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# ON THE GENERAL THEORY OF THIN AIRFOILS FOR NONUNIFORM MOTION

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## SUMMARY

General thin-airfoil theory for a compressible fluid is formulated as boundary problem for the velocity potential, without recourse to the theory of vortex motion.

On the basis of this formulation the integral equation of lifting-surface theory for an incompressible fluid is derived with the chordwise component of the fluid velocity at the airfoil as the function to be determined. It is shown how by integration by parts this integral equation can be transformed into the Biot-Savart theorem. A clarification is gained regarding the use of principal value definitions for the integrals which occur.

The integral equation of lifting-surface theory is used as the starting point for the establishment of a theory for the nonstationary airfoil which is a generalization of lifting-line theory for the stationary airfoil and which might be called "lifting-strip" theory. Explicit expressions are given for section lift and section moment in terms of the circulation function, which for any given wing deflection is to be determined from an integral equation which is of the type of the equation of lifting-line theory. The results obtained are for airfoils of uniform chord. They can be extended to tapered airfoils. One of the main uses of the results should be that they furnish a practical means for the analysis of the aerodynamic span effect in the problem of wing flutter. The range of applicability of "lifting-strip" theory is the same as that of lifting-line theory so that its results may be applied to airfoils with aspect ratios as low as three.

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# INTRODUCTION

The theory of thin airfoils may be characterized by the statement that it is the linear theory, obtained through simplifications from an exact formulation of airfoil theory for a nonviscous fluid. It is based on the assumptions of infinitely thin airfoil sections, infinitesimal angle of attack, and infinitesimal camber. In spite of these restrictive assumptions it is plausible and generally accepted that the theory reproduces the characteristic behavior of actual wings with finite angle of attack and camber rather accurately in many practical problems.

First contributions to the theory of thin airfoils are Prandtl's theory of the lifting line (reference 1) for the determination of the aerodynamic span effect for stationary airfoils of not too small aspect ratio, and Munk's two-dimensional theory of stationary airfoils (reference 2).

Shortly after Munk's work investigations were made on the two-dimensional linear theory of nonstationary motion independently by Birnbaum (reference 3) and by Wagner (reference 4). Birnbaum reduces the problem to an integral equation by means of theorems on vortex motion while Wagner obtains an integral equation by way of formulating the boundary problem for the velocity potential. Birnbaum's formulation (in common with the formulation of more general problems by means of the "acceleration potential" presently to be discussed) possesses the disadvantage of leading to an integral equation with a considerably more complicated kernel than in the velocity potential formulation of the same problem. Solutions of this integral equation were obtained by Birnbaum by means of numerical methods restricted, however, to a far too small range of values of the important "reduced frequency" parameter which was introduced by him. In Wagner's work attention is focussed on transient problems for which the integral equation of the problem is solved by series developments. Both authors deal only with the motion of a rigid straight-line profile.

Subsequently Glauert (reference 5), on the basis of Wagner's work, obtains an explicit solution for the case of a simple harmonic motion of a rigid profile. This solutions depends on certain definite integrals which are functions of the reduced frequency parameter. The definite integrals are evaluated numerically in Glauert's paper for still too small a range of the reduced frequency parameter.

In 1935 and 1936 Theodorsen (reference 6), Cicala reference 7), Ellenberger, and Von Borbely (references 8 and 9), and Küssner (reference 10), independently and publishing in this chronological order, gave the solution of the two-dimensional problem for arbitrary motion and deformation of the airfoil and obtained explicit expressions for air forces and moments. They also found that the definite integrals occurring in Glauert's special and their own general solution could be expressed in terms of certain tabulated Bessel functions of the reduced frequency parameter.

Modified derivations of these results and applications have subsequently been published by Garrick (reference 11), Von Kármán and Sears (reference 12), Dietze (reference 13), Schwarz (reference 14), Söhngen (reference 15), and others.

Approximate solutions for the two-dimensional motion of a rigid profile in a slightly compressible fluid have been given in 1938 by Possio (reference 16).

An account of the work regarding the effect of spanwise variation of the flow (three-dimensional theory) may be subdivided into two parts.

The first part includes investigations having the purpose of improving lifting-line theory for stationary motion on the basis of various formulations of lifting-surface theory. Workers in this field have been Blenk (reference 17), Burgers (reference 18), Von Kármán (reference 19), Schlichting (reference 20), Fuchs (reference 21), Bollay (reference 22), Wieghardt (reference 23), Kinner (reference 24), and Krienes (reference 25).

The second part includes investigations having the purpose of obtaining generalizations of lifting-line theory to problems of nonstationary motion. Attempts of such generalizations on the basis of vortex-filament considerations were made independently by Cicala (references 26 and 27) and Von Borbely (reference 28). A study of the special case of an infinite airfoil with uniform chord, undergoing bending deformations varying sinusoidally in the direction of the span is due to Sears (reference 29). Results based on the theory of the acceleration potential were given by Küssner (reference 30). The case of a rigid airfoil with elliptical plan form was investigated by Jones (references 31 and 32). The author believes that all these attempts contain assumptions, basically or analytically, which may lead to considerable errors in the determination of the aerodynamic span effect. This is discussed in the body of the present paper.

Lastly, mention is made of two general formulations of thin-airfoil theory by Prandtl (reference 33) and Küssner (reference 30) whereby an integral equation is obtained for the distribution of pressure over the airfoil. Since, according to the equations of motion for a nonviscous fluid, the pressure may be thought of as the potential for the acceleration field this approach has become known as the acceleration-potential method. An important point of this method is the avoidance of the explicit introduction of the trailing surface of velocity discontinuity ("trailing vortex sheet"). It is felt, however, that nonetheless this method possesses serious disadvantages when applied to problems of nonstationary motion, compared with a method making use of the velocity potential. In this connection it is the author's opinion that the failure of Birnbaum (reference 3) to obtain a complete solution of the two-dimensional problem is largely due to the fact that a method was used by him which is identical with the two-dimensional form of the acceleration potential method.

The present paper is composed of three parts. In part I the known fundamental differential equations and boundary conditions of thin airfoil theory are rederived. On the basis of these equations the boundary problem for the velocity potential for a slightly compressible fluid is formulated. In this formulation any reference to the theory of vortex motion is avoided by means of a simple symmetry consideration.

In part II an integral equation is derived for the chordwise component of the velocity of the fluid at the airfoil, restricting attention to the case of an incompressible fluid. This integral equation has the important property that its kernel is the same for problems of stationary and nonstationary motion. In contrast to what occurs in the integral equation for the acceleration potential. It is shown how the integral equation obtained can be transformed, by integration by parts, into a form equivalent to the Biot-Savart theorem. It may be mentioned that a clarification is gained regarding the use of principal value definitions for the integrals occurring in the different forms of the integral equation of the problem.

In part III the integral equation of lifting-surface theory is used to establish a theory for the nonstationary airfoil which is a generalization of lifting-line theory for the stationary airfoil and for which the name of lifting-strip theory is proposed. This new theory, which includes the known two-dimensional theory for the nonstationary airfoil as well as three-dimensional lifting-line theory for the stationary airfoil as special cases is believed to be the first correct theory of this kind. It permits determination of the aerodynamic span effect for airfoils of not too small aspect ratio in a manner which is a combination of the known procedures in the two-dimensional theory for the nonstationary airfoil and in the three-dimensional theory for the stationary airfoil. Explicit expressions are given for section lift and section moment in terms of the circulation function which for any given wing deflection is to be determined from an integral equation which is of the type of the equation of lifting-line theory. The calculations may be extended so as to obtain an explicit expression for the aileron hinge moment.

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## LIST OF SYMBOLS

$\vec{U}$ velocity vector, giving the difference of fluid velocity and velocity of flight

$\vec{U}_\infty$ velocity of flight ("undisturbed" velocity)

$U, V, W$ velocity components

$P + P_\infty$ pressure

$P_\infty$ undisturbed pressure

$\rho + \rho_\infty$ density

$\rho_\infty$ undisturbed density

$X, Y, Z$ Cartesian coordinates

$T$ time

$F$ implicit equation of the airfoil surface: $F(X,Y,Z,T) = 0$

$H$ explicit equation of the airfoil surface: $Z = H(X,Y,T)$

$a$ velocity of sound in undisturbed flow

$R_a$ region occupied by projection of airfoil surface on $X,Y$ – plane

$X_t$ X–coordinate of trailing edge of airfoil region

$x, y, z$ dimensionless Cartesian coordinates, $x = X/b$, $y = Y/b$, $z = Z/b$

$b$ reference length, for airfoils with uniform chord identified with the semi–chord

$t$ dimensionless time, $t = \omega T$

$\omega$ reference frequency

$h$ dimensionless explicit equation of airfoil surface, $h = H/b$

$\vec{u}$ dimensionless velocity vector, $\vec{u} = \vec{U}/|\vec{U}_\infty|$

$p$ dimensionless pressure, $p = P/\frac{1}{2}\rho_\infty U_\infty^2$

$k$ "reduced frequency" parameter, $k = b\omega/U_\infty$

$\beta$ Mach's number of the undisturbed flow, $\beta = U_\infty/a$

| | |
|---|---|
| $x_t$ | x-coordinate of trailing edge of airfoil region |
| $l\,dy$ | dimensionless air force associated with strip $dY$ |
| $m\,dy$ | dimensionless moment of air force associated with strip $dY$ |
| $L\,dY$ | air force associated with strip $dY$, $L = (2\rho_\infty U_\infty^2 b)l$ |
| $M\,dY$ | moment of air force associated with strip $dY$, $M = (2\rho_\infty U_\infty^2 b^2)\,m$ |
| $\Phi$ | velocity potential |
| $h_k$ | amplitude function defined by $h = e^{it} h_k$ |
| $\phi_k$ | defined by $\phi = e^{it} \phi_k$ |
| $p_k$ | defined by $p = e^{it} p_k$ |
| $x_l$ | x-coordinate of leading edge of airfoil region |
| $R_w$ | wake region, being the semi-infinite strip in the x,y-plane extending from the trailing edge in the direction of the main flow |
| $R_r$ | region of x,y-plane which is not part of airfoil and wake region |
| $\Gamma$ | circulation function defined by $\frac{1}{2}\Gamma = \int_{x_l}^{x_t} \frac{\partial\phi(x,y,+o)}{\partial x}\,dx$ |
| $u_o e^{it}$ | chordwise fluid velocity component at airfoil, $u_o = \frac{\partial\phi_{lk}(x,y,+o)}{\partial x}$ |
| $\Gamma_k$ | defined by $\Gamma = e^{it}\,\Gamma_k$ |
| $w_o e^{it}$ | normal fluid velocity component at airfoil, $w_o = \left(\frac{\partial\phi_k(x,y,z)}{\partial z}\right)_{z=o}$ |

| | |
|---|---|
| $I_n$ | a symbol designating various integrals in the course of evaluation or transformation |
| $l_k$ | defined by $l = e^{it}\, l_k$ |
| $m_k$ | defined by $m = e^{it}\, m_k$ |
| s | ratio of semi-span and semi-chord for airfoil of rectangular plan form |
| $y^*$ | dimensionless coordinate defined by $y^* = y/s = Y/sb$ |
| ~ | a sign designating quantities of the two-dimensional (section-force) theory |
| S | a function defined by equation (133) |
| Q | a function defined by equation (135) |
| $K_0, K_1$ | modified Bessel functions of the second kind |
| $J_0, J_1, J_2$ | Bessel functions of the first kind |

# I.– THE BOUNDARY PROBLEM FOR THE VELOCITY POTENTIAL IN NONSTATIONARY THIN-AIRFOIL THEORY

## FORMULATION OF THE PROBLEM

Neglecting finite thickness and finite angle-of-attack effects, thin-airfoil theory treats wings as almost flat plates, possessing no thickness. Flow of an ideal compressible fluid is assumed, the uniform velocity of which is disturbed by the presence of the airfoil, which is only slightly inclined against the direction of the undisturbed velocity. The velocity change $\vec{U}$ caused by the presence of the airfoil is considered small compared with the undisturbed velocity $\vec{U}_\infty$ and the changes in density and pressure, $\rho$ and $P$ are considered small compared with the undisturbed density and pressure $\rho_\infty$ and $P_\infty$. On the basis of these assumptions a linearized form of the problem is obtained.

Before linearization the differential equations of the problem are the Euler equation

$$\frac{\partial(\vec{U}+\vec{U}_\infty)}{\partial T} + (\vec{U}+\vec{U}_\infty)\cdot \text{grad } (\vec{U}+\vec{U}_\infty) = -\frac{\text{grad } (P+P_\infty)}{\rho+\rho_\infty} \quad (1)$$

and the equation of continuity

$$\frac{\partial(\rho+\rho_\infty)}{\partial T} + \text{div } [(\rho+\rho_\infty)\ (\vec{U}+\vec{U}_\infty)] = 0 \quad (2)$$

As boundary condition it is prescribed that on the surface of the airfoil the normal component of the fluid velocity equals the normal component of the velocity of the element of the airfoil with which it is in contact. If $F(X,Y,Z,T) = 0$ is the equation of the surface representing the airfoil the boundary condition has the form,

$$F(X,Y,Z,T) = 0; \quad \frac{\partial F}{\partial T} + (\vec{U}+\vec{U}_\infty)\cdot \text{grad } F = 0 \quad (3)$$

Formulation of the conditions at infinity and of the conditions along the edge of the airfoil surface is postponed until after the problem has been linearized.[1]

The undisturbed velocity $\vec{U}_\infty$ may be taken as parallel to the X-axis, so that

$$\vec{U}_\infty = U_\infty \vec{i} \tag{4}$$

where $\vec{i}$ represents a unit vector in the X-direction.

