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STABILITY OF ELASTICALLY SUPPORTED COLUMNS

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SUMMARY

A criterion is developed for the stiffness required of elastic lateral supports at the ends of a compression member to provide stability. A method based on this criterion is then developed for checking the stability of a continuous beam-column. A related method is also developed for checking the stability of a member of a pin-jointed truss against rotation in the plane of the truss.

INTRODUCTION

One important task in airplane structural design is the investigation of the elastic stability of a continuous member subjected to axial compression and provided with elastic support against lateral buckling. A typical member of this class is a longitudinal fuselage stiffener with the transverse fuselage rings acting as the elastic transverse supports. If the supports are assumed infinitely rigid, and they are not too numerous, the critical load can be determined by the methods of reference 1 (art. 14:6). When, however, as in actual structures, the supports are not infinitely rigid, the critical load for the continuous member is reduced, but the problem of determining its magnitude becomes much more complex. Practical solutions of the problem have been obtained for a few simple cases like those investigated by Klemperer and Gibbons (reference 2) and by Schwartz and Bogert (reference 3), but no satisfactory procedure has been developed for determining the critical loading for a continuous beam-column with elastic supports of arbitrary number, location, and stiffnesses. Conversely no satisfactory procedure has been developed to determine the stiffnesses required of arbitrarily located supports to ensure that the critical load of the supported member will exceed some desired minimum value.

Schwarz has shown (reference 3) that, with a single elastic support, if the stiffness of the support is but slightly in excess of that required for stability and there is initial deflection of the support point or play in the connection, the supported member is likely to deflect excessively and to fail plastically. It appears reasonable to assume that this condition would be just as likely to be present with more than one elastic support, and in practice the stiffnesses of such supports should be considerably in excess of the theoretical minimum requirements. From the practical point of view, therefore, it is not essential that the criterion for support stiffness indicate the theoretical minimum allowable values. The formulation of a criterion indicating stiffnesses definitely on the safe side would furnish the designer with an appreciated tool of analysis, provided that it did not prove excessively and needlessly conservative. The chief purpose of this paper is to show how such a criterion can be developed and applied, if the actually continuous beam-column is treated as a series of rigid links mutually pin-connected at the locations of lateral support.

The chief apparent defect of the criterion developed on this basis is that the stiffnesses attributed to the supporting members are effective stiffnesses that are influenced by the rigidity of whatever is provided to support those supporting members. Further investigation is therefore required to improve its practicability. The problem appears capable of solution and it is expected that the basic procedure outlined in this paper can be adapted to the solution of various important practical problems. The only extension that has been carried out thus far is that of developing the procedure described in this paper for checking the stability of a pin-ended truss member against rotation in the plane of the truss. Steps are being taken to obtain an experimental validation of this extension, but progress along this line is as yet insufficient to justify a report.

STABILITY OF LINKS ELASTICALLY SUPPORTED AT ONE END AND RIGIDLY SUPPORTED AT THE OTHER

The simplest type of elastically supported member is that represented by the link AB, shown in figure 1, which is completely restrained from both horizontal and vertical movement, though free to rotate, at its left end A and restrained from vertical movement at its right end B by the spring BC. This link is obviously in equilibrium when in the indicated horizontal position, regardless of the magnitude of the axial compression P. Whether that equilibrium is stable, neutral, or unstable depends on what would happen if the link were slightly displaced, and this condition would depend on the magnitude of the axial compression. If the system remained in equilibrium in spite of the displacement, it would be in neutral equilibrium. The magnitude of P associated with this condition is called the critical load and may be designated P_{cr}. If, however, $P < P_{cr}$, the displacement would develop forces tending to restore the original conditions, and the equilibrium would be stable. On the other hand, if $P > P_{cr}$, the forces developed by the displacement would tend to cause increased displacement and the equilibrium would be unstable. The practical problem is therefore to determine the critical load, and that load may be defined as the load under which the system would be in equilibrium in spite of a very small change in the position of AB.

For any practical structure there will be a critical load associated with each geometrically possible type of change in shape, but attention will be limited to that associated with rotation of the link AB in the vertical plane. This critical load can be most readily determined by the method described by Timoshenko in reference 4, based on the proposition that when $P = P_{cr}$ no change in the potential energy of a system would result from a small change in its configuration. Since that method is to be extended to investigations of more complex systems, it is desirable to review it at this point.

