NACA TN 3362

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3362

ESTIMATES OF PROBABILITY DISTRIBUTION OF ROOT-MEAN-SQUARE GUST VELOCITY OF ATMOSPHERIC TURBULENCE FROM OPERATIONAL GUST-LOAD DATA BY RANDOM-PROCESS THEORY

By Harry Press, May T. Meadows, and Ivan Hadlock

Langley Aeronautical Laboratory
Langley Field, Va.

Washington
March 1955

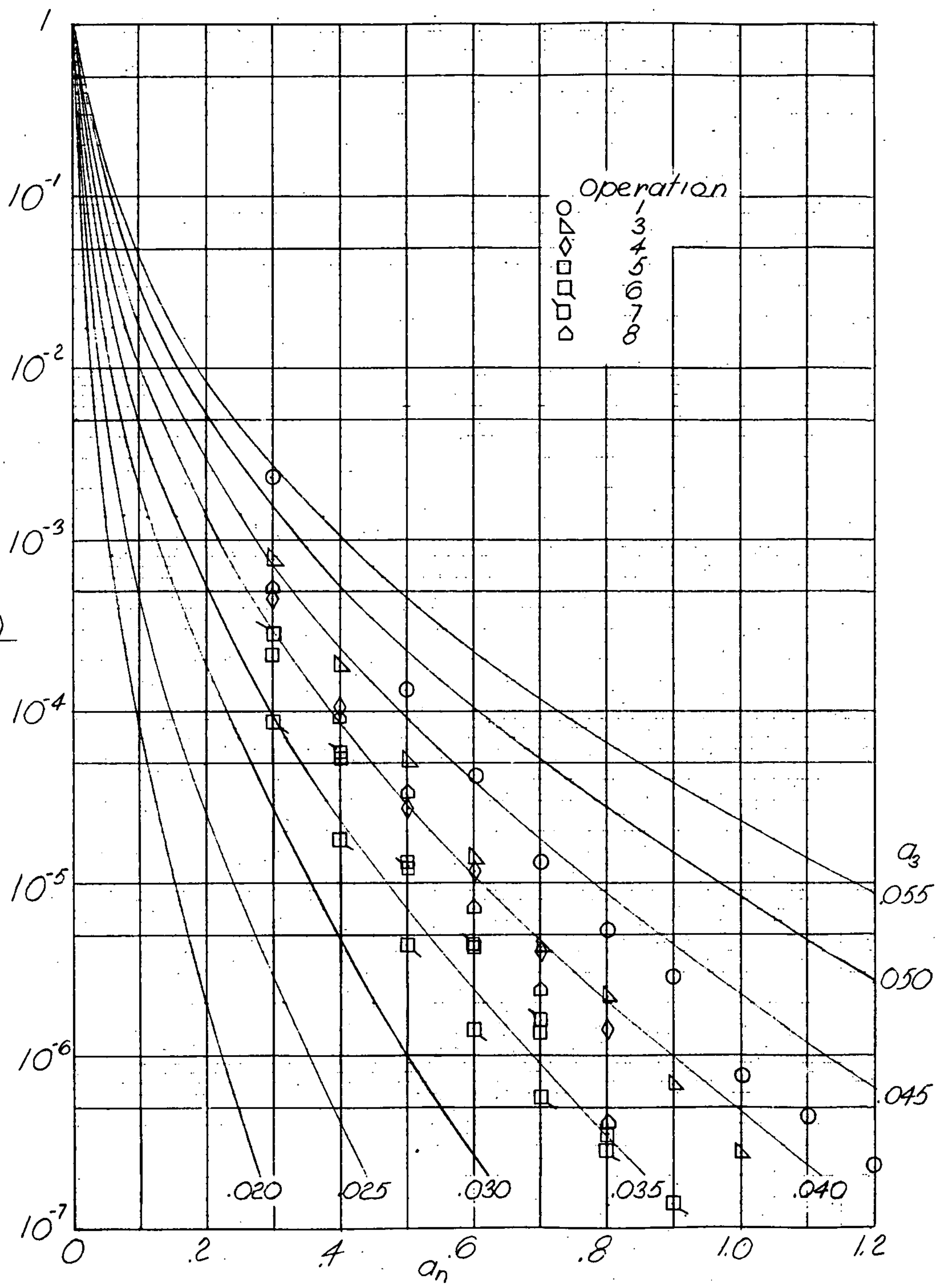


Figure 4.- Number of peak accelerations exceeding given values for case c.

ERRATA

NACA TN 3362

ESTIMATES OF PROBABILITY DISTRIBUTION OF
ROOT-MEAN-SQUARE GUST VELOCITY OF ATMOSPHERIC TURBULENCE
FROM OPERATIONAL GUST-LOAD DATA
BY RANDOM-PROCESS THEORY

By Harry Press, May T. Meadows, and Ivan Hadlock

March 1955

Pages 46 and 47: Figures 3 and 4 should be replaced with revised figures 3 and 4 attached hereto. The new figures contain a revision of the original figures in which the curves for $M(a_n)/C$ for various values of the parameters a_2 and a_3 were in error at the higher values of a_n. The revised figures also contain extensions to the ranges of values covered for the parameters a_2 and a_3. The data points are shown and are unchanged. As a consequence of the revision of the curves of figures 3 and 4, the derived values of a_2 and a_3 and the corresponding values of b_2 and b_3 given in Table IV should be slightly reduced (about 5 percent).

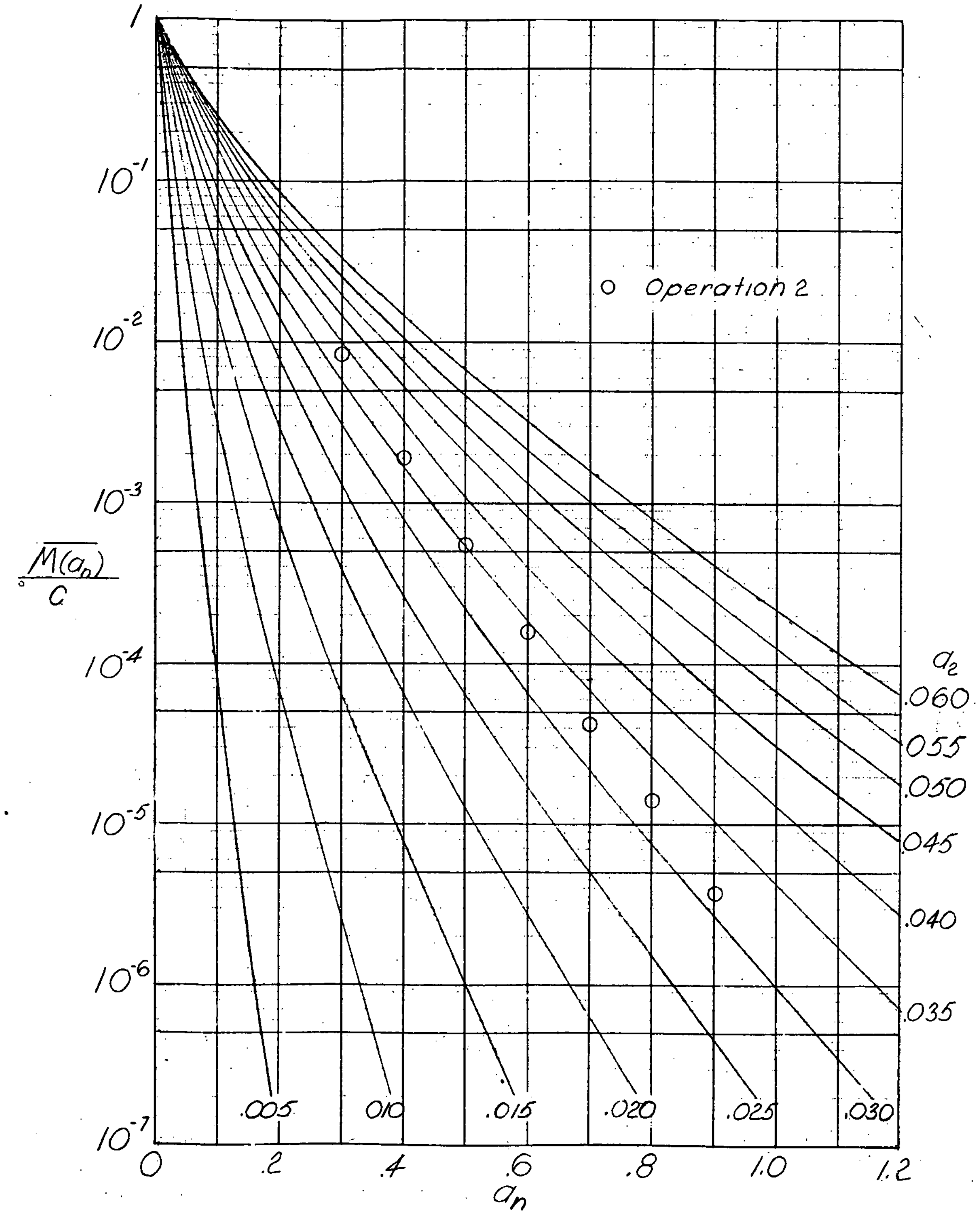


Figure 3.- Number of peak accelerations exceeding given values for case b.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3362

ESTIMATES OF PROBABILITY DISTRIBUTION OF ROOT-MEAN-SQUARE GUST VELOCITY OF ATMOSPHERIC TURBULENCE FROM OPERATIONAL GUST-LOAD DATA BY RANDOM-PROCESS THEORY

By Harry Press, May T. Meadows, and Ivan Hadlock

SUMMARY

Under the assumption that the operational gust or gust-load history of an airplane is a Gaussian random process with a single parameter, the root-mean-square value, relations are derived between the probability distribution of the root-mean-square acceleration and the associated number of peak accelerations above given values. These relations are then used in the analysis of available operational gust-load data in the form of peak counts to derive estimates of the probability distributions of root-mean-square acceleration. These probability distributions are then transformed on the basis of airplane-gust-response theory in order to derive the associated probability distribution of root-mean-square gust velocity. The application of these results to the calculation of load histories is also considered briefly.

INTRODUCTION

During the last few years, advances have been made in the analysis of airplane behavior in rough air through the application of the techniques of generalized harmonic analysis (refs. 1 to 7). The application of these techniques is based upon the representation of atmospheric turbulence as a continuous random disturbance characterized by power-spectral-density functions and certain probability distributions. The power spectrum of the turbulence is then used along with the airplane response characteristics to determine the power spectra and other statistical characteristics of the airplane loads or motions in rough air. The application of this approach to the problem of calculating load histories for operational flight requires detailed information on the spectrum of turbulence in the atmosphere. The information required may be considered of two types: detailed information on the spectrum of turbulence and its variations, and information on the probability of encountering the various spectra.

A few measurements of the power spectrum of atmospheric turbulence have so far been made; most of these are summarized in reference 5. These measurements indicate that, over most of the frequency range of interest, the spectra are inversely proportional to the square of the frequency and may be approximated by simple analytical expressions such as have been used in wind-tunnel studies of isotropic turbulence. Also, the intensity of the turbulence as given by the root-mean-square gust velocity varied appreciably with the weather conditions. These results thus appear to provide some information on the power spectrum and its variations. They do not, however, provide any information of the second type required, that is, information on the probability of encountering the various spectra in actual operations.

The only source of information on the probability of encountering the various conditions of atmospheric turbulence appears to be the considerable body of statistical data concerning atmospheric turbulence and airplane loads in rough air that has been collected by the National Advisory Committee for Aeronautics in the last 20 years. (See, for example, refs. 8 to 12). These data have, in most cases, been obtained from airplane acceleration measurements in normal operations, although, in some cases (ref. 10), the data were obtained in special flight investigations. These data are generally given in the form of the number of peak accelerations or effective (or "derived") gust velocities per second which exceeded given values and in this form do not appear applicable to spectral methods of analysis.

Fortunately, in the theory of random processes, relations have been derived (ref. 13) between peak counts (such as have been made for normal acceleration) and the associated power spectra. These relations apply to the case of a stationary Gaussian random process, the stationarity of the process implying that its characteristics do not change with time and the term Gaussian designating a process characterized by a Gaussian probability distribution for the amplitude of the disturbance as well as for its time derivatives. The approximately Gaussian character of turbulent velocity fluctuations has been noted and, for the case of atmospheric turbulence, results reported, for example, in reference 3 appear to support such an assumption. Inasmuch as the intensity of turbulence is known to vary widely with weather conditions, the overall gust and load experience in operations cannot be considered a stationary Gaussian process. If the operational gust history is considered to be a nonstationary Gaussian process varying only in intensity or root-mean-square gust velocity, the problem of specifying the gust history is reduced to that of specifying the probability distribution of the root-mean-square gust velocity. For this case, it appears possible to extend the results of reference 13 for the stationary Gaussian process in order to derive a basis for estimating the distribution of root-mean-square gust velocity from counts of acceleration peaks.

In the present paper, some techniques are developed and applied for the estimation of the probability distributions of root-mean-square acceleration and root-mean-square gust velocity from data on peak gust accelerations obtained in airplane operations. As an initial approach, it is assumed that the root-mean-square gust velocity has only several discrete and unknown values. On this basis, a graphical procedure is devised for the determination from data on peak accelerations of several root-mean-square accelerations and their associated probabilities of occurrence. This procedure is applied to data from a number of operational load histories and the appropriate root-mean-square accelerations and associated probabilities are determined. These root-mean-square accelerations are then transformed in order to obtain the associated values of root-mean-square gust velocities on the basis of an assumed turbulence spectral shape and average values for the parameters of each of the airplanes.

As a further extension, the gust experience is considered to consist of a continuous variation in root-mean-square gust velocity. Based on the earlier results on discrete values of root-mean-square gust velocity, estimates are made of the form of the probability distribution of the root-mean-square gust velocity experienced in operations. Three single-parameter probability distributions are considered and the associated number of peak loads per second exceeding given levels is calculated. These results are then compared with available experimental data. The application of the foregoing results to the calculation of airplane gust-load and other response histories in rough air is also considered briefly.