The airfoil surface is assumed to lie very nearly in the X,Y-plane (fig. 1) and its equation may be written as

$$F \equiv Z - H(X,Y,T) = 0 \tag{5}$$

The assumption of small disturbances is equivalent to the following order of magnitude relations, for the velocity changes,

$$|\vec{U}| \ll U_\infty, \qquad \frac{\partial H}{\partial T} \ll U_\infty \tag{6}$$

For the slope of the wing surface,

$$\frac{\partial H}{\partial X} \ll 1, \qquad \frac{\partial H}{\partial Y} \ll 1 \tag{7}$$

The condition of small density change is

$$\rho \ll \rho_\infty \tag{8}$$

permitting the linear pressure change density change relation

$$\rho = \frac{1}{a^2} P \tag{9}$$

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[1]The stationary problem $(\partial/\partial T \equiv 0)$ in its general form has been discussed by R. von Mises in a paper given before the American Mathematical Society in April 1942.

where

$$a^2 = \left(\frac{dP}{d\rho}\right)_\infty \qquad (10)$$

is the square of the velocity of sound of the undisturbed flow.

Introducing equations (6) to (10) into the differential equations (1) and (2) and neglecting terms small of higher order leads to the following linear equations,

$$\frac{\partial \vec{U}}{\partial T} + \vec{U}_\infty \cdot \text{grad } \vec{U} = -\frac{1}{\rho_\infty} \text{ grad } P \qquad (11)$$

$$\frac{\partial \rho}{\partial T} + \rho_\infty \text{ div } \vec{U} + \text{div } (\rho \vec{U}_\infty) = 0 \qquad (12)$$

while the boundary condition (3) becomes with $\text{grad } Z = \vec{k}$ where $\vec{k}$ stands for a unit vector in the Z-direction,

$$-\frac{\partial H}{\partial T} + \vec{U} \cdot \vec{k} - \vec{U}_\infty \cdot \text{grad } H = 0 \qquad (13)$$

In view of the fact that the airfoil surface lies very nearly in the X,Y - plane it is permissible to satisfy this condition instead of at the surface itself at the projection of the surface onto the X,Y-plane. Denoting by $R_a$ the region occupied by the projection of the airfoil surface, which will henceforth be called the airfoil region, and introducing for grad H its value $\vec{i}\,\frac{\partial H}{\partial X} + \vec{j}\,\frac{\partial H}{\partial Y}$ and writing W for the Z-component of the velocity, the boundary condition takes on its final form

$$Z = 0; \quad X,Y \text{ in } R_a; \quad W = \frac{\partial H}{\partial T} + U_\infty \frac{\partial H}{\partial X} \qquad (14)$$

The region $R_a$ is to be considered as the limit of closed surfaces surrounding $R_a$, the boundary condition (14) holding on both sides, $Z = \pm 0$.

Concerning the shape of the region $R_a$ the restriction is made that straight lines having the direction of $U_\infty$

intersect its boundary either in two points, are tangent to it, or are entirely outside.

In addition to the differential equations (11) and (12) and the boundary condition (14) the fundamental condition is imposed that along the trailing edge of the airfoil the velocity remains finite (Kutta-Joukowsky condition)

$$X_t (Y) = 0; \quad \vec{U} \text{ finite} \tag{15}$$

It should be remembered that this condition is motivated in the following way. Experiments show that for airfoils of finite thickness with rounded leading edge and sharp trailing edge the effect of viscosity manifests itself in such a way that the flow pattern is quickly developed into very nearly the same as that for an ideal fluid with the condition imposed that the velocity remains finite at the trailing edge, which is the only place where it could mathematically become infinite. Conditions for the existence of this close connection between viscous and ideal fluid flow theory involve limits on the thickness-chord ratio of the airfoil sections and on the magnitude and direction of the undisturbed velocity $\vec{U}_\infty$ which are satisfied for conventional airfoils with angle of attack below the stalling angle and velocity of flight not too close to the velocity of sound.

No condition is imposed in thin-airfoil theory on the velocity at the leading edge. A sharp leading edge is considered as the limit of rounded edges for which the velocity at the leading edge becomes, in general, infinite. To the extent that this happens, linearized thin-airfoil theory must be considered inconsistent. The excuse for permitting such inconsistency is furnished by the fact that it is restricted to a zone of small width adjacent to the leading edge. This makes plausible that the effect is insignificant so far as it concerns the calculation of the resultant forces and moments which the flowing fluid exerts on the airfoil. (In the two-dimensional stationary theory this has been confirmed by comparing the results of the linear theory with the known results of exact theory.)

Conditions at infinity are undisturbed flow far in front of the airfoil - that is,

$$X = -\infty; \quad \vec{U} = 0, \quad P = 0 \tag{16}$$

while, as regards conditions far behind the airfoil ($X=+\infty$)

the omission of the viscous terms in the Euler equations is, in general, responsible for a persistence of the disturbing influence of the airfoil.

## DIMENSIONLESS FORM OF THE EQUATIONS OF THE PROBLEM

Before proceeding further it is convenient to make the system of equations (11) to (16) dimensionless. Dimensionless coordinates are introduced by putting

$$x = \frac{X}{b}, \quad y = \frac{Y}{b}, \quad z = \frac{Z}{b} \tag{17}$$

where b is a reference length which in the two-dimensional theory will be identified with the semi-chord of the airfoil.

Dimensionless time is introduced by putting

$$t = \omega T \tag{18}$$

where $\omega$ is a reference frequency which in the case of harmonic oscillations will be identified with the frequency of oscillation.

A dimensionless camber surface equation is introduced by putting

$$h = \frac{H}{b} \tag{19}$$

Dimensionless velocity and pressure changes are introduced by putting

$$\vec{u} = \frac{\vec{U}}{U_\infty} \tag{20}$$

$$p = \frac{P}{\frac{1}{2}\rho_\infty {U_\infty}^2} \tag{21}$$

When equations (17) to (21) are introduced into the differential equations (11) and (12) and into the boundary conditions (14), it is found that it is convenient to define the following two dimensionless parameters,

$$k = \frac{\omega b}{U_\infty} \qquad (22)$$

$$\beta = \frac{U_\infty}{a} \qquad (23)$$

The "reduced frequency" parameter $k$ is of basic importance in the theory of nonstationary motion. The parameter $\beta$ represents Mach's number of the undisturbed flow, the ratio of the velocity of the undisturbed flow and of the velocity of sound in the undisturbed flow.

The differential equations and boundary conditions have now the following form, if use is made of the relation, $\rho/\rho_\infty = \frac{1}{2} p\beta^2$

$$k \frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{u}}{\partial x} = -\frac{1}{2} \operatorname{grad} p \qquad (24)$$

$$\frac{1}{2} \beta^2 \left(k \frac{\partial p}{\partial t} + \frac{\partial p}{\partial x}\right) + \operatorname{div} \vec{u} = 0 \qquad (25)$$

$$z = 0;\ x,y \text{ in } R_a; \quad w = k \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} \qquad (26)$$

$$z = 0; \quad x = x_t(y); \quad \vec{u} \text{ finite} \qquad (27)$$

$$x = -\infty; \quad \vec{u} = 0, \quad p = 0 \qquad (28)$$

The analytical part of thin-airfoil theory consists in solving this system of equations (24) to (28) so that expressions may be obtained for forces and moments exerted by the fluid on specified portions of the airfoil. These forces and moments manifest themselves through discontinuities of pressure within the airfoil region $R_a$. A convenient notation is to write $p_u$ for the pressure on the side $z = +0$ of $R_a$ and $p_l$ on the side $z = -0$ of $R_a$.

Denoting by $l dy$ and $m dy$ dimensionless forms of forces and moments associated with strips $dy$ of the airfoil, a convenient way of writing forces and moments per unit of span length is

$$l\ (y;\ x_1,\ x_2) = \frac{1}{4}\int_{x_1}^{x_2} [p_l\ (x,y) - p_u(x,y)]\ dx \qquad (29)$$

$$m(y; x_0, x_1, x_2) = \frac{1}{4}\int_{x_1}^{x_2} (x - x_0)\ [p_l(x,y) - p_u(x,y)]\ dx \quad (30)$$

where from now on the variables $y$, $x_1$, $x_2$ in $l$ and $m$ will not be written unless necessary to avoid ambiguities.

The relations between dimensionless forces and moments and the corresponding actual forces and moments $L$ and $M$ are established by means of equations (21) and (17),

$$L = (2\rho_\infty U_\infty^2 b)l \qquad (31)$$

$$M = (2\rho_\infty U_\infty^2 b^2)m \qquad (32)$$

## REDUCTION OF THE BOUNDARY PROBLEM TO A PROBLEM FOR THE HALF SPACE

So far the problem has been formulated as boundary problem for the exterior of an infinitely thin closed surface. It may be observed, however, that in equations (24) to (28) of the problem no distinction occurs between the portion of the fluid "above" the airfoil surface and the portion of the fluid "below" the airfoil surface. This indicates that the flow must possess properties of symmetry and antisymmetry with respect to the x,y-plane, in the sense that the components of the velocity vector $\vec{u}$ and the pressure $p$ are either even or odd functions of the z-coordinate. The boundary equation (26) which is to be understood as holding for $z = \pm 0$ indicates that the velocity component $w$ is an even function of $z$. From the z-component equation of equation (24) it follows then that also $\partial p/\partial z$ is an even function of $z$ and consequently $p$ an odd function of $z$. From the remaining component equations of equation (24) it follows then that also the velocity components $u$ and $v$ are odd functions of $z$.

The fact that the pressure $p$ is an odd function of $z$ in conjunction with the fact that $p$ is continuous within the fluid indicates that $p$ vanishes over the part of the x,y-plane outside the region $R_a$.

$$z = 0; \quad x, y \text{ outside } R_a: \; p = 0 \tag{33}$$

Combining equation (33) with equations (26) to (28) there is obtained a system of conditions for the entire x,y-plane which may then be considered as the boundary of one of the half spaces, say the half space $z > 0$.

## INTRODUCTION OF VELOCITY POTENTIAL

Further treatment of the problem is carried out here in terms of a velocity potential $\phi$, the existence of which follows from the fact that equation (24) implies

$$\left(k \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \text{rot } \vec{u} = 0 \tag{34}$$

Since $\text{rot } \vec{u} = 0$ in the region of undisturbed flow, it follows from equation (34) that throughout, the interior of the half spaces $z > 0$ and $z < 0$,

$$\text{rot } \vec{u} = 0 \tag{35}$$

From this it is concluded

$$\vec{u} = \text{grad } \phi \tag{36}$$

Introducing equation (36) into the continuity equation (25) leads to

$$\nabla^2 \phi + \frac{1}{2} \beta^2 \left(k \frac{\partial p}{\partial t} + \frac{\partial p}{\partial x}\right) = 0 \tag{37}$$

Introducing equation (36) into the equation of motion (24) leads to

$$\text{grad} \left(k \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\right) = -\frac{1}{2} \text{ grad } p \tag{38}$$

which is equivalent to

$$-\frac{1}{2} p = k \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} \tag{39}$$

with an additive arbitrary function of $t$ incorporated in $\phi$.

Introducing equation (39) into equation (37) furnishes the differential equation for $\phi$,

$$\nabla^2 \phi - \beta^2 \left(k \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 \phi = 0 \tag{40}$$

In terms of $\phi$ the boundary conditions (26) to (28) and (33) assume the following form:

$$z = 0; \; x,y \text{ inside } R_a: \; \frac{\partial \phi}{\partial z} = k \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} \tag{41}$$

$$z = 0; \quad x = x_t(y): \; \text{grad } \phi \text{ finite} \tag{42}$$

$$z = 0; \quad x = -\infty : \; \phi = 0 \tag{43}$$

$$z = 0; \quad x,y \text{ outside } R_a: \; k \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} = 0 \tag{44}$$

To simplify the work from here on the case of simple harmonic motion is assumed by putting

$$h(x,y,t) = h_k(x,y)\, e^{it} \tag{45}$$

Because of the linearity of the problem more general solutions may be obtained from the solution for this case by means of superposition, for instance, solutions for transient problems through the means of Fourier-Laplace integrals. The functions $h_k$ may be complex themselves so as to account for phase differences between different $h_k$.

With $h$ given by equation (45a) the potential $\phi$ will be of the form

$$\phi = \phi_k (x,y,z) e^{it} \qquad (46)$$

where again $\phi_k$ may be a complex function.