If the link AB were to rotate in the vertical plane, there would be no movement of A, on account of the rigid support assumed at that point. End B, however, would move both vertically and horizontally. If the angle of rotation α is assumed small, it is permissible to assume $\sin \alpha = \alpha$ and $\cos \alpha = 1 - \alpha^2/2$. Then δ, the vertical movement of B, will be equal to $L\alpha$, and γ, the horizontal movement of that point, will be

$$\gamma = \frac{\delta^2}{2L} \qquad (1)$$

Therefore, one result of the rotation of link AB would be to cause the axial load P to move through the distance γ and do the work

$$U_e = P\,\gamma = \frac{P\,\delta^2}{2\,L} \tag{2}$$

If the link AB and the support at C be assumed rigid, equation (2) is also a measure of the strain energy that must be stored in the spring BC for the net change of potential energy of the system to be zero. An alternative measure of the strain energy of the spring is provided by K, the spring constant of that member and the magnitude of its change in length. The last-mentioned quantity is identical with δ, the vertical movement of B, so the strain energy of the spring may also be written

$$U_i = \frac{K\,\delta^2}{2} \tag{3}$$

The equating of these alternative measures of the strain energy of the spring and the simplifying of them gives

$$P_{cr} = K\,L \tag{4}$$

as the relation that must be satisfied if the link is to be in neutral equilibrium.

In the previous discussion it is assumed that the link AB is rigid, but equation (4) would still be obtained as the criterion for neutral equilibrium with respect to rotation in the vertical plane even though the change in length, $\Delta L = PL/AE$ where A is the cross-sectional area and E is the modulus of elasticity, were taken into account. If it should happen that P_{cr} as obtained from equation (4) were greater than the Euler load, $P_e = \pi^2 EI/L^2$, actual failure would take place by buckling unaccompanied by rotation of the link as soon as the axial load P became equal to P_e. If, however, $P_{cr} < P_e$, failure would be by rotation under the former load.

In design the problem is often to determine the stiffness required for stable equilibrium when the axial load is of a specified magnitude P, less than P_e, rather than to determine the critical load P_{cr} associated with a spring of known stiffness. The value of K associated with neutral stability under a given load P may be called the critical spring constant for that load and be designated K_{cr}. Then by rearrangement of equation (4), the critical spring constant for the system of figure 1 is obviously

$$K_{cr} = P/L \tag{5}$$

The critical spring constant for the two-link system of figure 2 can be obtained in the same manner as that for a single link. As the result of small simultaneous rotations of the two links, point C would move horizontally through the distance

$$\gamma = \frac{\delta^2}{2\,L_1} + \frac{\delta^2}{2\,L_2} \tag{6}$$

and the force P would do work equal to

$$U_e = \frac{P\,\delta^2}{2}\left(\frac{1}{L_1} + \frac{1}{L_2}\right) \tag{7}$$

while the strain energy of the spring would again be represented by equation (3). The equating of equations (3) and (7) gives for the critical spring constant

$$K_{cr} = P\left(\frac{1}{L_1} + \frac{1}{L_2}\right) \tag{8}$$

From inspection of equation (8) it can be seen that the stiffness required of a single spring to maintain stability of the two-link system is the sum of the stiffnesses that would be required of the springs in the two single-link systems into which it might be resolved. In other words, the single spring at B might be assumed to be composed of a pair of associated springs with the same total stiffness, one of these partial springs being assigned to act with each of the links, and each link being investigated separately as though it were part of a system like that shown in figure 1. Thus, if the axial load P were specified, the critical spring constants for links AB and BC would be computed separately and then added to determine the minimum allowable spring constant for a single spring located at B.

CRITERION FOR THE STABILITY OF A LINK ON TWO ELASTIC SUPPORTS

In the preceding section it is assumed that one end of each link is rigidly supported against translation normal to the axial load, but in practice it is more likely that the restraint against such movement will be elastic in character. In order to extend the method of attack foreshadowed in that section, it is therefore necessary to develop a criterion for the stability of a link elastically supported at both ends, such as the link AB shown in figure 3.

In this system when the link AB rotates about any point along its length the work done by the axial forces P is

$$U_e = \frac{P}{2\,L}\,(\delta_1 + \delta_2)^2 \tag{9}$$

The resulting strain energy of the springs is

$$U_i = \frac{K_1 \delta_1{}^2}{2} + \frac{K_2 \delta_2{}^2}{2} \tag{10}$$

where K_1 and K_2 are the spring constants of the elastic supports at A and B, respectively. When the right-hand sides of equations (9) and (10) are equated

$$\frac{P}{2\,L}\,(\delta_1 + \delta_2)^2 = \frac{K_1 \delta_1{}^2}{2} + \frac{K_2 \delta_2{}^2}{2} \tag{11}$$

From the geometry of figure 3

$$\frac{\delta_2}{\delta_1} = \frac{b}{a} = n \quad \text{and} \quad \delta_2 = n\,\delta_1 \tag{12}$$