SYMBOLS

$\bar{A} = \frac{\rho VSm}{2W}\sqrt{\frac{I(K,s)}{\pi}}$

a_n acceleration, g units

a_1,a_2,a_3 scale parameters in distributions, $f\left(\sigma_{a_n}\right)$

b_1,b_2,b_3 scale parameters in distributions, $\hat{f}(\sigma_U)$

$\bar{c}$ average chord, ft

$f(\sigma_{a_n})$ probability density distribution of σ_{a_n}

$\hat{f}(\sigma_U)$ probability density distribution of σ_U

$\left[\frac{I(K,s)}{\pi}\right]^{1/2}$ airplane gust-response factor

K airplane mass parameter, $\frac{4W}{g\pi\rho S\bar{c}}$

L scale of turbulence

$\overline{M(a_n)}$ average number of maximums per second exceeding given value of a_n in operations

m slope of lift curve per radian

$N(\ )$ average number of maximums per second exceeding given value of specified argument for Gaussian disturbance

P proportion of total flight time

S wing area, sq ft

s ratio of chord to turbulence scale, $\bar{c}/L$

T specified time, sec

t time, sec

U gust velocity, ft/sec

V true airspeed, fps

W airplane weight, lb

$y(t)$ random function of time

$\sqrt{\overline{\ddot{Z}^2}}$ root-mean-square normal acceleration, ft/sec^2

ρ air density, slugs/cu ft

σ root-mean-square deviation, $\left(\overline{y^2(t)}\right)^{1/2}$

$\Phi(\omega)$ power-spectral-density function,

$$\lim_{T \to \infty} \frac{1}{2\pi T}\left|\int_{-T}^{T} y(t)e^{-i\omega t}\,dt\right|^2$$

Ω frequency, $\frac{\omega}{V}$, radians/ft

ω frequency, radians/sec

METHOD AND ANALYSIS

Basic Method

In the present analysis, use will be made of some concepts and results in the theory of random processes. The theory of random processes is a recently developed branch of probability theory and the aspects of the theory pertinent to the present applications are described in some detail in reference 13. In particular, the relations between peak counts and spectra for a stationary Gaussian random process which will be used in the present study are derived therein. Some general aspects of random-process theory are also covered in references 14 and 15.

In many recent studies of airplane behavior in rough air, atmospheric turbulence is generally considered a stationary Gaussian random process. The assumption that turbulence is a Gaussian random process appears warranted by the approximate Gaussian character of turbulent velocity fluctuations. The assumption of stationarity implies that the statistical characteristics of the turbulence are invariant with space and time and also appears warranted for many purposes. For present purposes (in which the overall gust and load experience of an airplane in operational flight is of concern), however, the process cannot be considered a simple stationary one, inasmuch as the turbulence characteristics of the atmospheric are well known to vary widely with weather conditions, particularly in regard to the intensity of the turbulence.

In order to account for the variations in atmospheric turbulence with weather condition, it will be assumed that turbulence is only locally Gaussian and stationary; that is, its statistical characteristics are Gaussian and invariant in a given restricted region and for a short time but vary, particularly in intensity, from time to time and place to place. This assumption implies that the region or time is small relative to the entire flight path or flight duration but large enough for statistical equilibrium to be achieved. On this basis, the overall turbulence experienced by an airplane in given operations is not a simple Gaussian process but consists of the summation for appropriate exposure times of a series of elemental Gaussian processes. If the turbulence is taken to vary in intensity only, this scheme may be considered to be a single-parameter Gaussian process.

If the airplane response to turbulence is assumed to be linear, as assumed in the present analysis, the response such as the load history to each elemental turbulence process is likewise a Gaussian process and the overall operational load history may in turn also be considered to consist of a summation of the loads for the various elemental Gaussian turbulence disturbances. The particular advantage of this scheme for present purposes is that it permits the use of relations between peak counts and spectra already derived for the stationary Gaussian case in reference 13.

The foregoing considerations form the basis for the present analysis. They will be applied in order to estimate the probability distribution of root-mean-square acceleration from operational data on peak accelerations. These estimates of the distribution of root-mean-square acceleration will then be used in order to obtain the associated probability distributions of root-mean-square gust velocity.

Relations Between Number of Peaks and Spectra

Simple Gaussian case.- The asymptotic relation between the average number of maximums per second exceeding a given value and the spectrum of a stationary Gaussian disturbance $y(t)$ has been derived in reference 13 and is for large values of y given by (see appendix)

$$N(y) = \frac{1}{2\pi}\left[\frac{\int_0^\infty \omega^2\Phi(\omega)d\omega}{\int_0^\infty \Phi(\omega)d\omega}\right]^{1/2} e^{-y^2/2\sigma^2} \tag{1}$$

where

$N(y)$ average number of maximums per second exceeding given values of y

ω frequency argument, radians/sec

$\Phi(\omega)$ power-spectral-density function of random disturbance $y(t)$

$$\sigma^2 = \int_0^\infty \Phi(\omega)d\omega$$

Equation (1) is the exact expression for the number of crossings per second with positive slope of given values of y but is an approximate expression for the number of maximums above a given value of y. Equation (1), however, appears to be an adequate approximation to the number of peaks for present purposes as indicated in the appendix and will form the basis for the present analysis.

Examination of equation (1) indicates that the number of peaks per second above given values depends upon the spectrum $\Phi(\omega)$ and upon σ^2, which is the area under the spectrum. If it is assumed that the spectral shape of turbulence is invariant with weather conditions (as suggested by available measurements) and varies only in intensity or root-mean-square gust velocity, the output spectrum $\Phi(\omega)$ for acceleration a_n for a given airplane under given operating conditions is likewise invariant in shape. Under these conditions, the coefficient of the exponential term in equation (1) is a fixed constant and equation (1) may be written as

$$N(a_n) = Ce^{-a_n^2/2\sigma^2} \tag{2}$$

where the constant C is

$$C = \frac{1}{2\pi}\left[\frac{\int_0^\infty \omega^2 \ \Phi(\omega)d\omega}{\int_0^\infty \Phi(\omega)d\omega}\right]^{1/2} \tag{3}$$

The constant C gives the expected number of crossings per second with positive slope by $y(t)$ of the value of zero. Because of the invariance of the shape of $\Phi(\omega)$, the quantity C is independent of the turbulence intensity. It has the dimensions of a frequency and in the present applications can be considered a characteristic frequency of the airplane response to turbulence.

Taking the logarithm of both sides of equation (2) yields

$$L = C_1 - \frac{{a_n}^2}{2\sigma^2} \tag{4}$$

where $L = \log N(a_n)$ and $C_1 = \log C$. Equation (4) indicates that the $\log N(a_n)$ is a linear function of a_n^2 with slope equal to $-1/2\sigma^2$. Thus, for a stationary Gaussian disturbance, the root-mean-square value σ may be obtained simply from the slope of the line for the number of peaks when plotted on semilogarithmic paper as a function of a_n^2.

A representative distribution of the number of peak accelerations per second (both positive and negative peaks) obtained from operations of a transport airplane is shown as the solid curve of figure 1(a) as a function of a_n^2. The data are given, as is generally the case, for a threshold value of 0.3g. The distribution appears concave upward and departs considerably from the straight lines that would be expected for a simple stationary Gaussian disturbance on this plot. It is thus clear that the overall distribution of peak accelerations cannot be adequately represented by the simple Gaussian case. In order to obtain a more adequate representation of the operational load history, it is necessary to account for the wide variations of turbulence intensity associated with different weather conditions. It appears possible to account for these wide variations in turbulence intensity by considering the load history as a composite of Gaussian processes differing in intensity. This approach is applied in the following paragraphs.

Composite Gaussian case.- If the overall operational load history (in terms of the average number of acceleration peaks per second exceeding given values) is considered to consist of various exposure times to different Gaussian disturbances, the average number of peaks per second exceeding given values is given by

$$\overline{M(a_n)} = \sum_{i=1}^{k} P_i N_i(a_n) \tag{5}$$

where P_i is the proportion of total flight at the ith condition and $N_i(a_n)$ is the number of peak accelerations per second exceeding given values of acceleration at the ith condition. In this form, equation (5) is general and permits accounting not only for variation in the spectrum of turbulence but also for variations in airplane response characteristics.

If equation (2) is substituted into equation (5), there is obtained

$$\overline{M(a_n)} = \sum_{i=1}^{k} C_i P_i e^{-a_n^2 / 2\sigma_i^2} \tag{6}$$

In order to simplify equation (6) to a form suitable for the present purposes, it will be assumed, as previously indicated, that the spectrum of atmospheric turbulence is invariant in shape but varies in intensity or in root-mean-square gust velocity. This assumption is suggested by the available measurements of turbulence spectra and appears reasonable for present purposes. It will be assumed further that the response characteristics for a particular airplane in operation may be represented by a single response function such as obtained for average values of the airplane and operating parameters of weight, airspeed, and density. Under these conditions, equation (6) may be written for this, the discrete case, as

$$\overline{M(a_n)} = C \sum_{i=1}^{k} P_i e^{-a_n^2 / 2\sigma_i^2} \tag{7}$$

where the quantity C is, as in equation (2), fixed for a given airplane. The continuous form of equation (7) is given by

$$\overline{M(a_n)} = C \int_0^\infty f(\sigma) e^{-a_n^2 / 2\sigma^2} d\sigma \tag{8}$$

where $f(\sigma)$ is the probability distribution of σ and the quantity $f(\sigma)d\sigma$ can be considered to represent the proportion of total flight time spent at root-mean-square values σ to $\sigma + d\sigma$.

The relations between the number of peaks and the root-mean-square values of the random process given by equation (7) for the discrete case and equation (8) for the continuous case will form the basis for the analysis of operational data on peak loads in the following sections. A number of operational load histories will first be described in terms of the proportions of flight time P_i spent at various levels of σ_{a_n} for the discrete case and then in terms of continuous distributions $f(\sigma)$. These distributions of root-mean-square accelerations will then be transformed in order to obtain the associated distributions of root-mean-square gust velocity.

Distribution of Root-Mean-Square Acceleration for Discrete Case

Graphical procedure.- As previously indicated, the operational load history in terms of the number of peaks per second exceeding given values is for a particular airplane given by equation (7). Each of the terms of the summation of equation (7) is given by

$$\overline{M_i(a_n)} = CP_i e^{-a_n^2 / 2\sigma_i^2} \tag{9}$$

which, as previously noted, yields a straight line of slope $-1/2\sigma_i^2$ if $\log \overline{M(a_n)}$ is plotted as a function of a_n^2. This condition suggests that the overall operational loads for a given operation when plotted in this specified form are built up of straight-line or Gaussian components. A simple trial shown in figure 1(a) indicates that a good approximation to $\overline{M(a_n)}$ can be obtained with only a few components. The values for the three components shown add up to give the dashed curve which is seen to be a close approximation to the overall load history. The procedure devised for the determination of these straight-line components consists of the following steps: First, line (1) is obtained by taking a tangent to the tail of the observed distribution; line (2) is then taken from the point on line (1) which underestimates $\overline{M(a_n)}$ by one-half and drawn tangent to the upper part of the overall load-history curve. The third line, if required, is then obtained from line (2) in the same manner that line (2) was obtained from line (1). The sum of the values of the lines obtained in this manner will generally yield a good approximation to the observed distribution, as will be seen subsequently.

The procedure outlined in the preceding paragraph for obtaining the linear components of the observed distribution is somewhat arbitrary. Several alternative procedures were also considered and discarded. These procedures included the selection of the first component at the upper end of the curve and the use of specified combinations of slopes corresponding to given values of root-mean-square gust velocities (which might be considered averages for various weather conditions such as clear-air turbulence and thunderstorms). These alternative procedures appeared to offer no significant advantages and had the additional undesirable characteristic of yielding a poorer approximation at the larger values of acceleration which are of greatest interest.

It will be recalled that the slopes of the lines in figure 1(a) are equal to $-1/2\sigma_1^2$, $-1/2\sigma_2^2$, and $-1/2\sigma_3^2$ where σ_1, σ_2, and σ_3, respectively, represent the root-mean-square acceleration for the three effective Gaussian components. The values for the case represented in figure 1(a) are $\sigma_1 = 0.430$, $\sigma_2 = 0.247$, and $\sigma_3 = 0.147$. These values, σ_1, σ_2, and σ_3, for the acceleration components are used subsequently to obtain the associated root-mean-square gust velocities for these Gaussian components.

In order to estimate the values of the proportion of total flight time P_i for the ith condition, it will be noted from equation (9) that for each component

$$P_i = \frac{\overline{M_i(a_n)}}{Ce^{-a_n^2/2\sigma_i^2}} \tag{10}$$

where $\overline{M_i(a_n)}$ is the number of accelerations per second exceeding a_n for the ith condition or component. The numerator of equation (10) may be obtained directly from the line for each component in figure 1(a).

The denominator of equation (10), as would be expected, is the total number of peak accelerations per second exceeding a_n that would be experienced, if the airplane spent all its flight time at the indicated root-mean-square value. The value of the denominator is seen to depend upon the value of the constant C. The determination of the value of C for a given case from its definition (eq. (3)) appears impractical for present purposes, since it depends upon the acceleration power spectrum which cannot be determined from the type of records available. The value of C may, however, be estimated from short samples of the acceleration time history in homogeneous rough air since it is equal to the average number of crossings per second of zero acceleration. Unfortunately, the film speed for the records available (2 to 8 feet per hour) was too slow to permit counts of the number of zero crossings with any degree of reliability. As an alternative procedure, it was found more convenient to obtain values for the related quantity C_2 defined by

$$\left.\begin{aligned} C_2 &= \frac{1}{2\pi}\left[\frac{\int_0^\infty \omega^2\Phi(\omega)\,d\omega}{\int_0^\infty \Phi(\omega)\,d\omega}\right]^{1/2} e^{-2} \\ &= 0.135C \end{aligned}\right\} \tag{11}$$

which is seen from equation (2) to be the number of maximums per second exceeding a value of acceleration equal to 2σ. On this basis, the proportion of the total time spent at a given root-mean-square acceleration σ_i is given by the alternative expression

$$P_i = \frac{\overline{M_i(2\sigma_i)}}{2C_2} \tag{12}$$

where $\overline{M_i(2\sigma_i)}$ is the average number of exceedances per second of $2\sigma_i$ for the linear component. (The factor 2 is included in the denominator since the operational-loads data, as is generally true, include minimums as well as maximums.) This relation is used in the subsequent analysis of the operational-loads data. For the illustration of figure 1(a), the values of $\overline{M_1(2\sigma_1)}$, $\overline{M_2(2\sigma_2)}$, and $\overline{M_3(2\sigma_3)}$ are indicated by ticks on the figure and are 4.3×10^{-6}, 3.2×10^{-4}, and 7.4×10^{-3}. The sum of the P's will in any case be less than one, the remaining time being in either smooth air or very light rough air which does not contribute many peak values above the threshold value.

Application of graphical procedure to operational data.- The procedure outlined provides a method for the representation of the operational-load history in terms of a few quantities: two or more root-mean-square accelerations σ_1, σ_2, . . ., and the associated proportions of flight time P_1, P_2, . . . spent at these root-mean-square acceleration values. This procedure has been applied to a series of eight operational load histories obtained from NACA VGH records. The scope of the data considered is summarized in table I. The data were obtained from six different types of transport airplanes with the samples 5, 6, and 7 representing the same airplane type flown by different operators. The pertinent airplane characteristics and operating conditions are given in table II. These data include estimates of the average flight altitude, average flight speed, and average weight. The distributions of peak acceleration measured in flight are given in table III. Values of C and C_2 were estimated in most cases from sections of record obtained in continuous rough air by counting the number of peak accelerations exceeding 2σ and are given in table IV. The measured distributions of acceleration increment for the eight samples are shown in figures 1(a) to 1(g), the illustration previously considered being sample 1 shown in figure 1(a). The subdivision of the distribution into Gaussian components in the manner previously described is also indicated by the dashed lines in each case. The values of σ_i and P_i for each operation obtained from the slopes of the linear components and the values of $\overline{M(2\sigma_i)}$ were determined in each case and are summarized in table IV.

