = \int_A w \, dA \tag{18}$$

The variational principle (ref. 4) states that, when F is written for the true solution to the boundary-value problem,

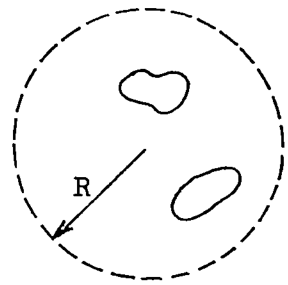
$$\delta F = 0 \tag{19}$$

for all stress variations $\delta\sigma_r$, $\delta\sigma_\theta$, and $\delta\tau_{r\theta}$ that satisfy equilibrium and provide no stress resultants on the boundaries. This principle is readily verified by noticing that, by the use of the definition (17) for w,

$$\delta F = \int_A (\epsilon_r \, \delta\sigma_r + \epsilon_\theta \, \delta\sigma_\theta + \gamma_{r\theta} \, \delta\tau_{r\theta}) dA \tag{20}$$

But since the strains are compatible and the stress variations satisfy equilibrium and provide zero boundary tractions, the principle of virtual work implies that the right-hand side of equation (20) must vanish. Furthermore, if $\delta^2 w > 0$ for all stress variations, $\delta^2 F > 0$, and hence F is a relative minimum.

Now suppose that the external boundary of A recedes to infinity in all directions, and that the boundary-value problem specifies a constant stress state at infinity and vanishing tractions along all internal boundaries. Define A_R as the area between a large circle C_R of radius R and the internal boundaries. (See sketch.) Then, by the principle of virtual work,



$$\delta \int_{A_R} w \, dA = \int_{C_R} \left(u_r \, \delta\sigma_r + u_\theta \, \delta\tau_{r\theta}\right) ds$$

where u_r and u_θ are the true displacements in the radial and circumferential directions. Hence

$$\delta\left[\int_{A_R} w \, dA - \int_{C_R}\left(u_r\sigma_r + u_\theta\tau_{r\theta}\right) ds\right] = 0 \tag{21}$$

for all admissible stress variations. Unfortunately, this variational equation is not of much practical use as a tool for finding the stresses because, for any finite value of R, the displacements u_r and u_θ are not known a priori. Furthermore, it is useless to let R become infinite inasmuch as the quantity in the brackets would then become infinite.

The situation may be remedied by subtracting from the bracketed quantity of equation (21) certain integrals that are independent of the plastic solution and that keep the expression finite as R becomes infinite. Thus, if the stresses of the solution to the elastic problem are denoted by σ_r^o, σ_θ^o, and $\tau_{r\theta}^o$, the modified equation may be written as

$$\delta\left\{\int_{A_R}\left[w(\sigma_r,\sigma_\theta,\tau_{r\theta}) - w(\sigma_r^o,\sigma_\theta^o,\tau_{r\theta}^o)\right] dA - \int_{C_R}\left[u_r(\sigma_r - \sigma_r^o) + u_\theta(\tau_{r\theta} - \tau_{r\theta}^o)\right] ds\right\} = 0 \tag{22}$$

Equation (22) is valid for any given value of R because the extra quantities in this equation, which depend only on the elastic solution, contribute nothing to the variation. If the differences between the stresses of the elastic and plastic solution are assumed to vanish at infinity as $1/r^2$ or faster, it can be shown that

$$F_{mod} = \lim_{R \to \infty} \left\{ \int_{A_R} \left[w(\sigma_r, \sigma_\theta, \tau_{r\theta}) - w(\sigma_r^o, \sigma_\theta^o, \tau_{r\theta}^o) \right] dA - \int_{C_R} \left[u_r(\sigma_r - \sigma_r^o) + u_\theta(\tau_{r\theta} - \tau_{r\theta}^o) \right] ds \right\} \tag{23}$$

is finite. Furthermore, there is no difficulty in evaluating the line integral in equation (23) since the displacements at infinity must be asymptotically equivalent to the easily calculable displacements that would occur for the infinite plate without any interior holes. Thus, since equation (22) is valid for all values of R, it follows that $\delta F_{mod} = 0$ for all admissible stress variations, where now the conditions of admissibility must be extended to permit only stress variations that vanish at infinity as $1/r^2$ or faster.

It is now possible to apply the direct methods of the variational calculus (for example, the Rayleigh-Ritz method) by substituting into equation (23) expressions for the stresses that contain undetermined parameters and then applying the condition that F_{mod} must be stationary with respect to these parameters. If it is known that $\delta^2 w > 0$ for all stress variations, the stronger condition that F_{mod} be a minimum with respect to the parameters can be used; this condition for a relative minimum is actually satisfied by the simple deformation theory (ref. 4).

The general expression for F_{mod} given by equation (23) can be simplified; the reduction is most economically performed with the help of tensor notation and is given in appendix A. The result is

$$F_{mod} = \lim_{R \to \infty} \left\{ \int_{A_R} \left[w^P(\sigma_r, \sigma_\theta, \tau_{r\theta}) - w^P(\sigma_r^o, \sigma_\theta^o, \tau_{r\theta}^o) + \frac{1}{2E}\left(\bar{\sigma}_r^2 + \bar{\sigma}_\theta^2 - \bar{\sigma}_r\bar{\sigma}_\theta + 3\bar{\tau}_{r\theta}^2\right) \right] dA - \int_{C_R} \left(\bar{u}_r\bar{\sigma}_r + \bar{u}_\theta\bar{\tau}_{r\theta} \right) ds \right\} \tag{24}$$

where

$$w^P(\sigma_r,\sigma_\theta,\tau_{r\theta}) = \int_0^{\sigma_r,\sigma_\theta,\tau_{r\theta}}\left(\epsilon_r{}^P d\sigma_r + \epsilon_\theta{}^P d\sigma_\theta + \gamma_{r\theta}{}^P d\tau_{r\theta}\right)$$

$$\bar{u}_r = u_r - u_r{}^O$$

$$\bar{u}_\theta = u_\theta - u_\theta{}^O$$

Here $u_r{}^O$ and $u_\theta{}^O$ denote the displacements associated with the elastic solution to the problem. The superscript P on the strains denotes the plastic part; that is, $\epsilon_r{}^P$, $\epsilon_\theta{}^P$, and $\gamma_{r\theta}{}^P$ are given by the second term in each of equations (1), respectively.

The expression for F_{mod} can now be reduced to the form appropriate to the present hole problem and the simple deformation theory. From equation (2) it follows that

$$d(\sigma_e{}^2) = 2\sigma_r\ d\sigma_r + 2\sigma_\theta\ d\sigma_\theta - \sigma_r\ d\sigma_\theta - \sigma_\theta\ d\sigma_r + 6\tau_{r\theta}\ d\tau_{r\theta}$$

or

$$\sigma_e\ d\sigma_e = \left(\sigma_r - \frac{1}{2}\sigma_\theta\right)d\sigma_r + \left(\sigma_\theta - \frac{1}{2}\sigma_r\right)d\sigma_\theta + 3\tau_{r\theta}\ d\tau_{r\theta} \qquad (25)$$

Hence, by equations (1) and (2),

$$\epsilon_r{}^P d\sigma_r + \epsilon_\theta{}^P d\sigma_\theta + \gamma_{r\theta}{}^P d\tau_{r\theta} = \left(\frac{1}{E_s} - \frac{1}{E}\right)\sigma_e\ d\sigma_e$$

so that, with the use of equation (4) for E_s,

$$\epsilon_r{}^P d\sigma_r + \epsilon_\theta{}^P d\sigma_\theta + \gamma_{r\theta}{}^P d\tau_{r\theta} = \frac{3}{7}\left(\frac{\sigma_e}{\sigma_1}\right)^{n-1}\frac{\sigma_e}{E}\ d\sigma_e$$

Consequently,

$$w^P(\sigma_r,\sigma_\theta,\tau_{r\theta}) = \int_0^{\sigma_e} \frac{3}{7}\left(\frac{\sigma_e}{\sigma_1}\right)^{n-1} \frac{\sigma_e}{E}\, d\sigma_e$$

whence

$$w^P(\sigma_r,\sigma_\theta,\tau_{r\theta}) = \frac{3}{7(n + 1)}\left(\frac{\sigma_e}{\sigma_1}\right)^{n-1} \frac{{\sigma_e}^2}{E} \tag{26}$$

The line integral in equation (22) may be evaluated by noting that, very far from the hole, to within a rigid-body displacement,

$$u_x \sim x \frac{\sigma_\infty}{E_s(\sigma_\infty)}$$

$$u_y \sim -y\left[\frac{\nu\sigma_\infty}{E} + \frac{1}{2}\left(\frac{1}{E_s(\sigma_\infty)} - \frac{1}{E}\right)\sigma_\infty\right]$$

where x and y are rectangular coordinates in the direction of, and perpendicular to, σ_∞.