Substituting $\phi$ of equation (46) in equation (40)

$$\nabla^2 \phi_k - \beta^2 \left(i k + \frac{\partial}{\partial x}\right)^2 \phi_k = 0 \qquad (47)$$

Substituting $\phi$ of equation (46) in equation (39),

$$-\frac{1}{2} p = -\frac{1}{2} p_k e^{it} = \left(i k \phi_k + \frac{\partial \phi_k}{\partial x}\right) e^{it} \qquad (48)$$

The boundary conditions (41) to (44) become

$$z = 0; \; x,y \text{ inside } R_a: \; \frac{\partial \phi_k}{\partial z} = ikh_k + \frac{\partial h_k}{\partial x} \qquad (49)$$

$$z = 0; \quad x = x_t(y): \text{ grad } \phi_k \text{ finite} \qquad (50)$$

$$z = 0; \quad x = -\infty: \quad \phi_k = 0 \qquad (51)$$

$$z = 0; \quad x,y \text{ outside } R_a: \; ik\phi_k + \frac{\partial \phi_k}{\partial x} = 0 \qquad (52)$$

It is important to note that equation (52) may be integrated to

$$\phi_k (x,y,0) = c (y) e^{-ikx} \qquad (53)$$

when $x$ and $y$ are outside the region $R_a$. Using equation (51) it is seen that $c$ vanishes along lines $y =$ constant which do not pass through the region $R_a$ and that c vanishes along the part of lines $y = \text{constant}$ passing through $R_a$ reaching from $x = -\infty$ to the leading edge coordinate $x_l$. Hence if in addition to the airfoil region $R_a$ there is defined (1) a wake region $R_w$ being the semi-infinite strip extending from the trailing edge in the direction of the main flow and (2) a region $R_r$ denoting the remaining part of the x,y-plane, the boundary condition (52) may be formulated in the following form (see also fig. 2)

$$z = 0;\ x,y \ \text{in}\ R_r:\ \phi_k = 0 \tag{54}$$

$$z = 0;\ x,y \ \text{in}\ R_w:\ \phi_k = \frac{1}{2}\Gamma_k(y)e^{-ik[x-x_t(y)]} \tag{55}$$

where $\frac{1}{2}\Gamma_k$ stands for the value of $\phi_k$ at the trailing edge

$$\frac{1}{2}\Gamma_k = \phi_k[x_t, y, +0] = \int_{x_l}^{x_t} \frac{\partial \phi_k(x, y, +0)}{\partial x} dx \tag{56}$$

and where, since $\phi_k(x,y,+0) = -\phi(x,y,-0)$, $\Gamma = \Gamma_k e^{it}$ is the value of what is known as the circulation.

The problem is now to determine the solution of the differential equation (47) for the half space $z > 0$ subject to the boundary conditions (49), (50), (54), and (55). The notable feature of this mixed boundary problem is that it contains the undetermined function $\Gamma_k$ which has to be found in such a form that the finiteness condition (50) is satisfied.

## II.- THE INTEGRAL EQUATION OF LIFTING-SURFACE THEORY FOR AN INCOMPRESSIBLE FLUID

Equation (47) indicates that for an incompressible fluid $\phi_k$ and also $\partial\phi_k/\partial x$ are harmonic functions. The function $\partial\phi_k/\partial x$ may be represented in the interior of the half space $z > 0$ by means of its values at the boundary $z = 0$ which may be denoted by $u_o$. Dispensing in what follows with the subscript $k$ this representation is

$$\frac{\partial\phi}{\partial x} = \frac{1}{2\pi} \iint \frac{u_o(\xi, \eta)\ z\ d\xi\ d\eta}{\left\{(x-\xi)^2 + (y-\eta)^2 + z^2\right\}^{3/2}} \tag{57}$$

The kernel

$$K = \frac{1}{2\pi} \frac{z}{\left\{(x-\xi)^2 + (y-\eta)^2 + z^2\right\}^{3/2}} \tag{58}$$

may be obtained either by way of introducing spherical coordinates about the point $(\xi, \eta, 0)$ in the differential equation for $\partial\phi/\partial x$ whence a solution possessing the proper singularity is found by separation of variables, or by means of a Fourier integral solution for $\partial\phi/\partial x$ for the case that $u_o$ vanishes everywhere except over an infinitesimal area $d\xi d\eta$ where it has the value $1/d\xi d\eta$.

To obtain on the left hand side of equation (57) the correct boundary values $u_o$ when $z = 0$ the integral must be defined properly. One such definition is obtained in the following way. Write equation (57) in the form

$$\frac{\partial\phi}{\partial x} = \iint_{R_1} + \iint_{R_2} \tag{59}$$

where $R_1$ is a small rectangular region surrounding the point $\xi = x$, $\eta = y$, and $R_2$ is the remaining region of integration. It is found that the correct boundary value $u_o$ is obtained from equation (59) when the sides of $R_1$

and the value of $z$ are simultaneously made small in such a way that the value of $z$ tends more rapidly to zero than the smaller of the sides of the rectangle $R_1$. This result holds independently of the location of the point $x,y$ within $R_1$ so that no reason to prefer the principal value of the integral — which corresponds to symmetrical location of $x,y$ within $R_1$ — appears at this stage. The motivation for working with principal values appears at a later stage.

The integral equation for $u_o$ is obtained by means of the condition that over part of the boundary the values of the normal velocity $\partial\phi/\partial z$ are prescribed. The value of $\partial\phi/\partial z$ is determined by first taking

$$\frac{\partial^2\phi}{\partial x\partial z} = \frac{1}{2\pi}\int\int u_o(\xi,\eta)\,\frac{\partial}{\partial z}\left(\frac{z}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}\right)d\xi\,d\eta \quad (60)$$

where it is legitimate to differentiate inside the sign of integration as long as $z > 0$. The value of $\partial\phi/\partial z$ is obtained by integrating equation (60) with respect to $x$. This integration is facilitated if use is made of the following identity,

$$\frac{\partial}{\partial z}\left(\frac{z}{r}\right) + \frac{\partial}{\partial x}\left(\frac{x-\xi}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y-\eta}{r}\right) = 0, \quad r = \left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2} \quad (61)$$

Substituting equation (61) in equation (60) and integrating from $x = -\infty$, where $\partial\phi/\partial z = 0$, there follows

$$\frac{\partial\phi}{\partial z} = -\frac{1}{2\pi}\int\int u_o\left\{\frac{x'-\xi}{\left\{(x'-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}\Bigg|_{-\infty}^{x}\right.$$

$$\left. + \frac{\partial}{\partial y}\left[\int_{-\infty}^{x}\frac{(y-\eta)\,dx'}{\left\{(x'-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}\right]\right\}d\xi\,d\eta$$

and after evaluation of the inner integral,

$$\frac{\partial\phi}{\partial z} = -\frac{1}{2\pi}\int\int u_o\left\{\frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}\right.$$

$$\left.+\frac{\partial}{\partial y}\left[\frac{(y-\eta)\ (x-\xi)}{\left[(y-\eta)^2+z^2\right]\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{\frac{1}{2}}}+\frac{y-\eta}{(y-\eta)^2+z^2}\right]\right\}d\xi d\eta \quad (62)$$

In this integral the region of integration consists of the airfoil region $R_a$ and the wake region $R_w$ only, as equation (54) indicates that $u_o$ vanishes in $R_r$.

To obtain the integral equation for $u_o$ the boundary conditions (49), (54), and (55) have to be used and the limit of the integral (62) as $z$ tends to zero has to be taken. In principle the simplest procedure would be to evaluate the integral first for non-vanishing $z$ and then make $z = 0$ in the integrated expression. However, inasmuch as the integrand in equation (62) becomes of simpler form when in it $z$ is made equal to zero it is desirable to define the process of integration in equation (62), or in an equation derived from it, in such a way that the correct result is obtained if first $z$ is made equal to zero and then the integration is carried out. This, as will be shown, is possible without transformation as regards the first term in equation (62) while the second term has to be brought into a different form, in order to avoid singularities of too high order.

In this transformation two different cases are distinguished: (1) the case for which the leading edge is straight and perpendicular to the direction of the mainflow, (2) the case where this is not so. In the first case the second term may be integrated by parts with respect to $\eta$ and since in view of equation (54) $u_o$ vanishes at both ends of the $\eta$-integration interval there follows from equation (62)

$$\frac{\partial\phi}{\partial z} = -\frac{1}{2\pi}\iint_{R_a+R_w}\left\{u_o \frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}\right.$$

$$\left. + \frac{\partial u_o}{\partial \eta}\frac{y-\eta}{(y-\eta)^2+z^2}\left[\frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{\frac{1}{2}}} \pm 1\right]\right\} d\xi\, d\eta \quad (63)$$

In the second case the transformation of equation (62) involves first integration by parts with respect to $\xi$,

$$\int_{x_1}^{\infty} u_o \frac{\partial}{\partial y}\left[\ldots\right] d\xi = \int_{x_1}^{\infty}\frac{\partial\phi_o}{\partial \xi}\frac{\partial}{\partial y}\left[\ldots\right] d\xi$$

$$= \left\{\phi_o \frac{\partial}{\partial y}\left[\ldots\right]\right\}_{x_1}^{\infty} - \int_{x_1}^{\infty}\phi_o \frac{\partial^2}{\partial\xi\,\partial y}\left[\ldots\right] d\xi$$

From equation (54) follows that in the integrated part $\phi_o$ vanishes when $\xi = x_1$ while for $\xi = \infty$ the factor of $\phi_o$ vanishes. Thus instead of equation (62) there may be written.

$$\frac{\partial\phi}{\partial z} = -\frac{1}{2\pi}\iint\left\{u_o \frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}\right.$$

$$\left. -\phi_o \frac{\partial}{\partial y}\left[\frac{y-\eta}{(y-\eta)^2+z^2}\frac{\partial}{\partial \xi}\left(\frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{1/2}} + 1\right)\right]\right\} d\xi\, d\eta$$

In this expression the second term may be integrated by parts with respect to $\eta$ and, writing $\partial\phi_o/\partial\eta = v_o$ and observing that the integrated part vanishes at the limits, there follows

$$\frac{\partial\phi}{\partial z} = -\frac{1}{2\pi}\iint_{R_a+R_w}\left\{u_o\frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}} + v_o\frac{y-\eta}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}\right\}d\xi d\eta \qquad (64)$$

It may be noted that an equation equivalent to equation (64) could also have been obtained by applying the Biot-Savart theorem with a vortex sheet occupying the regions $R_a$ and $R_w$ in the x,y-plane.

The integral equation of the problem is now obtained by substituting either in equation (63) or in equation (64) the boundary conditions (49) and (55). There follows, from equation (63),

x,y in $R_a$ :

$$w_o(x,y) = -\frac{1}{2\pi}\lim_{z\to o}\left\{\iint_{R_a}\left[u_o\frac{(x-\xi)}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}} + \frac{\partial u_o}{\partial\eta}\frac{y-\eta}{(y-\eta)^2+z^2}\left(\frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{1/2}}+1\right)\right]d\xi d\eta\right.$$

$$-\frac{1}{2}ik\iint_{R_w}\left[\Gamma(\eta)\frac{e^{-ik[\xi-x_t(\eta)]}(x-\xi)}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}\right.$$

$$\left.\left.+\frac{e^{-ik\xi}\frac{d}{d\eta}\left[\Gamma(\eta)e^{ikx_t}\right](y-\eta)}{(y-\eta)^2+z^2}\left(\frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{\frac{1}{2}}}+1\right)\right]d\xi d\eta\right\} \qquad (65)$$

Equation (65) is valid when the leading edge of the airfoil is a straight line parallel to the y-axis. From equation (64) follows, without restriction concerning the leading edge curve,

x,y in $R_a$ :

$$w_o(x,y) = -\frac{1}{2\pi}\lim_{z\to o}\left\{\iint_{R_a}\left[u_o\frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}\right.\right.$$
$$\left. + v_o\frac{y-\eta}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}\right]d\xi d\eta$$
$$+\frac{1}{2}\iint_{R_w}\left[-ik\Gamma(\eta)\frac{e^{-ik(\xi-x_t)}(x-\xi)}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}\right.$$
$$\left.\left. +\frac{d}{d\eta}\left(\Gamma e^{ikx_t}\right)\frac{e^{-ik\xi}(y-\eta)}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}\right]d\xi d\eta\right\} \quad (66)$$

In both equations $\Gamma$ is defined, according to equation (56), by

$$\frac{1}{2}\Gamma(\eta) = \int_{x_l}^{x_t} u_o(\xi,\eta)\,d\xi \quad . \quad (67)$$

According to equation (50), the additional condition is imposed that

$$u_o(\xi_t,\eta)\text{ finite} \quad (68)$$

Equation (65) has the advantage that in it only $u_o$ occurs, but the disadvantage that its applicability is restricted to airfoils with leading edge straight and perpendicular to the direction of the main flow. Equation (66) is free of this restriction, but contains the two unknowns $u_o$ and $v_o$ (which, however, are both derivatives of the same function $\phi_o$). The developments of part III of this paper are based on equation (65) but it is certain that equivalent results can be obtained on the basis of equation (66).

So far as the integral equations (65) and (66) are concerned it is possible to put $z$ directly equal to zero in some of the wake integrals as the variable $x$ is always exterior to $R_w$. This is not the case with the integrals over the airfoil region where the integrand becomes infinite when $z = 0$, $\xi = x$, $y = \eta$. The order of this infinity is most easily recognized if cylindrical coordinates about the point x,y are introduced, that is,

$$\xi - x = \rho \cos \theta, \quad \eta - y = \rho \sin \theta, \quad d\xi d\eta = \rho d\rho d\theta$$

whence, for instance,

$$\frac{(x-\xi)\, d\xi d\eta}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}} = \frac{-\cos\theta\, \rho^2 d\rho\, d\theta}{\left\{\rho^2 + z^2\right\}^{3/2}}$$

On the basis of this formula the integrals over $R_w$ and $R_a$ could be considered as composed of the following two regions: (1) a small circle with center $\rho = 0$, (2) the region minus the small circular region. It is possible to show that $\lim\limits_{z\to 0} \iint$ has the value which would also be obtained by excluding from the region of integration the small circle, assuming in the integral over the remaining region $z$ equal to zero and by finally letting the radius of the excluded circle go to zero.