Also, since equation (11) implies that the link, though rotated slightly, remains in equilibrium, $K_1\delta_1 = K_2\delta_2 = n\,K_2\delta_1$. Therefore

$$n = K_1/K_2 \tag{13}$$

Substitution of equations (12) and (13) in equation (11) produces

$$\frac{P}{L}(1 + n)^2 \delta_1^2 = n K_2 \delta_1^2 + n^2 K_2 \delta_1^2 \qquad (14)$$

which simplifies to

$$P = \frac{n K_2 L}{1 + n} \qquad (15)$$

Again, the substitution of equation (13) in equation (15) and the simplifying of the equation obtained results in

$$P_{cr} = \frac{K_1 K_2 L}{K_1 + K_2} \qquad (16)$$

Equation (16) expresses the criterion for the critical load of a link supported at both ends, the supports being of either equal or of unequal stiffness. If they happen to be of equal stiffnesses, $K_1 = K_2$ and the equation reduces to

$$P_{cr} = KL/2 \qquad (17)$$

If one of the supports is rigid, its spring constant is infinite and equation (16) reduces, as would be expected, to equation (4). This result can be proved by the division of the numerator and the denominator of the right-hand side by K_1, which gives

$$P_{cr} = \frac{K_2 L}{1 + \frac{K_2}{K_1}} \qquad (18)$$

which becomes $P_{cr} = K_2 L$ when K_1 is equal to infinity.

It is of interest to visualize the effect on the critical load resulting from increasing the stiffness of one of the supports. Let P_1 be the critical load when $n = K_1/K_2 = 1$. As the ratio n increases, P_{cr} increases until it becomes equal to $2 P_1$ when n is infinite. This result can be seen from comparison of equations (4) and (17). The character of this variation of P_{cr} with n is shown graphically in figure 4.

If it is desired to find the minimum allowable stiffness for the support at B when the stiffness of the support at A and the axial load P are given, this calculation can be made by solving equation (16) for K_2. The result is

$$K_2 = \frac{K_1 P}{K_1 L - P} \tag{19}$$

STABILITY OF TWO LINKS ON THREE ELASTIC SUPPORTS

The same method can be extended to obtain a criterion for the stability of the structure indicated in figure 5 where two pin-connected links are subjected to the axial loads P_1 and P_2 and are restrained against rotation by elastic supports at A, B, and C. Rotation of the links would be associated with relative horizontal movements of A, B, and C and the forces P_1 and P_2 would do work equal to

$$U_o = \frac{P_1}{2L_1} (\delta_1 + \delta_2)^2 + \frac{P_2}{2L_2} (\delta_2 + \delta_3)^2 \tag{20}$$

At the same time the supports would store strain energy equal to

$$U_i = \frac{K_1 \delta_1^2}{2} + \frac{K_2 \delta_2^2}{2} + \frac{K_3 \delta_3^2}{2} \tag{21}$$

For the system to be in neutral equilibrium, it is necessary that $U_o = U_i$, or that

$$\frac{P_1}{2L_1} (\delta_1+\delta_2)^2 + \frac{P_2}{2L_2} (\delta_2+\delta_3)^2 = \frac{K_1 \delta_1^2}{2} + \frac{K_2 \delta_2^2}{2} + \frac{K_3 \delta_3^2}{2} \tag{22}$$

Let

$$\frac{\delta_1}{\delta_2} = \frac{a}{b} = m \quad \text{and} \quad \frac{\delta_3}{\delta_2} = \frac{d}{c} = n \tag{23}$$

Equation (22) then reduces to

$$P_1 L_2 (m+1)^2 + P_2 L_1 (n+1)^2 = (K_1 m^2 + K_2 + K_3 n^2) L_1 L_2 \tag{24}$$

In order to satisfy the requirements of equilibrium

$$K_1\delta_1 + K_3\delta_3 = K_2\delta_2 \tag{25}$$

The combining of equations (23) and (25) and the simplifying of the result will give

$$K_1 m + K_3 n = K_2 \tag{26}$$

Let the total stiffness K_2 of the support at B be resolved into two portions, $p\ K_2 = m\ K_1$ and $q\ K_2 = n\ K_3$, where $p + q = 1$. Then

$$m = p\ K_2/K_1 \qquad m + 1 = (pK_2 + K_1)/K_1 \tag{27a}$$

$$n = q\ K_2/K_3 \qquad n + 1 = (qK_2 + K_3)/K_3 \tag{27b}$$

By substitution of the relations of equations (27) in equation (24)

$$P_1 L_2\ (pK_2 + K_1)^2/K_1^{\ 2} + P_2 L_1\ (qK_2 + K_3)^2/K_3^{\ 2}$$

$$= (p^2 K_2^{\ 2}/K_1 + K_2 + q^2 K_2^{\ 2}/K_3)\ L_1 L_2 \tag{28}$$

If $P_2 = r\ P_1$ this equation reduces to

$$P_1 = \frac{K_1 K_3 L_1 L_2\ [pK_2 K_3 (K_1 + pK_2) + qK_1 K_2 (K_3 + qK_2)]}{K_3^{\ 2} L_2 (pK_2 + K_1)^2 + rK_1^{\ 2} L_1 (qK_2 + K_3)^2} \tag{29}$$