Hence,

$$\bar{u}_x \sim x\left(\frac{1}{E_s} - \frac{1}{E}\right)\sigma_\infty$$

$$\bar{u}_y \sim \frac{-y}{2}\left(\frac{1}{E_s} - \frac{1}{E}\right)\sigma_\infty$$

and, by a change of coordinates,

$$\bar{u}_r \sim r\left(\frac{1}{E_s} - \frac{1}{E}\right)\left(\frac{1}{4} + \frac{3}{4}\cos 2\theta\right)\sigma_\infty$$

$$\bar{u}_\theta \sim -r\left(\frac{1}{E_s} - \frac{1}{E}\right)\left(\frac{3}{4}\sin 2\theta\right)\sigma_\infty$$

or

$$\left.\begin{aligned}\bar{u}_r &\sim \frac{3r}{7E}\left(\frac{\sigma_\infty}{\sigma_1}\right)^{n-1}\left(\frac{1}{4} + \frac{3}{4}\cos 2\theta\right)\sigma_\infty \\ \bar{u}_\theta &\sim \frac{-3r}{7E}\left(\frac{\sigma_\infty}{\sigma_1}\right)^{n-1}\left(\frac{3}{4}\sin 2\theta\right)\sigma_\infty\end{aligned}\right\} \tag{27}$$

With the substitution of equations (27) and (26) into equation (24), the modified complementary energy expression appropriate to the present problem may be written nondimensionally

$$\frac{2F_{mod}E}{a^2\sigma_1{}^2} = \int_1^\infty \int_0^{2\pi} \left[\frac{6}{7(n+1)}\left(\frac{\sigma_e}{\sigma_1}\right)^{n+1} - \frac{6}{7(n+1)}\left(\frac{\sigma_e{}^o}{\sigma_1}\right)^{n+1} + \left(\frac{\bar{\sigma}_e}{\sigma_1}\right)^2\right]\rho\, d\rho\, d\theta -$$

$$\lim_{\rho \to \infty} \frac{3\lambda^n}{14}\int_0^{2\pi} \rho^2\left[\left(\frac{\bar{\sigma}_r}{\sigma_1}\right)(1 + 3\cos 2\theta) - 3\left(\frac{\bar{\tau}_{r\theta}}{\sigma_1}\right)\sin 2\theta\right]d\theta \tag{28}$$

where $\lambda = \sigma_\infty/\sigma_1$ and $\rho = r/a$.

The quantities σ_e, $\sigma_e{}^o$, and $\bar{\sigma}_e$ are given by equation (2) with the use, respectively, of the actual stress components, the stress components from the elastic solutions, and the correction stress components. The final and most convenient form of the energy expression is obtained from equation (28) by using the fact that the integration need be carried out only over one quadrant of the plate and by introducing the substitution $\eta = 1/\rho$. Then, the function to be minimized becomes

$$\Phi = \int_0^1 \int_0^{\pi/2} \left[\frac{6}{7(n+1)}\left(\frac{\sigma_e}{\sigma_1}\right)^{n+1} - \frac{6}{7(n+1)}\left(\frac{\sigma_e{}^o}{\sigma_1}\right)^{n+1} + \left(\frac{\bar{\sigma}_e}{\sigma_1}\right)^2\right]\frac{1}{\eta^3}\, d\eta\, d\theta -$$

$$\lim_{\eta \to 0} \frac{3\lambda^n}{14}\int_0^{\pi/2} \left[\left(\frac{\bar{\sigma}_r}{\sigma_1}\right)(1 + 3\cos 2\theta) - 3\left(\frac{\bar{\tau}_{r\theta}}{\sigma_1}\right)\sin 2\theta\right]\frac{1}{\eta^2}\, d\theta \tag{29}$$

It is of interest to note that this expression and, hence, the solution for the stresses do not depend on the elastic Poisson's ratio.

Application of Rayleigh-Ritz Procedure

Analytical minimization.- An approximate analytical solution for the stress distribution could, in principle, be effected by retaining a specified number of undetermined coefficients in the stress function (15), calculating the corresponding forms of the correction stress from equations (16), substituting in the expression (29) for Φ, and then minimizing Φ with respect to the coefficients. Such a process would then yield a finite number of simultaneous nonlinear algebraic equations for the unknown coefficients. In practice, the analytical evaluation of Φ constitutes a very laborious calculation because of the $(n + 1)$ powers of σ_e that appear. The work required increases very sharply with increasing values of n and with increases in the number of undetermined coefficients that are taken into account. Consequently, an analytical solution was carried out only for the case $n = 3$ — corresponding to a gently sloping stress-strain curve — and with only three undetermined coefficients.

The contribution to Φ due to the term containing the single integration with respect to θ in equation (29) is very easily evaluated, and turns out to be (see appendix B)

$$\frac{3\pi\lambda^n}{28}\left(a_{00} + 3a_{01}\right) \tag{30}$$

This result is valid regardless of how many coefficients a_{pq} are taken into account and for all values of n. The calculation of the double integral for $n = 3$ and with only a_{00}, a_{01}, and a_{10} considered in the expression (15) for the stress function was, on the other hand, tedious and time consuming. The three simultaneous nonlinear equations obtained from the conditions $\frac{\partial\Phi}{\partial a_{00}} = \frac{\partial\Phi}{\partial a_{01}} = \frac{\partial\Phi}{\partial a_{10}} = 0$ applied to the form of Φ thus found were solved (by successive approximation) for various values of $\lambda = \sigma_\infty/\sigma_1$, with the results shown in table I.

The primary use made of these results was to gain insight into the range of values assumed by the coefficients, and, because of the great labor involved, the analytical approach was abandoned in favor of the numerical process to be described in the following section.

Numerical minimization.- The essential idea involved in the numerical minimization of Φ is its numerical evaluation for systematically varied sets of coefficients a_{pq} that lead (monotonically) to lower and lower values of Φ. The numerical evaluation of the double integral in the expression for Φ was made as follows: The rectangular domain $0 \leqq \eta \leqq 1$ and $0 \leqq \theta \leqq \pi/2$ in the η,θ plane was divided into a 10×10 grid. With the integrand denoted by $I(\eta,\theta)$, the double integral was then approximated by two successive applications of Simpson's rule as follows:

$$\int_0^1 \int_0^{\pi/2} I(\eta,\theta)d\eta\ d\theta = \frac{1}{30}\int_0^{\pi/2} I(0,\theta)d\theta + \frac{\pi}{1800}\sum_{i=1}^{10} s_i \sum_{j=0}^{10} s_j I\left(\frac{i}{10},\frac{\pi j}{20}\right) \tag{31}$$

where, according to Simpson's rule, $S_0 = 1$, $S_1 = 4$, $S_2 = 2$, $S_3 = 4$ . . . $S_7 = 4$, $S_8 = 2$, $S_9 = 4$, and $S_{10} = 1$. Special treatment was needed at $\eta = 0$, since direct numerical evaluation of the integrand is not feasible there. Fortunately, the integral $\int_0^{\pi/2} I(0,\theta)d\theta$ can be evaluated analytically, and turns out to be (see appendix B)

$$\int_0^{\pi/2} I(0,\theta)d\theta = \frac{3\pi\lambda^n}{56}\left(-2a_{00} - 6a_{01} + 2a_{10} + 3a_{11}\right) \tag{32}$$

This result is valid no matter how many coefficients a_{pq} are taken into account. Combining equations (30), (31), and (32) gives the final expression for numerical evaluation:

$$\Phi = \frac{\pi}{1800}\sum_{i=1}^{10}\sum_{j=0}^{10} s_i s_j I\left(\frac{i}{10},\frac{\pi j}{20}\right) + \frac{\pi\lambda^n}{560}\left(58a_{00} + 174a_{01} + 2a_{10} + 3a_{11}\right) \tag{33}$$

With the use of expression (33) for Φ, approximate numerical minimization was performed on the SEAC. The work was limited to the determination of only the four coefficients a_{00}, a_{10}, a_{01}, and a_{11}. A detailed description of the numerical procedure – essentially an application of a "method of steepest descent" – is given in appendix C. As has been stated, the method involves the repeated calculation of Φ for systematically varied sets of the coefficients a_{pq} that lead to a minimum value of Φ. Results were found for four stress-strain curves, described by values of 3, 5, 9, and 19 for the Ramberg-Osgood parameter n; for each n, several values of applied stress, as specified by $\lambda = \sigma_\infty/\sigma_1$, were considered.

The results found for the coefficients a_{pq} are given in table II. Convergence to the final results was generally slow; from 12 to 35 cycles of iteration were made for various values of λ. In each case, the final values of the coefficients agreed with those of several preceding iterations to within approximately 1 percent of the largest coefficient, but there was no absolute guarantee that many additional iterations would not have changed the results appreciably.