For the applications it is, however, more convenient to subdivide the region $R_a$ in a different way. This may be explained in detail for one of the integrals occurring in equations (65) and (66): namely,

$$I_o = \lim_{z \to o} \iint_{R_a} u_o \frac{(x-\xi)\, d\xi\, d\eta}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}} = \lim_{z \to o} I_z \qquad (69)$$

The integral $I_z$ may be written as

$$I_z = \iint_{R_1} + \iint_{R_2} \qquad (70)$$

where $R_1$ is a small rectangular region surrounding the point $\xi = x, y = \eta$ in the $\xi,\eta$-plane and $R_2$ the remainder of the airfoil region $R_a$. In equation (70) $R_1$ may be made so small that in it the function $u_o$ changes very little, so that

$$\iint_{R_1} = [u_o(x,y) + \delta_1] \iint_{R_1} \frac{(x-\xi)\, d\xi\, d\eta}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}} \qquad (71)$$

With $R_1$ thus determined make $z$ so small that

$$\iint_{R_2} \frac{u_o(\xi,\eta)\ (x-\xi)\ d\xi\, d\eta}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}} = \iint_{R_2} \frac{u_o(\xi,\eta)(x-\xi)\, d\xi\, d\eta}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}} + \delta_2 \qquad (72)$$

Thus

$$I_z = [u_o(x,y) + \delta_1] \iint_{R_1} \frac{(x-\xi)\, d\xi\, d\eta}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}$$

$$+ \iint_{R_2} \frac{u_o(\xi,\eta)(x-\xi)\, d\xi\, d\eta}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}} + \delta_2 \qquad (73)$$

Since in this definition $z$ approaches zero automatically as the size of the rectangle $R_1$ shrinks, there may be written

$$I_o = \lim_{z \to o} I_z = \lim_{R_1 \to o} I_z$$

$$= u_o(x,y) \lim_{R_1 \to o} \iint_{R_1} \frac{(x-\xi)\, d\xi\, d\eta}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}$$

$$+ \lim_{R_1 \to o} \iint_{R_2} \frac{u_o(\xi,\eta)\ (x-\xi) d\xi\, d\eta}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}} \tag{74}$$

The most convenient form of $I_o$ is obtained if the shape of the rectangle $R_1$ is chosen such that the first integral in equation (74) vanishes. This is the case when the rectangle $R_1$ is symmetrical about the line $\xi = x$, since the integrand is an odd function of $\xi - x$. This is the same as saying that the gap in the second integral is symmetrical so far as the $\xi$-integration is concerned, which is equivalent to saying that the principal value of the integral is taken in carrying out the integration with respect to $\xi$. This may be indicated by writing

$$I_o = \int_{R_a}\!\!⨍ \frac{u_o(\xi,\eta)\ (x-\xi)}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}}\, d\xi\, d\eta \tag{75}$$

In the same way it can be shown that the simplest form of the remaining singular integrals in equations (65) and (66) is obtained by making the rectangle $R_1$ symmetrical about the line $y = \eta$ — that is,

$$\lim_{z \to o} \iint_{R_a} \frac{\partial u_o}{\partial \eta} \frac{y-\eta}{(y-\eta)^2+z^2} \left(\frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{1/2}} + 1\right) d\xi\, d\eta$$

$$= ⨍\!\!\int_{R_a} \frac{\partial u_o}{\partial \eta} \frac{1}{y-\eta} \left(\frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{1/2}} + 1\right) d\xi\, d\eta \tag{76a}$$

and

$$\lim_{z \to 0} \iint_{R_a} \frac{v_0(\xi,\eta)(y-\eta)\, d\xi d\eta}{\left\{(x-\xi)^2+(y-\eta)^2+z^2\right\}^{3/2}}$$

$$= \int_{R_a} \oint \frac{v_0(\xi,\eta)(y-\eta)\, d\xi d\eta}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}} \qquad (76b)$$

As in one integral gap symmetry in the $\xi$-direction and in the other integral gap symmetry in the $\eta$-direction is desirable, it is most convenient to have in both integrals gap symmetry in both directions. With this understanding the integral equations (65) and (66) may be written

$x, y$ in $R_a$ :

$$w_0 = -\frac{1}{2\pi} \oint_{R_a} \oint \left\{ \frac{u_0(\xi,\eta)\ (x-\xi)}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}} + \frac{\partial u_0}{\partial \eta} \frac{1}{y-\eta} \left[ \frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{1/2}} + 1 \right] \right\} d\xi d\eta$$

$$+ \frac{ik}{4\pi} \int_{R_w} \oint \left\{ \frac{\Gamma(\eta) e^{-ik(\xi - x_t(\eta))}\ (x-\xi)}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}} \right.$$

$$\left. + \frac{\frac{d}{d\eta}\left[\Gamma e^{ikx_t}\right] e^{-ik\xi}}{y-\eta} \left[ \frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{1/2}} + 1 \right] \right\} d\xi d\eta \qquad (77)$$

or

$x, y$ in $R_a$ :

$$w_o = -\frac{1}{2\pi}\iint_{R_a}\left\{\frac{u_o(\xi,\eta)\,(x-\xi)}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}} + \frac{v_o(\xi,\eta)\,(y-\eta)}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}}\right\}d\xi\,d\eta$$

$$-\frac{1}{4\pi}\iint_{R_w}\left\{\frac{-ik\Gamma(\eta)e^{-ik(\xi-x_t)}\,(x-\xi)}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}} + \frac{\frac{d}{d\eta}\left[\Gamma e^{ikx_t}\right]e^{-ik\xi}\,(y-\eta)}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}}\right\}d\xi\,d\eta \tag{78}$$

In equations (77) and (78) $\Gamma$ is defined, as before, by equation (67) and $u_o$ is, as before, subject to the condition of finiteness of equation (68).

In what follows equation (77) will be specialized to the following cases (leaving deductions from equation (78) for future work):

(1) The two-dimensional theory, where no new results are obtained;

(2) The stationary theory for airfoils of rectangular plan form from which in a manner similar to that used by Burgers (reference 18) there is obtained Prandtl's equation of lifting-line theory. Also obtained are expressions for the spanwise variations of total moment and aileron hinge moment which are equivalent to earlier, apparently not well known, results of Glauert.

(3) The nonstationary theory for airfoils of rectangular plan form. Here a thory is obtained for the aerodynamic span effect which is a generalization of the Prandtl theory for the stationary airfoil and is considered just as reliable for the nonstationary airfoil as lifting-line theory for the stationary airfoil.

Before proceeding with this program expressions may be obtained for lift and moments in terms of the function $u_o$, which is the function to be determined from the form of the integral equation of lifting-surface theory as given in this paper. Substituting in equations (29) and (30) the value of $p$ from equation (48) and taking into account that $p_l = - p_u$ there follows first

$$l(y;\ x_1, x_2) = \int_{x_1}^{x_2} \left(ik\phi_k + \frac{\partial\phi_k}{\partial x}\right)e^{it}\ dx \tag{79}$$

and

$$m(y; x_o, x_1, x_2) = \int_{x_1}^{x_2} (x-x_o)\left(ik\phi_k + \frac{\partial\phi_k}{\partial x}\right)e^{it}\ dx \tag{80}$$

Writing

$$l = l_k\ e^{it} \qquad m = m_k\ e^{it} \tag{81}$$

there follows

$$l_k = \int_{x_1}^{x_2} (ik\phi_k + u_o)\ dx \tag{82}$$

and

$$m_k = \int_{x_1}^{x_2} (x-x_o)(ik\phi_k + u_o)\ dx \tag{83}$$

By integration by parts equations (82) and (83) may be transformed into expressions depending only on $u_o$.

$$l_k(x_1, x_2) = ik\left\{(x_2-x_1)\int_{x_1}^{x_1} u_o dx + \int_{x_1}^{x_2}(x_2-x)u_o dx\right\} + \int_{x_1}^{x_2} u_o dx \tag{84}$$

$$m_k(x_0, x_1, x_2) = ik\left\{(x_2-x_1)\left(\frac{x_2+x_1}{2} - x_0\right)\int_{x_l}^{x_1} u_0\,dx + \int_{x_1}^{x_2}(x_2-x)\left(\frac{x_2+x}{2} - x_0\right)u_0\,dx\right\} + \int_{x_1}^{x_2}(x-x_0)\,u_0\,dx \qquad (85)$$

From equations (84) and (85) there follows in particular for

(1) <u>the section lift</u> $(x_1 = x_l,\ x_2 = x_t)$

$$l_k = ik\int_{x_l}^{x_t}(x_t-x)u_0\,dx + \int_{x_l}^{x_t} u_0\,dx \qquad (86)$$

(2) <u>the section moment</u> about $x_0$ $(x_1 = x_l,\ x_2 = x_t)$

$$m_k(x_0) = ik\int_{x_l}^{x_t}(x_t-x)\left(\frac{x_t+x}{2} - x_0\right)u_0\,dx + \int_{x_l}^{x_t}(x-x_0)u_0\,dx \qquad (87)$$

(3) <u>the aileron hinge moment</u> $(x_1 = x_0 = c,\ x_2 = x_t)$

$$m_k(c, c) = ik\left\{\frac{1}{2}(x_t-c)^2\int_{x_l}^{c} u_0\,dx + \int_{c}^{x_t}(x_t-x)\left(\frac{x_t+x}{2} - c\right)u_0\,dx\right\} + \int_{c}^{x_t}(x-c)\,u_0\,dx \qquad (88)$$

It may be noted that equations (86) and (87) coincide with expressions previously used by Glauert (reference 5)

## III.- LIFTING-STRIP THEORY FOR THE NONSTATIONARY MOTION OF AN AIRFOIL OF FINITE SPAN

In what follows a theory is developed for the airfoil of finite span subjected to nonstationary motion which may be considered as a generalization of lifting-line theory for the stationary airfoil. No use is made in this development of the vortex filament motion. The starting point of the developments is the integral equation of lifting-surface theory in the form of equation (77). In this integral equation simplifying assumptions are introduced of which it is apparent that they are of the same nature for the stationary and for the nonstationary airfoil. Thus, the range of validity of the theory put forward here coincides with the range of validity of lifting-line theory for the stationary airfoil.

While derivation of the results for the wing in nonuniform motion depends on the same order of magnitude relations regarding aspect ratios as the derivation of the results for the wing in uniform motion, the steps involved in the solution of the uniform-motion case are naturally of a much simpler nature than the steps involved in the solution of the nonuniform-motion case.

The results obtained here consist in explicit expressions giving lift and moment intensity at every section of the span for any deflection of the wing in terms of the circulation function which has to be determined from an integral equation of the nature of the lifting-line equation. If the assumption of two-dimensionality is introduced into the results they reduce exactly to the known results of the two-dimensional theory.

No expressions are as yet given for aileron hinge moments. Such expressions may, however, be obtained from the present results.

Also, airfoils of rectangular plan form only have for the time being been considered, for the sake of perspicacity. It is certain that equivalent results can be obtained for tapered airfoils.

# THE INTEGRAL EQUATION OF LIFTING-SURFACE THEORY

# FOR AN AIRFOIL OF RECTANGULAR PLAN FORM

Taking as airfoil region $R_a$ the rectangle bounded by the lines $x = \pm 1$ and $y = \pm s$, the basic integral equation (77) becomes

$|x| \leq 1, |y| \leq s$:

$$w_0 = -\frac{1}{2\pi} \oint_{-1}^{1} \oint_{-s}^{s} \left\{ \frac{u_0(\xi,\eta)(x-\xi)}{\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{3/2}} + \frac{\partial u_0}{\partial \eta} \frac{1}{y-\eta} \left[ \frac{x-\xi}{\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{1/2}} + 1 \right] \right\} d\xi d\eta$$

$$+ \frac{ike^{ik}}{4\pi} \int_{1}^{\infty} \oint_{-s}^{s} e^{-ik\xi} \left\{ \frac{\Gamma(\eta)(x-\xi)}{\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{3/2}} + \frac{\frac{d\Gamma}{d\eta}}{y-\eta} \left[ \frac{x-\xi}{\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{1/2}} + 1 \right] \right\} d\xi d\eta \qquad (89)$$

with, according to equations (67) and (68)

$$\frac{1}{2} \Gamma(\eta) = \int_{-1}^{1} u_0(\xi,\eta)\, d\xi \qquad (90)$$

$$u_0(1,\eta) \text{ finite} \qquad (91)$$

Before considering the three-dimensional problem of the nonstationary airfoil this equation is specialized for the two-dimensional case and for the three-dimensional stationary case and some results are established pertaining to these cases which it is convenient to make use of later on.

## THE INTEGRAL EQUATION OF TWO-DIMENSIONAL THEORY

The airfoil region has now the form of the infinite strip $|x| \leq 1$ and the assumption of two-dimensional flow is expressed by

$$w_0(x,y) = w_0(x), \quad u_0(\xi, \eta) = u_0(\xi), \quad \Gamma(\eta) \text{ constant} \tag{92}$$

Equation (89) reduces to

$$w_0(x) = -\frac{1}{2\pi}\int_{-1}^{1}\int_{-\infty}^{\infty} \frac{u_0(\xi)(x-\xi)}{\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{3/2}}\,d\xi\,d\eta$$

$$+ \frac{ike^{ik}}{4\pi}\Gamma\int_{1}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-ik\xi}(x-\xi)}{\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{3/2}}\,d\xi\,d\eta \tag{93}$$

The integration with respect to $\eta$ can be carried out, leaving

$$w_0(x) = -\frac{1}{\pi}\int_{-1}^{1}\frac{u_0(\xi)}{x-\xi}\,d\xi + \frac{ike^{ik}}{2\pi}\Gamma\int_{1}^{\infty}\frac{e^{-ik\xi}}{x-\xi}\,d\xi \tag{94}$$

where $\Gamma$ is given in terms of $u_0$, by equation (90) and where the finiteness condition equation (91) has to be observed. Equations (94), (90), and (91) can be solved explicitly for $u_0$, the result may be substituted in equations (86) to (88) for lift and moments to obtain the well-known Theodorsen-Cicala-Ellenberger-Küssner results. These same results will appear as special cases of the new theory taking account of the aerodynamic span effect.

