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ON STOKES' STREAM FUNCTION IN COMPRESSIBLE SMALL-DISTURBANCE THEORY

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SUMMARY

Stokes' stream function is studied for subsonic or supersonic flow past axisymmetric bodies of small slope. The first-order equation is found to be nonlinear. It can be linearized if one seeks only the formal order of accuracy of the slender-body approximation. In any case, serious loss of accuracy results from imposing the condition of tangent flow at the body on the mass flux rather than on the velocity. In a second approximation, neglect of the nonlinearity leads to a false solution even in the slender-body expansion.

INTRODUCTION

Stokes' stream function would appear to offer attractive possibilities for treating subsonic or supersonic flow past bodies of revolution of small slope. The condition of tangent flow at the body surface is simpler for the stream function than for the velocity potential, because a condition is imposed on the dependent variable itself rather than on its derivative. Use of the stream function also permits streamlines away from the body to be determined easily. For these reasons, various investigators have adopted Stokes' stream function in linearized theory (refs. 1 to 3).

Another advantage might be anticipated if one proceeds to calculate higher approximations by iteration on the first, and considers practical shapes having stagnation points (rather than, say, cusped noses). Then it has been found that the failure of the first approximation in the vicinity of the nose may in some cases be compounded so that the second approximation is incorrect everywhere, and the danger of this is greater with the velocity potential φ than with the stream function ψ. Thus in the analogous problem of plane flow, straightforward iteration on the subsonic thin-airfoil solution for a sharp-edged profile (such as a biconvex section) yields a false second-order result even at mid-chord if one works with φ, but correct results (except very near the edges) if one uses ψ (ref. 4). Both methods fail, however, for round edges. One can then abandon the thin-airfoil approximation, and by using exact conformal mappings find the proper second approximation for a round-edged airfoil (which must still, however, be corrected near the edge, where it is not uniformly valid). Thus Hantzsche and Wendt treated the ellipse and Joukowski section by laborious computation, using the stream function (ref. 5). Recently a second-order compressibility rule has been discovered that reduces any subsonic airfoil problem to an incompressible one; and since these difficulties disappear at zero Mach number, they are no longer an obstacle in plane flow (ref. 4). They would, however, reappear if one proceeded to a third approximation.

For bodies of revolution, these matters are less thoroughly explored. The only relevant published solution is that of Schmieden and Kawalki who, working with Stokes' stream function, obtained a second approximation for subsonic flow past a slender unyawed ellipsoid of revolution, and gave also the asymptotic expansion for vanishing thickness that may be termed the second-order slender-body solution (ref. 6). They used conformal mapping, so that analogy with Hantzsche and Wendt's treatment of the ellipse would suggest that their treatment is valid. However, axisymmetric bodies induce weaker disturbances than do their plane counterparts, so that stagnation points play a more dominant role, and questions of validity are accordingly more delicate.

This is illustrated by the fact that, in contrast to the corresponding plane case, the second-order slender-body solution based upon φ fails at a round nose even in incompressible flow. Thus for the paraboloid whose radius is $r = \delta\sqrt{x}$, expansion of the exact incompressible solution gives to second order near the body

$$\frac{\varphi}{U} = x - \frac{1}{2}\delta^2 \ln\frac{\sqrt{2x}}{r} - \frac{1}{16}\delta^2\frac{r^2}{x^2} + \frac{1}{16}\frac{\delta^4}{x} + \ldots$$

whereas straightforward iteration on the slender-body theory of Heaslet and Lomax (ref. 7) misses the last term shown above. As in the plane problem for compressible flow, the missed term is an eigensolution associated with the blunt nose, being in fact the slender-body representation of a point source at the origin, of potential $\frac{1}{16}\delta^4(x^2 + r^2)^{-1/2}$. It induces no radial velocity on the axis, where the tangency condition is imposed, and so is overlooked if one works with φ. It does, however, induce a mass flux, so that one might hope to catch it by working with ψ, and this expectation is in fact realized. One might therefore be encouraged to attack the corresponding compressible problem using ψ.

For these reasons, the application of Stokes' stream function to compressible small-disturbance theory has been studied here in some detail. The results are surprising. Far from offering the expected advantages, the stream function is found to suffer from grave shortcomings. The proper first-order equation of motion is actually nonlinear. The nonlinearity can be disregarded in the first approximation if one seeks only the formal accuracy, as indicated by order estimates, of the asymptotic slender-body theory; but the numerical accuracy is in any case far inferior to that obtained by using φ. In the second approximation, neglect of the nonlinearity is disastrous, so that Schmieden and Kawalki's solution for the ellipsoid is in fact incorrect.