RESULTS AND DISCUSSION

Stress Concentration Factor

Once the values of the coefficients a_{pq} are known, the stress state at any point can be found from equations (5), (6), and (16). In terms of the four coefficients considered, the stresses are then given by

$$\left.\begin{aligned}
\frac{\sigma_r}{\sigma_1} &= \frac{\lambda}{2}\left[\left(1 - \eta^2\right) + \left(1 - 4\eta^2 + 3\eta^4\right)\cos 2\theta\right] - \eta^2\left[(1 - \eta)a_{00} + \left(\eta - \eta^2\right)a_{10} + \right. \\
&\qquad \left.\left(4 - 10\eta + 6\eta^2\right)a_{01}\cos 2\theta + \left(5\eta - 12\eta^2 + 7\eta^3\right)a_{11}\cos 2\theta\right] \\
\frac{\sigma_\theta}{\sigma_1} &= \frac{\lambda}{2}\left[\left(1 + \eta^2\right) - \left(1 + 3\eta^4\right)\cos 2\theta\right] + \eta^2\left[\left(1 - 2\eta\right)a_{00} + \left(2\eta - 3\eta^2\right)a_{10} - \right. \\
&\qquad \left.\left(4\eta - 6\eta^2\right)a_{01}\cos 2\theta + \left(2\eta - 12\eta^2 + 12\eta^3\right)a_{11}\cos 2\theta\right] \\
\frac{\tau_{r\theta}}{\sigma_1} &= -\frac{\lambda}{2}\left(1 + 2\eta^2 - 3\eta^4\right)\sin 2\theta - \eta^2\left[\left(2 - 8\eta + 6\eta^2\right)a_{01}\sin 2\theta + \right. \\
&\qquad \left.\left(4\eta - 12\eta^2 + 8\eta^3\right)a_{11}\sin 2\theta\right]
\end{aligned}\right\} \tag{34}$$

Of perhaps the greatest interest is the value of the stress concentration factor determined by the ratio of $\sigma_\theta(1,\pi/2)$ to the applied stress. This factor is found to be

$$K = 3 - \frac{a_{00} + 2a_{01} + a_{10} + 2a_{11}}{\lambda} \tag{35}$$

Figure 3 shows the variation of the stress concentration factor with $\lambda = \sigma_\infty/\sigma_1$ for each of the stress-strain curves considered. Also included in the figure is a limiting curve for the case $n = \infty$, corresponding to an elastic—ideally plastic material (fig. 2). This curve was obtained simply by assuming that the maximum stress at the hole is $3\sigma_\infty$ until σ_∞ reaches $\frac{1}{3}\sigma_1$, and thereafter remains at the value σ_1; thus, for $\frac{1}{3}\sigma_1 < \sigma_\infty < \sigma_1$ (or for $\frac{1}{3} < \lambda < 1$), the stress concentration factor is $K = 3/\lambda$. Values of λ greater than 1 are meaningless for this case inasmuch as the applied stress can never exceed σ_1. It is interesting to note that the curves calculated on SEAC for $n = 3, 5, 9,$ and 19, as well as the limiting curve for $n = \infty$, all intersect in a very small interval around $\lambda = 0.4$.

It is of particular interest to compare these results for stress concentration factors with those obtained from the following formula suggested by Stowell (ref. 1):

$$K = 1 + 2\frac{E_s(K\sigma_\infty)}{E_s(\sigma_\infty)} \tag{36}$$

For the Ramberg-Osgood stress-strain curve, Stowell's formula for K becomes

$$K = 1 + \frac{2\left(1 + \frac{3}{7}\lambda^{n-1}\right)}{1 + \frac{3}{7}K^{n-1}\lambda^{n-1}} \tag{37}$$

The variation of K with $\lambda = \sigma_\infty/\sigma_1$ obtained from equation (37) is given in figure 4 for each of the four stress-strain curves considered. The results of the present theoretical calculations and, in addition, the results obtained from the three-coefficient analytical solution for $n = 3$ are shown for comparison.

It is seen from figure 4 that the agreement of the present results with those obtained from Stowell's formula is only fair. Perhaps a more meaningful comparison is afforded by figure 5, which shows the variation with σ_∞/σ_1 of the maximum stress itself (nondimensionalized with the stress σ_1) for each of the four cases. It is interesting to note that as n increases a tendency for the curve to develop an inflection grows and, indeed, this inflection becomes so distinct at $n = 19$ as to indicate a very small reduction of stress with increasing load, at about

$\sigma_\infty/\sigma_1 = 0.7$. Such a reduction of stress, or "unloading," really invalidates the use of the stress-strain relations (1) but the magnitude of the unloading is so small (at least at the point of the plate presently considered) as to be probably not too important.

The results for the stress plotted according to Stowell's formula deviate from that of the present theory by amounts varying up to about 15 percent. It is impossible, however, to ascribe much significance to any agreement, or the lack thereof, between the two sets of results. While the present results stem from an approximate solution that satisfies equilibrium of stress exactly and compatibility of strain approximately, Stowell's formula is based on a treatment that ignores compatibility entirely and satisfies equilibrium in some average fashion over the entire region exterior to the hole. The present analysis clearly has a much more rigorous theoretical foundation, but its actual physical validity is not known inasmuch as it is based on an arbitrary plasticity stress-strain law as well as being approximate, even within the framework of the assumed theory. On the other hand, in reference 1 Stowell exhibits good agreement between the prediction of his formula and experimental results for stress concentration factors obtained by Griffith (ref. 7) for 2024-T (formerly 24S-T) aluminum alloy. A careful study of the stress-strain curve measured by Griffith for his specimen reveals that it cannot be described satisfactorily by a Ramberg-Osgood equation; consequently, no meaningful comparison between Griffith's results and the present analysis can be made. Certainly, Stowell's formula has much to recommend it by virtue of its simplicity; but the ultimate assessment of its validity, as well as that of the present analysis, must come from experiments on a variety of materials having both gently sloping and sharply breaking stress-strain curves.

Stress Distribution

Distributions of stress have been computed from equations (34) and are shown in figure 6 for the case $n = 9$ and $\sigma_\infty/\sigma_1 = 0.9$; these distributions are typical of those that occur for other cases in which substantial plastic flow occurs in the neighborhood of the hole. The separate sketches in figure 6 show the variations of the stress components along three radial lines: $\theta = \pi/2$ (the location of maximum tension at the hole), $\theta = \pi/4$, and $\theta = 0$. For comparison, the elastic stress distributions are also shown. The largest deviations between the two occur, as is to be expected, where the elastically computed stresses are highest. In addition, a rather striking difference between the elastic and plastic distributions of the circumferential stress σ_θ occurs at $\theta = \pi/4$; in the plastic case, the circumferential stress rises to a maximum and then drops, in contrast to the monotonic decrease in the elastic case.

The variation of σ_e, as given by equations (2) and (34), with increasing applied stress at infinity has been subjected to numerical study in order to determine whether the solution indicates unloading anywhere in the sheet. The very slight amount of unloading that occurs at the hole (where $\sigma_e = \sigma_\theta$) for $n = 19$ and $\theta = \pi/2$ has already been noted in figure 5; spot calculations of σ_e elsewhere in the sheet fail to indicate unloading except at values of σ_e well below σ_1, for which the behavior is still essentially elastic.

CONCLUDING REMARKS

The theoretical solution presented for the stresses in the plastic range around a circular hole in a plate subjected to uniaxial tension is very far from a final answer to the problem considered. The solution is based on a stress-strain relation of questionable validity; and it is only approximate, even within the framework of the postulated theory. However, it is felt that the solution has intrinsic theoretical interest since, except for problems with radial symmetry, little attention has hitherto been directed at plane stress problems for strain-hardening materials. Further, the solution fulfills to a limited extent the promise of variational principles as a useful tool in the solution of boundary-value problems of plasticity. On the negative side of the ledger is the fact that the numerical minimization process used in conjunction with the variational principle is not very efficient. Extension of the present approach to include many more degrees of freedom would sorely tax the capacity of even the largest high-speed computing machine presently available, and would probably require prohibitive amounts of machine time. The development of more efficient numerical minimization procedures would be a boon to the use of the variational approach to nonlinear problems of the kind considered in this paper.

Of great interest would be another treatment of the problem via the simple flow, or incremental, theory of plasticity, and the subsequent comparison with the present results. Although the flow and deformation theories may differ substantially from one another for arbitrary stress paths, little is known about the effect of such deviations on the stress distributions that would be predicted by the two theories in problems such as the presently considered one.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., August 16, 1955.