## THE INTEGRAL EQUATION OF THREE-DIMENSIONAL STATIONARY THEORY

With

$$k = 0$$

equation (89) reduces to

$$w_o(x,y) = -\frac{1}{2\pi}\oint_{-1}^{1}\oint_{-s}^{s}\left\{\frac{u_o(\xi,\eta)(x-\xi)}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}} + \frac{\partial u_o}{\partial \eta}\frac{1}{y-\eta}\left[\frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{1/2}}+1\right]\right\}d\xi d\eta \qquad (95)$$

while equations (90) and (91) remain unchanged.

Approximate solutions of equation (95) for airfoils without camber for which $w_o = \alpha =$ constant have been obtained by Blenk (reference 17) and Wieghardt (reference 23) with the aim of supplementing Prandtl's lifting-line theory. The inverse problem, to obtain convenient expressions for $w_o$ when $u_o$ is given, has been dealt with, using Fourier integrals, by Von Kármán (reference 19) and Fuchs (reference 21).

It appears that the task of obtaining quantitative solutions of equation (95) giving reliable corrections for the results of lifting-line theory is of considerable difficulty and requires work going beyond what has been accomplished by Blenk and Wieghardt.

Lifting-line theory may be obtained from equation (95), substantially according to Burgers (reference 18), in the following manner.

Substitute in equation (95) as new variables

$$y^* = \frac{y}{s}, \quad \eta^* = \frac{\eta}{s} \qquad (96)$$

which changes equation (95) into,

$$w_o(x,y) = -\frac{s}{2\pi}\oint_{-1}^{1}\oint_{-1}^{1}\left\{\frac{u_o(\xi,\eta^*)(x-\xi)}{\left\{(x-\xi)^2+s^2(y^*-\eta^*)^2\right\}^{3/2}} + \frac{1}{s^2}\frac{\partial u_o}{\partial \eta^*}\frac{1}{y^*-\eta^*}\left[\frac{x-\xi}{\left\{(x-\xi)^2+s^2(y^*-\eta^*)^2\right\}^{1/2}}+1\right]\right\}d\xi d\eta^* \qquad (97)$$

For sufficiently large $s$, practically for

$$s > 3 \tag{98}$$

the terms

$$\frac{1}{\left\{(x-\xi)^2 + s^2(y^* - \eta^*)^2\right\}^n}$$

have a steep maximum for $y^* = \eta^*$ so that, roughly, the main contribution to the value of the integrals containing these terms comes from the immediate neighborhood of the line $y^* = \eta^*$. In this neighborhood the values of $u_o$ and $\frac{\partial u_o}{\partial \eta}$ are thought to change sufficiently slowly to permit replacing their actual values by their values at $y^* = \eta^*$. If this approximation is accepted equation (97) becomes

$$w_o \approx -\frac{s}{2\pi}\int_{-1}^{1}\int_{-1}^{1}\left\{\frac{u_o(\xi,y^*)(x-\xi)}{\left\{(x-\xi)^2 + s^2(y^*-\eta^*)^2\right\}^{3/2}} + \frac{1}{s^2}\frac{\partial u_o}{\partial y^*}\frac{1}{y^*-\eta^*}\frac{x-\xi}{\left\{(x-\xi)^2 + s^2(y^*-\eta^*)^2\right\}^{1/2}} + \frac{1}{s^2}\frac{\partial u_o}{\partial \eta^*}\frac{1}{y^*-\eta^*}\right\}d\xi\, d\eta^* \tag{99}$$

In the first two integrals the integration with respect to $\eta^*$ may be carried out explicitly. Because the main contribution to the value of the integrals comes from the immediate neighborhood of $y^* = \eta^*$ the error introduced by integrating from $-\infty$ to $+\infty$ instead of from $-1$ to $+1$ is neglected. (This latter approximation evidently ceases to be good in the immediate vicinity of the tip sections $y^* = \pm 1$, and could not be made were it not for the fact that $u_o$ turns out to be small near the tip sections.) With the following values of the relevant two integrals,

$$\int_{-\infty}^{\infty}\frac{d\eta^*}{\left\{(x-\xi)^2 + s^2(y^*-\eta^*)^2\right\}^{3/2}} = \frac{2}{s(x-\xi)^2}$$

$$\int_{-\infty}^{\infty}\frac{d\eta^*}{(y^*-\eta^*)\left\{(x-\xi)^2 + s^2(y^*-\eta^*)^2\right\}^{1/2}} = 0$$

the integral equation (99) is reduced to

$$w_o \approx -\frac{1}{\pi}\oint_{-1}^{1}\frac{u_o(\xi,y^*)}{x-\xi}\,d\xi - \frac{1}{2\pi s}\oint_{-1}^{1}\int_{-1}^{1}\frac{\partial u_o}{\partial \eta^*}\,\frac{d\eta^* d\xi}{y^*-\eta^*} \tag{100}$$

and if in the second term use is made of equation (90) there follows as final form of the approximate integral equation of the stationary rectangular lifting surface,

$$w_o \approx -\frac{1}{\pi}\oint_{-1}^{1}\frac{u_o(\xi,y^*)}{x-\xi}\,d\xi - \frac{1}{4\pi s}\oint_{-1}^{1}\frac{d\Gamma}{d\eta^*}\,\frac{d\eta^*}{y^*-\eta^*} \tag{101}$$

It may be seen that the second term in equation (101) gives the finite-span correction, while neglecting the second term is equivalent to assuming two-dimensional flow at every section.

To determine the functions $u_o$ and $\Gamma$ use is made of a known inversion formula which is to be considered as a result of two-dimensional potential theory. (See, for instance, Söhngen, reference 34.) The inversion formula states that to the relation

$$g(x) = -\frac{1}{\pi}\oint_{-1}^{1}\frac{f(\xi)}{x-\xi}\,d\xi, \quad f(1) \text{ finite} \tag{102}$$

there corresponds the following inverse relation expressing $f$ in terms of $g$,

$$f(x) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,\frac{g(\xi)}{x-\xi}\,d\xi \tag{103}$$

Applying equations (102) and (103) to equation (101) in order to solve for $u_o$,

$$u_o(x,y^*) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\left\{w_o(\xi,y^*) + \frac{1}{4\pi s}\oint_{-1}^{1}\frac{d\Gamma}{d\eta^*}\,\frac{d\eta^*}{y^*-\eta^*}\right\}\frac{d\xi}{x-\xi}$$

and with

$$\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{d\xi}{x-\xi} = -\pi \tag{104}$$

there follows

$$u_o(x,y^*) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\left\{\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{w_o(\xi,y^*)}{x-\xi}d\xi - \frac{1}{4s}\oint_{-1}^{1}\frac{d\Gamma}{d\eta^*}\frac{d\eta^*}{y^*-\eta^*}\right\} \tag{105}$$

Equation (105) becomes, by integration, an equation for $\Gamma$,

$$\frac{1}{2}\Gamma(y^*) = \int_{-1}^{1}u_o(x,y^*)dx = -\frac{1}{4s}\oint_{-1}^{1}\frac{d\Gamma}{d\eta^*}\frac{d\eta^*}{y^*-\eta^*} + \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\left\{\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{w_o(\xi,y^*)}{x-\xi}d\xi\right\}dx$$

It is plausible and may be justified rigorously that in the second term on the right the order of integration may be interchanged. Then with

$$\oint_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{dx}{x-\xi} = -\pi \tag{106}$$

there follows as integral equation for $\Gamma$,

$$\frac{1}{2}\Gamma(y^*) = -\frac{1}{4s}\oint_{-1}^{1}\frac{d\Gamma}{d\eta^*}\frac{d\eta^*}{y^*-\eta^*} - \int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,w_o(\xi,y^*)d\xi \tag{107}$$

Since, according to equation (86)

$$l_o = \frac{1}{2}\Gamma \tag{108}$$

it follows that equation (107), the integral equation of lifting-line theory, determines directly the lift distribution in the stationary case.

In addition to the integral equation for the lift distribution further results may be deduced from equation (105). Introducing the value of $u_0$, as given by equation (105) into equation (87) for the section moment $m_0(x_0, -1, 1)$ there follows

$$m_0(x_0, -1, 1) = \tilde{m}_0(x_0, -1, 1) - \frac{1}{4\pi s}\int_{-1}^{1}\sqrt{\frac{1 - x}{1 + x}}(x - x_0)dx \oint_{-1}^{1}\frac{d\Gamma}{y^* - \eta^*} \tag{109}$$

where $\tilde{m}_0$ stands for the value of the moment in the absence of an aerodynamic span effect. From equation (109) follows that <u>there is no aerodynamic span effect for the moment about the quarter chord point</u> $x_0 = -\frac{1}{2}$ (which vanishes for the straight line profile)- that is,

$$m_0(-\tfrac{1}{2}, -1, 1) = \tilde{m}_0(-\tfrac{1}{2}, -1, 1) \tag{110}$$

A probably more important result of this nature concerns the value of the aileron hinge moment which is defined by equation (88). Introducing equation (105) into equation (88) gives

$$m_0(c,c,1) = \tilde{m}_0(c,c,1) - \frac{1}{4\pi s}\int_{c}^{1}(x - c)\sqrt{\frac{1 - x}{1 + x}}\, dx \oint_{-1}^{1}\frac{d\Gamma}{y^* - \eta^*} \tag{111}$$

Writing

$$f(c) = \int_{c}^{1}(x - c)\sqrt{\frac{1 - x}{1 + x}}\, dx = \left(1 + \frac{c}{2}\right)\sqrt{1 - c^2} - \left(\frac{1}{2} + c\right)\cos^{-1} c \tag{112}$$

and observing equations (107) and (108), there may be written instead of equation (111)

$$\frac{m_0(c,c,1) - \tilde{m}_0(c,c,1)}{\tilde{m}_0(c,c,1)} = \frac{l_0 - \tilde{l}_0}{\tilde{l}_0}\ \frac{f(c)\tilde{l}_0}{\tilde{m}_0(c,c,1)} \tag{113}$$

Equation (113) indicates that <u>the aerodynamic span effect for the hinge moment differs from the span effect for the lift</u> and in which manner the two are related.

## THE INTEGRAL EQUATION OF LIFTING-STRIP THEORY FOR NONSTATIONARY MOTION

The starting point is the complete integral equation (89) which may be written in the form

$$w_0 = I_1 + I_2 \tag{114}$$

where $I_1$ stands for the integral over the airfoil region and $I_2$ stands for the integral over the wake region. The appropriate approximation for $I_1$ has been obtained in the preceding section on the stationary airfoil. According to equation (101)

$$I_1 \approx -\frac{1}{\pi}\oint_{-1}^{1}\frac{u_0(\xi,y)}{x-\xi}\,d\xi - \frac{1}{4\pi}\oint_{-s}^{s}\frac{d\Gamma}{d\eta}\,\frac{d\eta}{y-\eta} \tag{115}$$

It remains to obtain an approximation for $I_2$ which corresponds to that obtained for $I_1$. It will be shown that several essential steps are involved in the derivation of this approximation.

Starting with the exact expression

$$I_2 = \frac{ike^{ik}}{4\pi}\int_1^{\infty}\oint_{-s}^{s} e^{-ik\xi}\left\{-\frac{\Gamma(\eta)(x-\xi)}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{3/2}} + \frac{\Gamma'(\eta)}{y-\eta}\left(\frac{x-\xi}{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{1/2}}+1\right)\right\}d\xi\,d\eta \tag{116}$$

the first step consists in separating from $I_2$ the value $I_3$ would possess if the two-dimensional theory were correct. According to equation (94) the value of $I_2$ in the two-dimensional theory is

$$I_3 = \frac{ike^{ik}}{2\pi}\Gamma(y)\int_1^{\infty}\frac{e^{-ik\xi}}{x-\xi}\,d\xi \tag{117}$$

The second step consists in writing

$$\Gamma(y) = \frac{1}{2}\int_{-s}^{s} \Gamma'(\eta)\, \frac{|y - \eta|}{y - \eta}\, d\eta \tag{118}$$

which may readily be verified in view of the fact that $\Gamma(\pm s) = 0$.

