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A SIMPLE METHOD FOR CALCULATING THE CHARACTERISTICS OF THE DUTCH ROLL MOTION OF AN AIRPLANE

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SUMMARY

A simple method for extracting the period and damping and the ratios of all variables of the Dutch roll motion of an airplane is found by arranging the lateral equations of motion in such form and order that a rapidly convergent iterative solution may be obtained. The method is proposed in order to circumvent the necessity for the solution of the classic biquadratic characteristic equation. Because of the simplicity of this procedure, the iterative method is believed to be particularly useful when no extensive computing facilities are available, though it may be used to reduce computation time on any type of digital computing equipment.

Primary effects of variation of the important stability derivatives on the period and damping can be seen more clearly from the iterative method than from the fourth-order characteristic equation.

INTRODUCTION

The importance of the short-period lateral oscillation or Dutch roll to the handling qualities of airplanes is recognized. (See, for example, refs. 1 and 2.) Although the complete solution of the lateral equations of motion to obtain transient response has long been possible by classical methods, the work involved was so great that most analytical studies were devoted to describing the period and damping and defining the stability boundaries of this oscillation (refs. 3 and 4). Through the use of Laplace transformations, systematized, though still rather tedious, solutions for the complete transient motion to a given disturbance have been presented in references 5 and 6.

Pilots have indicated (ref. 7) that Dutch roll characteristics are adequately described if the period and damping and the amplitude of the roll-to-sideslip ratios are known. The purpose of the present paper is to present a simple iterative method for just such a description of the Dutch roll that circumvents the necessity for the solution of the classic biquadratic characteristic equation.

Computational procedures and uses of the iterative method are discussed. Numerical solutions for Dutch roll characteristics of representative airplanes are given.

SYMBOLS

b wing span, ft

C_L trim lift coefficient, $\dfrac{W}{\frac{1}{2}\rho V^2 S}$

C_l rolling-moment coefficient, $\dfrac{\text{Rolling moment}}{\frac{1}{2}\rho V^2 S b}$

$C_{l_p} = \dfrac{\partial C_l}{\partial \frac{pb}{2V}}$

$C_{l_r} = \dfrac{\partial C_l}{\partial \frac{rb}{2V}}$

$C_{l_\beta} = \dfrac{\partial C_l}{\partial \beta}$

C_n yawing-moment coefficient, $\dfrac{\text{Yawing moment}}{\frac{1}{2}\rho V^2 S b}$

$C_{n_p} = \dfrac{\partial C_n}{\partial \frac{pb}{2V}}$

$C_{n_r} = \dfrac{\partial C_n}{\partial \frac{rb}{2V}}$

$C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$

C_Y lateral-force coefficient, $\frac{\text{Lateral force}}{\frac{1}{2}\rho V^2 S}$

$C_{Y_p} = \frac{\partial C_Y}{\partial \frac{pb}{2V}}$

$C_{Y_r} = \frac{\partial C_Y}{\partial \frac{rb}{2V}}$

$C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta}$

D differential operator, d/ds

h altitude, ft

K_{X_S} nondimensional radius of gyration in roll about longitudinal stability axis, $\sqrt{\left(\frac{k_X}{b}\right)^2 \cos^2\eta + \left(\frac{k_Z}{b}\right)^2 \sin^2\eta}$

K_{Z_S} nondimensional radius of gyration in yaw about vertical stability axis, $\sqrt{\left(\frac{k_Z}{b}\right)^2 \cos^2\eta + \left(\frac{k_X}{b}\right)^2 \sin^2\eta}$

K_{XZ} nondimensional product-of-inertia parameter, $\left[-\left(\frac{k_Z}{b}\right)^2 + \left(\frac{k_X}{b}\right)^2\right] \sin\eta \cos\eta$

k_X radius of gyration in roll about principal longitudinal axis, ft

k_Z radius of gyration in yaw about principal vertical axis, ft

m mass of airplane, slugs

p rolling angular velocity, $d\phi/dt$, radians/sec

r yawing angular velocity, $d\psi/dt$, radians/sec

S	wing area, sq ft
s	nondimensional time parameter based on span, Vt/b
t	time, sec
V	airspeed, ft/sec
v	sideslip velocity about lateral axis, ft/sec
W	weight of airplane, lb
X,Y,Z	stability coordinate axes (defined in fig. 1)
β	angle of sideslip, $\sin^{-1} \frac{v}{V}$, radians
η	inclination of principal longitudinal axis of airplane with respect to flight path, positive when principal axis is above flight path at nose, deg
μ	relative density factor, $m/\rho Sb$
ρ	mass density of air, slugs/cu ft
ϕ	angle of roll, radians
ψ	angle of yaw, radians