SMALL-DISTURBANCE EQUATIONS

The continuity equation for axisymmetric compressible flow is satisfied by introducing Stokes' stream function Ψ, according to which the axial and radial velocity components in cylindrical coordinates are given by

$$\left.\begin{aligned} U &= \frac{\rho_\infty}{\rho}\frac{\Psi_r}{r} \\ V &= -\frac{\rho_\infty}{\rho}\frac{\Psi_x}{r} \end{aligned}\right\} \tag{1}$$

where ρ is the density and ρ_∞ its value in the free stream. Uniform inviscid flow past a body of small slope is irrotational to at least third order, and to that approximation the equation of motion is

$$(c^2 - U^2)\Psi_{xx} + (c^2 - V^2)\Psi_{rr} - 2UV\Psi_{xr} - c^2\frac{\Psi_r}{r} = 0 \tag{2a}$$

where the speed of sound c is related to its free-stream value c_∞ and the flow speed U_∞ by

$$c^2 + \frac{\gamma-1}{2}(U^2 + V^2) = c_\infty{}^2 + \frac{\gamma-1}{2}U_\infty{}^2 \tag{2b}$$

Consider now perturbations about a uniform flow, introducing perturbation quantities (denoted by lower-case symbols) by setting

$$\left.\begin{aligned} \Psi &= U_\infty\left(\frac{1}{2}r^2 + \psi\right) \\ U &= U_\infty(1 + u) \\ V &= U_\infty v \end{aligned}\right\} \tag{3}$$

In terms of these perturbation quantities the equation of motion (2) becomes

$$\left[1 - \frac{\gamma-1}{2}M^2(2u + u^2 + v^2)\right]\left(\psi_{xx} + \psi_{rr} - \frac{\psi_r}{r}\right) = M^2\left[(1 + 2u + u^2)\psi_{xx} + v^2(1 + \psi_{rr}) + 2(1 + u)v\psi_{xr}\right] \tag{4}$$

The flow is isentropic to at least second order, and to that approximation the density ratio appearing in equations (1) is given by

$$\frac{\rho_\infty}{\rho} = \left(\frac{c_\infty}{c}\right)^{\frac{2}{\gamma-1}} = \left[1 - \frac{\gamma-1}{2} M^2(2u + u^2 + v^2)\right]^{-\frac{1}{\gamma-1}} \tag{5}$$

Expanding the right-hand side for small u and v, and combining with equations (1) and (3) leads to the following expressions for the perturbation velocity components (referred to free-stream speed) in terms of first derivatives of the perturbation stream function:

$$u = \frac{1}{\beta^2}\left[\frac{\psi_r}{r} + \frac{1}{2}M^2\left(\frac{\psi_x}{r}\right)^2\right] + \frac{M^2}{\beta^4}\left[\left(n + \frac{3}{2}\right)\left(\frac{\psi_r}{r}\right)^2 + \left(n + \frac{1}{2} + M^2\right)\frac{\psi_r}{r}\left(\frac{\psi_x}{r}\right)^2 + \frac{1}{4}\left(n + \frac{3}{2}M^2\right)\left(\frac{\psi_x}{r}\right)^4\right] + \ldots \tag{6a}$$

$$v = -\frac{\psi_x}{r} - \frac{M^2}{\beta^2}\frac{\psi_x}{r}\left[\frac{\psi_r}{r} + \frac{1}{2}\left(\frac{\psi_x}{r}\right)^2\right] + \ldots \tag{6b}$$

where

$$\left.\begin{aligned}\beta^2 &= 1 - M^2\\ n &= \frac{\gamma+1}{2}\frac{M^2}{\beta^2}\end{aligned}\right\} \tag{7}$$

Order of Perturbation Quantities

It is well known that the streamwise velocity perturbation induced by a body of revolution is, except near corners, of smaller order than the radial velocity. For a body of thickness ratio τ the radial component v is of order τ as indicated by the tangency condition, but u is only of order $\tau^2 \ln \tau$. It follows that the quantities (ψ_r/r) and $(\psi_x/r)^2$ bracketed in the first term of equation (6a) are $O(\tau^2 \ln \tau)$ and $O(\tau^2)$. Both are therefore of first order (even though one is linear and the other quadratic in perturbation quantities) because for slender bodies, terms of different order differ by factors of τ^2. Likewise, the terms in the second bracket are $O(\tau^4, \tau^4 \ln \tau, \tau^4 \ln^2 \tau)$ and hence all of second order. Near corners u is $O(\tau)$ because the flow is locally plane, and then the linear terms in equations (6) are of first order, the quadratic terms of second order, and the cubic and quartic terms are simply negligible.

