NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 4273

ON PAIRS OF SOLUTIONS OF A CLASS OF INTERNAL VISCOUS FLOW PROBLEMS WITH BODY FORCES

By Simon Ostrach and Lynn U. Albers

Lewis Flight Propulsion Laboratory
Cleveland, Ohio

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SUMMARY

In previous analyses of fully developed combined forced- and natural-convection flows, a few examples were presented of two distinct states of flow and heat transfer which were obtained for a given set of conditions if the frictional heating was taken into account. This report discusses these pairs of solutions in greater detail and shows how the solutions are affected by systematic variations of the basic physical parameters. It is also shown that a critical set of parametric values exists beyond which no fully developed solutions can be found.

INTRODUCTION

In recent analyses of combined forced- and natural-convection flow and heat transfer in enclosed regions, it was pointed out that the effects of frictional heating could, in practical cases, be important (refs. 1 to 4). Specifically, consideration was given to the fully developed flow between two parallel surfaces which were subject to various thermal conditions and oriented parallel to the body force direction. As a result of retaining the frictional heating terms in the basic equations, the final fourth-order ordinary differential equation was nonlinear. The boundary value problems specified by the nonlinear equation and appropriate boundary conditions were at first solved approximately by an analytical iteration technique in references 1, 3, and 4.

As a check of the accuracy of those solutions, the complete nonlinear problem was solved by numerical integration on an IBM Card Programmed Calculator (CPC). It was then discovered that the problems had either a pair of solutions or no solution, and examples of the pairs of solutions are given in references 1, 3, and 4. The solutions obtained analytically displayed no such behavior although they did approximate one of each pair of solutions in limited ranges of the parameters.

A somewhat analogous situation was noted by Hartree (ref. 5) in his study of boundary-layer flows with adverse pressure gradients. The second solutions to that problem are presented and discussed further in references 6 and 7.

Since these results are unusual and interesting, and only a few examples are presented in the literature, the purpose of the present report is to discuss in some detail the pairs of solutions to the convection problem. To this end, the method of obtaining these solutions is
presented, and the influence of the various physical parameters on these 4500
solutions is discussed.

BASIC EQUATIONS

Consideration is given herein to the fully developed laminar flow of a viscous fluid between two vertical plane surfaces and subject to a vertical body force (see fig. 1). Fully developed flow means that the velocity components are independent of the axial distance. Therefore, the solutions obtained are valid away from the channel ends and, hence, apply to channels with large length-to-gap ratios. It is further assumed that the physical properties of the fluids are constants, except that the essential influence of density variations on the flow is accounted for by the introduction of the fluid volumetric expansion coefficient in the body force term. The latter procedure is given in detail in appendix B of reference 1 and is justified for liquids and for gases only if pressure differences are small relative to temperature differences. The other influences of variable density and the variation of the expansion coefficient with temperature are also considered to be negligible. Fluids satisfying these assumptions will be called "quasi incompressible."

Under the conditions stated it follows that there is only one nonzero velocity component and that it is a function only of the transverse coordinate Y. Further, the temperature can be expressed as the sum of a linear function of the vertical (axial) coordinate and an arbitrary function of the horizontal (transverse) coordinate as

$$T^{*} = AX + T(Y) \tag{1}$$

where $A = \partial T^{*}/\partial X$ is the axial temperature gradient. (All symbols are defined in appendix A.) The temperatures of the two surfaces are specified either to be uniform or to vary linearly with the axial coordinate. In either case, the two surface temperatures can differ by a constant.

The reduction of the basic continuity, momentum, and energy equations by the above considerations is explicitly given in references 1, 3, and 4. The equations expressing the conservation of momentum and energy thus become

$$u'' + \tau = CK \tag{2}$$

$$\tau'' - (Ra)u + (u')^2 + \alpha K = 0 \tag{3}$$

The continuity equation is satisfied identically. The terms (from left to right) in equation (2) denote the viscous, buoyancy, and axial pressure forces and in equation (3) represent the conduction, convection, frictional heating, and heat-source effects. The parameter C essentially specifies the temperature level or axial pressure gradient, and K is the frictional heating parameter. The symbol Ra denotes the Rayleigh number (product of Prandtl and Grashof numbers), and α is the internal-heat-source parameter.