If K_2 is resolved into its components pK_2 and qK_2, the criterion for the critical value of P_1 becomes

$$P_1 = \frac{K_1 K_3 L_1 L_2\ [pK_2 K_3 (K_1 + pK_2) + qK_1 K_2 (K_3 + qK_2)]}{K_3^{\ 2} L_2 (pK_2 + K_1)^2 + rK_1^{\ 2} L_1 (qK_2 + K_3)^2} \tag{30}$$

The same procedure could be used to determine the critical loading for any number of links, but even for two links the expression for the critical loading becomes too unwieldy for practical use. In the development of that expression it may be noted that the stiffness of the central support K_2 is divided into two portions. This division is equivalent to assuming that the support is composed of two springs; (one attached to the link AB, and the other, to the link BC) in effect changing the system under consideration from that shown in figure 5 to that of figure 6, if the short link shown at B is assumed to be of zero length. In the system of figure 6, if each link is treated independently, and if P_1 is to be the critical load for span AB, and P_2, the critical load for span BC, from equation (19)

$$pK_2 = \frac{K_1 P_1}{K_1 L_1 - P_1} \qquad qK_2 = \frac{K_3 P_2}{K_3 L_2 - P_2} \tag{31}$$

Substitution of equations (31) in equation (30) leads to an identity. This result means that if equations (31) are satisfied, equation (30) is also satisfied, and, so far as stability against rotation of the links is concerned, the systems of figures 5 and 6 are equivalent.

THE SUCCESSIVE LINK METHOD OF INVESTIGATING STABILITY

In the investigation of the stability of a series of pin-connected links the most common problem is to determine whether the system is stable when the axial loads in the links and the spring constants of the elastic supports have specified values. This problem can readily be solved if each elastic support at the junction of two links is assumed to be composed of two supports, one attached to each of the links, between which the stiffness of the actual single support is partitioned. This set-up is illustrated by the analysis of the system shown in figure 7, in which five links are assumed rigidly supported against translation at the ends of the chain and elastically supported at the intermediate joints. For simplicity it is assumed that $P_1 = P_2 = P_3 = P_4 = P_5 = P$ and $L_1 = L_2 = L_3 = L_4 = L_5 = L$. It is also assumed that $P < \pi^2 EI/L^2$, so there will be no "Euler buckling" of the individual links between supports.

Beginning at the left support, from equation (5) the minimum allowable stiffness of the support at joint 1 is P/L, or

$$K_{1-0} = \frac{P}{L} = \frac{1000}{100} = 10 \text{ pounds per inch} \qquad (32)$$

Therefore, if K_1 is less than 10 pounds per inch, the system is obviously unstable; but if K_1 exceeds that value, there will be surplus stiffness that can be utilized to restrain rotation of link 1-2. If $K_1 = 20$ pounds per inch, this surplus is $K_{1-2} = 20 - 10 = 10$ pounds per inch, which is the value of K_1 to be used in the investigation of the second link. For that purpose equation (19) may be written

$$K_{2-1} = \frac{K_{1-2}\, P}{K_{1-2}\, L - P} \qquad (33)$$

or

$$K_{2-1} = \frac{10 \times 1000}{10 \times 100 - 1000} = \infty$$

In other words, if K_1 is only 20 pounds per inch, the support at station 2 must be rigid. If that support is to be elastic, K_1 must have a spring constant greater than 20 pounds per inch. If $K_1 = 40$ pounds per inch, $K_{1-2} =$ 30 pounds per inch, and from equation (33) the required value of K_{2-1} is

$$K_{2-1} = \frac{30 \times 1000}{30 \times 100 - 1000} = 15 \text{ pounds per inch}$$

If it is assumed that $K_2 = K_3 = K_4 = 40$ pounds per inch, the computations are as follows:

K_{2-3} available $= 40 - 15 = 25$ pounds per inch

K_{3-2} required $= \frac{25 \times 1000}{25 \times 100 - 1000} = 16.67$ pounds per inch

K_{3-4} available $= 40 - 16.67 = 23.33$ pounds per inch

K_{4-3} required $= 23{,}300/1{,}333 = 17.50$ pounds per inch

K_{4-5} available $= 40 - 17.50 = 22.50$ pounds per inch

K_{5-4} required $= 22{,}500/1{,}250 = 18$ pounds per inch

The support at station 5, however, is rigid; that is, its spring constant is infinite. Therefore, since the actual stiffness of the support at station 5 exceeds the required value, the system is in stable equilibrium. In fact, there is an excess of support stiffness and the structure would be stable even though the stiffnesses of some of the intermediate supports were reduced.