The results of figure 1 and table IV indicate that the operational load histories can, in most cases, be reproduced by numbers of the order of 1 percent of the flight time at $\sigma_{a_n} = 0.15g$ and 0.05 percent of the flight time at $\sigma_{a_n} = 0.3g$. In some cases, three components were found desirable. Actually, the total flight time in rough air is considerably higher than the 1 percent indicated by these values, as previously mentioned, and should probably include perhaps as much as 10 percent at a lower value of σ of the order of 0.05. This flight time would yield the primary contribution to the number of peak accelerations at values of a_n below 0.3g but contributes only a few peak accelerations above the threshold value of 0.3g used in the data evaluations considered herein.

Distribution of Root-Mean-Square Acceleration

for Continuous Case

The representation of the load experience in the previous section in terms of a few discrete values of root-mean-square acceleration is a simplification, since atmospheric turbulence may actually be expected to cover a continuous variation in intensity. In this section, the problem of estimating the associated continuous distribution of root-mean-square acceleration from the overall peak counts will be considered.

The determination of the actual probability distribution of root-mean-square acceleration $f(\sigma)$ from a given peak-load history requires the solution of the integral equation given by equation (8). Since it does not, in general, appear possible to represent the load experience, in terms of the number of peak accelerations above given values, by a simple function, an effort was made instead to estimate $f(\sigma)$ directly in simple form, based on the results obtained in the preceding section. Equation (8) is then integrated and the results obtained are compared with the operational data.

Consideration of the values of P and σ in table IV suggested that the distribution of σ decreases rapidly with increasing values of σ and might be approximated by simple exponential distributions. On this basis $(\sigma \geqq 0)$, three exponential-type probability density functions were considered, namely,

$$\left.\begin{aligned}
&\text{Case a:} \\
&\qquad f_1(\sigma) = \frac{1}{a_1}\sqrt{\frac{2}{\pi}}\, e^{-\sigma^2/2a_1^2} \\
&\text{Case b:} \\
&\qquad f_2(\sigma) = \frac{1}{a_2}\, e^{-\sigma/a_2} \\
&\text{Case c:} \\
&\qquad f_3(\sigma) = \frac{1}{2a_3^2}\, e^{-\sqrt{\sigma}/a_3}
\end{aligned}\right\} \tag{13}$$

For each case, a is a scale parameter, larger values of a representing more severe load histories. The three distributions are shown in figure 2 for values of $a = 1$ and, in order, are seen to have increasingly larger areas at the higher values of σ. These functions, by the definition of a probability density function, all have unit area.

Case a.- If it is assumed that $f(\sigma)$ is given by case a, equation (8) may be integrated in closed form and yields the following result for $\overline{M(a_n)}$:

$$\overline{M(a_n)} = Ce^{-a_n/a_1} \tag{14}$$

Equation (14) yields a simple result for the number of peaks per second exceeding given values of a_n, in terms of the single-scale parameter of the distribution of σ. The value of the parameter a_1 is inversely proportional to the slope of the line for $\overline{M(a_n)}/C$ when plotted on semilogarithmic paper and also gives the value of σ_{a_n} below which 68 percent of the airplane flight time is spent. The linear variation for the $\log \overline{M(a_n)}$ given by equation (14) is of considerable interest because gust-load flight-test data on peak accelerations, under some conditions, particularly where a limited range of weather conditions is represented, tend to exhibit linear trends when plotted on semilogarithmic paper $\left(\log \overline{M(a_n)}\right.$ as a function of $\left.a_n\right)$. This condition, however, does not generally apply to operational data.

Case b.- The distributions of $\overline{M(a_n)}$ observed from operational flights frequently exhibit a less rapid decrease of $\overline{M(a_n)}$ with increasing a_n than given by equation (14) and thus imply more variations in the intensity of turbulence sampled and in particular a larger proportion of flight under the more severe conditions of turbulence. The distribution of $f(\sigma)$ for these cases might be more adequately represented by case b or case c. Substituting the expression for case b in equation (8) yields the following expression for the number of maximums exceeding a_n:

$$\overline{M(a_n)} = \int_0^\infty \frac{C}{a_2} e^{-\frac{a_n^2}{2\sigma^2} - \frac{\sigma}{a_2}} d\sigma \tag{15}$$

A closed-form evaluation of the integral could not be found. Numerical evaluations of the integral, however, are made readily. If the substitution $s = \sigma/a_2$ is made into equation (15) and both sides of the equation are divided by C, there is obtained

$$\frac{\overline{M(a_n)}}{C} = \int_0^\infty e^{-\frac{a_n^2}{2a_2^2}\frac{1}{s^2} - s} ds \tag{16}$$

Equation (16) was evaluated for various values of $\left(a_n^2/2a_2^2\right)$ so that $\overline{M(a_n)}/C$ could be determined for various values of a_2. The results obtained are shown plotted in figure 3. Operational gust loads data can be plotted conveniently in figure 3 in order to determine whether they adhere to the present distribution shape. (The plotting requires the determination of the value of C which may be estimated from flight records as described previously.) Only one of the eight cases (sample 2) appeared to follow this shape and is shown in the figure. The measured distribution of peaks which included positive and negative peaks was divided by 2 for this comparison. The appropriate values a_2 can be obtained from the figure by interpolation and are seen to be roughly equal to 0.036 for the operation plotted. This operation was a low-level feeder-line operation which differed somewhat from the other operations. In particular, this operation appeared to be characterized by extensive flight time in low-altitude light turbulence with, however, less than the usual exposure to more severe turbulence. The remaining seven operations had a slower rate of decrease in $M(a_n)/C$ with increasing a_n; this condition suggested that case c might be more appropriate.

Case c.- For case c, equation (8) becomes

$$\overline{M(a_n)} = \frac{C}{2a_3^2}\int_0^\infty e^{-\frac{a_n^2}{2\sigma^2}-\frac{\sqrt{\sigma}}{a_3}}\,d\sigma \qquad (17)$$

Equation (17) could not be evaluated in closed form but was evaluated numerically for various values of a_3. The results obtained for several values of a_3 are shown in figure 4. The operational-loads data for all but one case of table I appear to be represented best by this case and are with the exception of sample 2 shown in figure 4. Again, one-half the measured distributions are plotted to be comparable with the calculated curves. The data for operation 1 do not appear to be too well represented by these curves at low values of a_n but were included here inasmuch as the data for higher values of a_n are more adequately represented by this case. This operation differed somewhat from the others shown in this figure, because it was more in the nature of a feeder-line operation with a good part of the flight time spent at low altitudes.

The data shown in figure 4 appear generally to be approximated adequately by the distribution shapes for this case. Values of a_3 for each of the operations were estimated by interpolation and are summarized in table IV. The acceleration histories for these operations are thus described by the probability distribution for case (3):

$$f(\sigma) = \frac{1}{2a_3^2}e^{-\sqrt{\sigma}/a_3} \qquad (\sigma \geqq 0)$$

where the appropriate values of a_3 for each operation are given in table IV.

Distribution of Root-Mean-Square Gust Velocity

In the foregoing, the overall airplane acceleration history is given in terms of either several discrete root-mean-square values of acceleration and their associated percentage exposure times to each level or in terms of a continuous distribution of root-mean-square acceleration. The conversion of these results in terms of the airplane gust experience would be desirable in order to provide a basis for the calculation of load histories for other airplanes. In the following paragraphs, this conversion problem is considered.

The root-mean-square gust acceleration in rough air is related to the turbulence spectrum and the airplane response characteristics by

$$\sigma_{a_n}{}^2 = \int_0^\infty \Phi_i(\omega)T^2(\omega)d\omega \tag{18}$$

where

Φ_i spectrum of vertical gust velocity

$T(\omega)$ amplitude of airplane acceleration response to sinusoidal gusts of unit amplitude

Since the root-mean-square acceleration obtained by the preceding evaluation is seen to depend on both the spectrum of turbulence and the airplane response characteristics, the root-mean-square acceleration is apparently not sufficient to fix the turbulence spectrum. Several procedures appear possible for the analysis of loads measurements for the present purpose of deducing root-mean-square gust velocities. These include the actual measurement of acceleration spectra and the calculation of the airplane frequency-response function. The turbulence spectrum is then given by

$$\Phi_i(\omega) = \frac{\Phi_o(\omega)}{T^2(\omega)} \tag{19}$$

This procedure is extremely laborious and requires extensive calculations for the output spectrum $\Phi_o(\omega)$; the reliability of the results in turn depends upon the reliability of the calculated airplane transfer functions. At the present time, it is highly unlikely that the large amount of work involved in this approach is warranted even if the available records were in a form which would permit this type of analysis. Actually, the film speed used for the available records was far too fast to permit this type of analysis.

As an alternative, some simplifications appear warranted. As a preliminary effort in this direction, it will be assumed that:

(1) The airplane is rigid.

(2) The airplane is free to move vertically only (not pitch).

(3) As a first approximation, it will also be assumed that the airplane flies at an average weight, altitude, and airspeed, although in particular cases a more detailed consideration of these factors might be desirable.

(4) The spectrum of vertical gust velocity is given by

$$\Phi_1(\Omega) = \sigma_U^2 \frac{L}{\pi} \frac{1 + 3\Omega^2 L^2}{\left(1 + \Omega^2 L^2\right)^2} \tag{20}$$

where Ω is a reduced frequency ω/V in radians per foot and L is the scale of turbulence. These assumptions, although crude, should nevertheless provide some reasonable approximations of the gust histories.

For the foregoing conditions, a useful result obtained by Y. C. Fung (ref. 2) is that

$$\overline{\ddot{Z}^2} = \sigma_U^2 \frac{16V^2}{\pi \bar{c}^2 (1 + K)^2} I(K,s) \tag{21}$$

where

$\overline{\ddot{Z}^2}$ root-mean-square acceleration

$\bar{c}$ mean chord

K airplane mass parameter

s ratio of mean chord to scale of turbulence, $\bar{c}/L$.

Since $K \gg 1$ in almost all cases of concern, equation (21) may be simplified to yield

$$\left.\begin{aligned} \sigma_{a_n} &= \sigma_U \frac{\rho V S m}{2W} \sqrt{\frac{I(K,s)}{\pi}} \\ &= \bar{A}\sigma_U \end{aligned}\right\} \tag{22}$$

where

$$\bar{A} = \frac{\rho VSm}{2W}\sqrt{\frac{I(K,s)}{\pi}}$$

In this representation, the three-dimensional slope of the lift curve m has been used to replace 2π in order to account for the over-all three-dimensional aerodynamic effects. In this form, observed values of σ_{a_n} may be used directly with the values of the airplane parameters to determine σ_U. This calculation requires the determination of the value of $\sqrt{\frac{I(K,s)}{\pi}}$ which may be considered to be a gust-response factor and is shown in figure 5 as a function of the mass parameter K for various values of s. For given operations (or, if necessary, portions of operations), an average value of mass parameter K would appear to be adequate since the value of $\sqrt{\frac{I(K,s)}{\pi}}$ varies slowly with small variations in K. Average values of K were determined for the eight sets of operations on the basis of an average altitude and an estimated average weight as indicated in table II. The use of average values for ρ, V, and W in equation (22) would also appear adequate for present purposes.

The value of $\sqrt{\frac{I(K,s)}{\pi}}$ as noted also depends on the ratio $\bar{c}/L$ and thus requires the choice of a representative value of L for present purposes. It might be expected that the scale of turbulence would vary somewhat with altitude and weather condition. Available spectral measurements suggest that the value of L varies from perhaps 300 to over 1,000 feet. Since the quantity $\sqrt{\frac{I(K,s)}{\pi}}$ can be seen to vary appreciably over this range of L values, two values of $L = 400$ feet and $L = 1{,}000$ feet will be taken as representative and used in the subsequent calculations. On the basis of these values of L and the mass-parameter values previously determined, the values of $\sqrt{\frac{I(K,s)}{\pi}}$ for the eight operations were determined from figure 5 and are given in table II.

For the case of discrete values of σ_{a_n}, equation (22) permits the direct evaluation of the associated values of root-mean-square gust velocity σ_U. The results obtained for σ_U for the eight operational samples are also given in table IV. The gust experience for each of the operations is thus seen to be represented by two or three root-mean-square gust velocities along with their associated values of P, the proportion of flight time associated with each of these values. This representation of the gust experience in terms of a few quantities, as will be discussed subsequently, is readily applicable to the calculation of loads for other airplanes operated in a similar manner.

For the case of a continuous distribution of root-mean-square acceleration, the appropriate distribution of root-mean-square gust velocity is obtained from equation (22) by the relation for a change of variables for probability distributions and in terms of $f\left(\sigma_{a_n}\right)$ is given by

$$\hat{f}\left(\sigma_U\right) = \bar{A}f\left(\bar{A}\sigma_U\right) \tag{23}$$

where $a_n = \bar{A}\sigma_U$. Thus, for example, for the continuous distribution of case b

$$\hat{f}_2\left(\sigma_U\right) = \frac{1}{b_2} e^{-\sigma_U/b_2} \qquad (\sigma \geqq 0) \tag{24}$$

where $b_2 = a_2/\bar{A}$. The one operational sample considered to be represented by case b, it will be recalled, yielded a value of a_2 equal to 0.036 which, in this case, gives a value of b_2 of 1.629 for $L = 400$ feet.

The application of equation (23) to the distribution $f_3\left(a_n\right)$ for case c yields

$$\left.\begin{aligned} \hat{f}_3\left(\sigma_U\right) &= \frac{\bar{A}}{2a_3^{\,2}} e^{-\sqrt{\bar{A}\sigma_U}/a_3} \\ &\text{or} \\ \hat{f}_3\left(\sigma_U\right) &= \frac{1}{2b_3^{\,2}} e^{-\sqrt{\sigma_U}/b_3} \end{aligned}\right\} \tag{25}$$

where

$$b_3 = \frac{a_3}{\sqrt{\bar{A}}}$$

The appropriate values of b_3 for the seven pertinent operations obtained by means of equation (25) are given in table IV for both values of L. These values and equation (25), thus, give the probability distributions of the root-mean-square gust velocity and define the proportion of total flight time at various values of σ_U for the different operations.

DISCUSSION

General Results

The foregoing analysis has served to indicate that the probability distribution of the root-mean-square gust velocity experienced in operations can be related to the overall distribution of peak loads. In general, this relation depends upon the power-spectral-density functions of the turbulence and the frequency-response characteristics of the airplane. By assuming that atmospheric turbulence is a Gaussian process with a single parameter, the root-mean-square gust velocity, a simple relation is derived between the number of peak accelerations experienced in operations by a given airplane and the probability distribution of the root-mean-square acceleration. This relation in conjunction with the relations between the root-mean-square gust velocity and acceleration provides a basis for both estimating the peak loads from known probability distributions of root-mean-square gust velocity and, conversely, for estimating the distribution of the root-mean-square gust velocities from data on peak loads.