DEVELOPMENT OF METHOD

Equations

A simple method for extracting the period and damping and the ratios of all variables of the Dutch roll oscillation is found by arranging the lateral equations of motion in such form and order that a rapidly convergent iterative solution may be obtained. In this section, the original equations of motion are given in standard and then in modified form and are operated on to yield the iterative equations expressed as

(1) The ratio of roll to yaw ϕ/ψ as a function of the operator D

(2) The ratio of sideslip to yaw β/ψ as a function of ϕ/ψ and D

(3) A quadratic in the operator D as a function of ϕ/ψ and β/ψ

Assumptions.- The usual assumptions of lateral-stability theory are made:

(1) There are only three degrees of freedom: sideslip β, roll ϕ, and yaw ψ.

(2) All aerodynamic controls are fixed.

(3) Small classic perturbations are allowed.

Modifications of standard equations of lateral motion.- The linear, second-order, simultaneous differential equations of lateral motion referred to stability axes (see fig. 1) for the condition of controls fixed are as follows:

Side force:

$$\beta\left(2\mu D - C_{Y_\beta}\right) + \phi\left(-C_L - \frac{1}{2}C_{Y_p}D\right) + \psi\left(2\mu D - \frac{1}{2}C_{Y_r}D\right) = 0 \qquad (1)$$

Rolling moment:

$$\beta\left(-C_{l_\beta}\right) + \phi\left(2\mu {K_X}_s{}^2D^2 - \frac{1}{2}C_{l_p}D\right) + \psi\left(-2\mu K_{XZ}D^2 - \frac{1}{2}C_{l_r}D\right) = 0 \qquad (2)$$

Yawing moment:

$$\beta\left(-C_{n_\beta}\right) + \phi\left(-2\mu K_{XZ}D^2 - \frac{1}{2}C_{n_p}D\right) + \psi\left(2\mu {K_Z}_s{}^2D^2 - \frac{1}{2}C_{n_r}D\right) = 0 \qquad (3)$$

In the process of solving these equations by classical methods (ref. 3), the characteristic equation is obtained by setting the determinant of the coefficients of equations (1) to (3) equal to zero. The characteristic equation is a biquadratic in the operator D, and it has been shown (ref. 4) that the roots of the characteristic equation describe the damping of any motion present and the period if the motion is oscillatory.

The operator D may be considered a variable in the three standard lateral equations of motion and still represent damping and the period if the motion is oscillatory. On this basis, there are now four variables: β, ϕ, ψ, and D. However, the operator D in equations (1) to (3) never appears except as a coefficient of one of the other variables. If each of the equations be divided by one of the freedoms (e.g., ψ), the number of unknowns is again reduced to three: β/ψ, ϕ/ψ, and D.

Dividing equations (1), (2), and (3) each by ψ yields the following equations:

Side force:

$$\frac{\beta}{\psi}\left(2\mu D - C_{Y_\beta}\right) + \frac{\phi}{\psi}\left(-C_L - \frac{1}{2}C_{Y_p}D\right) + 2\mu D - \frac{1}{2}C_{Y_r}D = 0 \qquad (4)$$

Rolling moment:

$$\frac{\beta}{\psi}\left(-C_{l_\beta}\right) + \frac{\phi}{\psi}\left(2\mu K_{X_s}{}^2D^2 - \frac{1}{2}C_{l_p}D\right) - 2\mu K_{XZ}D^2 - \frac{1}{2}C_{l_r}D = 0 \qquad (5)$$

Yawing moment:

$$\frac{\beta}{\psi}\left(-C_{n_\beta}\right) + \frac{\phi}{\psi}\left(-2\mu K_{XZ}D^2 - \frac{1}{2}C_{n_p}D\right) + 2\mu K_{Z_s}{}^2D^2 - \frac{1}{2}C_{n_r}D = 0 \qquad (6)$$

Equations (4) to (6) contain enough information to define adequately the Dutch roll. (See ref. 7.) In this form, however, solution is admittedly difficult.