It should be noted that while the assumption of equal lengths and axial loads for the links and equal stiffnesses for the intermediate springs simplified the numerical work of the example, it did not affect the essential characteristics of the computation method, which would be equally valid for any arbitrarily chosen values for those quantities. In the example it was found that the available spring constant at each station exceeded that required and the system was therefore adjudged stable. Had the available spring constant at any station been less than that required, the system would have been adjudged unstable.

In practical problems the spring constants of the various supports are usually not known to a high degree of precision, and there may be initial deflections of the stations. Therefore, it is desirable to provide supports with spring constants considerably in excess of those theoretically called for. In many problems it would also save labor and be conservative to assume that just half the spring constant at each intermediate support represented the stiffness available to each of the links at that station. This assumption would prevent the accumulation of errors due to lack of precision in the assumed values for the support spring constants.

The chain of links shown in figure 7 is not a common practical structure. The designer is much more likely to be confronted with the problem of investigating the stability of a continuous beam-column. If such a member is considered as a series of links with pin joints at the supports, the resistance to buckling due to continuity of the member is neglected and the application of this "successive-link method" of analysis gives conservative results. It is the belief of the writers, however, that the degree of conservatism is not excessive, and that the actual effect of continuity is to provide an extra margin of safety comparable to that required to absorb the possible effects of initial deflections and lack of precision in the estimated values of the support stiffnesses.

Whether or not this fact is true can be determined only by experience and further study of the problem.

STABILITY OF MEMBERS OF PIN-JOINTED TRUSSES

In the preceding sections criterions are developed for the stability of a link with varying degrees of elastic-support stiffness. A method is then shown for using these criterions for investigating the stability of a series of pin-connected links. This method, however, is based on the assumption that the supports are independent and the deflection of one has no effect on the deflections of the others. While this assumption is true for some structures, it is far from being generally true, and the method requires modification if it is to be applied to many practical types of structure. The rest of this report is therefore devoted to the development of a procedure by which the basic method under study can be used to investigate the stability against rotation of a member of a pin-jointed truss.

A pin-jointed truss, such as that shown in figure 8, may be treated as a system of elastically supported links, in which each link is restrained from rotation by the axial loads that would be developed in the truss members, including the one in question. It differs from the link of figure 3 in that the movements of its ends are not independent but mutually dependent. On this account the criterions developed previously cannot be applied directly, but related criterions that take into account this mutual dependence are needed. Such criterions can be developed for any specific member in terms of two truss properties that may be called the rotational spring constant and the induced rotational spring constant for the member.

The rotation of any truss member with respect to a line through the truss supports can be determined by established methods. The use of the method of virtual work (reference 1, ch. XII) to determine the rotation due to a unit couple applied to the member and resisted by a unit couple composed of forces acting on the supports, results in the expression

$$\alpha_{ab} = \Sigma \frac{p_{ab}{}^2 L}{AE} \qquad (34)$$

where p_{ab} is the axial load produced in a member by the specified external force system, L the length of the member, A its cross-sectional area, and E the modulus of elasticity of the material. The ratio of the moment of an imposed couple to the resulting rotation of a member, which is

$$k_{ab} = \frac{1}{\alpha_{ab}} \tag{35}$$

can therefore be computed for any given member. This ratio is what is here termed the "rotational spring constant" of the member.

When a unit couple is applied to one member, each of the other members will rotate, and those rotations may be expressed by the relation

$$\alpha_{xy} = \Sigma \frac{p_{ab} p_{xy} L}{AE} \tag{36}$$

where α_{xy} is the rotation of any arbitrarily chosen member, XY; p_{xy}, the axial load produced in a member by a unit couple imposed on member XY; and p_{ab}, A, E, and L are the same as in equation (34).