The applications of these relations to operational gust-load data have provided estimates of the distribution of root-mean-square acceleration and root-mean-square gust velocity for several sets of operations. These distributions are given in two forms: first, for the discrete case, in the simple form of percent of the total flight time at two or three discrete values of root-mean-square normal acceleration and root-mean-square gust velocity; and second, in a more detailed form of a continuous distribution of the values of the root-mean-square acceleration and gust velocity.

Quantitative Results

For the discrete case, the results obtained indicated that the distribution of root-mean-square gust velocity can be approximated by two or three levels of gust or load intensity and associated proportions of flight times. The quantitative results for the levels and proportions of flight time for the turbulence experienced in a number of operations varied considerably in this representation and are not comparable in simple terms. Comparison of the gust experience among the various operations will therefore be deferred until the results for the continuous case are discussed where such comparisons can be made directly.

Consideration of the values in table IV for the root-mean-square gust velocities σ_U and for the associated proportion of flight time P indicated considerable variation in the composition of the gust history. For the assumed value of the scale of turbulence, $L = 400$ feet, the following values appear to be roughly representative for the present type of operations: 0.6 percent of the flight time at $\sigma = 8$ fps, and perhaps 0.02 percent at $\sigma_U = 16$ fps. For 10,000 hours of flight, these values yield 60 hours at σ_U of 8 fps, 2 hours at σ_U of 16 fps. In addition, a smaller amount of the total flight, as indicated by some of the results in the table, perhaps a few minutes, may be spent at around 25 fps. The feeder-line operations of sample 2 are represented by considerably larger flight time at a $\sigma_U = 8$ fps, 1.3 percent or 130 hours in 10,000 hours of flight, with one-half hour at a $\sigma_U = 13$ fps. For L = 1,000 feet, the exposure times are unchanged although the root-mean-square gust velocities obtained are increased by about 20 to 30 percent in the various operations.

The foregoing results were derived from gust statistics on accelerations above 0.3g. If a smaller threshold value had been used, say 0.1 g, it is estimated that an additional linear component of the order of perhaps 10 percent flight time at around $\sigma_U = 4$ feet per second would be required to approximate the load history to these lower thresholds.

For the continuous case, seven of the eight samples considered seemed to be represented best by the distribution of gust velocity given by

$$\hat{f}_3(\sigma_U) = \frac{1}{2b_3{}^2} e^{-\sqrt{\sigma_U}/b_3} \qquad (\sigma_U \geqq 0)$$

with values of b_3 varying from 0.294 to 0.312 for L = 400 and from 0.326 to 0.349 for L = 1,000 feet. In each case, the values of b_3 varied by about ±3 percent about an average value. This 3-percent variation in b_3, when applied to a given airplane to determine acceleration peaks, can be seen from figure 4 to give rise to 2:1 variations in the number of larger peak loads. The implications of such variations in loads experience are generally not considered significant. Thus, the results appear to indicate that the overall variations in the gust experience for these operations is not large.

The loads for one sample of data for low-altitude feeder-line operations was better approximated by a distribution of σ_U given by

$$\hat{f}_2(\sigma_U) = \frac{1}{b_2} e^{-\sigma_U/b_2} \qquad (\sigma \geqq 0)$$

with a value of $b_2 = 1.63$. This difference is attributed to the low-altitude nature of this feeder-line operation which appears from table IV to include a larger percentage of flight time in moderately rough air but an unusually small percentage of the flight time at the more severe levels of turbulence.

Statistical Reliability of Results

Some remarks should be made in regard to the statistical reliability of the quantitative results derived for the gust experience. Since the present results are based on relatively small operational samples (of the order of 1,000 hours), the statistical reliability of the desired distributions of root-mean-square value is dependent upon the statistical reliability of the acceleration data. Past experience has indicated that these distributions are reliable for samples of this size at the lower levels of acceleration (0.3g to 0.5g) but have poor statistical reliability at the higher acceleration values. As a consequence, it may be expected that the derived probabilities for the higher root-mean-square gust velocities are only rough estimates and should be used only as a guide. More reliable information regarding the higher root-mean-square gust velocities requires more extensive flight data although it may be possible to supplement the present results by use of available NACA V-G records. The extension of the present applications to include such other data is, however, beyond the scope of the present report.

Application to Load Calculation

The application of the derived distributions of root-mean-square gust velocity to calculations of loads and other responses for new airplanes requires considerable care in regard to both the airplane response and operational considerations. It will be recalled that it was assumed in the derivation of the root-mean-square gust velocities that the airplane was rigid and restrained in pitch although free to move vertically. These assumptions are admittedly rough approximations and their effects on the derived gust data require additional study. Extensions of the present results to include the effects of these two additional degrees of freedom appear possible, although they would involve a considerably larger number of airplane and operating parameters. It should, however, be possible in the meantime to make some rough overall corrections to the root-mean-square gust velocities for some of these effects. For example, for the airplanes considered in the present study, available information suggests that the neglect of the dynamic structural response effects on the center-of-gravity accelerations might be expected to have resulted in roughly a 10-percent overestimation of the root-mean-square gust velocities for most of the airplanes considered. The effects of the airplane pitching motions on the root-mean-square gust velocities for the present airplanes are also generally considered small. It might be expected that the pitching motions for the present airplanes would tend to decrease the gust accelerations and thus tend to lead to some underestimation of the root-mean-square gust velocities. A rough estimate of about 10 percent appears reasonable for this effect. Thus it is suggested that the overall effects of pitch and flexibility might largely cancel each other. However, additional study of these effects is needed.

The effects of assuming average conditions of weight, altitude, and speed may also be expected to introduce some errors in the derived root-mean-square gust velocities. The errors resulting from the assumptions of average weight and altitude may be expected to be small because these errors should largely average out. The effects of speed variations are in a somewhat different category inasmuch as efforts are normally made to reduce speed in rough air. For the operations considered herein, it appears that the reductions in speed from normal operating speeds were generally small and negligible at the lighter levels of turbulence. At the more severe levels of turbulence, the airspeeds were, on the average, somewhat lower than normal operating speeds, but the reductions in most cases were small, perhaps 5 to 10 percent. Thus, the distribution of σ_U at the higher levels may be biased to this extent.

In the present analysis, no consideration has been given to the effects of gust averaging introduced by the finite span. If the ratio of the span to the scale of turbulence is large, these effects, as has been indicated in reference 6, may be appreciable. For example, for a span of 150 feet and a value of $L = 450$, the results of reference 6 suggest that the root-mean-square value of σ_U may be underestimated in the present analysis by perhaps as much as 15 percent. For smaller values of the ratio of the span to the scale of turbulence, the magnitude of this discrepancy is considerably smaller. If desirable, the incorporation of these effects on an average basis would also appear straightforward.

Considerations in Applications

In view of the foregoing considerations, the distribution of root-mean-square gust velocity given herein may be considered a reasonable first-order estimate of the characteristics of atmospheric turbulence that are essentially independent of the characteristics of the airplanes involved. Thus, these gust spectra and root-mean-square gust-velocity distributions can be reasonably applied in gust-load calculations in which the effects of pitching motions and flexibility are included in the determination of the airplane transfer functions. The relations obtained in these cases between σ_{a_n} and σ_U would then replace the result for the one-degree-of-freedom case given by equation (22) and would be used in the transformation of the distribution of σ_U to the distributions of σ_{a_n} as given by equation (23).

Load calculations based on a single-degree-of-freedom vertical motion can be made simply by reversing the procedures followed in the foregoing derivation of root-mean-square gust velocities. The steps involved are the selection of the appropriate distribution of $f(\sigma_U)$ (using either the results for the discrete or continuous case) and the transformation of this distribution in accordance with the rule for probability distributions in order to obtain the associated distribution $f(\sigma_{a_n})$. For the case of discrete values of σ_U, the contributions of all the turbulence components to the number of peak accelerations must be summed to obtain the overall distribution of peak accelerations. For the case of continuous variations of σ_U, the number of peak loads per second above given values can be obtained directly from figure 3 or 4 from the parameter value of the distribution of root-mean-square acceleration, a_2 or a_3, respectively.

Operational Considerations

The direct application of the foregoing results to new operations assumes that the flight paths and the operational procedures would be similar to those from which the gust data were derived. Actually, new operations may be expected to incorporate modifications in operational procedures which may modify the gust experience. Some possibilities in this regard include higher operating altitudes, higher rates of climb and descent which would result in less time exposure to the frequent turbulence at low altitudes, turbulence avoidance through the use of radar, and longer flights with associated larger proportions of flight time at the relatively less turbulent higher altitudes. For such variations in flight plan, the overall probability distribution of σ_U may be expected to differ from those obtained for the operations considered herein. In order to obtain the required flexibility for calculations of gust-load histories for such variations in operational procedures, information on the variations of the probability distribution of σ_U is required in greater detail than given herein. Information on the variations of the distribution of σ_U with altitude, geography, and possibly with the type of weather condition may be necessary. For trend studies, it might be possible in the meanwhile to make reasonable estimates of the effects on the distribution of σ_U of some variations in flight plan from available gust-loads data. For example, for a high-altitude jet-transport operation, the proportion of the total flight time at the various discrete values of σ_U might be expected to be decreased somewhat from that obtained for the present operations. The amount of decrease can perhaps be estimated from a comparison of the altitude flight plan with that for the present operations.

The effects on gust loads of reductions in airspeed in rough air may also be important in some cases. Although the present data, as previously mentioned, are to some extent biased by reductions in airspeed in rough air in the present operations, the speed reductions appear generally small. In some operations, it may be possible to achieve pronounced airspeed reductions under the more severe conditions of turbulence by use of radar or low airspeeds in climb and descent. The effects of such airspeed reductions would, of course, be important and require consideration in calculations. The effects of such variations in airspeed practices can be accounted for by permitting the quantity $\bar{A}$ in equation (23) to vary with σ_U. Similar means could presumably also be used to account for wide variations in weight and altitude. These operational considerations warrant additional study in order to determine the best methods of incorporating these factors in load calculations.

CONCLUDING REMARKS

The foregoing analysis has served to indicate that the probability distribution of the root-mean-square gust velocity experienced by an airplane in normal operations can be related to the overall distribution of peak loads experienced in operations. These relations were then used to derive estimates of the probability distribution of root-mean-square gust velocity from available measurements of peak accelerations in normal operations. The procedure used was to derive the probability distributions of root-mean-square acceleration from the data on peak accelerations. By using the relation between root-mean-square gust velocity and acceleration, this probability distribution was then transformed in order to obtain the probability distribution of root-mean-square gust velocity. The airplane gust-response theory was limited in this investigation to one degree of freedom (vertical motion only) and a simple analytical expression for the power spectrum of atmospheric turbulence was used. The effects of other airplane degrees of freedom such as pitch and wing-bending flexibility on the derived results are also considered briefly and it is indicated that their overall effects on the reliability of these results is probably small. On this basis, it appears reasonable to consider the derived probability distributions of root-mean-square gust velocity to be largely independent of the airplane gust-response characteristics. Thus, these results may be applied to the calculation of loads and other airplane responses in operational flight in which the effects of pitching motions and airplane elasticity are included in the determination of the airplane transfer function.

The application of the distributions of root-mean-square gust velocity derived herein to calculations for new operations is also considered briefly and it is indicated that in such applications care is required in regard to such operational factors as altitude flight plan and airspeed practices, inasmuch as these data were derived from specific types of operations. As a consequence, new operations which incorporate different operational patterns particularly in regard to altitude may be expected to experience somewhat different turbulence histories. Also, modifications in regard to airspeed practices in rough air may require more detailed considerations. The best means of accounting for such operational factors appear to warrant further study.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., October 22, 1954.

APPENDIX

RELATIONS BETWEEN PEAK COUNTS AND SPECTRA FOR A GAUSSIAN RANDOM PROCESS

In the present analysis, use is made of the relations between the number of maximums per second and the spectra for a Gaussian random process. These relations are derived in reference 13 and are summarized herein in order to permit the examination of the reliability of the approximate expression used in the body of the paper.

Number of maximums.- The probability p_m that a Gaussian random process $y(t)$ will have a maximum intensity ranging from y_1 to $y_1 + dy_1$ in the time interval t to $t + dt$ is given in reference 13 as

$$p_m = dy_1\, dt\, \frac{(2\pi)^{1/2}}{M_{33}}\left[|M|^{1/2} e^{-M_{11}y_1^2/2|M|} + M_{13}y_1\left(\frac{\pi}{2M_{33}}\right)^{1/2} e^{-y_1^2/2\psi_0}\left(1 + \operatorname{erf}\frac{M_{13}y_1}{\left(2|M|M_{33}\right)^{1/2}}\right)\right] \tag{A1}$$

where the error function is

$$\operatorname{erf} Z_1 = \frac{2}{\sqrt{\pi}}\int_0^{Z_1} e^{-Z^2}\, dZ$$

and the coefficients M_{ij} can be expressed in terms of the value at zero of the autocorrelation function $\psi(\tau)$ and its derivatives as

$$\left.\begin{aligned} M_{11} &= -\psi_0''\psi_0^{(4)} \\ M_{13} &= \left(\psi_0''\right)^2 \\ M_{33} &= -\psi_0\psi_0'' \\ |M| &= -\psi_0''\left[\psi_0\psi_0^{(4)} - \left(\psi_0''\right)^2\right] \end{aligned}\right\} \tag{A2}$$

where

$$\psi(\tau) = \lim_{T \longrightarrow \infty} \frac{1}{2T}\int_{-T}^{T} y(t)y(t + \tau)dt$$

and ψ_0'' and $\psi_0^{(4)}$ are, respectively, the values at 0 of the second and fourth derivative of the autocorrelation function $\psi(\tau)$. The values at zero of the autocorrelation function and its derivates are in turn related to the power-spectral-density function $\Phi(\omega)$ by the following relations:

$$\left.\begin{aligned} \psi_0 &= \int_0^\infty \Phi(\omega)d\omega \\ -\psi_0'' &= \int_0^\infty \omega^2\Phi(\omega)d\omega \\ \psi_0^{(4)} &= \int_0^\infty \omega^4\Phi(\omega)d\omega \end{aligned}\right\} \tag{A3}$$

Equation (A1) is rather unwieldy, but two simple results of interest may be obtained from it. The first result, the expected number of maximums per second, N_p, is obtained by integrating over y_1 from $-\infty$ to ∞ and over t for a time of 1 second to obtain

$$\left.\begin{aligned} N_p &= \frac{1}{2\pi}\left(\psi_0^{(4)}\Big/-\psi_0''\right)^{1/2} \\ &= \frac{1}{2\pi}\left[\frac{\int_0^\infty \omega^4\Phi(\omega)\,d\omega}{\int_0^\infty \omega^2\Phi(\omega)\,d\omega}\right]^{1/2} \end{aligned}\right\} \qquad \text{(A4)}$$

The second result is an asymptotic expression for large y_1 for the number of maxima per second $N(y_1)$ exceeding given values of y_1, which can be obtained by integrating an asymptotic approximation to p_m from y_1 to ∞; the result, given in reference 13, is

$$\left.\begin{aligned} N(y_1) &= \frac{1}{2\pi}\left(-\psi_0''\Big/\psi_0\right)^{1/2} e^{-y_1^2/2\psi_0} \\ &= \frac{1}{2\pi}\left[\frac{\int_0^\infty \omega^2\Phi(\omega)\,d\omega}{\int_0^\infty \Phi(\omega)\,d\omega}\right]^{1/2} e^{-y_1^2/2\psi_0}. \end{aligned}\right\} \qquad \text{(A5)}$$

It is of interest to note that the right-hand side of equation (A5) is also the exact expression for the number of crossings per second with positive slope of given values of y_1.