The orders of ψ and its derivatives near the body can be determined from equations (6) together with the facts that r is $O(\tau)$ and that integration or differentiation with respect to x does not affect the order of a term. Aside from powers of $\ln \tau$, which are $O(1)$ for all practical purposes, the result is

$$\left.\begin{aligned} u &= O(\tau^2) & \psi &= O(\tau^2) & \psi_{xx} &= O(\tau^2) \\ v &= O(\tau) & \frac{\psi_x}{r} &= O(\tau) & \psi_{xr} &= O(\tau^3) \\ & & \frac{\psi_r}{r} &= O(\tau^2) & \psi_{rr} &= O(\tau^2) \end{aligned}\right\} \tag{8}$$

First-order and Linearized Equations

Substituting the expressions (6) for u and v into equation (4) and retaining only leading terms (of order τ^2) yields the first-order equation of motion:

$$(1 - M^2)\psi_{xx} + \psi_{rr} - \frac{\psi_r}{r} = M^2\left(\frac{\psi_x}{r}\right)^2 \tag{9}$$

An unexpected result is that the first-order equation is nonlinear.

All previous investigators (refs. 1 to 3, and 6) have based their first approximation upon the linearized equation obtained by dropping the right-hand side of equation (9):

$$(1 - M^2)\psi_{xx} + \psi_{rr} - \frac{\psi_r}{r} = 0 \tag{10}$$

and have also linearized the connection between ψ and u by omitting the quadratic first-order term in equation (6a). This procedure actually yields results correct to first order, as can be verified by introducing a modified perturbation stream function $\bar{\psi}$ according to

$$\psi = \bar{\psi} - \frac{1}{2} M^2\bar{\psi}_x{}^2 \ln r \tag{11}$$

The order estimates of equations (8) apply to $\bar{\psi}$ as well as ψ. Hence substitution shows that the first-order problem for ψ implies, to first order, the linearized problem for $\bar{\psi}$. However, the relation (6a) connecting u and ψ must be linearized as well as the equation of motion (10); to linearize either alone would lead to incorrect results. This suggests that the linearization may be somewhat of a coincidence, which cannot be relied upon to have its counterpart in higher approximations.

It should be noted that, aside from enormous mathematical simplification, linearization leads to a problem satisfying the similitude of Göthert's rule (ref. 8), which is not true of the first-order problem based upon equation (9).

Two theories that are asymptotically of equal order may differ considerably in numerical accuracy. Whether the nonlinear first-order equation or the linearized equation yields greater accuracy, and how they compare with the more conventional treatment using φ are questions to be considered in a subsequent example.

Second-order Equation

Retaining terms of $O(\tau^4)$ in equation (4) yields the second-order equation of motion which, written in a form suited to iteration on the first-order solution, is

$$(1 - M^2)\psi_{xx} + \psi_{rr} - \frac{\psi_r}{r} - M^2\left(\frac{\psi_x}{r}\right)^2 = M^2\left[\begin{aligned}&2(n + 1)\frac{\psi_r}{r}\psi_{xx} - 2\frac{\psi_x}{r}\psi_{xr} + \\ &n\left(\frac{\psi_x}{r}\right)^2\psi_{xx} + \left(\frac{\psi_x}{r}\right)^2\psi_{rr} + \\ &2n\left(\frac{\psi_x}{r}\right)^2\frac{\psi_r}{r} + n\left(\frac{\psi_x}{r}\right)^4\end{aligned}\right] \tag{12}$$

Here the right-hand side has been simplified by using the fact that the left-hand side is to be equated to zero in the first approximation.

The counterpart of linearizing the first-order equation is the retention here only of quadratic terms, which yields, in a form suited to iteration on the linearized solution

$$(1 - M^2)\psi_{xx} + \psi_{rr} - \frac{\psi_r}{r} = M^2\left[2(n + 1)\frac{\psi_r}{r}\psi_{xx} - 2\frac{\psi_x}{r}\psi_{xr} + \left(\frac{\psi_x}{r}\right)^2\right] \tag{13}$$

and this is the basis of Schmieden and Kawalki's second-order solution for subsonic flow past an ellipsoid of revolution (ref. 6). Correspondingly, they retain only quadratic terms in the expressions (6) for velocity components.