The Rayleigh number is the natural-convection equivalent of the Peclet number (product of Prandtl and Reynolds numbers), and K is analogous to the Eckert number $U_R^2/c_p\theta_{w_0}$, which is a measure of the dissipation in forced flows.

Elimination of τ between equations (1) and (2) yields the aforementioned fourth-order nonlinear ordinary differential equation:

$$u^{iv} + (Ra)u - (u')^2 - \alpha K = 0 \tag{4}$$

Equation (4) applies for the three problems treated in references 1, 3, and 4. For the constant-wall-temperature case, $Ra = 0$ (see eq. (1)); heating from below is simulated by the case $Ra < 0$, whereas $Ra > 0$ is the conventional flow with linearly varying wall temperatures.

BOUNDARY CONDITIONS

The no-slip condition for viscous fluids is imposed in all cases, that is,

$$u(0) = u(1) = 0 \tag{5}$$

In the first problem treated in reference 1, it was specified that the temperature is constant along each wall but that the two walls are not necessarily at the same temperature. Hence, $A = 0$ and the thermal boundary conditions for this case can, by use of equation (2), be written as

$$u''(0) = (C - 1)K \tag{6a}$$

$$u''(1) = m(C - 1)K \tag{7a}$$

where m, which is essentially a measure of the asymmetry of the thermal boundary conditions, is given by

$$m = (C - \theta_{w_1}/\theta_{w_0})/(C - 1) \tag{8a}$$

The thermal boundary conditions for linearly varying wall temperatures are

$$u''(0) = CK \tag{6b}$$

$$u''(1) = mCK \tag{7b}$$

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where

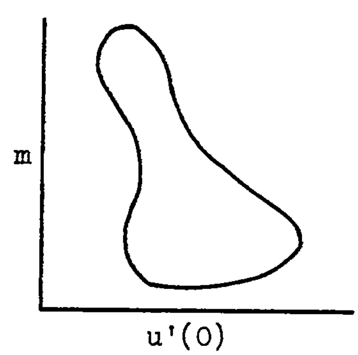
$$m = 1 - \theta_{w_1}/CAd \tag{8b}$$

The cases $A > 0$ and $A < 0$ (the latter corresponds to heating from below) are discussed in references 3 and 4, respectively. For the varying-wall-temperature cases the reference temperature in K and elsewhere (denoted by θ_R for the purpose of unifying past work in the present report) is Ad and not θ_{w_0}.

METHOD OF SOLUTION

It can be seen from equations (2), (3), (6), and (7) that to specify a problem the parameters CK, Ra, αK, and m must be known. However, the solution of the problem is more easily obtained if the boundary condition on the second derivative at $y = 1$ is replaced by an arbitrary choice of the first derivative at $y = 0$. Then the third derivative at zero is varied until integration yields $u(1) = 0$. The resultant $u''(1)$ yields the m to which the solution just obtained applies. The iteration method used in obtaining the solutions is discussed in detail in appendix B.

The variation of m with $u'(0)$ is represented on a curve called an m-curve. Such a curve is illustrated in the following schematic sketch:



In general, only the uppermost part of an m-curve is of practical interest, and only this part is presented in subsequent figures.

Each point on an m-curve corresponds to a solution of the problem with the prescribed CK, Ra, and αK. The m-curves indicate that maximum and minimum values of m exist for each CK, Ra, and αK beyond which no solutions of the type considered in the analysis (i.e., fully developed) exist and between which pairs of solutions exist. Thus, arbitrary values of the parameters cannot be specified a priori, and two
4500 distinct states of flow and heat transfer are indicated for each given set of conditions within the limits discussed above. Similar results are reported in reference 7 for boundary-layer flows.