APPENDIX A

SIMPLIFICATION OF MODIFIED COMPLEMENTARY ENERGY EXPRESSION

In tensor notation, with the use of the summation convention, the complementary energy density for plane stress may be written

$$w(\sigma_{\alpha\beta}) = \int_0^{\sigma_{\alpha\beta}} \epsilon_{\alpha\beta}\, d\sigma_{\alpha\beta}$$

where the indices take on only the values 1 and 2. The modified complementary energy expression (23) is, in tensor form,

$$F_{mod} = \lim_{R \longrightarrow \infty} \left[\int_{A_R}\left(\int_0^{\sigma_{\alpha\beta}} \epsilon_{\alpha\beta}\, d\sigma_{\alpha\beta} - \int_0^{\sigma_{\alpha\beta}{}^o} \epsilon_{\alpha\beta}\, d\sigma_{\alpha\beta}\right) dA - \int_{C_R}\left(T_\alpha - T_\alpha{}^o\right) u_\alpha\, ds\right] \tag{A1}$$

where T_α denotes the traction on C_R, u_α is the displacement vector, and the superscript o denotes the elastic solution. Introducing the notations $\bar{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} - \sigma_{\alpha\beta}{}^o$, $\bar{u}_\alpha = u_\alpha - u_\alpha{}^o$, and $\overline{T}_\alpha = T_\alpha - T_\alpha{}^o$ leads to

$$F_{mod} = \lim_{R \longrightarrow \infty} \left\{\int_{A_R}\left[\int_{\sigma_{\alpha\beta}{}^o}^{\sigma_{\alpha\beta}{}^o + \bar{\sigma}_{\alpha\beta}} \left(\epsilon_{\alpha\beta}{}^E + \epsilon_{\alpha\beta}{}^P\right) d\sigma_{\alpha\beta}\right] dA - \int_{C_R} \overline{T}_\alpha\left(u_\alpha{}^o + \bar{u}_\alpha\right) ds\right\}$$

where

$$\epsilon_{\alpha\beta}{}^E = \frac{1}{E}\left[(1 + \nu)\sigma_{\alpha\beta} - \nu\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right]$$

and

$$\epsilon_{\alpha\beta}{}^{P} = \frac{3}{2}\left(\frac{1}{E_s} - \frac{1}{E}\right)\left(\sigma_{\alpha\beta} - \frac{1}{3}\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right)$$

Now

$$\int_{\sigma_{\alpha\beta}{}^{O}}^{\sigma_{\alpha\beta}{}^{O}+\bar{\sigma}_{\alpha\beta}} \epsilon_{\alpha\beta}{}^{E}\, d\sigma_{\alpha\beta} = \frac{1}{E}\left[(1 + \nu)\sigma_{\alpha\beta}{}^{O}\bar{\sigma}_{\alpha\beta} - \nu\sigma_{\alpha\alpha}{}^{O}\bar{\sigma}_{\beta\beta}\right] + \frac{1}{2E}\left[(1 + \nu)\bar{\sigma}_{\alpha\beta}\bar{\sigma}_{\alpha\beta} - \nu\bar{\sigma}_{\alpha\alpha}\bar{\sigma}_{\beta\beta}\right] \tag{A2}$$

But

$$\frac{1}{E}\left[(1 + \nu)\sigma_{\alpha\beta}{}^{O}\bar{\sigma}_{\alpha\beta} - \nu\sigma_{\alpha\alpha}{}^{O}\bar{\sigma}_{\beta\beta}\right] = \bar{\sigma}_{\alpha\beta}\epsilon_{\alpha\beta}{}^{O}$$

and, by the principle of virtual work,

$$\int_{A_R} \bar{\sigma}_{\alpha\beta}\epsilon_{\alpha\beta}{}^{O}\, dA = \int_{C_R} \bar{T}_{\alpha}u_{\alpha}{}^{O}\, ds$$

Hence,

$$F_{mod} = \lim_{R\to\infty}\left(\int_{A_R}\left\{\int_{\sigma_{\alpha\beta}{}^{O}}^{\sigma_{\alpha\beta}{}^{O}+\bar{\sigma}_{\alpha\beta}} \epsilon_{\alpha\beta}{}^{P} d\sigma_{\alpha\beta} + \frac{1}{2E}\left[(1 + \nu)\bar{\sigma}_{\alpha\beta}\bar{\sigma}_{\alpha\beta} - \nu\bar{\sigma}_{\alpha\alpha}\bar{\sigma}_{\beta\beta}\right]\right\} dA - \int_{C_R} \bar{T}_{\alpha}\bar{u}_{\alpha}\, ds\right) \tag{A3}$$

Now

$$(1 + \nu)\bar{\sigma}_{\alpha\beta}\bar{\sigma}_{\alpha\beta} - \nu\bar{\sigma}_{\alpha\alpha}\bar{\sigma}_{\beta\beta} = \frac{3}{2}\left(\bar{\sigma}_{\alpha\beta}\bar{\sigma}_{\alpha\beta} - \frac{1}{3}\,\bar{\sigma}_{\alpha\alpha}\bar{\sigma}_{\beta\beta}\right) + \left(\nu - \frac{1}{2}\right)\left(\bar{\sigma}_{\alpha\beta}\bar{\sigma}_{\alpha\beta} - \bar{\sigma}_{\alpha\alpha}\bar{\sigma}_{\beta\beta}\right)$$

The first of these two terms is precisely $(\sigma_e)^2$, as defined by equation (2), and the quantity $\int_0^{\sigma_{\alpha\beta}} \epsilon_{\alpha\beta}{}^{P}\, d\sigma_{\alpha\beta}$ is w^P, as defined immediately after equation (24). Hence, equation (A3) is the tensor equivalent of equation (24), with the additional term

$$\lim_{R \to \infty} \int_{A_R} \frac{\nu - \frac{1}{2}}{2E} \left(\bar{\sigma}_{\alpha\beta} \bar{\sigma}_{\alpha\beta} - \bar{\sigma}_{\alpha\alpha} \bar{\sigma}_{\beta\beta} \right) dA \tag{A4}$$

It remains only to show that this term vanishes to establish the validity of equation (24).

In terms of the stress function φ,

$$\bar{\sigma}_{\alpha\beta} = \varphi_{,\gamma\gamma} \delta_{\alpha\beta} - \varphi_{,\alpha\beta}$$

where, as is usual in tensor notation, commas denote differentiation. Then

$$\int_{A_R} \left(\bar{\sigma}_{\alpha\beta} \bar{\sigma}_{\alpha\beta} - \bar{\sigma}_{\alpha\alpha} \bar{\sigma}_{\beta\beta} \right) dA = \int_{A_R} \left(\varphi_{,\alpha\beta} \varphi_{,\alpha\beta} - \varphi_{,\alpha\alpha} \varphi_{,\beta\beta} \right) dA$$

$$= \int_{A_R} \left[\left(\varphi_{,\alpha\beta} \varphi_{,\alpha} \right)_{,\beta} - \left(\varphi_{,\alpha\alpha} \varphi_{,\beta} \right)_{,\beta} \right] dA \tag{A5}$$

Now since $\bar{\sigma}_{\alpha\beta}$ provides zero resultant stress at all interior boundaries, $\varphi_{,\alpha}$ is a single-valued function (see ref. 3, p. 191); hence, Green's theorem can be applied to the area integral (A5), with the result that it equals the line integral

$$I_R = \int_{\Gamma} \left(\varphi_{,\alpha\beta} \varphi_{,\alpha} - \varphi_{,\alpha\alpha} \varphi_{,\beta} \right) n_\beta \, ds$$

where the line Γ includes both the circle C_R and the internal boundaries, and n_β is the exterior unit normal to Γ. But

$$I_R = \int_\Gamma \varphi_{,\alpha}\left(\varphi_{,\alpha\beta} - \varphi_{,\gamma\gamma}\delta_{\alpha\beta}\right)n_\beta \, ds$$

$$= -\int_\Gamma \varphi_{,\alpha}\bar{\sigma}_{\alpha\beta}n_\beta \, ds$$

$$= -\int_\Gamma \varphi_{,\alpha}\bar{T}_\alpha \, ds$$

Since $\bar{T}_\alpha$ vanishes at internal boundaries,

$$I_R = -\int_{C_R} \varphi_{,\alpha}\bar{T}_\alpha \, ds$$

But since $\sigma_{\alpha\beta}$, by assumption, dies out at infinity as $1/r^2$ or faster, it follows that

$$\lim_{R \to \infty} I_R = 0$$

Thus, it is established that the expression (A4) is zero and, therefore, the simplified form of F_{mod} given by equation (24) is equivalent to equation (23).

APPENDIX B

EVALUATION OF TWO LINE INTEGRALS

The term

$$-\lim_{\eta\to 0}\frac{3\lambda^n}{14}\int_0^{\pi/2}\left[\frac{\bar{\sigma}_r}{\sigma_1}\left(1 + 3\cos 2\theta\right) - 3\left(\frac{\bar{\tau}_{r\theta}}{\sigma_1}\right)\sin 2\theta\right]\frac{1}{\eta^2}\,d\theta \tag{B1}$$

appearing in the expression (29) for Φ is readily evaluated. From equations (34) which give the stresses in terms of the coefficient a_{pq} in the stress function, it is seen that

$$\frac{\bar{\sigma}_r}{\sigma_1} = -\eta^2\left(a_{00} + 4a_{01}\cos 2\theta\right) + O(\eta^3)$$

$$\frac{\bar{\tau}_{r\theta}}{\sigma_1} = -\eta^2\left(2a_{01}\sin 2\theta\right) + O\left(\eta^3\right)$$