FIRST APPROXIMATIONS FOR CONE IN SUPERSONIC FLOW

It happens that the first-order equation (9), though nonlinear, can be integrated in closed form for the standard example of supersonic flow past an unyawed circular cone. This example can, therefore, serve as a test case for assessing the relative accuracies of the various possible first approximations. The proper choice of the relation between perturbation velocities and pressure coefficient is a possibly controversial question that will be avoided by restricting attention to the streamwise velocity increment on the cone.

First-order Solution

For flow past a cone, the stream function is homogeneous of order 2 in the space coordinates, so that

$$\psi(x,r) = r^2 f(t) \tag{14a}$$

where

$$\left.\begin{aligned} t &= \frac{Br}{x} \\ B^2 &= M^2 - 1 \end{aligned}\right\} \tag{14b}$$

Hence the first-order equation of motion (9) becomes the nonlinear equation for $f(t)$:

$$(1 - t^2)f'' + \frac{3-2t^2}{t} f' = \frac{M^2}{B^2} t^2 f'^2 \tag{15}$$

This is Bernoulli's equation for f', which is a linear equation for $1/f'$, with solution

$$f'(t) = \frac{B^2}{M^2} \frac{\sqrt{1-t^2}}{t^3(k + \operatorname{sech}^{-1} t)} \tag{16}$$

The integration can be carried out in terms of exponential integrals. Imposing the condition that the disturbances vanish at the Mach cone, $f(1) = 0$, leads to

$$f(t) = \frac{1}{4} \frac{B^2}{M^2} \left\{ 2 \ln\left(1 + \frac{\operatorname{sech}^{-1} t}{k}\right) - e^{-2k}\left[\overline{Ei}(2k + 2 \operatorname{sech}^{-1} t) - \overline{Ei}(2k)\right] - e^{2k}\left[Ei(-2k - 2 \operatorname{sech}^{-1} t) - Ei(-2k)\right]\right\} \tag{17a}$$

where

$$Ei(-x) = -\int_x^\infty \frac{e^{-s}}{s}\,ds\ , \qquad \overline{Ei}(x) = \int_{-\infty}^x \frac{e^s}{s}\,ds\ , \qquad x > 0 \tag{17b}$$

First-order Solution With Tangency on Mass Flux

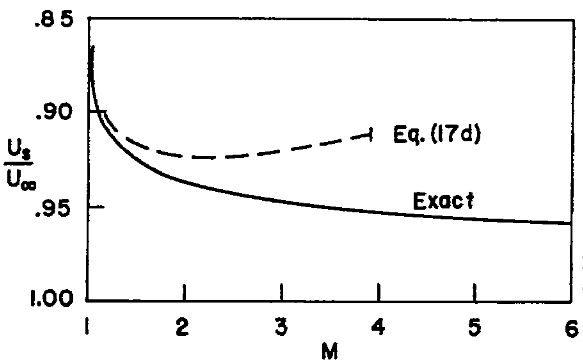
The condition of tangent flow at the surface can now be imposed on either the mass flux or the velocity. In the full theory these are of course equivalent, but in the approximate theory they yield results differing in higher-order terms. Zero mass flux through the surface is assured by requiring the stream function to vanish there. For a cone of semivertex angle ϵ this yields the transcendental equation

$$f(B\epsilon) = -\frac{1}{2} \tag{17c}$$

for determining k. Then the surface increment in streamwise velocity is given by

$$u_s = \frac{\Delta U_s}{U_\infty} = -\frac{1}{B^2}\left[-1 + tf' + \frac{1}{2}\frac{M^2}{B^2}(t^2 f')^2\right]_{t=B\epsilon} \tag{17d}$$

The transcendental equation (17c) has been solved numerically for a cone of 10° semivertex angle. The resulting variation of u with free-stream Mach number is compared in sketch (a) with the exact results from Kopal's table (ref.9).