Because limits on parametric values were indicated, it became necessary to determine these limits in addition to computing velocity and temperature profiles. To this end, regions of the limit curves near the maximum m values were investigated in detail, and only those portions are included herein (i.e., see fig. 2). Some investigations were carried far enough to exhibit the general nature of these curves as discussed previously. The minimum values of m indicated therefrom were generally very large negative numbers, and these were not considered to be of much interest from a practical viewpoint.

RESULTS

The dependence of the limit curves on the parameters K, Ra, and α can be discussed first. The constant C was arbitrarily taken to be zero for the constant-wall-temperature case (eqs. (6a) and (7a)) (essentially fixing the temperature level); for the varying-wall-temperature case, C was given the value -1 (eqs. (6b) and (7b)) so that the boundary conditions for all cases were the same. For $\alpha = 0$ the value of the parameter CK is considered to be determined by a combination of individual C or K values.

The effect of different K is shown in figure 2 plotted from table I for the case $Ra = \alpha = 0$ (i.e., constant wall temperature and no internal heat sources). The m-curves for other Ra and α are of the same form. Recall that only parts of the m-curves are presented here. In general, the limiting value of m increases algebraically as K decreases algebraically. In other words, to increase the probability of a problem having a solution decrease the value of K in magnitude. Physically lower K values imply less frictional heating. On the basis of this observation and similar behavior for Ra and α variations (which also indicate the amount of heat input to the channel), it appears that the limit beyond which no solutions of the type considered herein exist depends on the heat input to the channel. The reason for this result is not as yet evident. However, another example where the heat input is limited is in the familiar one-dimensional channel flow where thermal choking occurs.

The effect of variation of the heat-source parameter α plotted from tables I and II is shown in figure 3. This shows that the greater the heat source, the less likely a solution exists for a particular problem. This trend is also in accord with the physical discussion given previously.

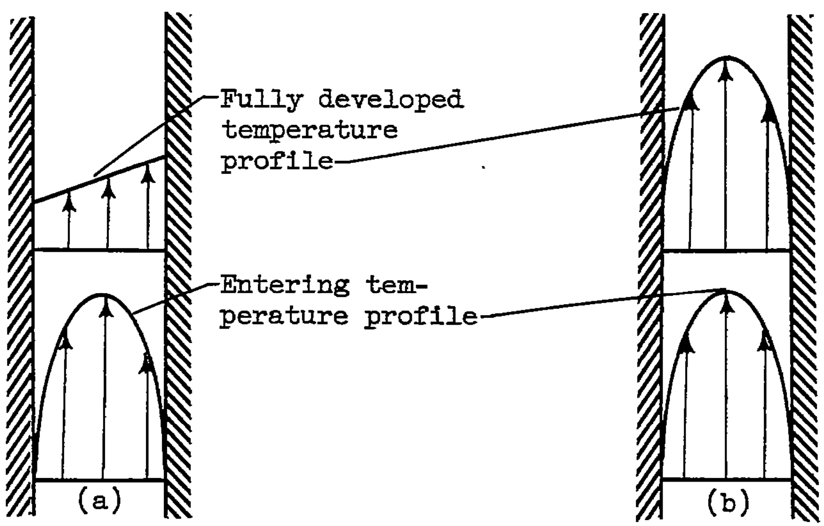
The influence of Ra variation on the limit curves is shown in 4500
figure 4 plotted from table III. Increasing Ra makes it more likely that a particular problem has a solution. This, too, is in accord with the previous discussion, because larger Ra implies increased axial temperature gradients and, hence, a large transfer of heat along the channel axis. This axial heat transfer should permit the fluid to be subject to greater heat input.