With equations (117) and (118) $I_2$ of equation (116) may be written as

$$I_2 = I_3 + \frac{ike^{ik}}{4\pi}\int_1^\infty \int_{-s}^{s} e^{-ik\xi}\left\{\frac{\Gamma(\eta)(x - \xi)}{\left\{(x - \xi)^2 + (y - \eta)^2\right\}^{3/2}} + \frac{\Gamma'(\eta)}{y - \eta}\left(\frac{x - \xi}{\left\{(x - \xi)^2 + (y - \eta)^2\right\}^{1/2}} + 1 - \frac{|y - \eta|}{x - \xi}\right)\right\} d\xi\, d\eta \tag{119}$$

A further transformation is accomplished by the following integration by parts

$$\int_{-s}^{s} \frac{\Gamma(\eta)\,d\eta}{\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{3/2}} = \Gamma(\eta)\, \frac{\eta - y}{(x-\xi)^2\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{1/2}}\Bigg/_{-s}^{s} - \int_{-s}^{s} \frac{\Gamma'(\eta)(\eta - y)\,d\eta}{(x - \xi)^2\left\{(x - \xi)^2 + (y - \eta)^2\right\}^{1/2}} \tag{120}$$

The integrated part of equation (120) vanishes as $\Gamma$ vanishes at both limits. Introducing equation (120) into equation (119), there follows

$$I_2 = I_3 + \frac{ike^{ik}}{4\pi}\int_1^\infty \int_{-s}^{s} e^{-ik\xi}\,\Gamma'(\eta)\left\{\frac{y - \eta}{(x - \xi)\left\{(x - \xi)^2 + (y - \eta)^2\right\}^{1/2}} + \frac{1}{y - \eta}\left(\frac{x - \xi}{\left\{(x - \xi)^2 + (y - \eta)^2\right\}^{1/2}} + 1 - \frac{|y - \eta|}{x - \xi}\right)\right\} d\xi\, d\eta \tag{121}$$

and combining the first two terms within the braces,

$$I_2 = I_3 + \frac{ike^{ik}}{4\pi}\int_1^{\infty}\oint_{-s}^{s}\frac{e^{-ik\xi}\Gamma'(\eta)}{y-\eta}\left\{1 - \frac{|y-\eta|}{x-\xi} + \frac{\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{1/2}}{x-\xi}\right\}d\xi\, d\eta \tag{122}$$

The next step consists in separating in equation (122) the integration with respect to $\xi$ in two parts as follows:

$$\int_1^{\infty} = \int_x^{\infty} - \int_x^{1}$$

Then

$$I_2 = I_3 + I_4 + I_5 \tag{123}$$

where

$$I_4 = \frac{ike^{ik}}{4\pi}\int_x^{\infty}\oint_{-s}^{s}\frac{e^{-ik\xi}\Gamma'(\eta)}{y-\eta}\left\{1 - \frac{|y-\eta|}{x-\xi} + \frac{\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{1/2}}{x-\xi}\right\}d\xi\, d\eta \tag{124}$$

$$I_5 = -\frac{ike^{ik}}{4\pi}\int_x^{1}\oint_{-s}^{s}\frac{e^{-ik\xi}\Gamma'(\eta)}{y-\eta}\left\{1 - \frac{|y-\eta|}{x-\xi} + \frac{\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{1/2}}{x-\xi}\right\}d\xi\, d\eta \tag{125}$$

Equation (124) may be simplified by introducing a new variable of integration

$$k(\xi - x) = \lambda, \quad k\,d\xi = d\lambda \tag{126}$$

whence

$$I_4 = \frac{ie^{ik(1-x)}}{4\pi}\oint_{-s}^{s}\frac{\Gamma'(\eta)}{y-\eta}\left\{\int_0^{\infty} e^{-i\lambda}\left[1 + \frac{k|y-\eta|}{\lambda} - \frac{\left\{\lambda^2 + k^2(y-\eta)^2\right\}^{1/2}}{\lambda}\right]d\lambda\right\}d\eta \tag{127}$$

So far, no approximations have been made in the treatment of the wake integral. The place where they are made is in the remaining term $I_5$. This is possible owing to the fact that the $\xi$-region of integration, $(x,1)$, in $I_5$ is always smaller than the chord of the airfoil and that part of the factor multiplying $e^{-ik\xi}\Gamma'(\eta)$ in the integrand is, as was the case previously in the integrals extended over the airfoil region, of appreciable magnitude in the neighborhood of the line $\eta = y$ only. The terms in question are

$$\frac{\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{1/2} - |y-\eta|}{(x-\xi)(y-\eta)} \approx \frac{1}{2}\,\frac{x-\xi}{(y-\eta)|y-\eta|} \tag{128}$$

and the $\approx$ sign holds as soon as $|y - \eta|$ is somewhat larger than $|x - \xi|$ which, considering the assumption regarding aspect ratios as expressed by equation (98) means over most of the span. With this observation the approximation for $I_5$ is

$$I_5 \approx -\frac{ike^{ik}}{4\pi}\int_x^1 \oint_{-s}^{s} e^{-ik\xi}\,\frac{\Gamma'(\eta)}{y-\eta}\,d\eta\,d\xi$$

$$-\frac{ike^{ik}}{4\pi}\,\Gamma'(y)\int_x^1 \oint_{-\infty}^{\infty} \frac{e^{-ik\xi}}{y-\eta}\left\{\frac{\left\{(x-\xi)^2+(y-\eta)^2\right\}^{1/2}}{x-\xi} - \frac{|y-\eta|}{x-\xi}\right\}d\eta\,d\xi \tag{129}$$

In the first term of equation (129) the integration with respect to $\xi$ may be carried out, the second term cancels because the integrand is an odd function of $y - \eta$. Hence

$$I_5 \approx \frac{1 - e^{ik(1-x)}}{4\pi}\oint_{-s}^{s} \frac{\Gamma'(\eta)}{y-\eta}\,d\eta \tag{130}$$

Introducing $I_5$ from equation (130), $I_4$ from equation (127) and $I_3$ from equation (117) into equation (123) there follows

$$I_2 \approx \frac{ike^{ik}}{2\pi}\,\Gamma(y)\int_1^{\infty} \frac{e^{-ik\xi}}{x-\xi}\,d\xi + \frac{ie^{ik(1-x)}}{4\pi}\oint_{-s}^{s}\frac{\Gamma'(\eta)}{y-\eta}\left\{\int_0^{\infty} e^{-i\lambda}\left[1 + \frac{k|y-\eta|}{\lambda} - \frac{\left\{\lambda^2 + k^2(y-\eta)^2\right\}^{1/2}}{\lambda}\right]d\lambda\right\}d\eta + \frac{1-e^{ik(1-x)}}{4\pi}\oint_{-s}^{s}\frac{\Gamma'(\eta)}{y-\eta}\,d\eta \tag{131}$$

Introducing $I_2$ from equation (131) and $I_1$ from equation (115) into equation (114) there follows as approximate integral equation of lifting-surface theory for the rectangular nonstationary airfoil

$$w_0 \approx -\frac{1}{\pi}\oint_{-1}^{1}\frac{u_0(\xi,y)}{x-\xi}\,d\xi + \frac{ike^{ik}}{2\pi}\Gamma(y)\int_1^{\infty}\frac{e^{-ik\xi}}{x-\xi}\,d\xi$$
$$-\frac{e^{ik(1-x)}}{4\pi}\oint_{-s}^{s}\frac{\Gamma'(\eta)}{y-\eta}\left\{1-i\int_0^{\infty}e^{-i\lambda}\left[1+\frac{k|y-\eta|}{\lambda}\right.\right.$$
$$\left.\left.-\frac{\left\{\lambda^2+k^2(y-\eta)^2\right\}^{1/2}}{\lambda}\right]d\lambda\right\}d\eta \qquad (132)$$

Putting as an abbreviation

$$S(k|y-\eta|) = i\int_0^{\infty}e^{-i\lambda}\left[1+\frac{k|y-\eta|-\left\{\lambda^2+k^2(y-\eta)^2\right\}^{1/2}}{\lambda}\right]d\lambda \qquad (133)$$

equation (132) may be written in the form

$$w_0 \approx -\frac{1}{\pi}\oint_{-1}^{1}\frac{u_0(\xi,y)}{x-\xi}\,d\xi + \frac{ike^{ik}}{2\pi}\Gamma(y)\int_1^{\infty}\frac{e^{-ik\xi}}{x-\xi}\,d\xi$$
$$-\frac{e^{ik(1-x)}}{4\pi}\oint_{-s}^{s}\frac{\Gamma'(\eta)}{y-\eta}\left\{1-S(k|y-\eta|)\right\}d\eta \qquad (134)$$

Equations (134) and (133) are the generalization of equation (94) of the two-dimensional theory and of equation (101) of the three-dimensional stationary theory. They reduce to these special cases when $\Gamma'(\eta) = 0$ or $k = 0$, respectively. It may be noted that according to equation (134) the induced velocity due to the finite-span effect varies across the chord in contrast to the result of the stationary theory in which this velocity is uniform. In addition to this the cumulative effect of the spanwise rate of change of $\Gamma$ is modified as compared with the result of the stationary theory by the occurrence of the function $S$ which can be tabulated once for all.

It is because of the chordwise variation of the induced velocity that it is no longer possible to speak of a lifting line. To express the fact that the derivation of the integral equation (134) depends on the assumption of the span being rather longer than the chord the name of lifting-strip theory is proposed for the basic equation (134) and for the consequences derived therefrom.

Further treatment of equation (134) is possible by means of a combination of the known procedures of the two-dimensional theory and of the three-dimensional stationary theory.

As in the three-dimensional stationary theory an integral equation for $\Gamma$ is obtained which may be solved by numerical methods.

As in the two-dimensional theory explicit expressions for the section lift and the section moments can be obtained in terms of the function $\Gamma$. This will be done in what follows for lift and moment of the entire chord, leaving calculation of the aileron hinge moment for future work.

The first step in this program consists in the determination of $u_o$ from equation (134) by means of the inversion formulas (102) and (103). The result is, if as a further abbreviation there is put

$$Q = \oint_{-s}^{s} \frac{\Gamma'(\eta)}{y - \eta} \left\{ 1 - S \right\} d\eta \tag{135}$$

$$u_o(x,y) = \frac{1}{\pi}\sqrt{\frac{1 - x}{1 + x}} \oint_{-1}^{1} \sqrt{\frac{1 + \xi}{1 - \xi}} \left\{ w_o(\xi, y) - \frac{ike^{ik}}{2\pi} \Gamma(y) \int_{1}^{\infty} \frac{e^{-ik\lambda}}{\xi - \lambda} d\lambda + \frac{e^{ik(1-\xi)}}{4\pi} Q(y) \right\} \frac{d\xi}{x - \xi} \tag{136}$$

The section lift and the section moments may be calculated by introducing equation (136) into equations (86) to (88). From the two-dimensional theory it is known that these calculations lead to explicit results in terms of known functions and in terms of $\Gamma$ for part of equation (136). It will be shown that also the remaining terms can be expressed in terms of known functions and in terms of Q.

THE EQUATION DETERMINING THE CIRCULATION FUNCTION

Integrating equation (136)

$$\frac{1}{2}\Gamma = \int_{-1}^{1} u_0\, dx = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\left\{\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\left[w_0(\xi,y) + \frac{ike^{ik}}{2\pi}\Gamma\int_{1}^{\infty}\frac{e^{-ik\lambda}}{\lambda-\xi}\,d\lambda + \frac{e^{ik(1-\xi)}}{4\pi}Q\right]\frac{d\xi}{x-\xi}\right\}dx \tag{137}$$

To evaluate equation (137) the order of integration with respect to $x$ and $\xi$ is interchanged and use is made of equation (106). There follows

$$\frac{1}{2}\Gamma = -\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\left[w_0 + \frac{ike^{ik}}{2\pi}\Gamma\int_{1}^{\infty}\frac{e^{-ik\lambda}}{\lambda-\xi}\,d\lambda + \frac{e^{ik(1-\xi)}}{4\pi}Q\right]d\xi \tag{138}$$

It remains to evaluate the integrals

$$I_6 = \int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\left[\int_{1}^{\infty}\frac{e^{-ik\lambda}}{\xi-\lambda}\,d\lambda\right]d\xi \tag{139}$$

$$I_7 = \int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,e^{-ik\xi}\,d\xi \tag{140}$$

To obtain $I_6$, interchange the order of integration

$$I_6 = \int_{1}^{\infty} e^{-ik\lambda}\left[\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,\frac{d\xi}{\xi-\lambda}\right]d\lambda$$

$$= \pi\int_{1}^{\infty} e^{-ik\lambda}\left[1-\sqrt{\frac{\lambda+1}{\lambda-1}}\right]d\lambda \tag{141}$$

The value of this integral is known in terms of modified Bessel functions (see Durand, vol. II, p. 295)

$$I_6 = -\pi\left[K_o(ik) + K_1(ik) - \frac{c^{-ik}}{ik}\right] \tag{142}$$

To obtain $I_7$, use is made of the following known formula (see, for instance, Gray and Mathews Treatise on Bessel functions, p. 46)

$$\int_{-1}^{1} \frac{e^{-ik\xi}}{\sqrt{1-\xi^2}}\, d\xi = \pi\, J_o(k) \tag{143}$$

From this there follows

$$I_7 = \int_{-1}^{1} \frac{1+\xi}{\sqrt{1-\xi^2}}\, e^{-ik\xi}\, d\xi = \pi\left[J_o(k) - iJ_1(k)\right] \tag{144}$$

Substituting equations (142) and (144) in equation (138),

$$\frac{1}{2}\Gamma = -\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\, w_o\, d\xi - \frac{ike^{ik}}{2}\Gamma\left[K_o(ik) + K_1(ik) - \frac{e^{-ik}}{ik}\right]$$

$$- \frac{e^{ik}}{4}Q\left[J_o(k) - iJ_1(k)\right] \tag{145}$$

and canceling the term on the left against one of the terms on the right, rearranging and introducing the value of $Q$ from equation (135) there follows as integral equation for $\Gamma$,

$$\frac{1}{2}\Gamma(y) + \frac{J_o(k) - iJ_1(k)}{4ik[K_o(ik) + K_1(ik)]}\oint_{-s}^{s}\frac{\Gamma'(\eta)}{y-\eta}\left\{1 - S(k|y-\eta|)\right\}d\eta$$

$$= -\frac{\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\, w_o(\xi,y)\, d\xi}{ike^{ik}[K_o(ik) + K_1(ik)]} \tag{146}$$

A similar equation has been obtained by Cicala (reference 27) by means of considerations involving vortex filaments. Cicala's equation distinguishes itself from equation (159) by the fact that the factor $J_0(k) - iJ_1(k)$ is replaced by a factor $e^{-ik}$. Since for large $k$

$$J_0(k) - iJ_1(k) \approx \sqrt{\frac{2}{\pi k}}\, e^{i\frac{\pi}{4}}\, e^{-ik} \tag{147}$$

it is seen that finite span corrections of a different order of magnitude in $k$ are to be expected from Cicala's equation and from the one given here. The difference is large even for relatively small values of $k$ well within the practical range as can be seen by comparing the two terms as follows,

$$k = \frac{\pi}{4} \approx 0.78; \quad e^{-ik} \approx 0.7 - i\,0.7; \quad J_0(k) - iJ_1(k) = 0.85 - i\,0.36 \tag{148}$$

It may be noted that Cicala's result would follow from the basic equation (134) if in the third term of this equation the factor $\exp[ik(1-x)]$ were missing. This means that in Cicala's work the chordwise variation of the finite-span-effect contribution to the induced velocity has been left out of consideration. Further discrepancies which, it appears, cannot all be accounted for in this manner are found between Cicala's expressions for section lift and moment and the expressions given in what follows. It may be emphasized that while there is a formal resemblance between part of the present results and Cicala's results, the present work and Cicala's work are fundamentally different. Cicala's approach to the problem does not permit a rational determination of all the factors of importance in the problem.