Sketch (a)

A curious feature of the solution is that beyond a certain Mach number the constant k becomes negative. If k is slightly negative the factor $(k + \operatorname{sech}^{-1}t)$ in equation (16) will vanish on the cone $t = \operatorname{sech}(-k)$ which lies within the flow field, so that f' is singular there. This would mean that the radial velocity v was infinite in the flow field, which is intolerable. The solution must accordingly be regarded as having broken down when k becomes zero. There is no apparent physical interpretation of this limit. It arises at a lower Mach number than the more familiar cutoff of linearized theory at $B\epsilon = 1$, which corresponds to the free-stream Mach cone's having been forced down onto the surface. Thus, for the 10° cone, k vanishes at $B\epsilon = 0.699$ or $M = 3.92$, as indicated by the cutoff in sketch (a).

First-Order Solution With Tangency on v

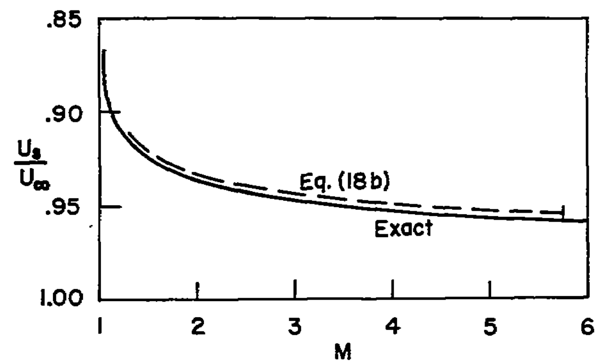
Instead of requiring the stream function to vanish, one can impose the tangency condition on the velocity at the surface. In the approximation usually adopted in linearized theory, the small increment in streamwise velocity is neglected, so that the condition is simply that v equal the slope of the body. For the cone, this determines the constant k according to

$$k = \frac{\sqrt{1-B^2\epsilon^2}}{M^2\epsilon^2} - \operatorname{sech}^{-1}B\epsilon \tag{18a}$$

Then on the surface

$$u_s = -\frac{1}{B^2}\left[1 + 2f(B\epsilon) + \frac{1}{2}M^2\epsilon^2\right] \tag{18b}$$

This is compared in sketch (b) with the exact solution for a 10° cone. The approximate solution is terminated at $B\epsilon = 1$, which occurs at $M = 5.76$.



Sketch (b)

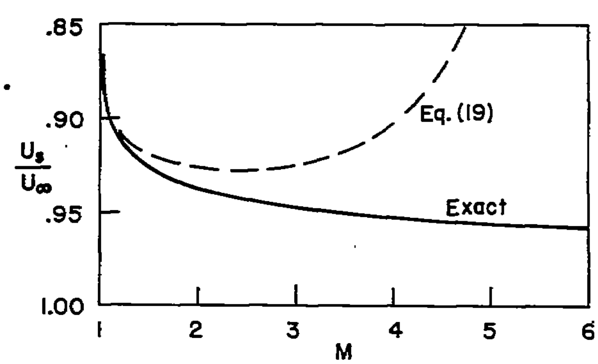
Comparison With Other Approximations

The linearized stream function for a cone has been found by Moore (ref. 3) by solving equation (10) and requiring Ψ to vanish on the surface. The streamwise velocity increment on the surface is

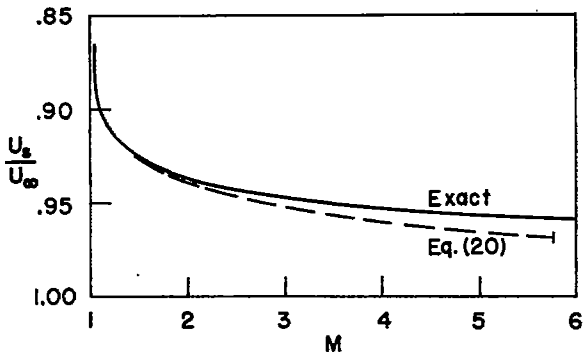
$$u_s = -\epsilon^2 \frac{\operatorname{sech}^{-1}T}{\sqrt{1-T^2-T^2\operatorname{sech}^{-1}T}} \tag{19}$$

where $T = B\epsilon$. Sketch (c) compares this result with the exact solution. Alternatively, one can impose the tangency condition on the velocity, which gives (with the approximate condition $v = \epsilon$ discussed previously)

$$u_s = -\epsilon^2 \frac{\operatorname{sech}^{-1}T}{\sqrt{1-T^2}} \tag{20}$$



Sketch (c)



Sketch (d)

The comparison with exact theory is given in sketch (d).