Equation (A5) is the basic relation used in the present analysis for the number of peaks per second exceeding given values of y_1. Since it is an approximation for this purpose, the magnitude of the errors introduced in the present analysis by its use is of interest and is considered in the remainder of the appendix.

Reliability of the approximation.- Past experience has indicated that in gust-load applications, the approximation given by equation (A5) is in most cases good for values of $y_1/\sigma > 2$. For values of $y_1/\sigma < 2$, equation (A5) tends to underestimate the number of peak loads to some extent. The magnitude of these errors does not, however, appear to be large and, as will be indicated, has only a very small effect on the reliability of the present analysis.

The magnitude of the error introduced by the approximation for small values of y_1/σ may be indicated by considering the ratio $N(0)$ to the total number of peaks N_p. From equations (A4) and (A5), this ratio is given by

$$\frac{N(0)}{N_p} = \frac{\int_0^\infty \omega^2\Phi(\omega)d\omega}{\left(\int_0^\infty \omega^4\Phi(\omega)d\omega\right)^{1/2}\left(\int_0^\infty \Phi(\omega)d\omega\right)^{1/2}} \tag{A6}$$

For a low-pass filter, which appears to describe roughly the response of a relatively rigid airplane to turbulence, equation (A6) reduces to

$$\frac{N(0)}{N_p} = \frac{\sqrt{5}}{3} \approx 0.75 \tag{A7}$$

The quantity N_p, which gives all the maximums, includes some maximums at negative values of y_1. For the low-pass-filter case, the results of reference 13 indicate that about 15 percent of all maximums are at negative values of y_1. Thus, the approximation of equation (A5) appears to be roughly 10 percent low for the low-pass-filter case at $y_1/\sigma = 0$. For increasing values of y_1/σ, this error decreases rapidly and is less than 3 percent at $y_1/\sigma = 0.5$.

For moderately flexible airplanes of the type considered in the present study, the degree of underestimation of the asymptotic formula is somewhat larger than for the band-pass case. In this case, calculations indicate that equation (A5) appears to underestimate the number of peaks by about 30 percent at $y_1/\sigma = 0$, 10 to 15 percent at $y_1/\sigma = 1$, and 2 to 3 percent at $y_1/\sigma = 2$. The effect of these errors is, however, considerably mitigated in the present applications for the following reason. The total number of peak accelerations exceeding given values is seen from equation (8) to depend in principle upon the whole distribution of root-mean-square values. Actually, for the particular exponential-type functions for $f(\sigma)$ considered in the present analysis, the principal contributions to $\overline{M(a_n)}$ arise from values of a_n/σ_{a_n} that range from about 1.5 to 3.0. Thus, the asymptotic expression is being applied principally over the region where the underestimation is only a few percent. Therefore, the errors introduced in the present analysis by the use of the asymptotic formula (equation (A5)) can be considered negligible.

REFERENCES

1. Liepmann, H. W.: On the Application of Statistical Concepts to the Buffeting Problem. Jour. Aero. Sci., vol. 19, no. 12, Dec. 1952, pp. 793-800, 822.

2. Fung, Y. C.: Statistical Aspects of Dynamic Loads. Jour. Aero. Sci., vol. 20, no. 5, May 1953, pp. 317-330.

3. Press, Harry, and Mazelsky, Bernard: A Study of the Application of Power-Spectral Methods of Generalized Harmonic Analysis to Gust Loads on Airplanes. NACA TN 2853, 1953.

4. Clementson, Gerhardt C.: An Investigation of the Power Spectral Density of Atmospheric Turbulence. Ph. D. Thesis, M.I.T., 1950.

5. Press, Harry, and Houbolt, John C.: Some Applications of Generalized Harmonic Analysis to Gust Loads on Airplanes. Jour. Aero. Sci., vol. 22, no. 1, Jan. 1955, pp. 17-26.

6. Diederich, Franklin Wolfgang: The Response of an Airplane to Random Atmospheric Disturbances. Ph. D. Thesis, Calif. Inst. Tech., 1954.

7. Chilton, Robert G.: Some Measurements of Atmospheric Turbulence Obtained From Flow-Direction Vanes Mounted on an Airplane. NACA TN 3313, 1954.

8. Press, Harry, and McDougal, Robert L.: The Gust and Gust-Load Experience of a Twin-Engine Low-Altitude Transport Airplane in Operation on a Northern Transcontinental Route. NACA TN 2663, 1952.

9. Walker, Walter G.: Summary of Revised Gust-Velocity Data Obtained From V-G Records Taken on Civil Transport Airplanes From 1933 to 1950. NACA TN 3041, 1953.

10. Binckley, E. T., and Funk, Jack: A Flight Investigation of the Effects of Compressibility on Applied Gust Loads. NACA TN 1937, 1949.

11. Coleman, Thomas L., Copp, Martin R., Walker, Walter G., and Engel, Jerome N.: An Analysis of Accelerations, Airspeeds, and Gust Velocities From Three Commercial Operations of One Type of Medium-Altitude Transport Airplane. NACA TN 3365, 1955.

12. Coleman, Thomas L., and Walker, Walter G.: Analysis of Accelerations, Gust Velocities, and Airspeeds From Operations of a Twin-Engine Transport Airplane on a Transcontinental Route From 1950 to 1952. NACA TN 3371, 1955.

13. Rice, S. O.: Mathematical Analysis of Random Noise. Pts. I and II. Bell Syst. Tech. Jour., vol. XXIII, no. 3, July 1944, pp. 282-332; Pts. III and IV, vol. XXIV, no. 1, Jan. 1945, pp. 46-156.

14. Lawson, James L., and Uhlenbeck, George E., eds.: Threshold Signals. McGraw-Hill Book Co., Inc., 1950, ch. 3.

15. James, Hubert M., Nichols, Nathaniel B., and Phillips, Ralph S.: Theory of Servomechanisms. McGraw-Hill Book Co. Inc., 1947, ch. 6.

TABLE I

SCOPE OF OPERATIONS

Operation	Route	Flight hours	Flight miles
1	Northern trans-continental	834.28	186,120
2	Rocky Mountains – North and South	331.10	49,231
3	Southern trans-continental	766.45	148,774
4	90 percent east of Mississippi River	770.77	173,693
5	New York to Europe – New York to South America	1078.5	284,000
6	San Francisco to Honolulu	1958.0	488,000
7	Northern trans-continental	875.5	235,300
8	Southern trans-continental	706.45	193,378

TABLE II

AIRPLANE AND OPERATIONAL CHARACTERISTICS

Operation	Gross weight, W, lb	Average weight, $\bar{W}$, lb	Wing area, S, sq ft	Average chord, $\bar{c}$, ft	Average flight altitude, ft	Average air density, ρ, slugs/cu ft	Airspeed, V, ft/sec	m, per radian	Mass parameter, K	L = 400; $\sqrt{\frac{I(K,s)}{\pi}}$	L = 1,000; $\sqrt{\frac{I(K,s)}{\pi}}$
1	39,900	33,915	864	10.1	5,000	0.002049	327.2	5.0	75.00	0.550	0.411
2	25,200	21,420	987	10.4	3,000	.002176	218.1	4.92	37.91	.411	.293
3	40,500	34,425	817	9.7	5,000	.002049	281.0	5.03	83.83	.570	.426
4	90,000	76,500	1,650	14.7	10,000	.001756	330.5	4.93	72.12	.586	.458
5	142,500	121,125	1,720	12.9	12,500	.001622	386.2	5.12	133.08	.672	.567
6	142,500	121,125	1,720	12.9	12,500	.001622	365.6	5.12	133.08	.672	.567
7	142,500	121,125	1,720	12.9	12,500	.001622	394.2	5.12	133.08	.672	.567
8	89,900	76,415	1,463	13.7	12,100	.001653	401.5	4.95	91.20	.621	.495

TABLE III

NUMBER OF ACCELERATION PEAKS EXCEEDING GIVEN VALUES

a_n	Cumulative Frequency for Various Operations[1]							
	1	2	3	4	5	6	7	8
0.3	20,609	19,483	5,593	1,888	659	612	909	1,287
.4	------	4,632	1,350	427	152	132	178	232
.5	1,203	1,288	365	118	40	31	40	60
.6	377	370	104	44	13	10	14	18
.7	124	100	31	17	5	4	5	6
.8	47	35	16	6	2	2	-----	1
.9	26	9	5	2	1	1	-----	-----
1.0	7	3	2	-----	1	------	-----	-----
1.1	4	------	2	-----	1	------	-----	-----
1.2	2	------	-----	-----	------	------	-----	-----
1.3	1	------	-----	-----	------	------	-----	-----
Total flight hours	834	331.1	676.5	770.8	1078.5	1953.4	875.5	706.5

[1]Number includes both positive and negative peaks.

TABLE IV

RESULTS FOR VARIOUS OPERATIONS

(a) Summary of acceleration experience

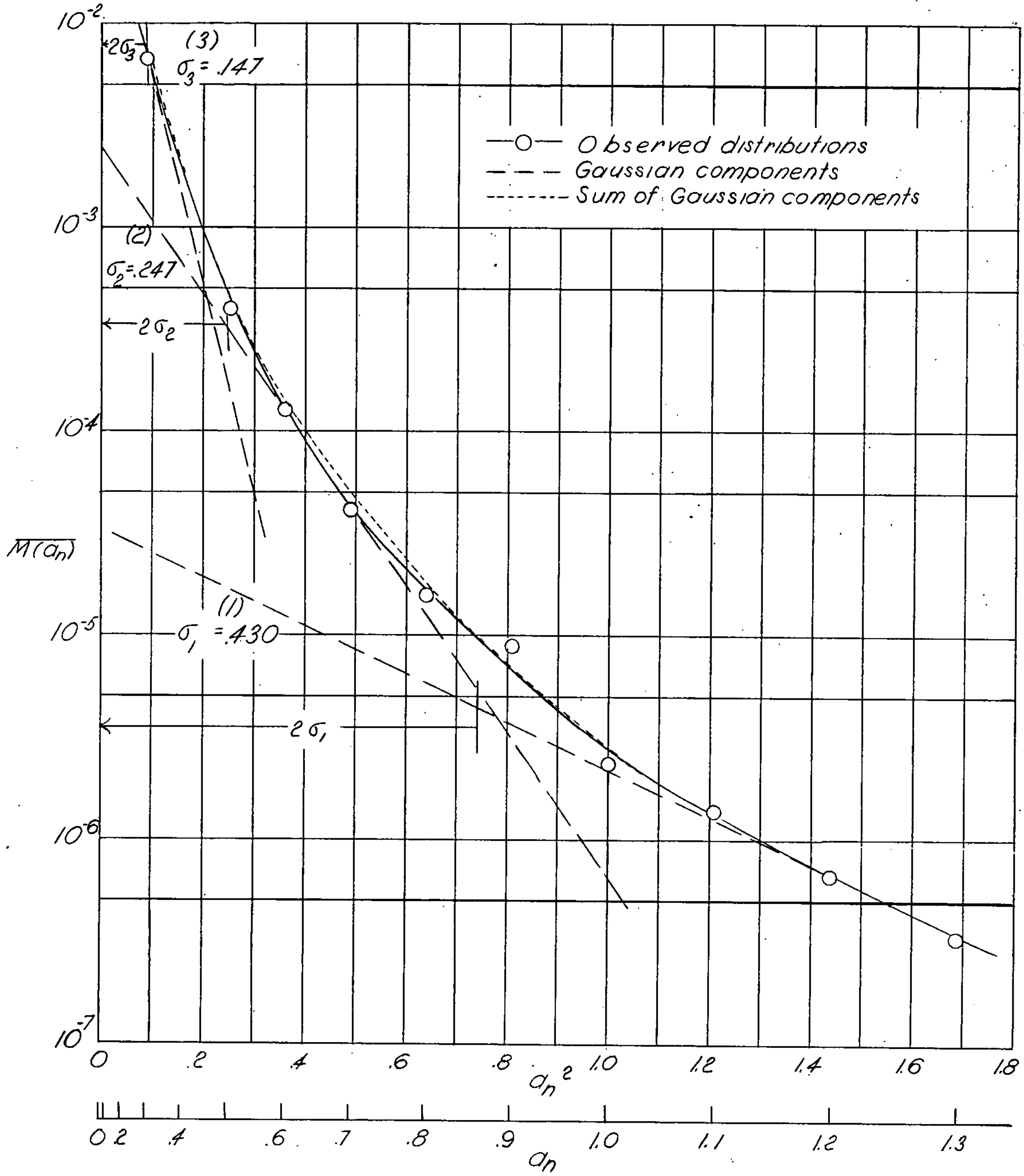
	Acceleration experience for operation –							
	1	2	3	4	5	6	7	8
$\sigma_{a_{n_1}}$	0.430	0.287	0.323	0.278	0.349	0.364	0.255	0.226
$\sigma_{a_{n_2}}$	0.247	0.181	0.181	0.147	0.178	0.171	0.132	0.128
$\sigma_{a_{n_3}}$	0.147	-----	-----	-----	-----	0.104	-----	-----
$M_1(2\sigma_1)$	4.3×10^{-6}	1.35×10^{-4}	1.35×10^{-5}	1.9×10^{-5}	9.5×10^{-7}	4.6×10^{-5}	9.5×10^{-6}	2.85×10^{-5}
$M_2(2\sigma_2)$	3.2×10^{-4}	6.2×10^{-3}	8.5×10^{-4}	8.2×10^{-4}	6.6×10^{-5}	3.7×10^{-5}	5.4×10^{-4}	1.15×10^{-3}
$M_3(2\sigma_3)$	7.4×10^{-3}	-----------	-----------	-----------	-----------	3.0×10^{-4}	-----------	-----------
C	1.5	1.0	1.5	0.5	0.5	0.5	0.5	0.5
$2C_2$	0.406	0.270	0.406	0.135	0.135	0.135	0.135	0.135
P_1	1.06×10^{-5}	4.99×10^{-4}	3.33×10^{-5}	1.41×10^{-4}	7.04×10^{-6}	3.41×10^{-6}	7.04×10^{-5}	2.11×10^{-4}
P_2	7.88×10^{-4}	1.29×10^{-2}	2.09×10^{-3}	6.07×10^{-3}	4.89×10^{-4}	2.74×10^{-4}	4.00×10^{-3}	8.52×10^{-3}
P_3	1.82×10^{-2}	-----------	-----------	-----------	-----------	2.22×10^{-3}	-----------	-----------
a_2	-----------	0.036	-----------	-----------	-----------	-----------	-----------	-----------
a_3	0.045	-----------	0.0425	0.0415	0.0385	0.036	0.039	0.041