The quantity (B1) is therefore

$$\frac{3\lambda^n}{14}\int_0^{\pi/2}\left[\left(a_{00} + 4a_{01}\cos 2\theta\right)\left(1 + 3\cos 2\theta\right) - 6a_{01}(\sin 2\theta)^2\right]d\theta =$$

$$\frac{3\pi\lambda^n}{28}\left(a_{00} + 3a_{01}\right)$$

The evaluation of the integral $\int_0^{\pi/2} I(0,\theta)d\theta$ is a little more involved. From equation (29),

$$\int_0^{\pi/2} I(0,\theta)d\theta = \lim_{\eta\to 0}\int_0^{\pi/2}\left[\frac{6}{7(n + 1)}\left(\frac{\sigma_e}{\sigma_1}\right)^{n+1} - \frac{6}{7(n + 1)}\left(\frac{\sigma_e{}^o}{\sigma_1}\right)^{n+1} + \left(\frac{\bar{\sigma}_e}{\sigma_1}\right)^2\right]\frac{1}{\eta^3}\,d\theta$$

Since the correction stresses $\bar{\sigma}_{\alpha\beta}$ are $O(\eta^2)$, then $\bar{\sigma}_e^{\ 2}$ is $O(\eta^4)$ and therefore the last term in the integrand makes no contribution in the limit as η approaches zero. The first two terms in the integrand can be expanded into

$$\frac{6}{7(n+1)}\left\{\left[\left(\frac{\sigma_r^{\ o}+\bar{\sigma}_r}{\sigma_1}\right)^2+\left(\frac{\sigma_\theta^{\ o}+\bar{\sigma}_\theta}{\sigma_1}\right)^2-\left(\frac{\sigma_r^{\ o}+\bar{\sigma}_r}{\sigma_1}\right)\left(\frac{\sigma_\theta^{\ o}+\bar{\sigma}_\theta}{\sigma_1}\right)+3\left(\frac{\tau_{r\theta}^{\ o}+\bar{\tau}_{r\theta}}{\sigma_1}\right)^2\right]^{\frac{n+1}{2}}-\left[\left(\frac{\sigma_r^{\ o}}{\sigma_1}\right)^2+\left(\frac{\sigma_\theta^{\ o}}{\sigma_1}\right)^2-\right.\right.$$

$$\left.\left.\left(\frac{\sigma_r^{\ o}}{\sigma_1}\right)\left(\frac{\sigma_\theta^{\ o}}{\sigma_1}\right)+3\left(\frac{\tau_{r\theta}^{\ o}}{\sigma_1}\right)^2\right]^{\frac{n+1}{2}}\right\}=\frac{6}{7(n+1)}\left\{\left[\left(\frac{\sigma_r^{\ o}}{\sigma_1}\right)^2+\left(\frac{\sigma_\theta^{\ o}}{\sigma_1}\right)^2-\left(\frac{\sigma_r^{\ o}}{\sigma_1}\right)\left(\frac{\sigma_\theta^{\ o}}{\sigma_1}\right)+3\left(\frac{\tau_{r\theta}^{\ o}}{\sigma_1}\right)^2+2\left(\frac{\sigma_r^{\ o}}{\sigma_1}\right)\left(\frac{\bar{\sigma}_r}{\sigma_1}\right)+\right.\right.$$

$$\left.2\left(\frac{\sigma_\theta^{\ o}}{\sigma_1}\right)\left(\frac{\bar{\sigma}_\theta}{\sigma_1}\right)-\left(\frac{\sigma_r^{\ o}}{\sigma_1}\right)\left(\frac{\bar{\sigma}_\theta}{\sigma_1}\right)-\left(\frac{\sigma_\theta^{\ o}}{\sigma_1}\right)\left(\frac{\bar{\sigma}_r}{\sigma_1}\right)+6\left(\frac{\tau_{r\theta}^{\ o}}{\sigma_1}\right)\left(\frac{\bar{\tau}_{r\theta}}{\sigma_1}\right)+O\left(\bar{\sigma}_{\alpha\beta}^{\ 2}\right)\right]^{\frac{n+1}{2}}-\left[\left(\frac{\sigma_r^{\ o}}{\sigma_1}\right)^2+\left(\frac{\sigma_\theta^{\ o}}{\sigma_1}\right)^2-\left(\frac{\sigma_r^{\ o}}{\sigma_1}\right)\left(\frac{\sigma_\theta^{\ o}}{\sigma_1}\right)+\right.$$

$$\left.\left.3\left(\frac{\tau_{r\theta}^{\ o}}{\sigma_1}\right)^2\right]^{\frac{n+1}{2}}\right\}$$

Now, terms of the order $\bar{\sigma}_{\alpha\beta}^{\ 2}$ will make no contribution to the integral as $\eta \longrightarrow 0$ because $\bar{\sigma}_{\alpha\beta}$ is $O(\eta^2)$. Hence, using the binomial expansion formula leads to

$$\int_0^{\pi/2} I(0,\theta)d\theta = \lim_{\eta\longrightarrow 0}\int_0^{\pi/2}\frac{6}{7(n+1)}\left(\frac{n+1}{2}\right)\left[\left(\frac{\sigma_r^{\ o}}{\sigma_1}\right)^2+\left(\frac{\sigma_\theta^{\ o}}{\sigma_1}\right)^2-\right.$$

$$\left.\left(\frac{\sigma_r^{\ o}}{\sigma_1}\right)\left(\frac{\sigma_\theta^{\ o}}{\sigma_1}\right)+3\left(\frac{\tau_{r\theta}^{\ o}}{\sigma_1}\right)^2\right]^{\frac{n-1}{2}}\left[2\left(\frac{\sigma_r^{\ o}}{\sigma_1}\right)\left(\frac{\bar{\sigma}_r}{\sigma_1}\right)+2\left(\frac{\sigma_\theta^{\ o}}{\sigma_1}\right)\left(\frac{\bar{\sigma}_\theta}{\sigma_1}\right)-\right.$$

$$\left.\left(\frac{\sigma_r^{\ o}}{\sigma_1}\right)\left(\frac{\bar{\sigma}_\theta}{\sigma_1}\right)-\left(\frac{\sigma_\theta^{\ o}}{\sigma_1}\right)\left(\frac{\bar{\sigma}_r}{\sigma_1}\right)+6\left(\frac{\tau_{r\theta}^{\ o}}{\sigma_1}\right)\left(\frac{\bar{\tau}_{r\theta}}{\sigma_1}\right)\right]\frac{1}{\eta^3}\,d\theta$$

As $\eta \longrightarrow 0$, the first bracketed quantity in the integrand approaches

$$\left[\frac{\sigma_e(\infty)}{\sigma_1}\right]^2 = \left(\frac{\sigma_\infty}{\sigma_1}\right)^2 = \lambda^2$$

Thus,

$$\int_0^{\pi/2} I(0,\theta)d\theta = \lim_{\eta\to 0}\int_0^{\pi/2} \frac{3\lambda^{n-1}}{7}\left[2\left(\frac{\sigma_r^{\,o}}{\sigma_1}\right)\left(\frac{\bar{\sigma}_r}{\sigma_1}\right) + 2\left(\frac{\sigma_\theta^{\,o}}{\sigma_1}\right)\left(\frac{\bar{\sigma}_\theta}{\sigma_1}\right) - \left(\frac{\sigma_r^{\,o}}{\sigma_1}\right)\left(\frac{\bar{\sigma}_\theta}{\sigma_1}\right) - \left(\frac{\sigma_\theta^{\,o}}{\sigma_1}\right)\left(\frac{\bar{\sigma}_r}{\sigma_1}\right) + 6\left(\frac{\tau_{r\theta}^{\,o}}{\sigma_1}\right)\left(\frac{\bar{\tau}_{r\theta}}{\sigma_1}\right)\right]\frac{1}{\eta^3}\,d\theta$$

Now $\sigma_{\alpha\beta}^{\,o}$ is $O(1)$; furthermore, terms in the stress function containing a_{pq} lead to terms in the correction stresses of the order η^{p+2}. Hence, consideration need be limited only to $p = 0$ and 1. Furthermore, $\sigma_{\alpha\beta}^{\,o}$ has only terms that are either independent of θ, or contain $\cos 2\theta$ or $\sin 2\theta$. Hence, by the orthogonality properties of these trigonometric functions in the range $(0,\pi/2)$, only values of $q = 0$ and 1 provide non-vanishing results for the integrals of products of the elastic stress and the correction stresses. Thus, no matter how many coefficients a_{pq} are taken into account, only a_{00}, a_{01}, a_{10}, and a_{11} can possibly contribute to the integral being evaluated. Using the appropriate parts of equations (34) for $\sigma_{\alpha\beta}^{\,o}$ and $\bar{\sigma}_{\alpha\beta}$, and legitimately ignoring terms of the order η^4 or higher in their products, yields

$$\int_0^{\pi/2} I(0,\theta)d\theta = \lim_{\eta\to 0}\int_0^{\pi/2} \frac{3\lambda^n}{7}\left[a_{00}\left(-\frac{\eta^3}{2}\right) + a_{10}\left(\frac{\eta^3}{2}\right) + a_{01}\left(6\eta^2\cos^2 2\theta - 6\eta^2\sin^2 2\theta + 21\eta^3\cos^2 2\theta - 24\eta^3\sin^2 2\theta\right) + a_{11}\left(-\frac{21}{2}\eta^3\cos^2 2\theta + 12\eta^3\sin^2 2\theta\right)\right]\frac{1}{\eta^3}\,d\theta$$

Integrating with respect to θ and then letting $\eta \longrightarrow 0$ gives

$$\int_0^{\pi/2} I(0,\theta)d\theta = \frac{3\pi\lambda^n}{7}\left(-\frac{a_{00}}{4} + \frac{a_{10}}{4} - \frac{3}{4}a_{01} + \frac{3}{8}a_{11}\right)$$

$$= \frac{3\pi\lambda^n}{56}\left(-2a_{00} - 6a_{01} + 2a_{10} + 3a_{11}\right)$$

APPENDIX C

DESCRIPTION OF NUMERICAL MINIMIZATION PROCEDURE

The numerical minimization procedure used was essentially the same as that used in reference 8 in the solution of a different plasticity problem; much of the ensuing description parallels that contained in appendix C of reference 8, but is included herein for the sake of completeness of the present report.