Instead of working with the stream function, one can use the perturbation velocity potential φ, for which the linearized or first-order equation is

$$(1 - M^2)\varphi_{xx} + \varphi_{rr} + \frac{\varphi_r}{r} = 0$$

Then the approximate tangency condition $v = \epsilon$ leads again to equation (20). Alternatively, one can require zero mass flux through the surface. This condition has been adopted for mathematical simplicity by Parker in his analysis of minimum drag bodies (ref. 10), using a cylindrical control surface. The result is then equation (19). Thus it appears immaterial whether one uses the linearized equation of motion for φ or ψ; the result depends only upon the form of tangency condition adopted.

It can be verified that each of the approximations considered so far has the same asymptotic form for small $T = B\epsilon$, namely the slender-body solution, according to which

$$u_s = -\epsilon^2 \ln T + O(\epsilon^4 \ln^2 T) \tag{21}$$

This approximation is compared with the exact solution for a 10° cone in figure 1, together with all the other first approximations discussed previously. The various approximations differ only in the subsequent terms of the asymptotic series, which are not correct in any case because they are affected by the neglected nonlinear terms. Despite occasional statements to the contrary, however, this does not rule out the possibility that one approximation is consistently better than another.

The most striking result is the inaccuracy associated with imposing the tangency condition on mass flux, whether in the linearized equation for φ or ψ (eq. (19) and sketch (c)) or the nonlinear first-order equation for ψ (eq. (17d) and sketch (a)). The error is never less than twice that in simple slender-body theory, and rises rapidly with Mach number.

On the other hand, imposing the tangency condition on velocity leads in every case to results more accurate than those of slender-body theory. Although the nonlinear solution is the most accurate at high Mach numbers (eq. (18b) and sketch (b)), it falls behind at lower speeds; and in any case it could not reasonably be extended to other bodies of revolution.

There remains the linearized solution (using either φ or ψ) with tangency imposed on v (eq. (20) and sketch (d)) as the most accurate approximation applicable to other shapes. Experience has shown that it consistently yields accuracy of the sort encountered in this example. Hence it seems unlikely that Stokes' stream function offers any advantages in first-order compressible flow theory, except possibly for determining streamlines in subsonic flow, where the inaccuracy would probably be no greater than it is at a Mach number of $\sqrt{2}$ in supersonic flow.

HIGHER APPROXIMATIONS

It was suggested in the introduction that the stream function might have the advantage in higher approximations of applying to blunt bodies for which the velocity potential fails. However, a practical iteration scheme could probably only be based upon the linearized approximation. The discovery that the true first-order equation of motion is actually nonlinear casts some doubt upon the validity of such a scheme. This matter will be studied by considering again the special case of a cone, for which the correct second-order solution is known.

Iteration on Linearized ψ for Cone

The second approximation for a cone in supersonic flow will be sought by proceeding in strict analogy with Schmieden and Kawalki's treatment of an ellipsoid in subsonic flow (ref. 6). Thus the slender-body expansion is introduced only at the last stage, which eliminates the well-known difficulty with the distant boundary condition in slender-body theory.

With the previous substitution (14) for the perturbation stream function, the linearized equation (10) becomes

$$(1 - t^2)f_1'' + \frac{3 - 2t^2}{t} f_1' = 0 \tag{22}$$

which is just equation (15) without its nonlinear term. The solution that vanishes at the Mach cone $t = 1$ is

$$f_1 = -\frac{1}{2} A\left(\frac{\sqrt{1-t^2}}{t^2} - \operatorname{sech}^{-1}t\right) \tag{23a}$$

The condition that the full stream function vanish on the cone of semi-vertex angle ϵ determines the constant A as

$$A = \frac{T^2}{\sqrt{1-T^2} - T^2\operatorname{sech}^{-1}T} \tag{23b}$$

These are just Moore's results (ref. 3).