If the rotation due to the hypothetical unit couple applied to member AB is small, the axial loads produced by the actual loading system on the truss may be assumed unchanged in magnitude, though slightly changed in direction. If the actual axial load in any member be P, it may be resolved into two components, $P \cos \alpha$ parallel to the direction of the member prior to the imposition of the unit couple, and $P \sin \alpha$ perpendicular to that direction. Since α is assumed small, these components may be assumed equal to P and $P\alpha$, respectively. Since each truss member would be designed to withstand its axial load P, and since $\Sigma P = 0$ at each point, there is no danger of instability resulting from the action of the parallel components. On the other hand, since the rotations of the various members entering a joint will not, in general, be identical, it cannot be assumed that at each joint $\Sigma P\alpha = 0$; but there will be a finite resultant of the perpendicular components of the forces acting at each joint. These resultants may be termed the induced loads since they are induced by the rotations resulting from applying the unit couple on member AB. These induced loads would tend to produce additional rotations of the truss members, which may be termed the "induced rotations."

The magnitude of the induced rotation of member AB may be obtained from

$$\alpha_{ab}' = \Sigma \frac{p_{ab} p_{ind} L}{AE} \tag{37}$$

where p_{ab}, L, A, and E are the same as previously given and p_{ind} is the axial force in a member produced by the induced loads. From equation (37) the ratio of induced loading to resulting rotation, which may be termed the "induced spring constant" of a member AB would be

$$k_{ab}' = \frac{1}{\alpha_{ab}'} \tag{38}$$

The rotational spring constant of a member may be interpreted physically as a measure of the resistance offered by the truss to the rotation of a member due to a unit couple applied to that member. Similarly, the induced rotational spring constant may be interpreted as a measure of the resistance offered by the truss to rotation of the member as a result of the induced loads.

Therefore, it may be deduced that if k_{ab}' exceeds k_{ab} so that the resistance to rotation caused by the induced loads is greater than the resistance to rotation caused by the unit couple that would produce those induced loads, the member is in stable equilibrium. Vice versa, if k_{ab}' is less than k_{ab}, indicating that the resistance to rotation caused by the induced loads is smaller than the resistance to rotation caused by the original unit couple, the member is in unstable equilibrium and the structure would collapse.

This criterion may be applied to the truss of figure 8, which is simple enough to show clearly and completely the steps involved in such an analysis, and yet is of such design that by study of this approach it can be seen how the criterion can be applied to more complex structures. Purely for convenience in calculation and presentation, all members are assumed to be of the same material and of the same cross-sectional area. The external loading assumed consists of a vertical load W applied at joint F and the necessary reactions at D and H, but the conditions under any other loading could be treated in essentially the same manner.

By application of equation (34) the rotation, with respect to a line through the supports, of member AB due to a unit couple applied to that member is found to be

$$\alpha_{ab} = \frac{0.1296}{AE} \text{ radians per inch-pound}$$

Therefore, the rotational spring constant of member AB, as defined by equation (35) is

$$k_{ab} = \frac{AE}{0.1296} \text{ inch-pounds per radian}$$

By successive use of equation (36) the associated rotations of the other members with respect to the line through the supports can be computed. The multiplying of each rotation by the load imposed on the member by the external force W, the resolving of the products into horizontal and vertical components, and the combining of those components, produce the induced loads shown in figure 9. Values of the rotations and the corresponding induced loads are listed in table I. The rotation of member AB that would be produced by the induced-load system of figure 9 is found by application of equation (37) to be

$$\alpha_{ab}' = \frac{0.3274\ W}{A^2E^2} \text{ radians per inch-pound}$$

From equation (38) the induced rotational spring constant is therefore

$$k_{ab}' = \frac{A^2E^2}{0.3274\ W} \text{ inch-pounds per radian}$$

If it be assumed that

$A = 2.00$ square inches

$E = 30{,}000{,}000$ pounds per square inch

$W = 80{,}000$ pounds

$$k_{ab} = \frac{2 \times 30,000,000}{0.1296} = 463,000,000 \text{ inch-pounds per radian}$$

and

$$k_{ab}' = \frac{4 \times 900,000,000,000,000}{0.3274 \times 80,000} = 137,400,000,000 \text{ inch-pounds per radian}$$

Since $k_{ab}' > k_{ab}$ it may be concluded that member AB is in stable equilibrium.

TABLE I

TABULATED VALUES OF α AND $P\alpha$

Member	AE α	AE $P\alpha$
AB	0.1296	0.0972W
BC	-.0648	-.0486W
DE	-.0230	.0086W
EF	.0852	-.0319W
FG	-.0204	.0073W
GH	-.0418	.0157W
AD	-.0092	-.0057W
AE	.0094	.0000
AF	.0435	-.0272W
BF	.0322	.0000
CF	-.0065	.0041W
CG	.0094	.0000
CH	-.0279	-.0175W

A positive value of α indicates that the member rotated in the same direction as the applied unit couple, with respect to the reference line.