Regarding Küssner's work (reference 10) on the same subject the following may be said. On the basis of the integral equation of lifting-surface theory set up in terms of the acceleration potential Küssner obtains an integral equation for a quantity which, in the notation of the present paper, is the section lift $l_k$. Küssner states that this equation is correct only when $w_0(x,y) = \overline{w}(y)e^{-ikx}$ and for other functions $w_0$ an equivalent $\overline{w}$ might be determined by means of the equation

$$\overline{w}(y)\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\, e^{-ik\xi}\, d\xi = \int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\, w_0(\xi,y)\, d\xi$$

It appears that Küssner' result cannot be correct inasmuch as it does not reduce to the value of the section lift of the two-dimensional theory when the assumption of two-dimensionality is introduced into the equation. Furthermore, no provision is made in Küssner's work for the determination of the finite-span effect on the values of the section moments.

The comment that his result does not reduce to the appropriate two-dimensional result when it should do so applies also to Sears' special solution (reference 29) for the infinite periodically bent airfoil. Sears' paper also contains no formula for the effect of three-dimensional flow on the values of the section moment.

As Jones' work (references 31 and 32) deals with the transient problem of a rigid airfoil with elliptical form view no direct comparison is possible of his results with the ones given here. Inasmuch, however, as his developments make essential use of vortex-filament notions it appears desirable to compare his results with results which may be obtained on the basis of the notions of the present paper.

## DETERMINATION OF THE EXPRESSION FOR THE SECTION LIFT

According to equation (86)

$$l_k = ik\int_{-1}^{1}(1 - x)\ u_o\ dx + \frac{1}{2}\Gamma \tag{149}$$

and it remains to evaluate the integral

$$I_s = ik\int_{-1}^{1}(1 - x)\ u_o\ dx \tag{150}$$

Substituting equation (136) in equation (150)

$$I_s = \frac{ik}{\pi}\int_{-1}^{1}(1 - x)\sqrt{\frac{1 - x}{1 + x}}\left\{\oint_{-1}^{1}\sqrt{\frac{1 + \xi}{1 - \xi}}\left[w_o - \frac{ike^{ik}}{2\pi}\Gamma\int_{1}^{\infty}\frac{e^{-ik\lambda}}{\xi - \lambda}\,d\lambda + \frac{e^{ik(1-\xi)}}{4\pi}Q\right]\frac{d\xi}{x - \xi}\right\}dx \tag{151}$$

Interchanging the order of integration with respect to $\xi$ and $x$ leads to the integral

$$\oint_{-1}^{1} (1 - x)\sqrt{\frac{1 - x}{1 + x}}\frac{dx}{x - \xi} = \pi\,(\xi - 2) \tag{152}$$

Hence

$$I_8 = ik\int_{-1}^{1}(\xi - 2)\sqrt{\frac{1 + \xi}{1 - \xi}}\left[w_0 - \frac{ike^{ik}}{2\pi}\Gamma\int_1^{\infty}\frac{e^{-ik\lambda}}{\xi - \lambda}\,d\lambda + \frac{e^{ik(1-\xi)}}{4\pi}Q\right]d\xi \tag{153}$$

It is convenient (but not essential) to make use of equation (138) by writing equation (153) in the form

$$I_8 = \frac{ik}{2}\Gamma - ik\int_{-1}^{1}\sqrt{1 - \xi^2}\left[w_0 - \frac{ike^{ik}}{2\pi}\Gamma\int_1^{\infty}\frac{e^{-ik\lambda}}{\xi - \lambda}\,d\lambda + \frac{e^{ik(1-\xi)}}{4\pi}Q\right]d\xi \tag{154}$$

It remains to evaluate

$$I_9 = \int_{-1}^{1}\sqrt{1 - \xi^2}\left[\int_1^{\infty}\frac{e^{-ik\lambda}}{\xi - \lambda}\,d\lambda\right]d\xi \tag{155}$$

and

$$I_{10} = \int_{-1}^{1}\sqrt{1 - \xi^2}\,e^{-ik\xi}\,d\xi \tag{156}$$

$I_9$ is found by interchanging the order of integration

$$I_9 = \int_1^{\infty} e^{-ik\lambda}\left[\int_{-1}^{1}\frac{\sqrt{1 - \xi^2}}{\xi - \lambda}\,d\xi\right]d\lambda$$

$$= -\pi\int_1^{\infty} e^{-ik\lambda}\left[\lambda - \sqrt{\lambda^2 - 1}\right]d\lambda$$

Integrating by parts

$$I_9 = -\pi\left\{\frac{e^{-ik}}{ik} + \frac{1}{ik}\int_1^\infty e^{-ik\lambda}\left[1 - \sqrt{\frac{\lambda+1}{\lambda-1}} + \frac{1}{\sqrt{\lambda^2-1}}\right]d\lambda\right\}$$

Using equation (141) and the known formula

$$\int_1^\infty \frac{e^{-ik\lambda}}{\sqrt{\lambda^2-1}}\,d\lambda = K_0(ik) \qquad (157)$$

there follows

$$I_9 = \pi\left\{-\frac{e^{-ik}}{ik} + \frac{1}{ik}\left[K_1(ik) - \frac{e^{-ik}}{ik}\right]\right\} \qquad (158)$$

The integral $I_{10}$ is found by means of the following formula (see Bessel Functions by Gray and Mathews, p. 46 - reference 35)

$$J_1(x) = \frac{1}{\sqrt{\pi}}\frac{1}{\Gamma\left(\frac{3}{2}\right)}\left(\frac{x}{2}\right)\int_{-1}^{1} e^{-ix\xi}\sqrt{1-\xi^2}\,d\xi \qquad (159)$$

where now $\Gamma$ represents the classical $\Gamma$-function and $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$

Thus

$$I_{10} = \int_{-1}^{1}\sqrt{1-\xi^2}\,e^{-ik\xi}\,d\xi = \pi\frac{J_1(k)}{k} \qquad (160)$$

Introducing now $I_9$ from equation (158) and $I_{10}$ from equation (160) into equation (154) there follows after some cancellations

$$I_8 = -ik\int_{-1}^{1}\sqrt{1-\xi^2}\,w_0(\xi,y)\,d\xi$$

$$+\frac{ike^{ik}}{2}\Gamma\left[K_1(ik) - \frac{e^{-ik}}{ik}\right] - \frac{ike^{ik}}{4}Q\frac{J_1(k)}{k} \qquad (161)$$

Substituting $I_8$ from equation (161) in equation (149), there follows for the section lift

$$l_k = -ik\int_{-1}^{1}\sqrt{1-\xi^2}\,w_o(\xi,y)\,d\xi + \frac{ike^{ik}}{2}K_1(ik)\;\Gamma(y) - \frac{ike^{ik}}{4}\frac{J_1(k)}{k}Q(y) \tag{162}$$

where $\Gamma$ is to be determined from equation (146) and $Q$ is defined by equation (135). For two-dimensional motion equation (162) reduces to

$$\tilde{l}_k = -ik\int_{-1}^{1}\sqrt{1-\xi^2}\,w_o(\xi)\,d\xi - \frac{ike^{ik}}{2}K_1(ik)\,\tilde{\Gamma} \tag{163}$$

and, since according to equation (146) the two-dimensional circulation is given by

$$\frac{\tilde{\Gamma}}{2} = -\frac{\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,w_o(\xi)\,d\xi}{ike^{ik}[K_o(ik) + K_1(ik)]} \tag{164}$$

there follows as special case the known expression for the section lift of the two-dimensional theory

$$\tilde{l}_k = -ik\int_{-1}^{1}\sqrt{1-\xi^2}\,w_o(\xi)d\xi - \frac{K_1(ik)}{K_o(ik) + K_1(ik)}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,w_o(\xi)d\xi \tag{165}$$

The factor of the second integral has been designated by Theodorsen (reference 6) by $C(k) = F(k) + i\,G(k)$.

It is apparent that once the integral equation (146) for $\Gamma$ has been solved the calculation of the lift distribution is no more complicated in the three-dimensional theory than it is in the two-dimensional theory.

## DETERMINATION OF THE EXPRESSION FOR THE SECTION MOMENT

According to equation (87) the section moment about the semi-chord point is given by

$$m_k(o) = \frac{ik}{2}\int_{-1}^{1}(1 - x^2)\, u_o\, dx + \int_{-1}^{1} x\, u_o\, dx \tag{167}$$

The second term of this equation can be expressed in terms of $\Gamma$ and $I_8$ of equation (150)

$$\int_{-1}^{1} x\, u_o\, dx = -\frac{I_8}{ik} + \frac{\Gamma}{2} \tag{168}$$

Hence, with equation (161).

$$\int_{-1}^{1} x\, u_o\, dx = \int_{-1}^{1}\sqrt{1 - \xi^2}\, w_o\, d\xi - \frac{e^{ik}}{2} K_1(ik)\Gamma + \frac{\Gamma}{2ik} + \frac{\Gamma}{2} + \frac{e^{ik}}{4}\frac{J_1(k)}{k} Q \tag{169}$$

It remains to evaluate

$$I_{11} = \frac{ik}{2}\int_{-1}^{1}(1 - x^2)\, u_o\, dx \tag{170}$$

which, according to equation (136) is given by

$$I_{11} = \frac{ik}{2\pi}\int_{-1}^{1}(1 - x^2)\sqrt{\frac{1 - x}{1 + x}}\left\{\oint_{-1}^{1}\sqrt{\frac{1 + \xi}{1 - \xi}}\left[w_o - \frac{ike^{ik}}{2\pi}\Gamma\int_{1}^{\infty}\frac{e^{-ik\lambda}}{\xi - \lambda}\, d\lambda + \frac{e^{ik(1-\xi)}}{4\pi} Q\right]\frac{d\xi}{x - \xi}\right\}dx \tag{171}$$

Interchanging as before the order of integration with regard to $x$ and $\xi$ the following integral has to be evaluated first

$$\oint_{-1}^{1}(1 - x^2)\sqrt{\frac{1 - x}{1 + x}}\frac{dx}{x - \xi} = \pi\left(\xi^2 - \xi - \frac{1}{2}\right) \tag{172}$$

With this

$$I_{11} = \frac{ik}{2}\int_{-1}^{1}\left(\xi^2 - \xi - \frac{1}{2}\right)\sqrt{\frac{1 + \xi}{1 - \xi}}\left[w_0 - \frac{ike^{ik}}{2\pi}\,\Gamma\int_1^{\infty}\frac{e^{-ik\lambda}}{\xi - \lambda}\,d\lambda + \frac{e^{ik(1-\xi)}}{4\pi}\,Q\right]d\xi \tag{173}$$

Writing

$$\left(\xi^2 - \xi - \frac{1}{2}\right)\sqrt{\frac{1 + \xi}{1 - \xi}} = -\,\xi\sqrt{1 - \xi^2} - \frac{1}{2}\sqrt{\frac{1 + \xi}{1 - \xi}}$$

and making use of equation (138) $I_{11}$ may be written in the form

$$I_{11} = -\frac{ik}{2}\int_{-1}^{1}\xi\sqrt{1 - \xi^2}\left[w_0 - \frac{ike^{ik}}{2\pi}\,\Gamma\int_1^{\infty}\frac{e^{-ik\lambda}}{\xi - \lambda}\,d\lambda + \frac{e^{ik(1-\xi)}}{4\pi}\,Q\right]d\xi + \frac{ik}{8}\,\Gamma \tag{174}$$

There remains to be evaluated

$$I_{12} = \int_{-1}^{1}\xi\sqrt{1 - \xi^2}\left[\int_1^{\infty}\frac{e^{-ik\lambda}}{\xi - \lambda}\,d\lambda\right]d\xi \tag{175}$$

$$I_{13} = \int_{-1}^{1}\xi\sqrt{1 - \xi^2}\,e^{-ik\xi}\,d\xi \tag{176}$$

The value of $I_{12}$ is found by first interchanging the order of integration,

$$I_{12} = \int_1^{\infty}e^{-ik\lambda}\left[\int_{-1}^{1}\frac{\xi\sqrt{1 - \xi^2}}{\xi - \lambda}\,d\xi\right]d\lambda$$

$$= \pi\int_1^{\infty}e^{-ik\lambda}\left[\frac{1}{2} - \lambda\left(\lambda - \sqrt{\lambda^2 - 1}\right)\right]d\lambda \tag{177}$$

by then integrating by parts

$$I_{12} = \pi\left\{-\frac{e^{-ik}}{2ik} + \frac{1}{ik}\int_1^{\infty} e^{-ik\lambda}\left[-2\lambda + 2\sqrt{\lambda^2 - 1} + \frac{1}{\sqrt{\lambda^2 - 1}}\right]d\lambda\right\} \tag{178}$$