(b) Summary of gust experience for L = 400 ft

	Gust experience for operation –							
	1	2	3	4	5	6	7	8
$\bar{A}$	0.0235	0.0221	0.0196	0.0181	0.0153	0.0145	0.0156	0.0195
σ_{u_1}	18.298	12.990	16.480	15.359	22.810	25.103	16.346	11.590
σ_{u_2}	10.511	8.190	9.235	8.122	11.634	11.793	8.461	6.039
σ_{u_3}	6.255	-----------	-----------	-----------	-----------	7.172	-----------	-----------
b_2	----------	1.629	-----------	-----------	-----------	-----------	-----------	-----------
b_3	0.294	-----------	0.304	0.309	0.311	0.299	0.312	0.294

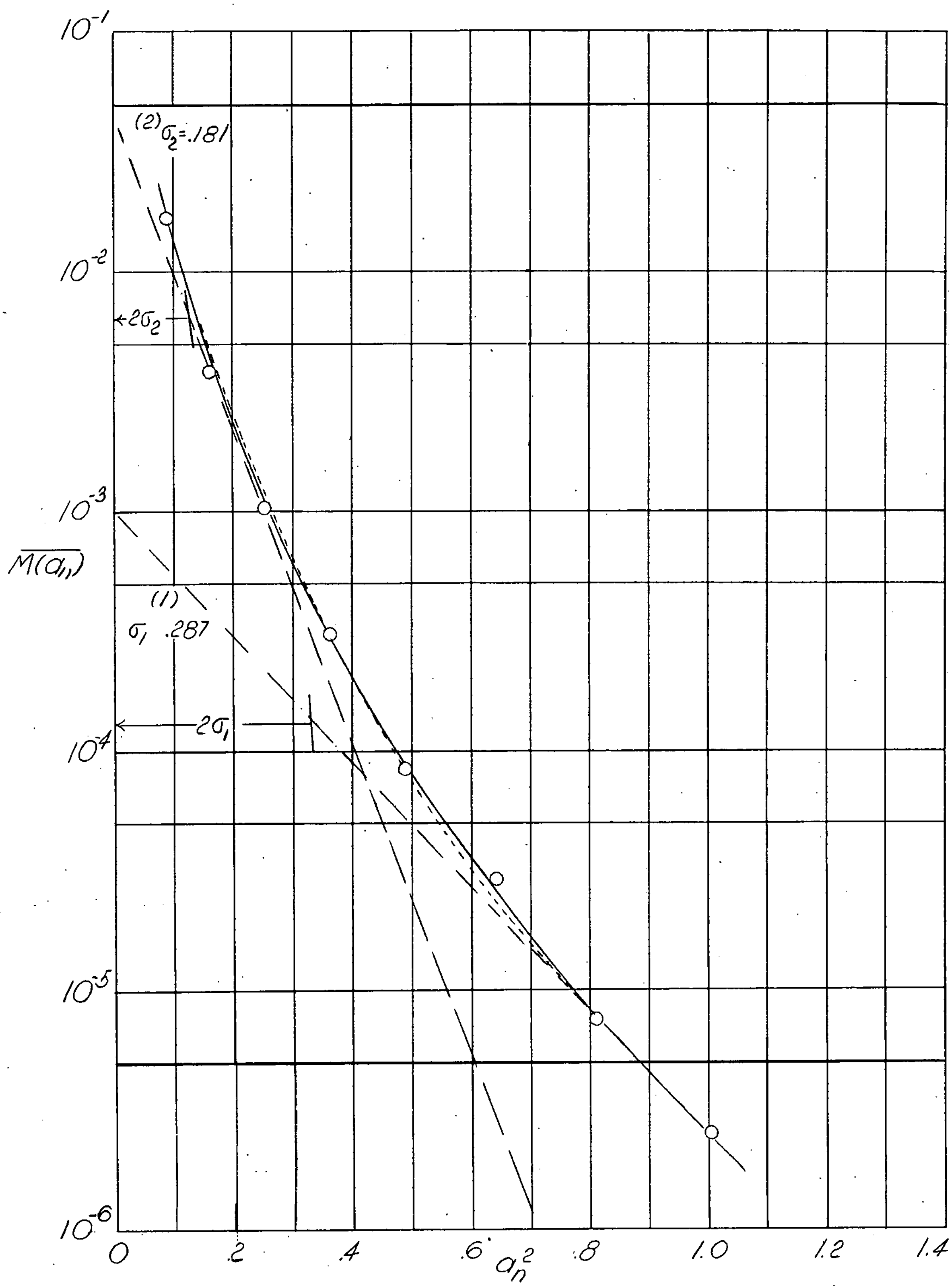
(c) Summary of gust experience for L = 1,000 ft

	Gust experience for operation –							
	1	2	3	4	5	6	7	8
$\bar{A}$	0.01755	0.0158	0.0146	0.0141	0.0129	0.0122	0.0132	0.0156
σ_{u_1}	24.501	18.211	22.063	19.674	27.033	29.787	19.347	14.515
σ_{u_2}	14.074	11.485	12.363	10.403	13.788	13.993	10.015	8.221
σ_{u_3}	8.376	-----------	-----------	-----------	-----------	8.511	-----------	-----------
b_2	___________	2.284	-----------	-----------	-----------	-----------	-----------	-----------
b_3	0.340	-----------	0.351	0.349	0.339	0.326	0.340	0.329



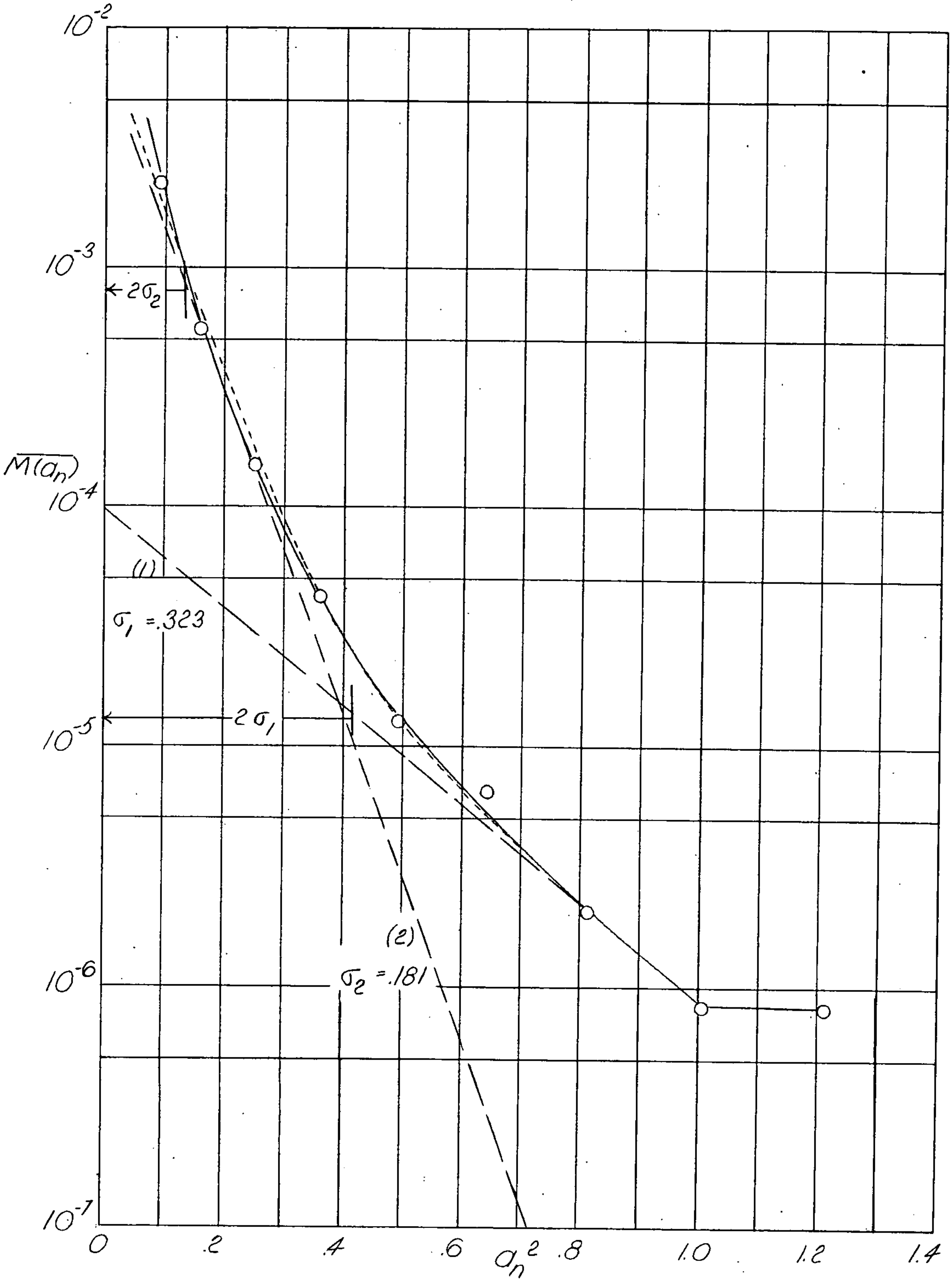
(a) Operation 1.

Figure 1.- Graphical separation into Gaussian components of distribution of peak acceleration.



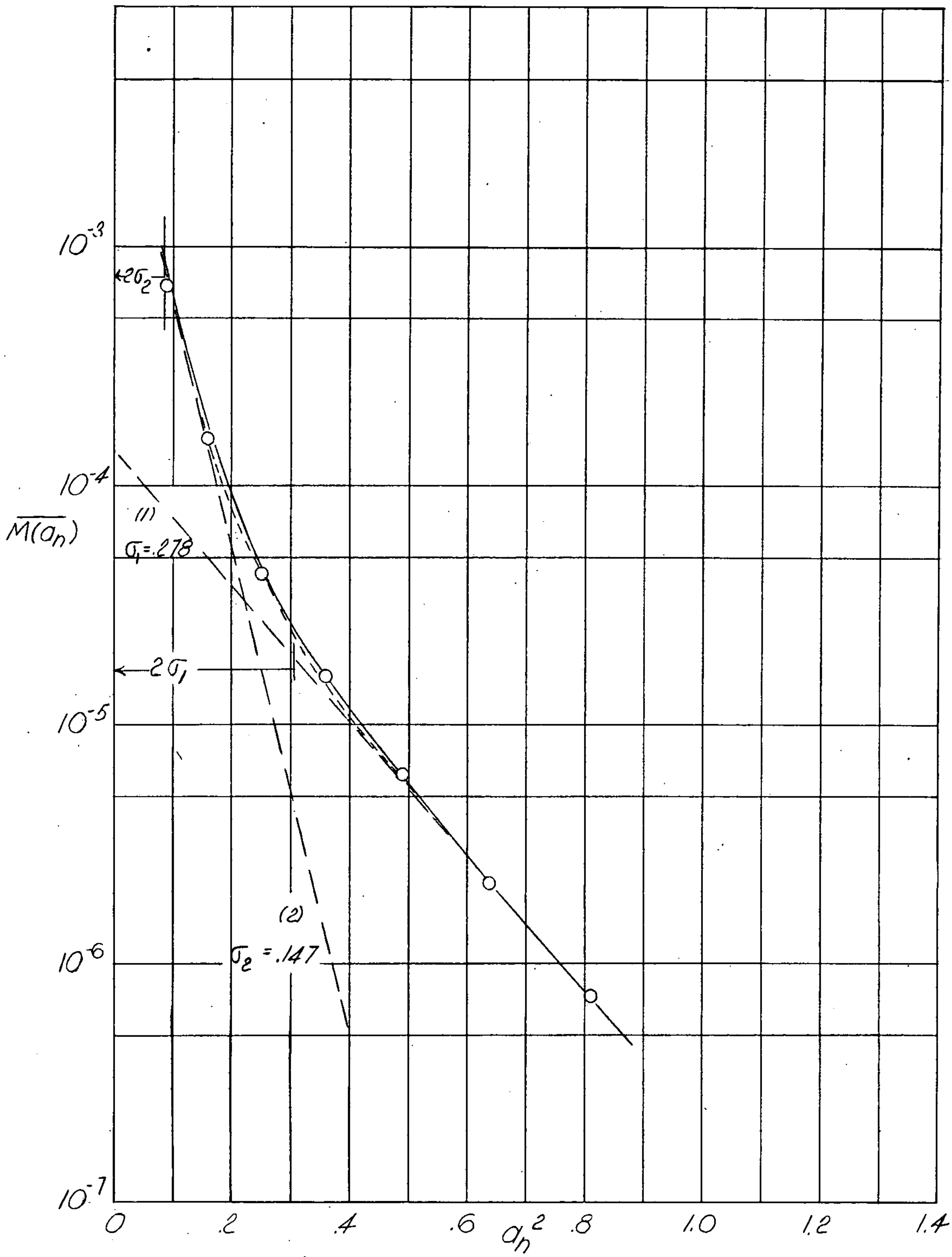
(b) Operation 2.

Figure 1.- Continued.