A negative value of α indicates that the member rotated in a direction opposite to that of the applied unit couple with respect to the reference line.

A negative value of $P\alpha$ indicates that they form a clockwise couple on the member.

A positive value of $P\alpha$ indicates that they form a counterclockwise couple on the member.

If in this example it were desired to determine the magnitude of the load at which member AB would become unstable, that could be done by equating the expressions for k_{ab} and k_{ab}' and solving for W. This would give

$$\frac{AE}{0.1296} = \frac{A^2E^2}{0.3274\ W}$$

whence

$$W = \frac{0.1296\ AE}{0.3274} = 0.396\ AE = 23,750,000 \text{ pounds}$$

In this example the critical value of W for member AB is nearly 300 times as great as the assumed load of 80,000 pounds, yet that assumed load subjects the member to an average unit stress of 30,000 pounds per square inch. If the material were any known variety of steel, member AB would fail in direct compression long before instability of the member against rotation became an important factor.

In a practical truss, however, the members would seldom be of equal cross-sectional area, and with the design of figure 8, in particular, since the load W imposed at F would produce no axial load in that member, BF would normally be made much lighter than AB. At times it might therefore be suspected that BF was so lightly designed that member AB was in danger of experiencing rotational instability failure, and a check by the method just outlined would be in order.

An alternative method of investigating the stability of truss members against rotation is described by Timoshenko (reference 4, art. 28), who credits it to von Mises and Ratzersdorfer. The system of attack just outlined, however, appears to the writers to be more easily applied to the investigation of trusses of arbitrary proportions and loading, particularly if the work must be done by personnel of only moderate experience.

In the development of this criterion for the stability of a truss, it is implicitly assumed that elastic failure would be the result of the rotation of a selected member. The critical load thus obtained might not, therefore, be the minimum critical load, since there might be a smaller critical load associated with rotations of some other member or group of members. It would be desirable to continue the study of this criterion to find out if the critical load obtained by its use is likely to be significantly in excess of the critical load obtained by the method of von Mises and Ratzersdorfer that is not subject to this defect. It would also be desirable to investigate various possibilities by which the criterion of this paper, or its practical application, could be simplified for use in routine design work.

Guggenheim Aeronautics Laboratory,
Stanford University, Calif., March 23, 1942.

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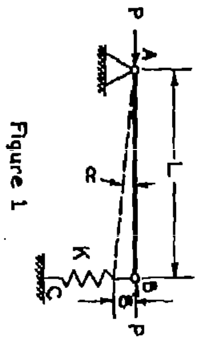