Substituting this solution into the right-hand side of equation (13) gives as the iteration equation for the second-order increment to f

$$(1 - t^2)f_2'' + \frac{3-2t^2}{t} f_2' = \frac{M^2A^2}{B^2}\left[2(N - 1)\frac{\operatorname{sech}^{-1}t}{t^2\sqrt{1-t^2}} + \frac{1+t^2}{t^4}\right] \tag{24a}$$

where

$$N = -n = \frac{\gamma+1}{2}\frac{M^2}{B^2} = \frac{\gamma+1}{2}\frac{M^2}{M^2-1} \tag{24b}$$

The general solution vanishing at the Mach cone is

$$f_2 = -\frac{1}{2}C\left(\frac{\sqrt{1-t^2}}{t^2} - \operatorname{sech}^{-1}t\right) + \frac{M^2A^2}{4B^2}\left[2\frac{\sqrt{1-t^2}\operatorname{sech}^{-1}t}{t^2} - (2N + 3)\frac{1-t^2}{t^2} + (1 - 2N)(\operatorname{sech}^{-1}t)^2\right] \tag{25a}$$

At this point it is appropriate to introduce the slender-body approximation by expanding asymptotically for small t. Then requiring the stream function to vanish on the surface to second order determines the constant C as

$$C = M^2\epsilon^2T^2\left(\ln\frac{2}{T} - N - \frac{3}{2}\right) \tag{25b}$$

The velocity components are related to the stream function by equations (6). Schmieden and Kawalki's analysis implies (falsely) that linear and quadratic terms are respectively of first and second order, which means that the streamwise velocity increment is given by

$$u = -\frac{1}{B^2}\left(\frac{\psi_{1r}}{r} + \frac{\psi_{2r}}{r}\right) - \frac{M^2}{2B^2}\left[\frac{2N-3}{B^2}\left(\frac{\psi_{1r}}{r}\right)^2 + \left(\frac{\psi_{1x}}{r}\right)^2\right] \tag{26}$$

On the surface of the cone, the linearized solution for f_1 gives, in the slender-body expansion

$$\frac{\psi_{1r}}{r} = T^2\left[\ln\frac{2}{T} + T^2\left(\ln^2\frac{2}{T} + \frac{1}{2}\ln\frac{2}{T} - \frac{1}{4}\right) + \ldots\right] \tag{27a}$$

$$\frac{\psi_{1x}}{r} = -\epsilon\left[1 + T^2\ln\frac{2}{T} + \ldots\right] \tag{27b}$$

and the second-order increment f_2 gives

$$\frac{\psi_{2r}}{r} = -\frac{1}{2}M^2\epsilon^2 + M^2\epsilon^2T^2\left[\left(\frac{3}{2} - N\right)ln^2\frac{2}{T} - \frac{7}{2}ln\frac{2}{T} + N + 1\right] + \ldots \quad (27c)$$

Note the apparent contradiction that this supposed second-order term actually starts off with a first-order term in ϵ^2. That contribution to u is, however, cancelled by the first-order nonlinear term in $(\psi_{1x}/r)^2$, which is just the coincidence discussed previously that permits the first-order problem to be linearized. Thus equation (26) gives as the supposed second approximation

$$u_S \overset{?}{=} 1 - \epsilon^2 ln\frac{2}{B\epsilon} - \epsilon^4\left[B^2 ln^2\frac{2}{B\epsilon} - \frac{1+4M^2}{2}ln\frac{2}{B\epsilon} + M^2N + \frac{1+3M^2}{4}\right] \quad (28)$$

This result is unfortunately not entirely correct. Broderick's solution of the same problem, using the velocity potential, gives $6M^2/4$ in place of $3M^2/4$ in the last term. His solution has been confirmed by independent analyses, and must be regarded as reliable. That the discrepancy is in a term proportional to M^2 is plausible since the linearized and nonlinear first-order problems differ by terms in M^2.

If one imposes the tangency condition on velocity rather than mass flux, the result is different but still incorrect. A discrepancy then appears also in the coefficient of $\epsilon^4 ln\ 2/B\epsilon$.

Discussion.- The preceding example shows that although the first-order problem for ψ can coincidentally be linearized, the second-order problem cannot be limited to quadratic terms and solved by iteration on the linearized solution. Three modifications, of successively greater complexity, suggest themselves as the correct procedure.

The simplest possibility is that the true second approximation can be found merely by retaining some cubic or quartic terms in the right-hand side of the iteration equation (13) or in the expressions (6) for velocity components. This is known to be the case if one works instead with φ, where the iteration equation must include a cubic term, and quadratic terms appear in the expression for pressure. However, attempts to isolate the proper combination (if it exists) from the many possibilities in the preceding example have not succeeded. Secondly, it may be that the remainder of the second-order terms would emerge from the next step of the procedure, involving cubic terms. Thirdly, it might be necessary to work with the nonlinear iteration equation (12), which would introduce enormous mathematical difficulties.