The so-called steepest-descent procedure (see, for example, refs. 9 and 10) which formed the basis of the numerical minimization may be described in general terms as follows: Consider a function $\Phi\left(x_1, x_2, \ldots x_n\right)$. The set of n independent parameters may be conveniently denoted by the n-component vector x_i $(i = 1, 2, 3, \ldots n)$. The value of x_i that minimizes Φ is sought. An initial trial vector $x_i^{(0)}$ is assumed, and the gradient of Φ, that is, $\frac{\partial\Phi}{\partial x_i}$, is calculated at $x_i^{(0)}$. The direction $-\frac{\partial\Phi}{\partial x_i}$ is then the direction of steepest descent of the function Φ; the function $\Phi\left[x_i^{(0)} - \delta\frac{\partial\Phi}{\partial x_i}\right]$ is then evaluated for various positive values of δ in an effort to find the value $\bar{\delta}$ that minimizes $\Phi\left[x_i^{(0)} - \delta\frac{\partial\Phi}{\partial x_i}\right]$. When this value of $\bar{\delta}$ is found (presumably, approximately), a new direction of steepest descent is determined by evaluation of the gradient of Φ at the point $x_i^{(1)} = x_i^{(0)} - \bar{\delta}\frac{\partial\Phi}{\partial x_i}\left[x_i^{(0)}\right]$. The process is continued until satisfactory convergence is obtained to the lowest possible value of Φ.

In the present problem, the function Φ is given by equation (33), and the four coefficients a_{00}, a_{01}, a_{10}, and a_{11} play the role of the components of the vector x_i. The basic procedure outlined was modified in several respects. The evaluation of the gradient of Φ was actually performed on the basis of a finite-difference approximation to each of the four partial derivatives required. Also, the value of $\bar{\delta}$ in any given cycle was found as the minimum point of a parabola through the three points $\Phi(x_i)$, $\Phi\left(x_i - \delta\frac{\partial\Phi}{\partial x_i}\right)$, and $\Phi\left(x_i - 2\delta\frac{\partial\Phi}{\partial x_i}\right)$ where δ was chosen to be of some convenient magnitude, preferably of the order expected for $\bar{\delta}$. The procedure was further modified by inserting after each two successive cycles of minimization, in the direction of the negative gradient, a third cycle of a different nature. In this extra cycle, minimization was performed in the direction determined by the difference between the last-obtained approximations to the unknowns and the approximations of two cycles before. The motivation for this extra cycle stems from the fact that the unmodified method of steepest descent often tends to furnish successive approximations to the minimizing vector that zigzag toward the true minimum; the extra cycle was an attempt to speed up convergence by moving in the direction determined by the mean of a zig and a zag.

The procedure outlined was coded for calculation on SEAC; once values of n, λ, and initial estimates for the coefficients a_{pq} were put into the machine, the iteration process proceeded automatically. Successive approximations to the minimizing coefficients were printed; the number of cycles of iteration performed varied from 12 to 35. In each case, the final results agreed with those of several preceding iterations to within approximately 1 percent of the largest coefficient. Although the nature of the numerical procedure is such that the accuracy of the final results can not be positively assessed, some confidence in their accuracy is lent by the smooth variations with λ obtained for the stress concentration factor.

REFERENCES

1. Stowell, Elbridge Z.: Stress and Strain Concentration at a Circular Hole in an Infinite Plate. NACA TN 2073, 1950.

2. Ramberg, Walter, and Osgood, William R.: Description of Stress-Strain Curves by Three Parameters. NACA TN 902, 1943.

3. Timoshenko, S., and Goodier, J. N.: Theory of Elasticity. Second ed., McGraw-Hill Book Co., Inc., 1951.

4. Greenberg, H. J.: On the Variational Principles of Plasticity. Rep. A11-S4 (Contract N7onr-358, T.O.1, NR-041-032), Graduate Div. Appl. Math., Brown Univ., Mar. 1949.

5. Wang, Chi-Teh: Variational Method in the Theory of Compressible Fluid. Jour. Aero. Sci., vol. 15, no. 11, Nov. 1948, pp. 675-685.

6. Wang, Chi-Teh: A Note on Bateman's Variational Principle for Compressible Fluid Flow. Quarterly Appl. Math., vol. IX, no. 1, Apr. 1951, pp. 99-102.

7. Griffith, George E.: Experimental Investigation of the Effects of Plastic Flow in a Tension Panel With a Circular Hole. NACA TN 1705, 1948.

8. Mayers, J., and Budiansky, Bernard: Analysis of Behavior of Simply Supported Flat Plates Compressed Beyond the Buckling Load Into the Plastic Range. NACA TN 3368, 1955.

9. Hartree, Douglas R.: Some Unsolved Problems in Numerical Analysis. Problems for the Numerical Analysis of the Future, Appl. Math. Series 15, National Bur. Standards, June 29, 1951, pp. 1-9.

10. Curry, Haskell B.: The Method of Steepest Descent for Non-Linear Minimization Problems. Quarterly Appl. Math., vol. II, no. 3, Oct. 1944, pp. 258-261.

TABLE I

COEFFICIENTS a_{pq}; ANALYTICAL SOLUTION, $n = 3$

λ	a_{00}	a_{01}	a_{10}
0.2	-0.00011	0.00458	0.00991
.5	.00521	.04647	.09033
.7	.01868	.09274	.16796
1.0	.05330	.17346	.28840

TABLE II

COEFFICIENTS a_{pq}; NUMERICAL SOLUTION

λ	a_{00}	a_{01}	a_{10}	a_{11}
n = 3				
0.2	0.0103	0.0146	-0.0027	-0.0109
.4	.0079	.0152	.0455	.0244
.7	.0304	.0688	.1457	.0445
1.0	.0666	.1551	.2666	.0338
n = 5				
0.2	0.0013	0.0004	0.0014	0.0017
.4	.0015	.0052	.0542	.0436
.7	.0703	.1221	.1986	.0632
.9	.1572	.2808	.2883	-.0230
1.1	.2287	.4489	.4013	-.1364
n = 9				
0.3	-0.0014	-0.0018	0.0082	0.0102
.5	-.0154	-.0006	.1437	.1159
.7	.1131	.1712	.2401	.0764
.9	.3349	.5130	.2548	-.1866
1.1	.4928	.8190	.2871	-.4256
n = 19				
0.4	-0.0116	-0.0158	0.0543	0.0672
.5	-.0286	-.0271	.1556	.1647
.7	.1617	.1895	.2610	.0993
1.0	.6440	1.1120	-.0240	-.6039