Iterative equations.- The modified equations of lateral motion (eqs. (4) to(6)) will be operated on to yield the iterative equations. The first iterative equation, the ratio of roll to yaw ϕ/ψ, may be found by solving the roll and yaw equations (eqs. (5) and (6)) simultaneously. Thus,

$$\frac{\phi}{\psi} = \frac{C_{n_\beta}\frac{1}{2}C_{l_r} - C_{l_\beta}\frac{1}{2}C_{n_r} + 2\mu D\left(C_{n_\beta}K_{XZ} + C_{l_\beta}K_{Z_s}{}^2\right)}{-C_{n_\beta}\frac{1}{2}C_{l_p} + C_{l_\beta}\frac{1}{2}C_{n_p} + 2\mu D\left(C_{n_\beta}K_{X_s}{}^2 + C_{l_\beta}K_{XZ}\right)} \qquad (7)$$

It may be noted that equation (7) is the simplest expression for any ratio formed from β, ϕ, and ψ inasmuch as it is the only expression in the lateral system that contains only first-order terms in the operator D and constant terms.

The ratio of sideslip to yaw β/ψ may be found by rearrangement of terms of the remaining lateral equation, the side-force equation (eq. (4)). The ratio β/ψ expressed in terms of the ratio ϕ/ψ, the operator D, and constant terms is

$$\frac{\beta}{\psi} = \frac{\left(2\mu - \frac{1}{2}C_{Y_r}\right)D + \frac{\phi}{\psi}\left(-C_L - D\frac{1}{2}C_{Y_p}\right)}{-2\mu D + C_{Y_\beta}} \tag{8}$$

The remaining step is to write the quadratic in D as a function of ϕ/ψ and β/ψ, where D is the Dutch roll root. Since both the yawing and rolling equations contribute significantly to the Dutch roll oscillation, writing both the rolling-moment and the yawing-moment equations (eqs. (5) and (6)) as quadratics in the operator D and adding yields the desired quadratic in D:

$$D^2\left(2\mu K_{Zs}^2 K_{Xs}^2 - 2\mu K_{XZ}^2\right) + D\left[-\frac{1}{2}C_{n_r}K_{Xs}^2 - \frac{1}{2}C_{l_r}K_{XZ} - \left(\frac{1}{2}C_{n_p}K_{Xs}^2 + \frac{1}{2}C_{l_p}K_{XZ}\right)\frac{\phi}{\psi}\right] - \left(C_{n_\beta}K_{Xs}^2 + C_{l_\beta}K_{XZ}\right)\frac{\beta}{\psi} = 0 \tag{9}$$

It should be noted that before the addition the rolling-moment equation was multiplied by K_{XZ} and the yawing-moment equation was multiplied by K_{Xs}^2. This procedure simplified considerably the resulting expression because it eliminated the complex parameter ϕ/ψ as a factor of D^2 that appeared in equations (5) and (6). In a somewhat arbitrary fashion, this procedure also actually "weighted" the importance of the contributions of the yawing and rolling equations to the Dutch roll roots. In the limiting case of zero principal-axis tilt, the weighting factor K_{XZ} is equal to zero and equation (9) reduces simply to the yawing-moment equation, equation (6). Equation (9) also reduces to

$$D = i\sqrt{\frac{1}{2\mu K_{Zs}^2}\left(C_{n_\beta} + C_{l_\beta}\frac{K_{XZ}}{K_{Xs}^2}\right)} \tag{10}$$

for two degrees of freedom (yaw and roll) where the system is undamped, $\beta = -\psi$, and second-order terms are neglected. Equation (10) is identical to the undamped natural frequency for two degrees of freedom as defined in reference 8 and thus is further justification for the weighting chosen.

Method of Solution

The three iterative equations derived from the lateral equations of motion are ϕ/ψ in terms of D (eq. (7)), β/ψ in terms of ϕ/ψ and D (eq. (8)), and D in terms of ϕ/ψ and β/ψ (eq. (9)) wherein ϕ/ψ, β/ψ, and D are all complex quantities.