Figure 1

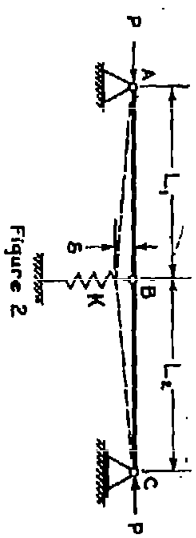


Figure 2

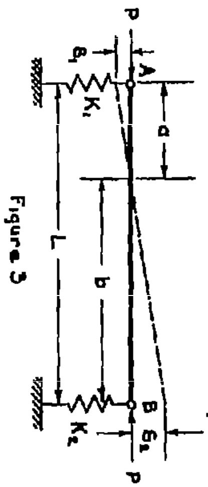


Figure 3

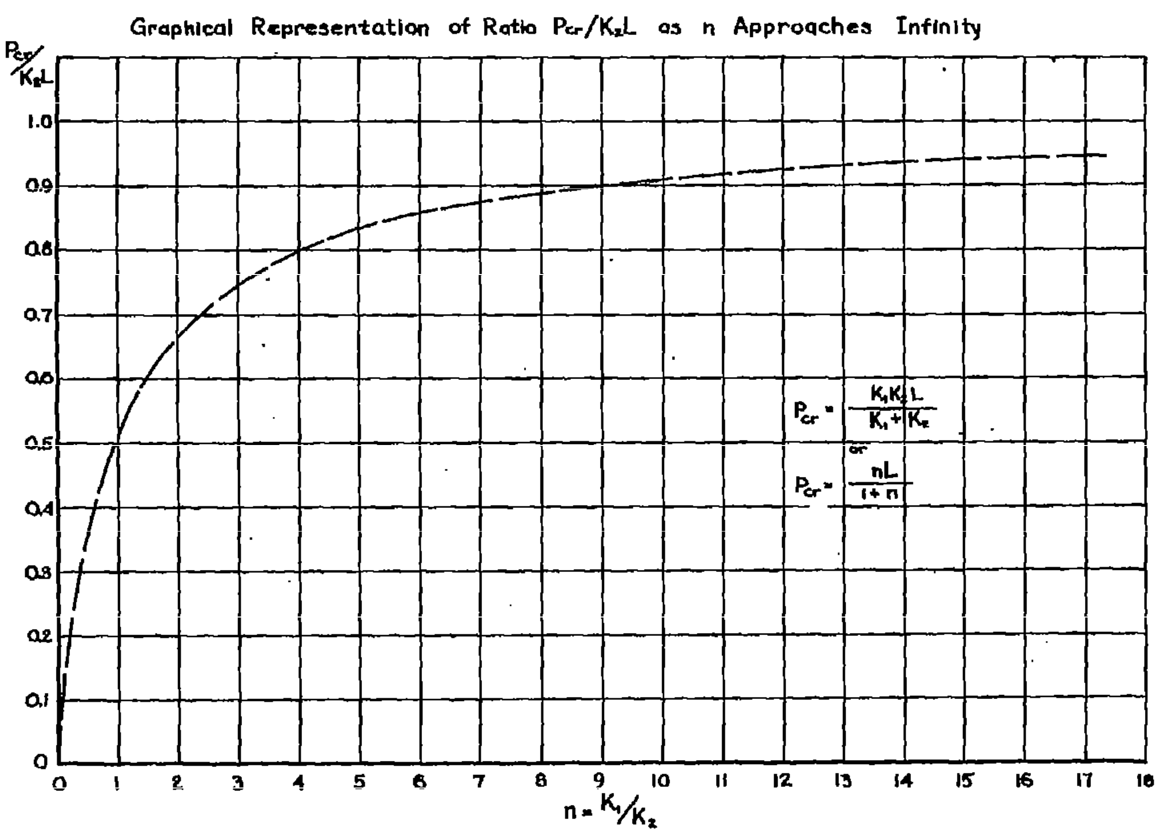


Figure 4

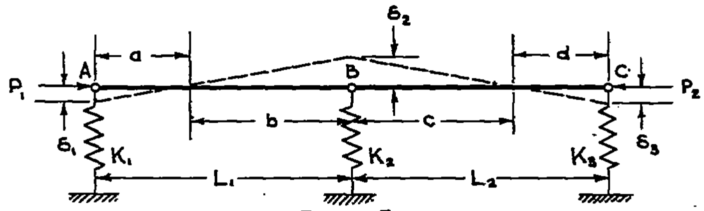
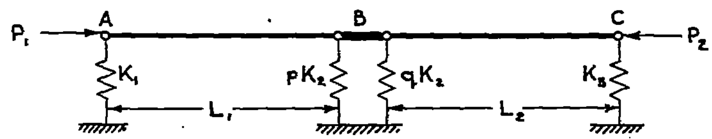


Figure 5



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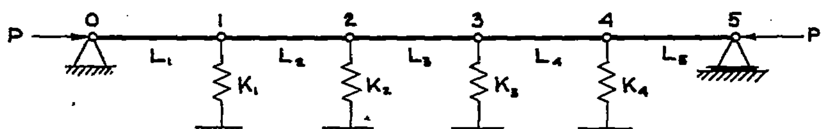


Figure 7

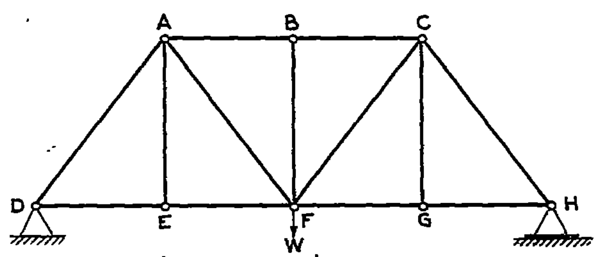
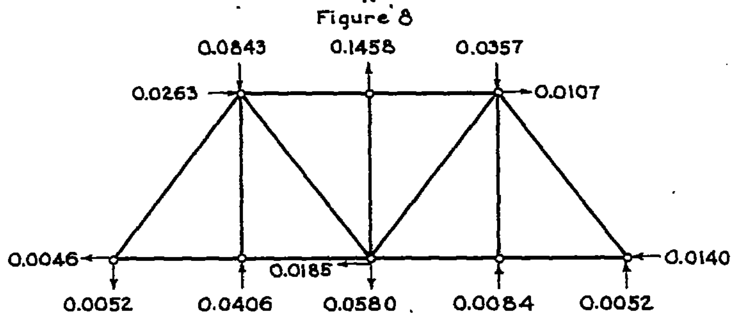


Figure 8



Loads shown should be multiplied by W/AE to obtain actual values of resultants of induced loads.

Figure 9