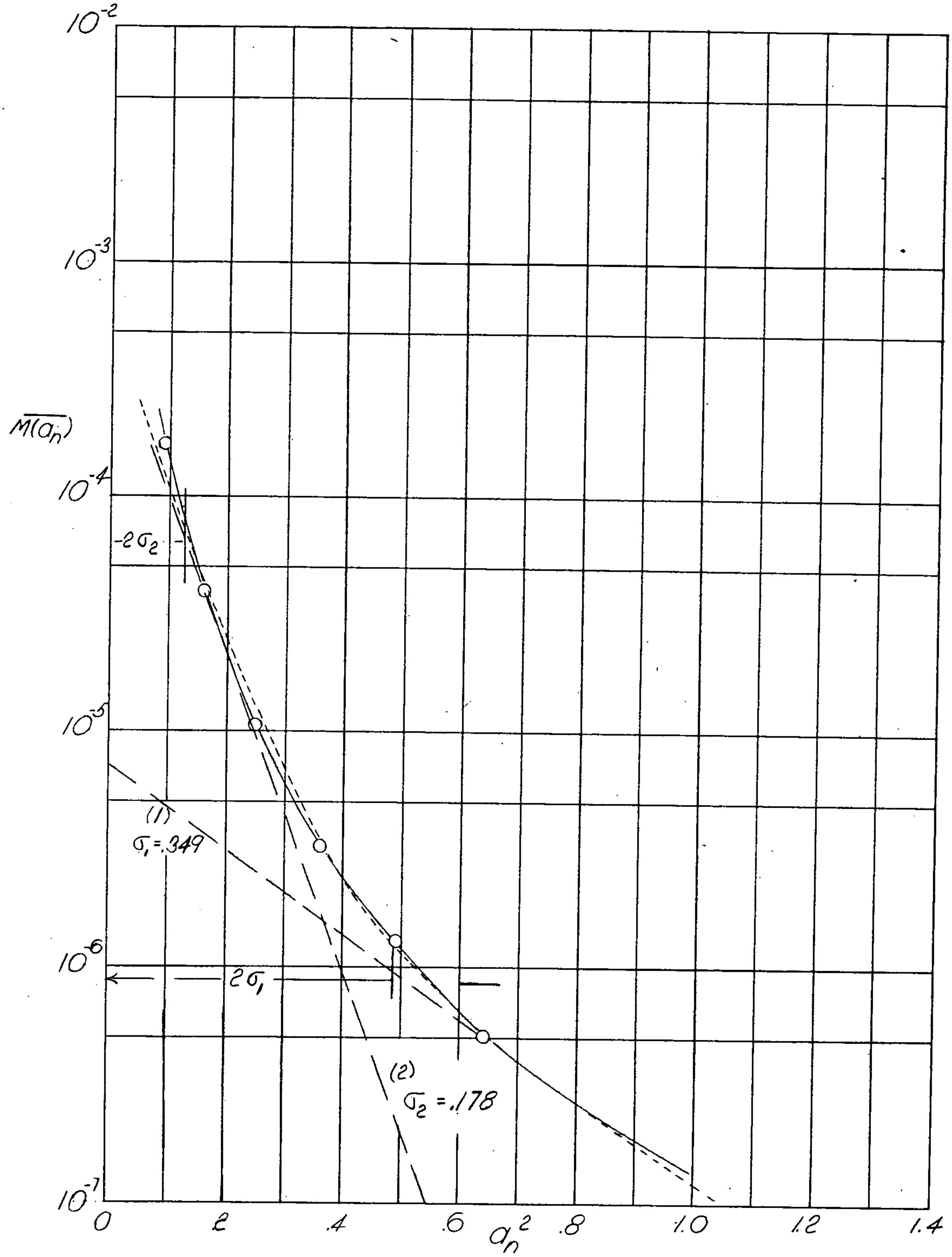
(c) Operation 3.

Figure 1.- Continued.



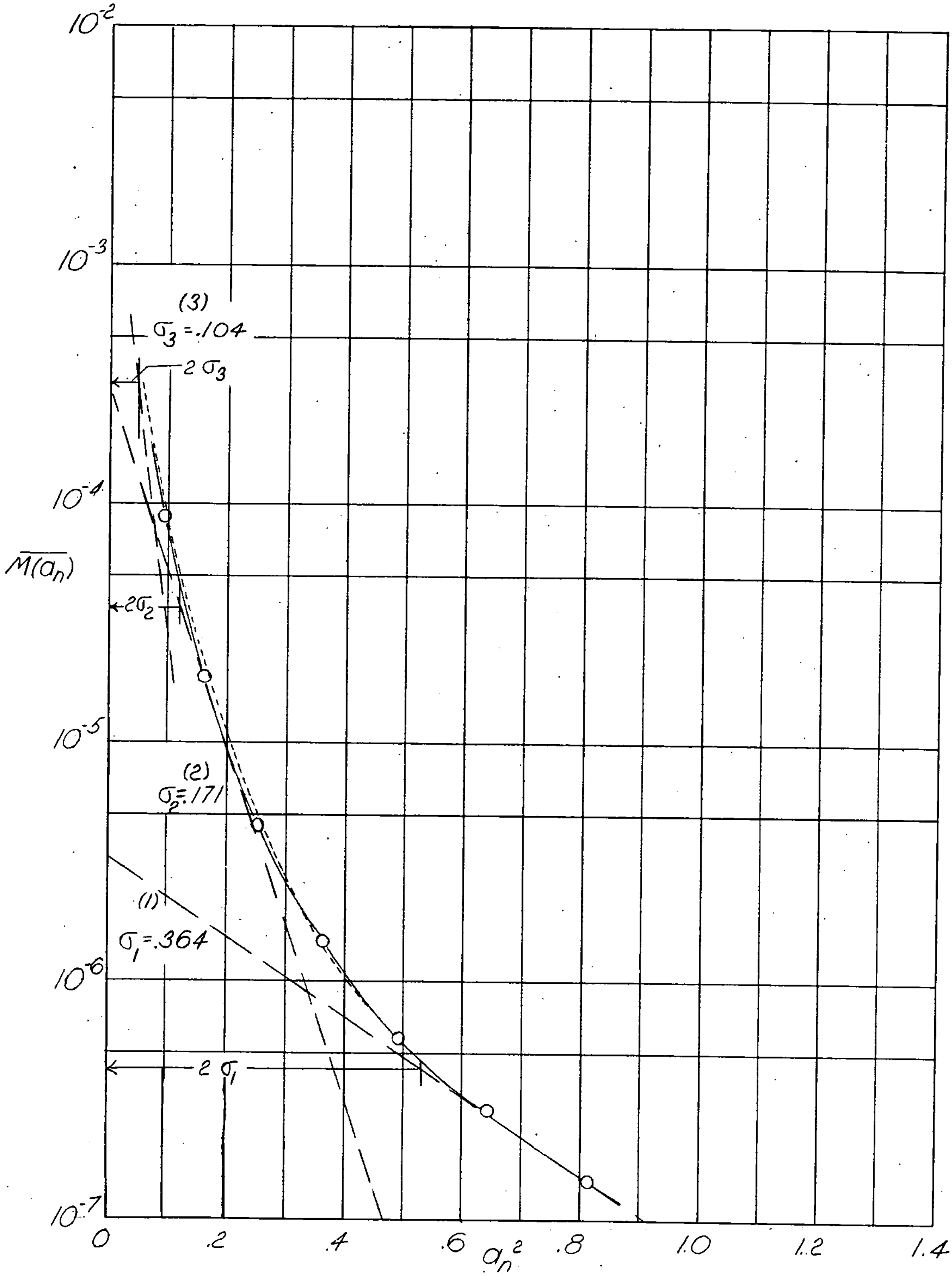
(d) Operation 4.

Figure 1.- Continued.



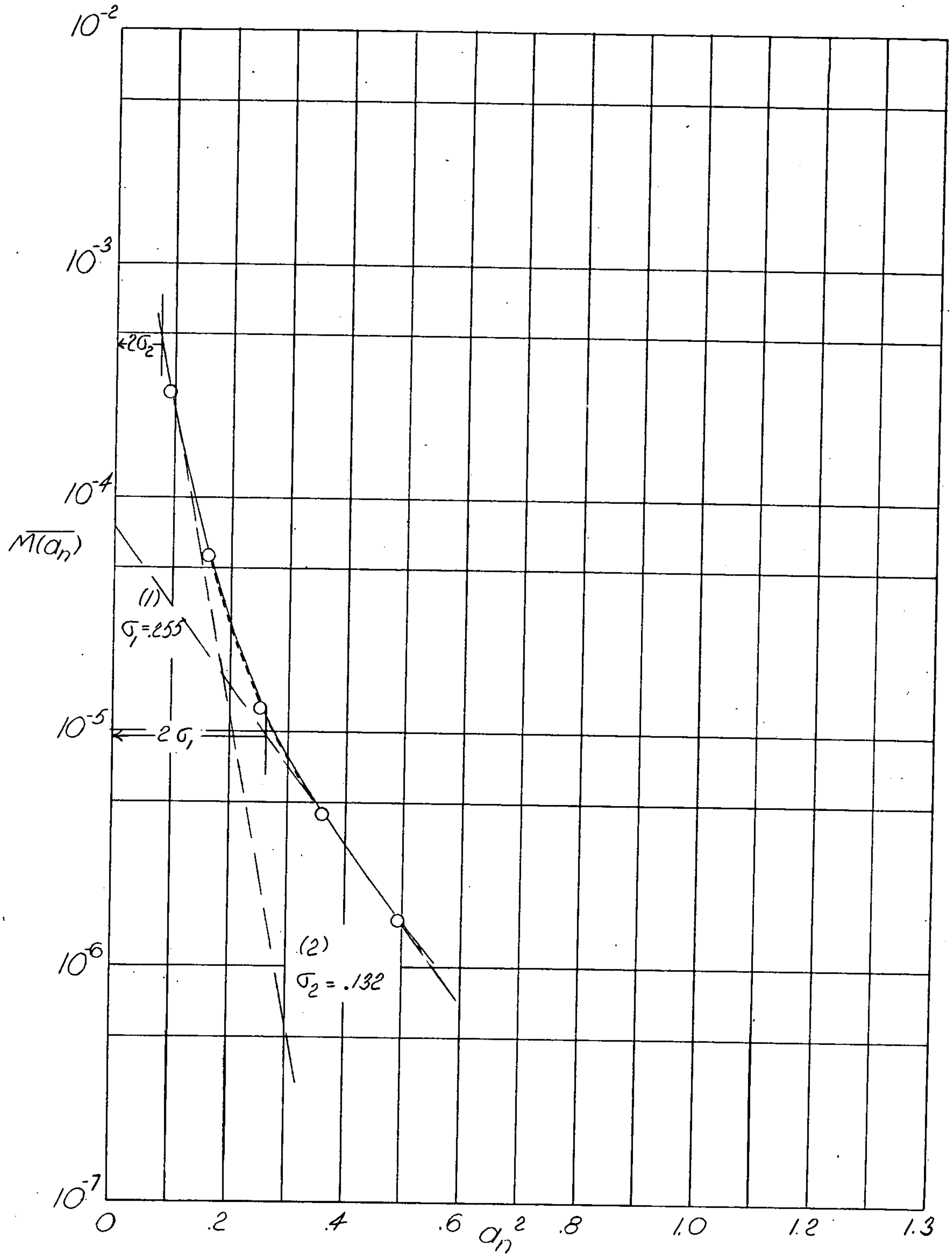
(e) Operation 5.

Figure 1.- Continued.



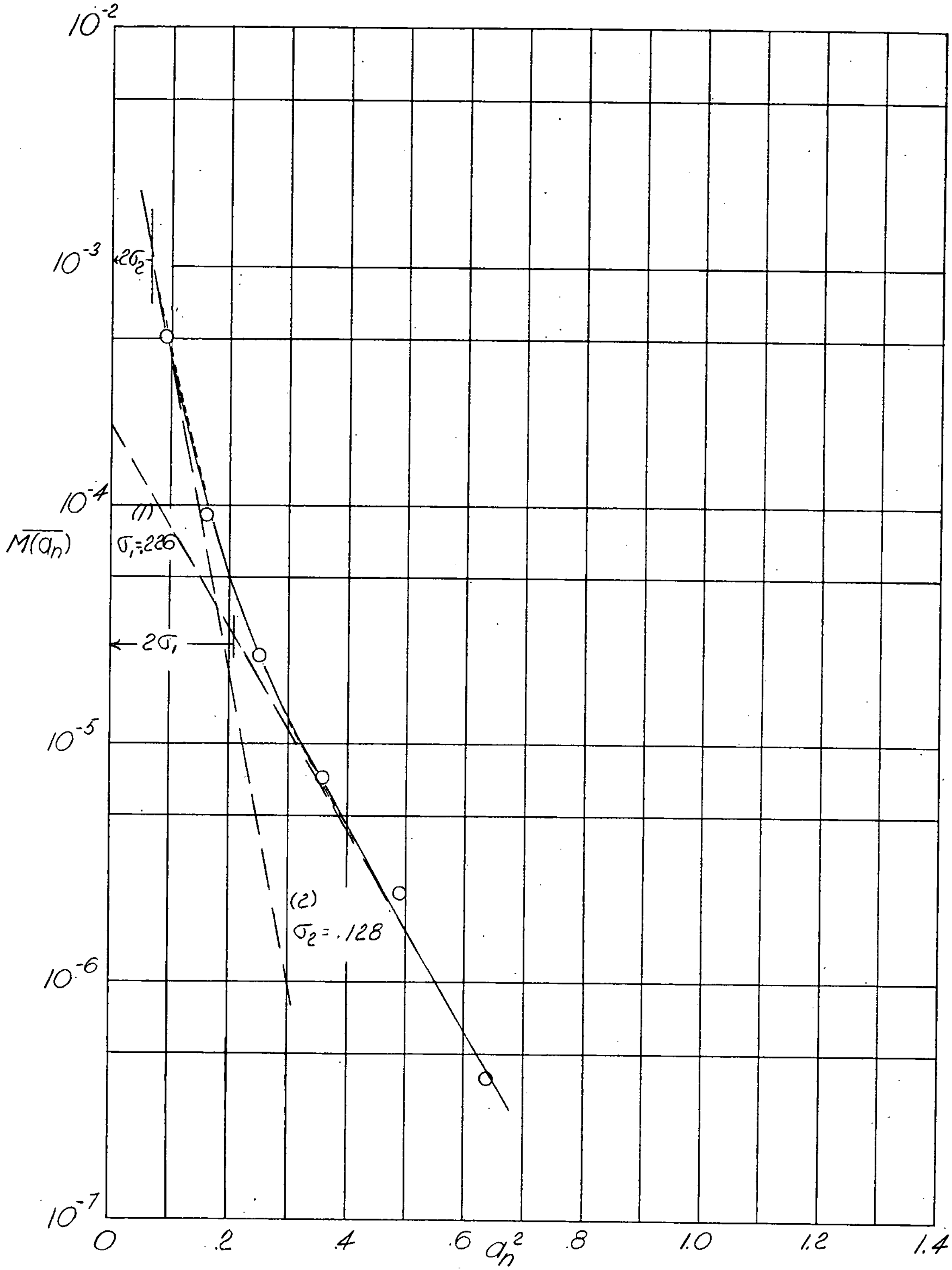
(f) Operation 6.

Figure 1.- Continued.



(g) Operation 7.

Figure 1.- Continued.



(h) Operation 8.

Figure 1.- Concluded.