Computational procedure.- The ratio ϕ/ψ is estimated from equation (7) with an assumed value of D. The ratio β/ψ is then determined from equation (8) with the assumed value of D and the estimated value of ϕ/ψ from equation (7) being used. Next, a new value of D is determined from equation (9) through use of the estimated values of ϕ/ψ from equation (7) and β/ψ from equation (8). This new value of D is used in equation (7) in place of the original assumed value, and thus the iterative process is well started.

A suggested original assumption for the value of D is the undamped period for the single degree of freedom in yaw wherein the airplane center of gravity travels in a straight line. This well-known approximation is

$$D = i\sqrt{\frac{C_{n_\beta}}{2\mu K_{Z_s}^2}} \tag{11}$$

It has been found through experience that the amplitude and phase angle of the roll-to-yaw ratio ϕ/ψ appear to be negligibly affected by damping and are insensitive to the period of the Dutch roll above the value of D given by equation (11). Thus, the actual choice of the initial D at the start of the iterative process is unimportant. Any reasonable approximation that presupposes an oscillation should produce a solution.

The Dutch roll roots of more than 70 airplanes were calculated by the iterative method and checked by the conventional method. Table I gives some idea of the range of airplane variables investigated. As a result of the experience gained from these calculations, answers of engineering usefulness are found with the iterative method by merely substituting an approximate value of D in equations (7) and (8) and then solving equations (7) to (9) for ϕ/ψ, β/ψ, and D, respectively. However, it is recommended that the process be iterated at least once because, if the iterated values of ϕ/ψ, β/ψ, and D agree with the values that were first calculated, the work is obviously numerically correct. Two possibilities exist if the values are not in close agreement - that of numerical error and that of failure to converge.

Infrequently, convergence does not occur (e.g., static directionally unstable airplanes and light airplanes that have extreme damping in yaw), and the conventional method must be used to evaluate the variables.

The equations of the iterative method have been derived from the conventional equations of motion without any additional assumptions. As a result, it is impossible for the iterative method to converge on answers different from those calculated by the conventional method.

Uses for the method.- The proposed iterative method is designed to circumvent the necessity for the solution of the biquadratic characteristic equation. The biquadratic is in effect replaced by a quadratic with complex coefficients (eq. (9)) that may be easily iterated. Thus, the iterative method is somewhat simpler mathematically and requires only about one-third the computational time of the conventional method. In addition, evaluation of the ratios of roll to yaw and sideslip to yaw which are necessary to describe adequately the Dutch roll is inherent with the iterative method.

Equation (9) may be rewritten as

$$D^2 + D\left[\frac{-\frac{1}{2}C_{n_r}K_{Xs}^2 - \frac{1}{2}C_{l_r}K_{XZ} - \left(\frac{1}{2}C_{n_p}K_{Xs}^2 + \frac{1}{2}C_{l_p}K_{XZ}\right)\frac{\phi}{\psi}}{2\mu\left(K_{Zs}^2K_{Xs}^2 - K_{XZ}^2\right)}\right] - \frac{\left(C_{n_\beta}K_{Xs}^2 + C_{l_\beta}K_{XZ}\right)\frac{\beta}{\psi}}{2\mu\left(K_{Zs}^2K_{Xs}^2 - K_{XZ}^2\right)} = 0 \tag{12}$$

Equation (12) is an extremely interesting equation in that it contains all the terms in a three-degree-of-freedom system that affect the Dutch roll, expressed as a quadratic in the operator D. Primary effects of variation of the important stability derivatives on the period and damping can be seen more clearly from equation (12) than from the fourth-order characteristic equation.

REPRESENTATIVE SOLUTIONS

Table II presents the characteristics of four representative airplanes. These airplanes were examined by means of the simple iterative method to determine their Dutch roll characteristics.

Example solutions by the iterative method are given in table III. A conventional modern bomber, an extreme-altitude fighter, and a sonic interceptor represent typical solutions for present and proposed airplanes, and a hypothetical delta-wing light airplane in the landing condition represents an atypical solution in which the iterative method does not converge. The high rate of convergence of the typical solutions of the Dutch roll characteristics is apparent. The atypical case illustrates the ample warning (lack of convergence) that is present when the iterative solution should not be used.