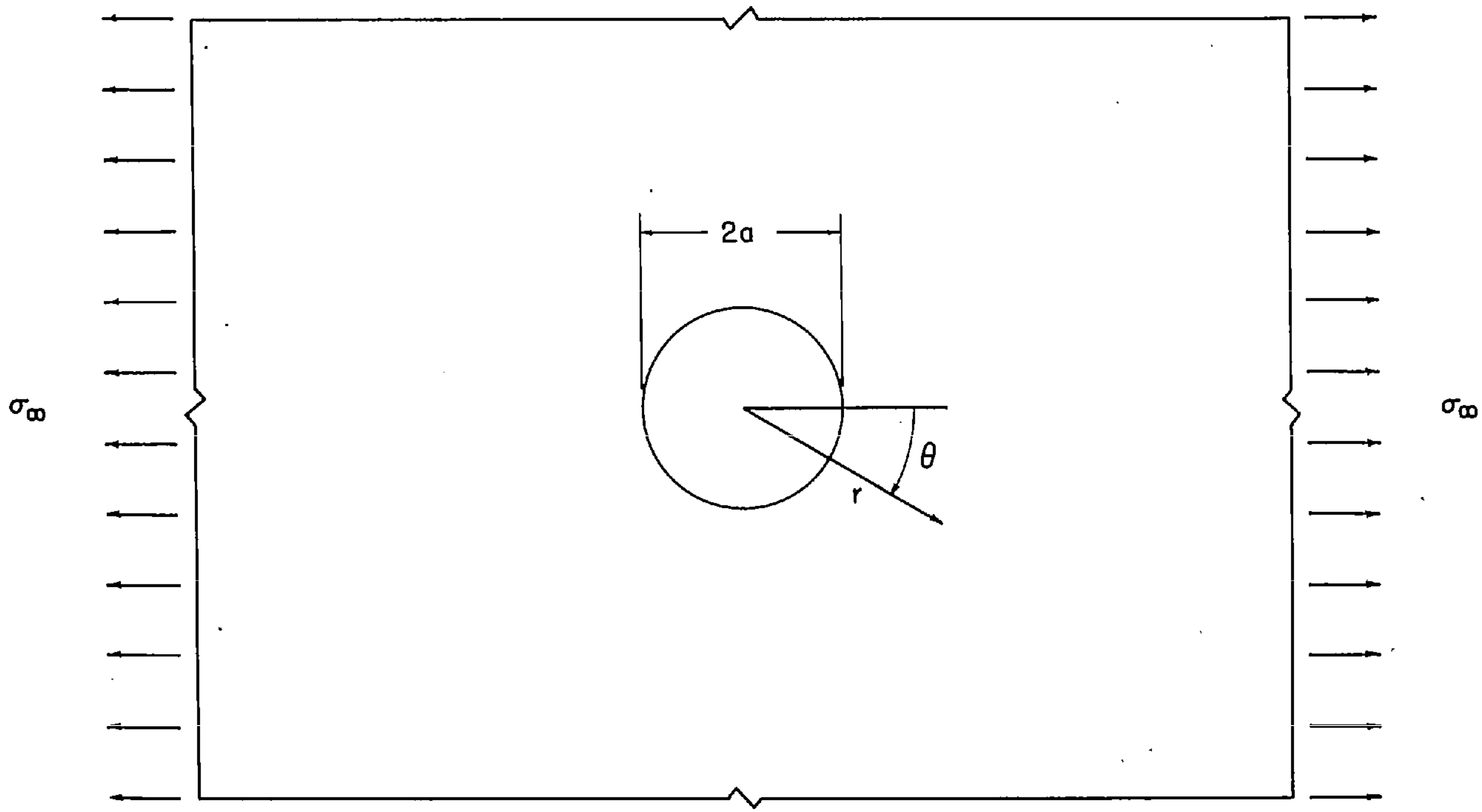


Figure 1.- Circular hole in an infinite plate subjected to uniaxial tension.

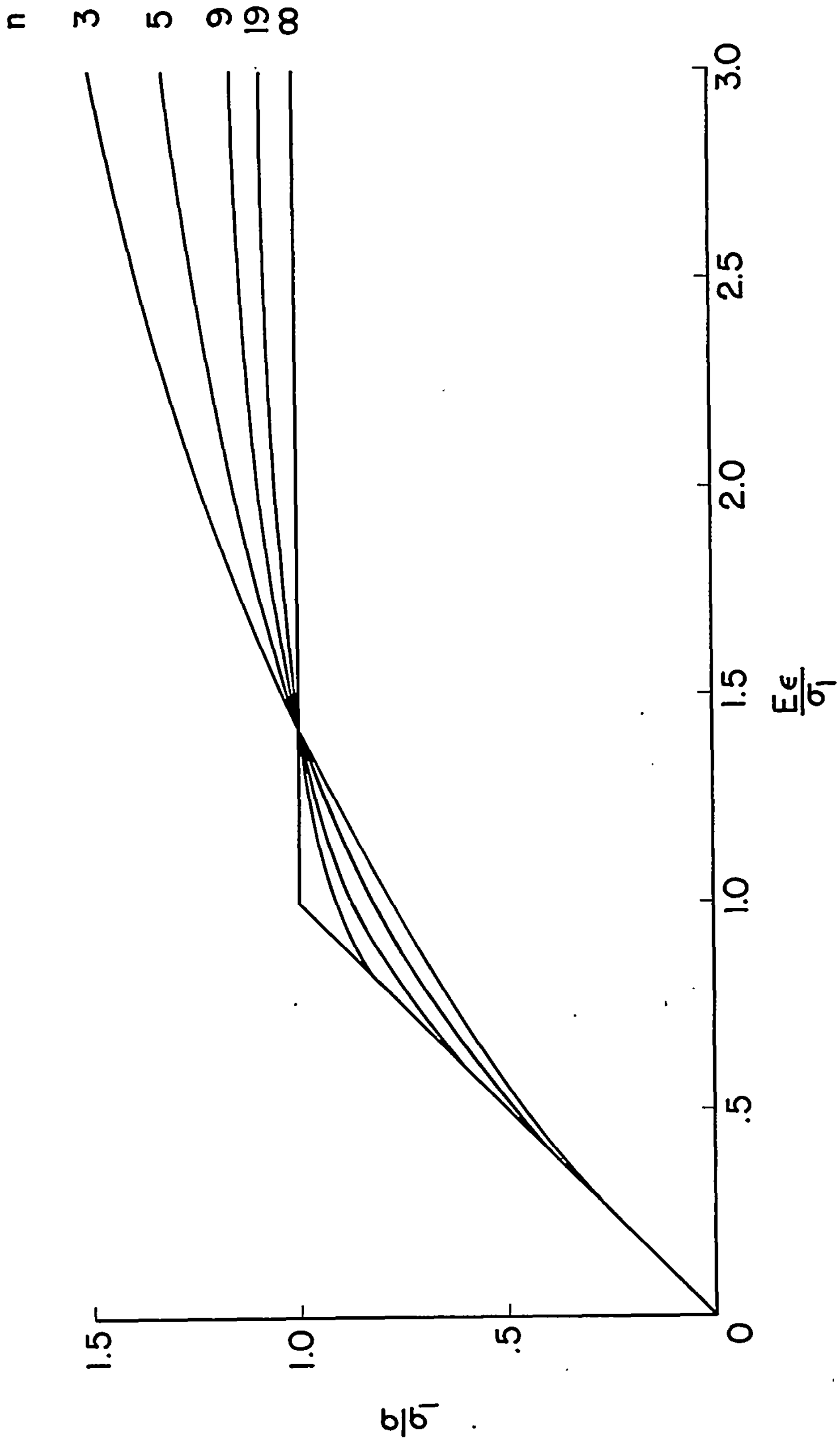


Figure 2.- Nondimensional Ramberg-Osgood stress-strain curves.

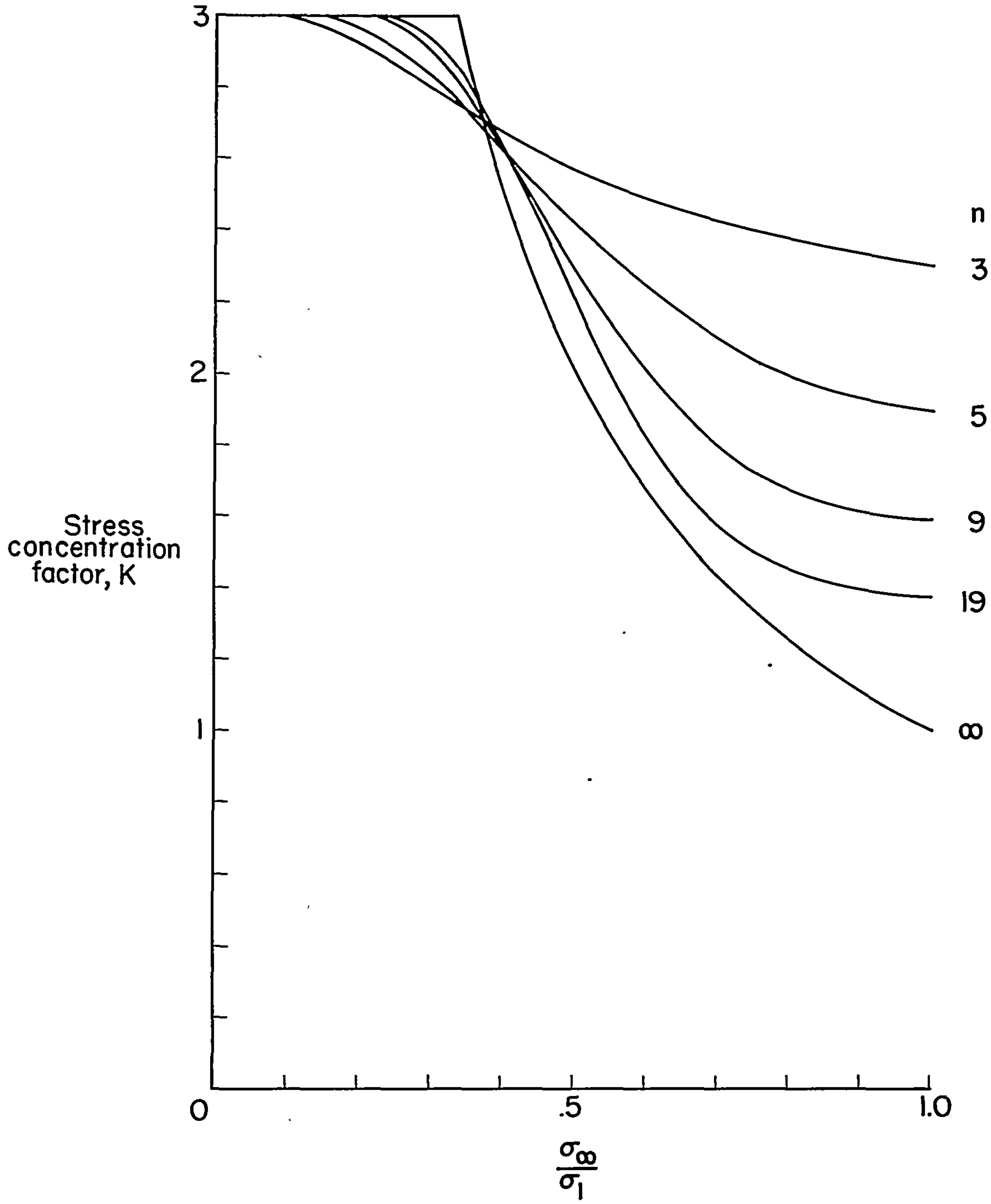


Figure 3.- Variation of stress concentration factor with σ_∞/σ_1.