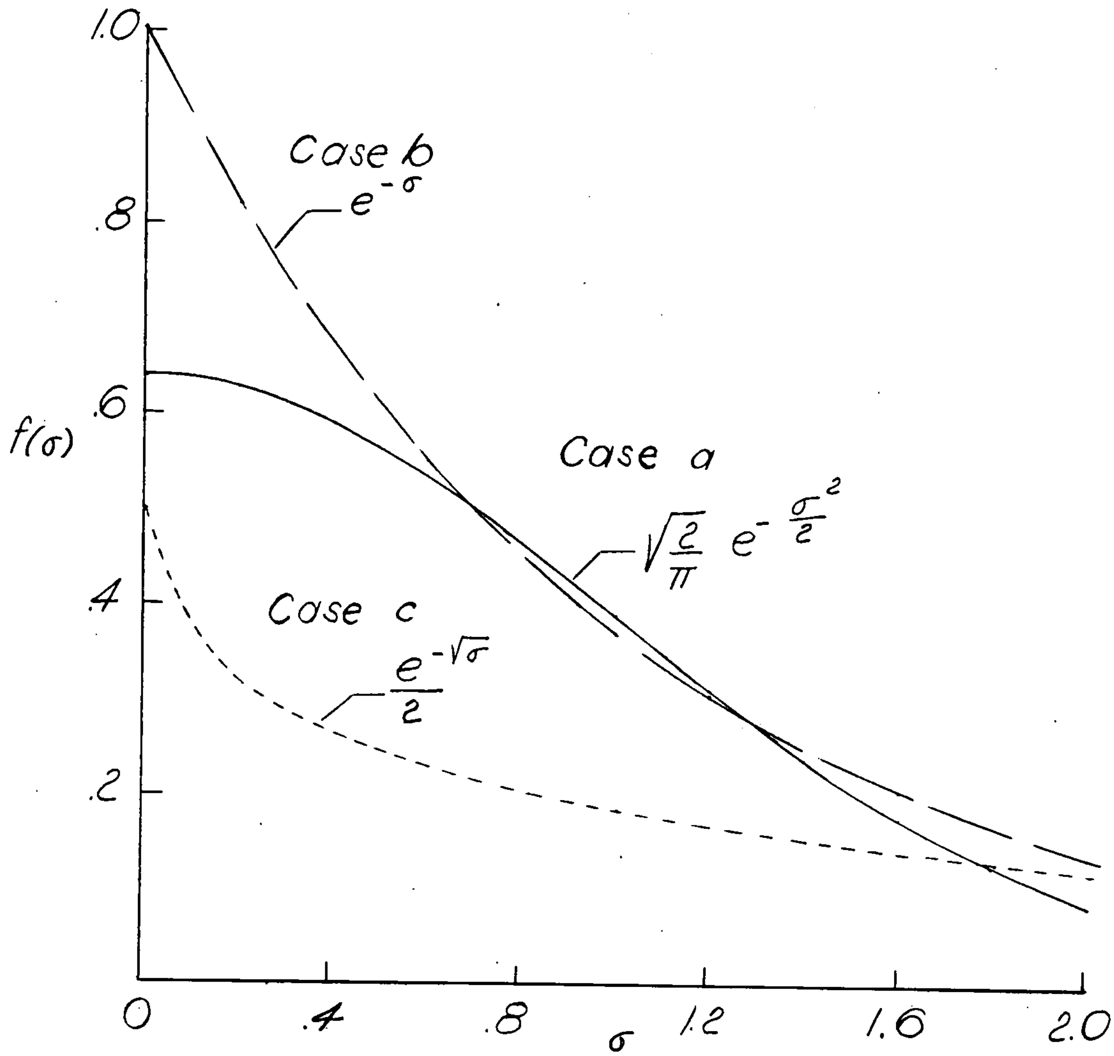


Figure 2.- Comparison of three assumed distributions.

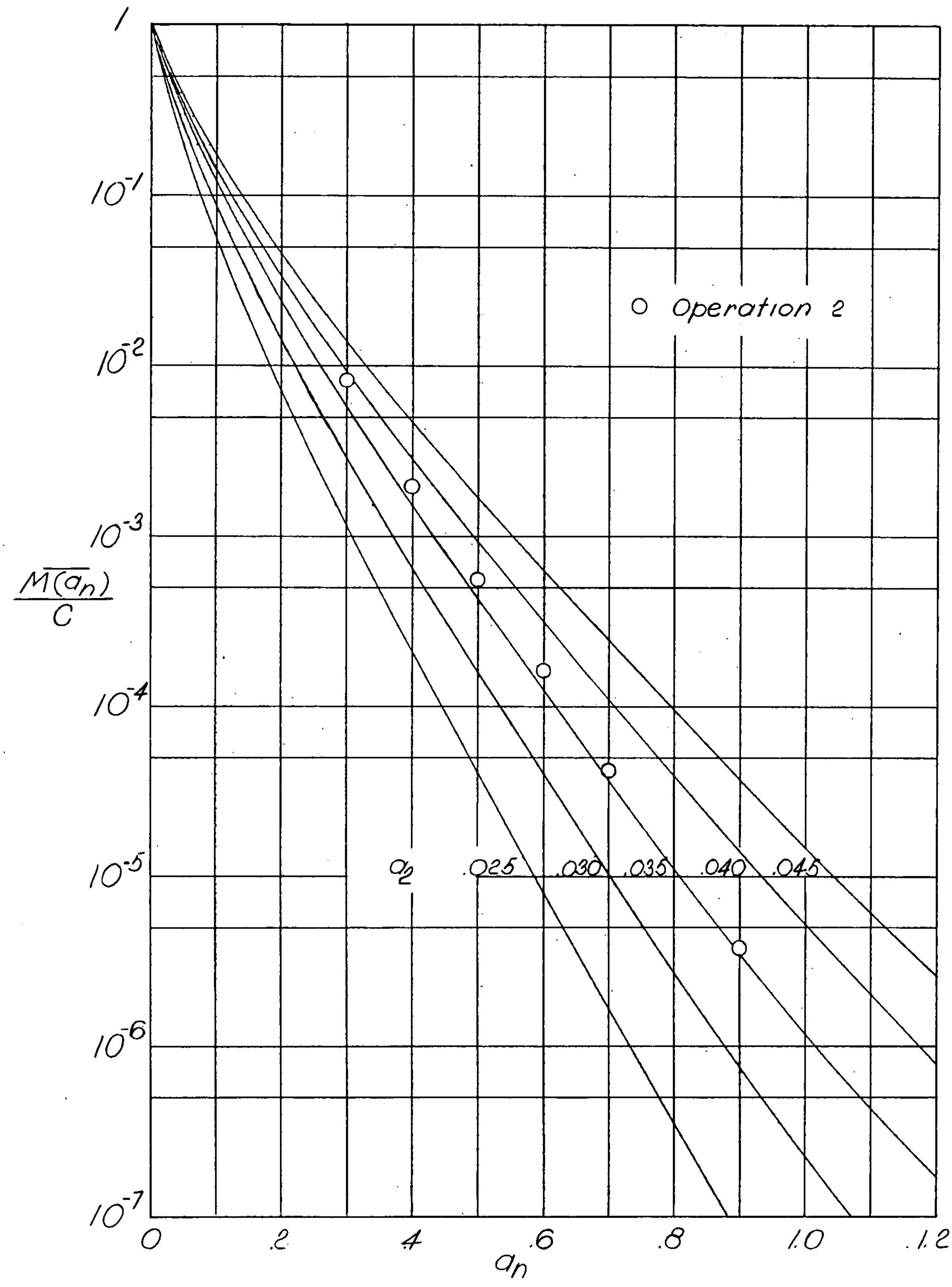


Figure 3.- Number of peak accelerations exceeding given values for case b.

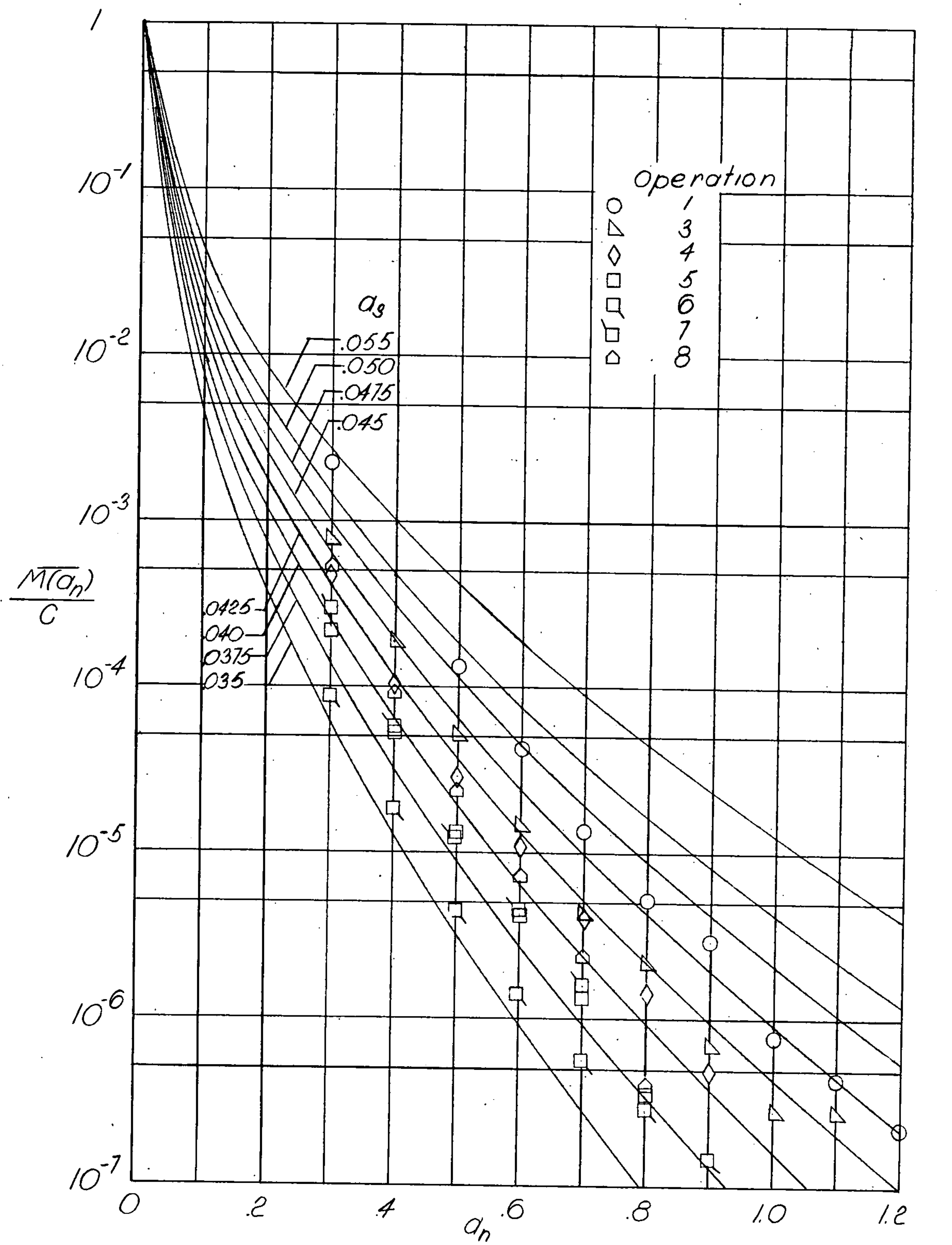


Figure 4.- Number of peak accelerations exceeding given values for case c.

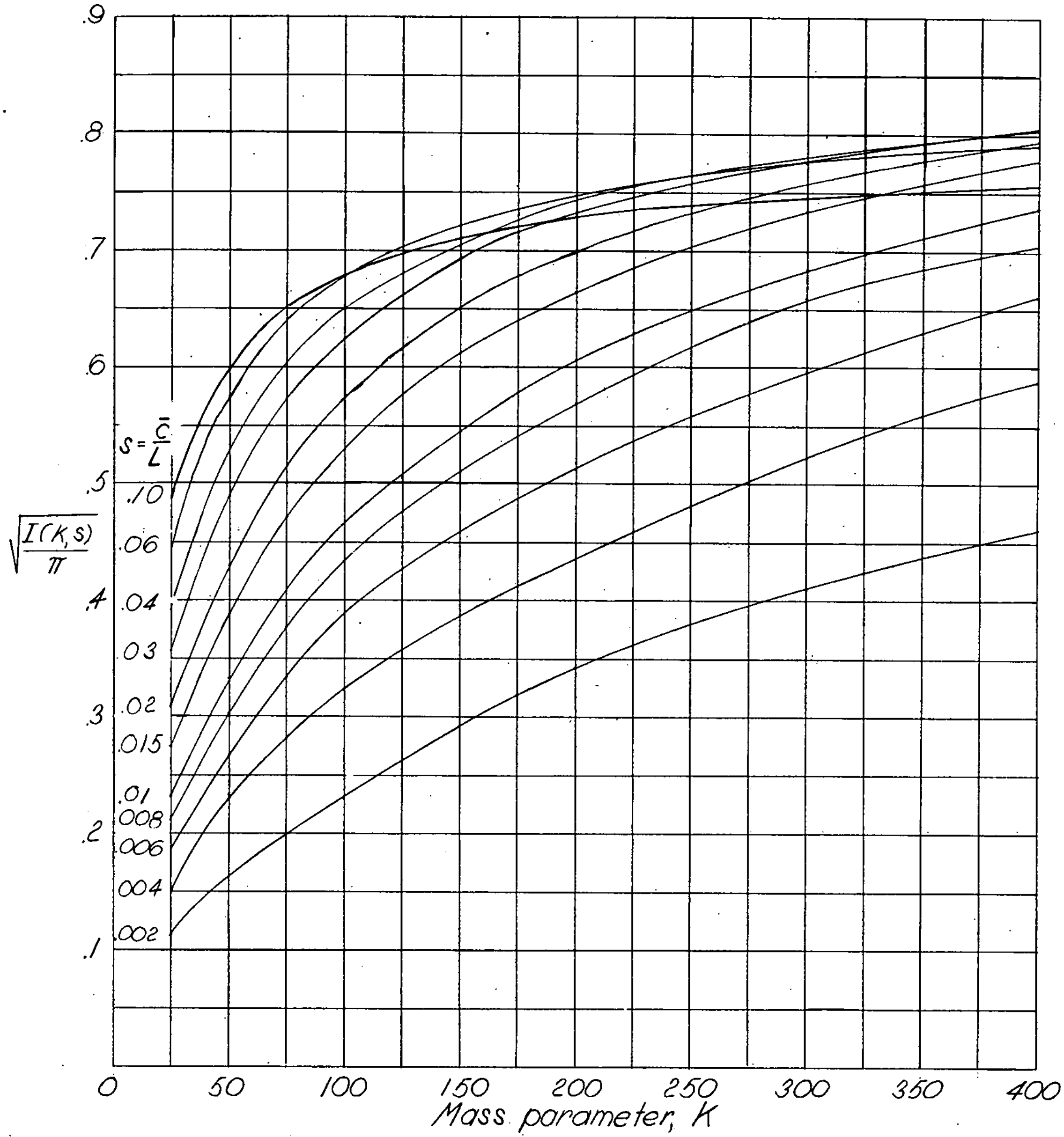


Figure 5.- Gust response factor $\sqrt{\frac{I(K,s)}{\pi}}$.