CONCLUSIONS

A simple method for calculating the characteristics of the Dutch roll motion of an airplane has been obtained by arranging the lateral equations of motion in such form and order that an iterative process is quickly convergent. The iterative method is believed to be particularly useful when no extensive computing facilities are available, although it may be used to reduce computational time on any type of digital computing equipment. Experience gained from the calculation of the Dutch roll characteristics of more than 70 airplanes by the iterative method as compared with the conventional method has indicated that

1. About one-third the computational time is required for the iterative method as was required for the conventional method.

2. The arithmetical processes are simpler in the iterative method because the need for the solution of a biquadratic equation is avoided.

3. In the iterative method the arithmetical processes are iterative, which permit running numerical checks.

4. Evaluation of the ratios of roll to yaw and sideslip to yaw is inherent in the iterative process.

5. Primary effects of aerodynamic derivatives on the Dutch roll roots can be more readily seen by use of the iterative method.

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Langley Field, Va., June 8, 1956.

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TABLE I

RANGE OF AIRPLANE VARIABLES CONSIDERED FOR WHICH

SATISFACTORY SOLUTIONS WERE OBTAINED

BY MEANS OF ITERATIVE METHOD

Primary variable	Numerical range
$\frac{{K_{X_s}}^2}{{K_{Z_s}}^2}$	0.2 to 1.0
μ	12 to 275
η, deg	-5 to 5
C_{n_r}	-1.2 to 3.6
C_{n_p}	-0.70 to 0.90
C_{l_r}	-2.90 to 3.10
C_{l_p}	-7.30 to 0.20
C_{n_β}	0.03 to 0.25
C_{l_β}	-0.22 to 0

TABLE II

CHARACTERISTICS OF REPRESENTATIVE

AIRPLANES USED IN CALCULATIONS

	Airplane			
	Bomber	Extreme-altitude fighter	Sonic interceptor	Hypothetical delta-wing airplane in landing condition
${K_{X_s}}^2$	0.0311	0.0156	0.010	0.030
${K_{Z_s}}^2$	0.072	0.156	0.065	0.0679
K_{XZ}	0	0.0020	-0.004	-0.030
μ	31.83	182	65.4	11.85
C_L	0.443	0.49	0.323	1.0
h, ft	35,000	50,000	32,000	0
V, ft/sec	700	776	691	99.8
b, ft	116	25	36.6	38.1
C_{Y_β}	-0.61	-0.58	-0.28	-0.286
C_{Y_p}	0	0	0	0
C_{Y_r}	0	0	0	0
C_{l_β}	-0.14	-0.18	-0.08	-0.0573
C_{l_p}	-0.44	-0.33	-0.15	-0.02
C_{l_r}	0.149	0.23	0.40	0.60
C_{n_β}	0.12	0.25	0.028	0.0573
C_{n_p}	-0.0276	-0.050	-0.20	-0.20
C_{n_r}	-0.156	-0.069	-0.40	-1.10