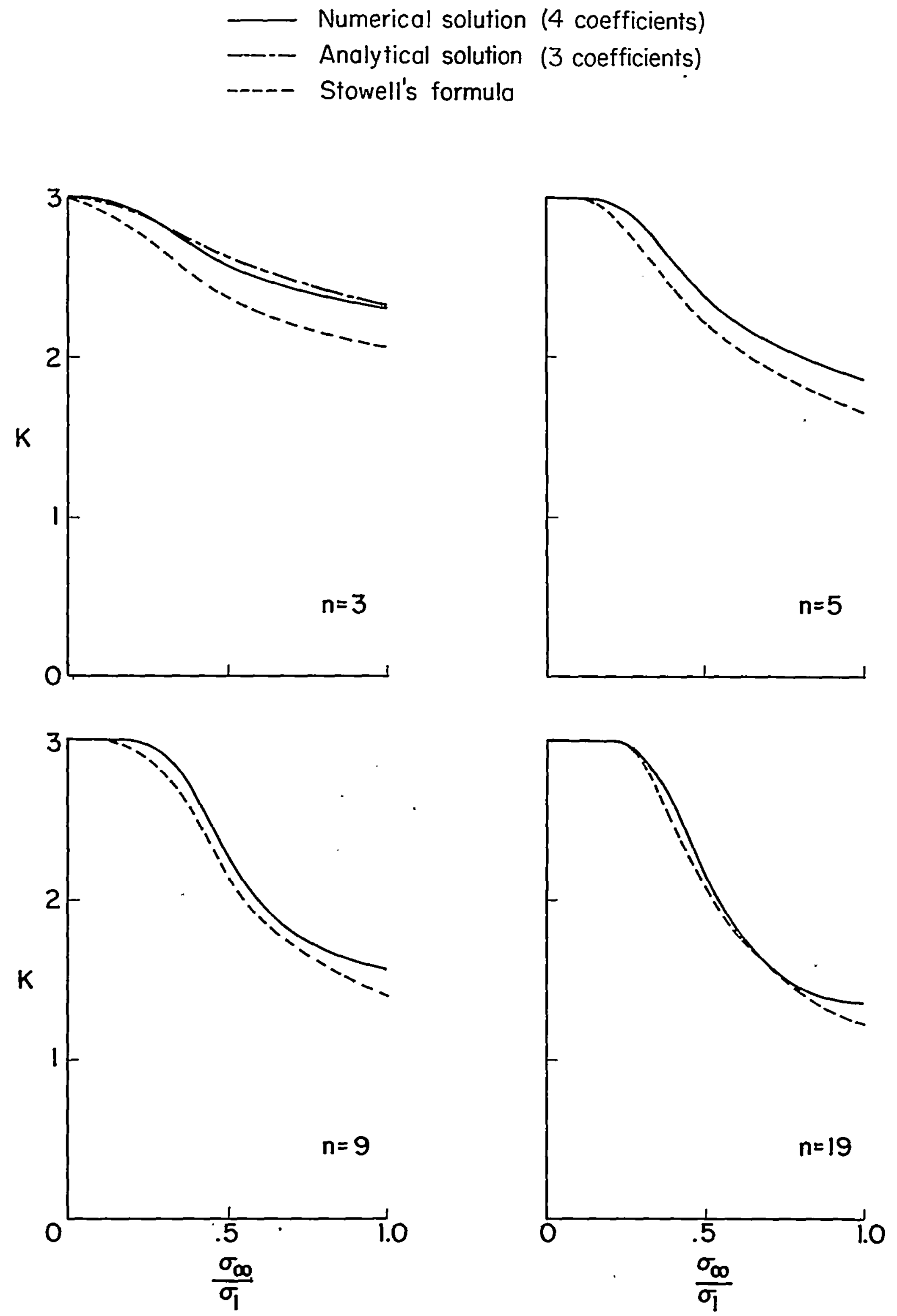


Figure 4.- Comparison of calculated stress concentration factors with those given by Stowell's formula.

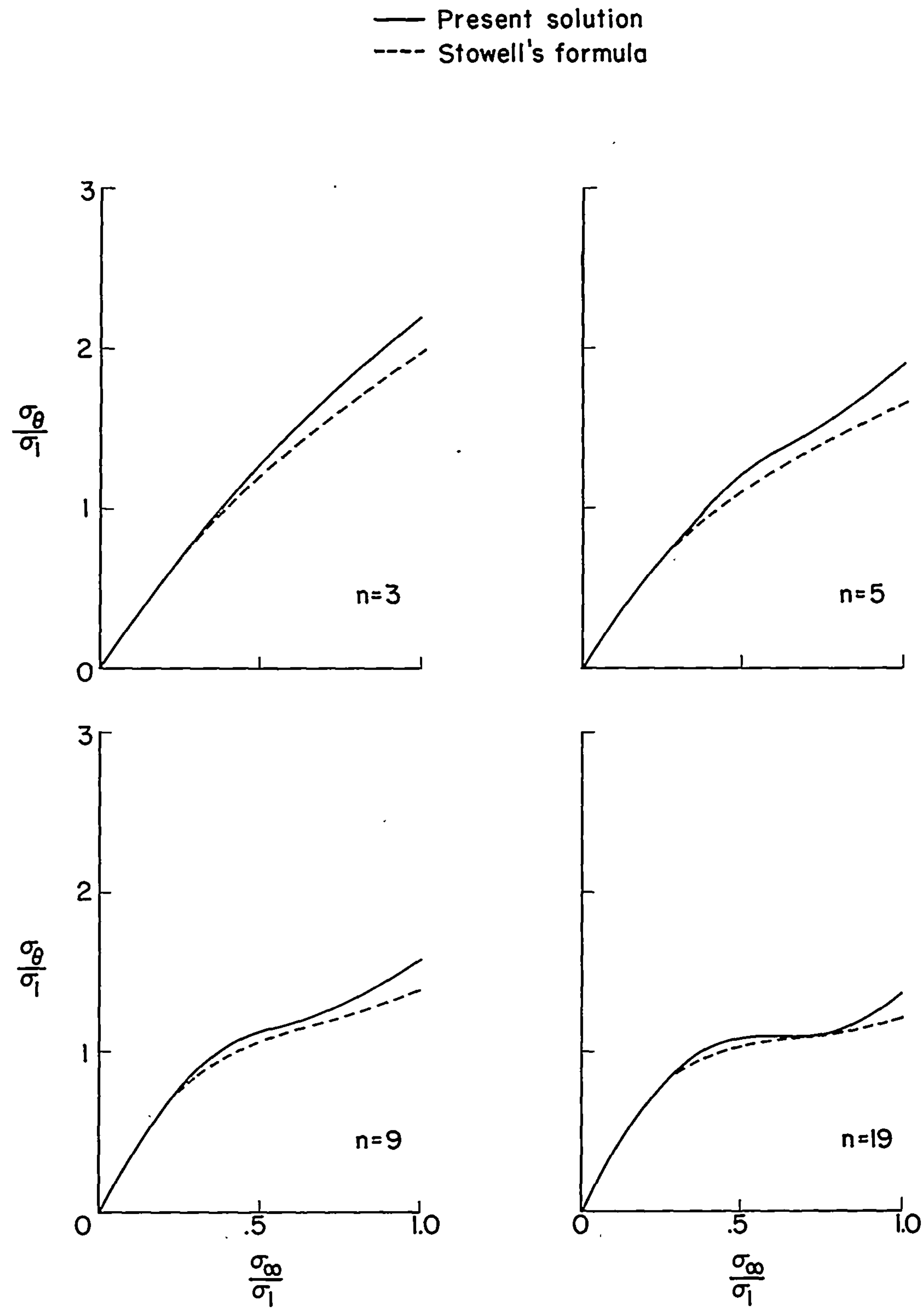


Figure 5.- Comparison of maximum hole stress predicted by Stowell's formula and the present solution.