Utilizing equations (142) and (157)

$$I_{12} = \pi\left\{-\frac{e^{-ik}}{2ik} + \frac{2}{ik}\left[-\frac{e^{-ik}}{ik} + \frac{1}{ik}\left(K_1(ik) - \frac{e^{-ik}}{ik}\right)\right] + \frac{K_0(ik)}{ik}\right\} \tag{179}$$

The value of $I_{13}$ is found with the help of equation (160)

$$I_{13} = \frac{\partial}{-i\partial k}\int_{-1}^{1}\sqrt{1 - \xi^2}\, e^{-ik\xi}\, d\xi = \pi i\,\frac{d}{dk}\left(\frac{J_1(k)}{k}\right) \tag{180}$$

Substituting now equations (179) and (180) in equation (174) there results after some cancellations

$$I_{11} = -\frac{ik}{2}\int_{-1}^{1}\xi\sqrt{1 - \xi^2}\, w_o\, d\xi + \frac{ike^{ik}}{4}\Gamma\left\{2\left[-\frac{e^{-ik}}{ik} + \frac{1}{ik}\left(K_1(ik) - \frac{e^{-ik}}{ik}\right)\right] + K_o(ik)\right\} + \frac{e^{ik}}{8}J_2(k)\, Q \tag{181}$$

Introducing equations (181) and (169) into equations (170) and (167), there follows, after some further cancellations and after making use of a recurrence formula for Bessel functions, as expression for the section moment distribution in the three-dimensional theory

$$m_k(o) = \int_{-1}^{1}\sqrt{1 - \xi^2}\, w_o\, d\xi - \frac{ik}{2}\int_{-1}^{1}\xi\sqrt{1 - \xi^2}\, w_o\, d\xi + \frac{ike^{ik}}{4}K_o(ik)\,\Gamma + \frac{e^{ik}}{8}J_o(k)\, Q \tag{182}$$

For the two-dimensional case this expression for the section moment reduces to

$$\tilde{m}_k(o) = \int_{-1}^{1}\sqrt{1-\xi^2}\, w_o\, d\xi - \frac{ik}{2}\int_{-1}^{1}\xi\sqrt{1-\xi^2}\, w_o\, d\xi + \frac{ike^{ik}}{4} K_o(ik)\tilde{\Gamma} \tag{183}$$

and with $\tilde{\Gamma}$ from equation (164)

$$\tilde{m}_k(o) = \int_{-1}^{1}\sqrt{1-\xi^2}\, w_o\, d\xi - \frac{ik}{2}\int_{-1}^{1}\xi\sqrt{1-\xi^2}\, w_o\, d\xi - \frac{1}{2}\,\frac{K_o(ik)}{K_1(ik)+K_o(ik)}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\, w_o\, d\xi \tag{184}$$

This is in agreement with the known results for this case.

## RÉSUMÉ OF BASIC FORMULAS OF LIFTING STRIP THEORY FOR AIRFOILS OF RECTANGULAR PLAN FORM

(1) The integral equation of lifting-surface theory as reduced for sufficiently large aspect ratio, $(s \geq 3)$

$$w_o(x,y) = -\frac{1}{\pi}\oint_{-1}^{1}\frac{u_o(\xi,y)}{x-\xi}\, d\xi + \frac{ike^{ik}}{2\pi}\Gamma_k(y)\int_{1}^{\infty}\frac{e^{-ik\xi}}{x-\xi}\, d\xi - \frac{e^{ik(1-x)}}{4\pi}\oint_{-s}^{s}\frac{\Gamma_k'(\eta)}{y-\eta}\left\{1 - S(k|y-\eta|)\right\} d\eta \tag{134}$$

where

$$w_o = ikh_k + \frac{\partial h_k}{\partial x},\quad h = e^{it}\, h_k,\quad \Gamma = \Gamma_k\, e^{it},\quad \frac{1}{2}\Gamma_k = \int_{-1}^{1} u_o\, dx$$

and

$$S(k|y-\eta|) = i\int_{0}^{\infty} e^{-i\lambda}\left[1 + \frac{k|y-\eta| - \sqrt{\lambda^2 + k^2(y-\eta)^2}}{\lambda}\right] d\lambda \tag{133}$$

The function S is related to a function F which is tabulated in equation (27), in the following way

$$S(x) = ix\,F(x)$$

(2) The integral representation for the chordwise velocity component at the airfoil

$$u_o(x,y) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\oint_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\left\{w_o - \frac{ike^{ik}}{2\pi}\Gamma_k\int_1^{\infty}\frac{e^{-ik\lambda}}{\xi-\lambda}\,d\lambda + \frac{e^{ik(1-\xi)}}{4\pi}\oint_{-s}^{s}\frac{\Gamma_k'(\eta)}{y-\eta}\left\{1-S\right\}d\eta\right\}\frac{d\xi}{x-\xi} \tag{136}$$

(3) The integral equation for the circulation function

$$\frac{1}{2}\Gamma_k(y) + \frac{J_0(k) - i\,J_1(k)}{4ik[K_0(ik) + K_1(ik)]}\oint_{-s}^{s}\frac{\Gamma_k'(\eta)}{y-\eta}\left\{1-S\right\}d\eta = -\frac{\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,w_o d\xi}{ike^{ik}[K_o(ik) + K_1(ik)]} \tag{146}$$

The modified Bessel functions of the second kind may be expressed in terms of Hankel functions,

$$K_o(ik) = -i\,\frac{\pi}{2}\,H_o^{(2)}(k), \quad K_1(ik) = -\frac{\pi}{2}\,H_1^{(2)}(k)$$

(4) The formula for the section lift $l = l_k\,e^{it}$

$$l_k = -ik\int_{-1}^{1}\sqrt{1-\xi^2}\,w_o\,d\xi + \frac{ike^{ik}}{2}K_1(ik)\,\Gamma_k(y) - \frac{ike^{ik}}{4}\,\frac{J_1(k)}{k}\oint_{-s}^{s}\frac{\Gamma_k'(\eta)}{y-\eta}\left\{1-S\right\}d\eta \tag{162}$$

(5) The formula for the section moment about the semi-chord point $m(o) = m_k(o)\,e^{it}$:

$$m_k(o) = \int_{-1}^{1} \sqrt{1 - \xi^2}\, w_o\, d\xi - \frac{ik}{2}\int_{-1}^{1} \xi\sqrt{1 - \xi^2}\, w_o\, d\xi$$

$$+ \frac{ike^{ik}}{4} K_o(ik)\, \Gamma_k(y) + \frac{e^{ik}}{8} J_o(k) \oint_{-s}^{s} \frac{\Gamma_k{}'(\eta)}{y - \eta}\left\{1 - s\right\} d\eta \qquad (182)$$

## CONCLUDING REMARKS

The developments of this paper indicate that use of the velocity potential is preferable in important respects to the use of the acceleration potential in thin-airfoil theory.

Recognition of this fact is considered as an essential aid in the establishment of "lifting-strip" theory for a nonstationary airfoil. While this theory has here been developed for airfoils of rectangular plan form it can be extended to tapered airfoils.

Application of the results of the present paper should permit, among other things, the investigation of the aerodynamic span effect in the problem of wing flutter. For this purpose it remains to establish a convenient scheme for the numerical solution of the integral equation for the circulation function.

While some investigators have stated as their opinion that the error in flutter calculations resulting from the assumption of two-dimensional flow is neglibibly small for all wings with aspect ratios above three, the author considers the available evidence as inconclusive. He believes that a decision on this question can be reached by applying the results of part III of this paper to a number of representative flutter cases.

It is emphasized that the manner in which lifting-strip theory is obtained indicates that its range of validity is no less than the range of validity of lifting-line theory for the stationary airfoil. Inasmuch as experiments have shown that lifting-line theory may be applied for wings with aspect ratios as low as three, the same must be true for the results obtained in part III of this paper for the nonstationary airfoil. It should be possible to apply lifting-strip theory to tail flutter problems.

Massachusetts Institute of Technology,
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[1]Available for reference or loan in the Office of Aeronautical Intelligence, NACA, Washington, D. C.

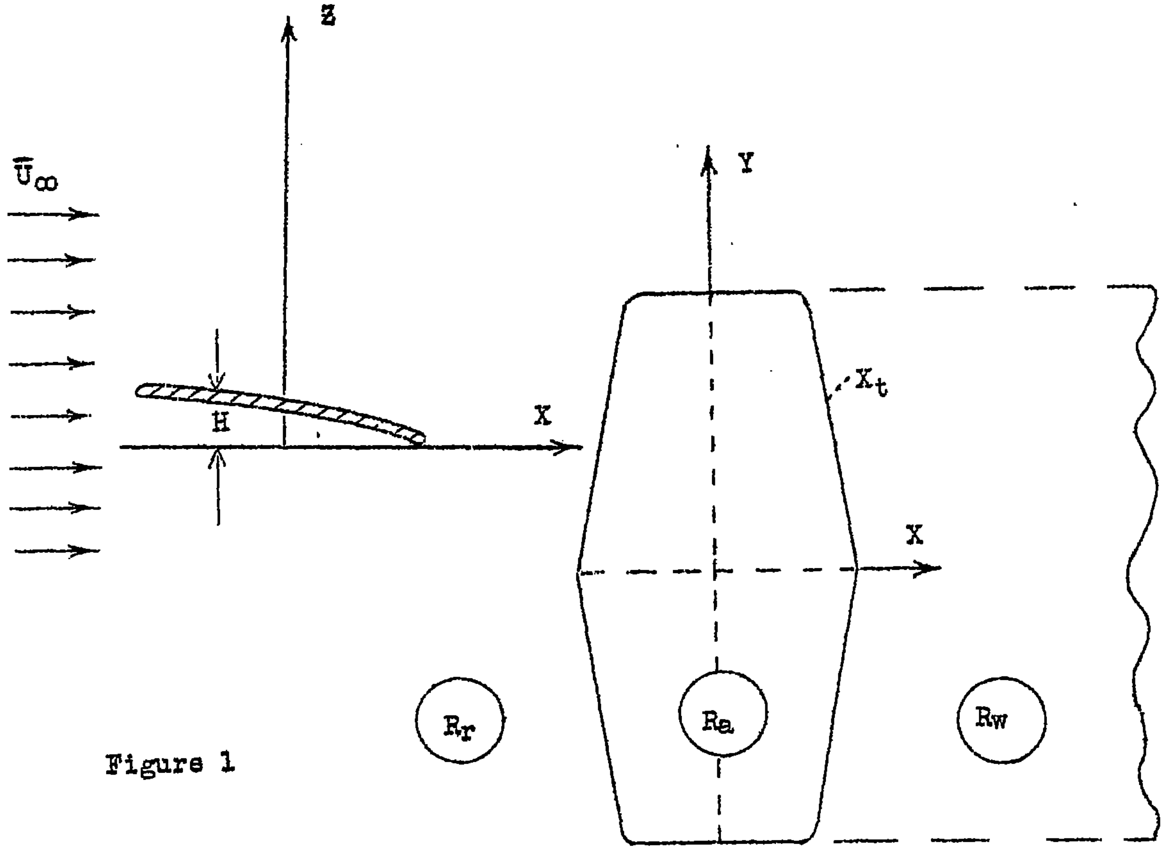


Figure 1

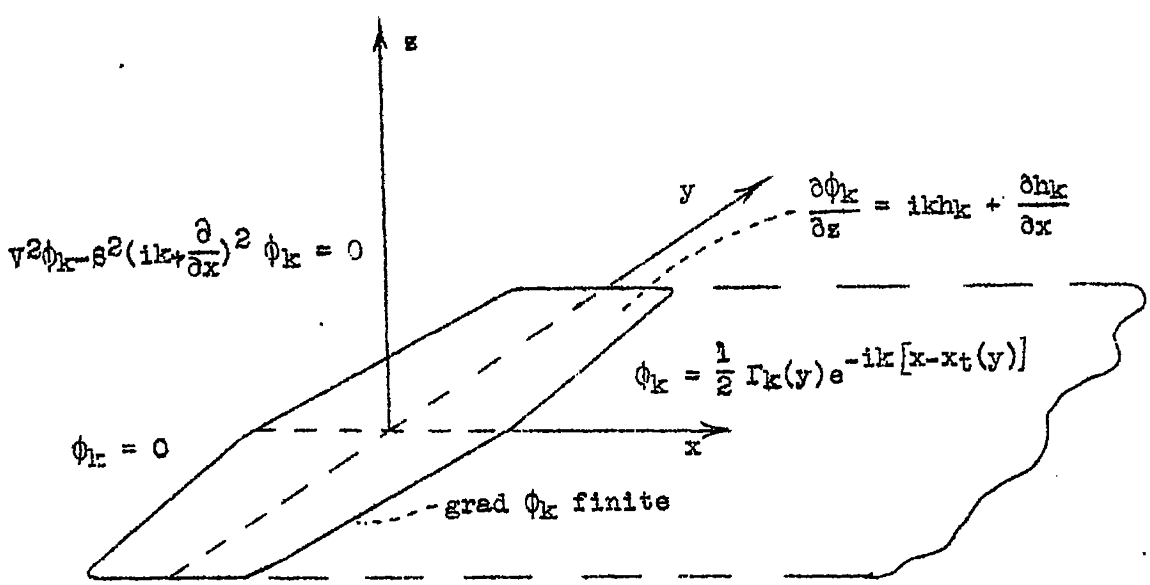


Figure 2