TABLE III

REPRESENTATIVE SOLUTIONS

Assumptions	Airplane		
	Conventional bomber		
	ϕ/ψ	β/ψ	D
	Iterated values		
$D = i\sqrt{\dfrac{C_{n_\beta}}{2\mu K_{Z_s}{}^2}} = 0.1620i$	-1.755 - 1.253i -1.840 - 1.235i -1.825 - 1.243i	-1.053 + 0.01317i -1.053 + 0.0177i -1.053 + 0.01692i	-0.00491 + 0.1679i -0.00441 + 0.1679i -0.00452 + 0.1679i
	Exact values		
	-1.825 - 1.242i	-1.053 + 0.01692i	-0.00447 + 0.1679i
	Extreme-altitude fighter		
	ϕ/ψ	β/ψ	D
	Iterated values		
$D = i\sqrt{\dfrac{C_{n_\beta}}{2\mu K_{Z_s}{}^2}} = 0.0664i$	-6.23 - 2.95i -6.17 - 2.84i -6.17 - 2.85i	-1.062 + 0.1008i -1.064 + 0.0968i -1.064 + 0.0970i	0.00271 + 0.0665i 0.00256 + 0.0665i 0.00257 + 0.0665i
	Exact values		
	-6.17 - 2.85i	-1.064 + 0.0970i	0.00258 + 0.0665i
	Sonic interceptor		
	ϕ/ψ	β/ψ	D
	Iterated values		
$D = i\sqrt{\dfrac{C_{n_\beta}}{2\mu K_{Z_s}{}^2}} = 0.0574i$	-2.58 - 2.27i -4.41 - 2.54i -4.33 - 2.46i	-1.102 + 0.0700i -1.045 + 0.0775i -1.047 + 0.0749i	-0.01393 + 0.1198i -0.00852 + 0.1182i -0.00886 + 0.1181i
	Exact values		
	-4.33 - 2.46i	-1.047 + 0.0751i	-0.00885 + 0.1181i
	Hypothetical delta-wing light airplane in landing condition		
	ϕ/ψ	β/ψ	D
	Iterated values		
$D = i\sqrt{\dfrac{C_{n_\beta}}{2\mu K_{Z_s}{}^2}} = 0.1887i$	-1.724 + 0.225i -1.411 + 0.231i -1.039 + 0.234i	-0.970 + 0.323i -0.781 + 0.0862i -0.764 - 0.0227i	-0.1845 + 0.1123i -0.1498 + 0.0286i -0.1389 + 0.000617i
	Exact values		
	-1.722 + 0.589i	-0.451 + 0.385i	-0.0647 + 0.0822i

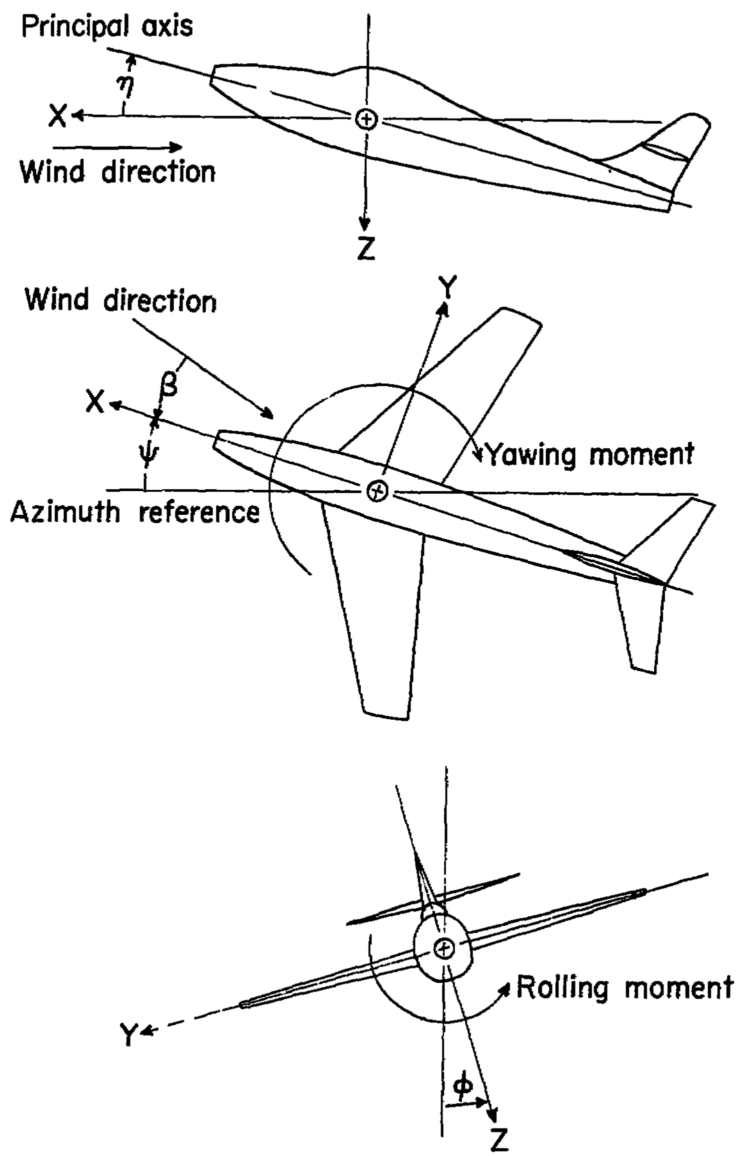

Figure 1.- Stability axes system employed with positive direction of forces, moments, and displacements shown.