The problem mentioned in the introduction of second-order subsonic flow past a paraboloid has recently been solved by the author in an indirect way, and the solution obtained by using ψ in the preceding fashion is found again to be in error by a term in M^2. Since a paraboloid is a limiting case of an ellipsoid, this confirms the suspicion that Schmieden and Kawalki's solution (ref. 6) is incorrect in the second-order terms in M^2.

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National Advisory Committee for Aeronautics
Moffett Field, Calif., Dec. 10, 1956

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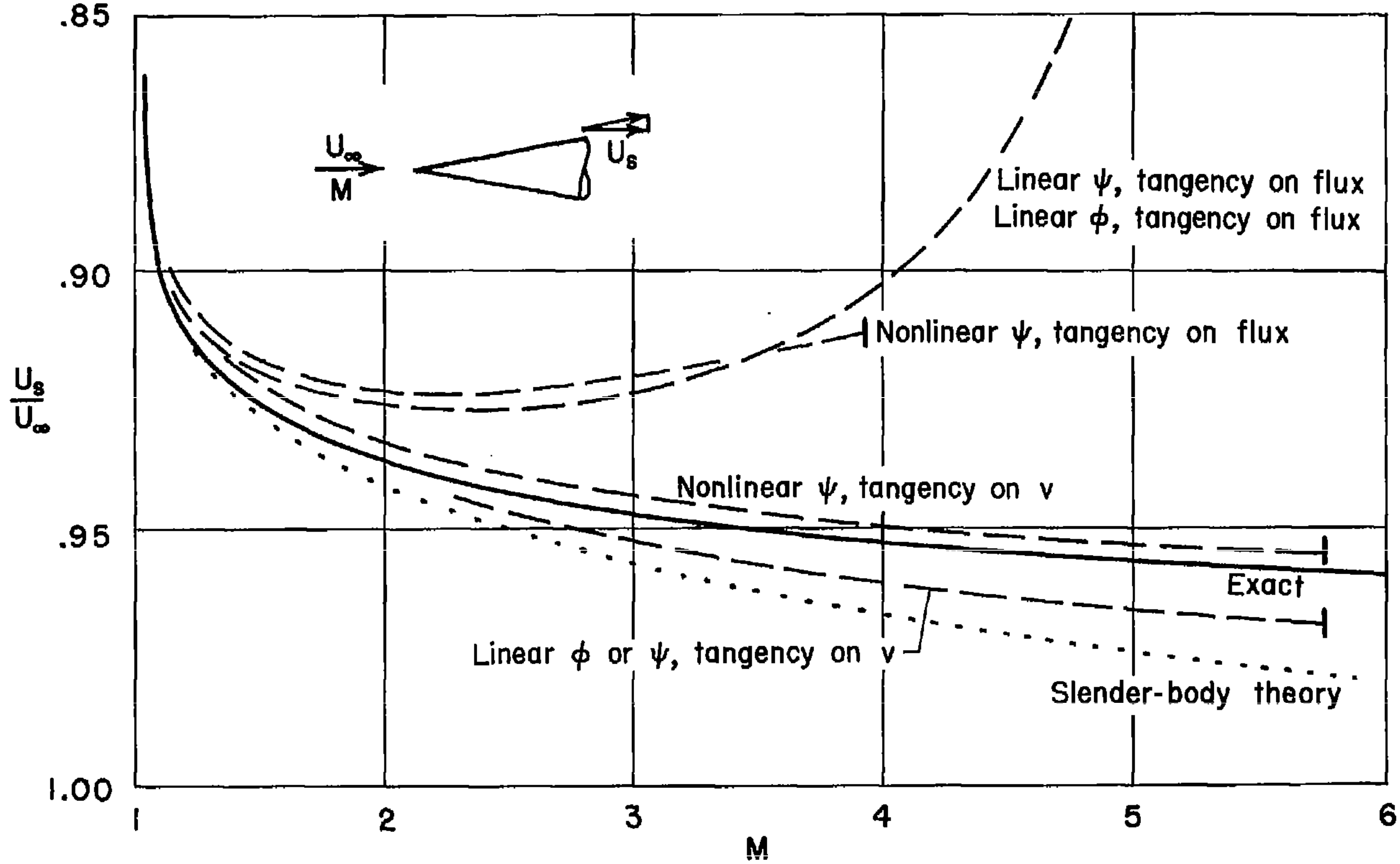

Figure 1.- Various first approximations for surface speed on cone in supersonic flow.

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