The negative Ra cases, simulating heating from below, were not studied in as great detail as those reported previously. However, results similar to those for positive Ra were found for the cases investigated and are discussed in some detail in reference 4.

The m-curves show that, if the effect of frictional heating is taken into account in internal combined forced- and natural-convection flows as considered herein, two unusual results occur which do not occur if frictional heating is neglected: (1) there exist critical sets of conditions beyond which no fully developed solutions exist and (2) where solutions exist, there are, in general, two solutions for each set of admissible parametric values. The first of these results has been discussed, but the second one should be considered in more detail. Figure 5 presents a set of representative pairs of velocity and temperature distributions taken from the previous work reported in reference 3 for $Ra = 10$, $K = 10$, $\alpha = 0$, and $m = 2$. These profiles are associated with the two circled points on the limit curve in figure 4.

Since each point on a limit curve corresponds to a solution of a given problem, the effect of parametric variations on the flow velocities can be observed in figures 1 to 3. For example, larger velocities are obtained for increasing values of the frictional heating parameter K; also, for a given m, increasing the internal-heat-source parameter α or decreasing the Rayleigh number Ra causes one of the pair of flows to have greater velocities and the other lower velocities. Note the broadness of the m-curve for large Ra. This broadness indicates that the profiles of each velocity pair and of each temperature pair differ greatly in magnitude except for m near the maximum.

The flow with the superscript (1) agrees reasonably well with that computed by the analytic method previously mentioned. In general, the analytic solution agrees with the one of the pair that has the lower velocity extremes because the iteration procedure employed to obtain the analytic solution started with the no-frictional-heating case. Because less frictional heating is associated with the less intense solution, this solution is the one to be expected in practice. The other solution, however, denoted with the superscript (2), represents a flow with velocities and temperature differences more than an order of magnitude greater than the first. The second solutions are connected intimately with the regenerative action of frictional heating in natural convection wherein frictional heating acts as a heat source in the fluid and thereby leads to larger flow velocities. As an illustration of the role that the dissipation plays in these problems, consider the simplest case, namely, the one wherein the walls are kept at the same constant temperatures and no heat sources are in the fluid ($\alpha = 0$). For this case, which is treated in reference 1, the less intense solution yields essentially the conduction (linear) temperature distribution. That is, no matter what kind of temperature profile exists in the fluid as it enters the channel, the profile becomes the conduction distribution by the time the flow has become fully developed because heat is extracted at the walls in the entrance region (see sketch (a)). For the more intense solution



the entering profile becomes parabolic and differs from the conduction profile considerably (see sketch (b) and fig. 5(b)). It is thus evident that the frictional heating acts as a heat source in the fluid. Another significant fact that can be seen from figure 5 and the other examples of pairs of solutions given in references 1, 3, and 4 is that the mass flow and energy associated with the more intense solution are greater than for the less intense solution. It, therefore, seems that, if the intense flow is to be obtained in practice, the flow would have to be started with the mass flow or energy at the higher level. This might be possible, for example, by heating the fluid considerably as it enters the channel.

The next question to arise in connection with the second solution concerns its stability; that is, since the frictional heat is dependent on the large velocities, would a small decrease in velocity, for example, decrease the frictional heating, and would this process keep repeating until the second solution attenuates. The stability of the symmetric case $(m = 1)$ has recently been analyzed, and the results, which will be published soon, show that this case is stable.

Although pairs of solutions were obtained for three somewhat different problems, all the analyses were restricted to fully developed flows in which the fluid was only "quasi incompressible." Recently, S. Maslen studied the same problems as discussed herein, but he considered the fluid to be compressible and to have property variations. He treated the complete nonlinear problem by Galerkin's method, and not only did he determine explicitly the conditions under which the present analysis is valid, but Maslen too obtained pairs of solutions. These results are to be published soon.

CONCLUDING REMARKS

It can be stated from the work reported herein and from the work by Maslen that two flow and heat-transfer states are predicted theoretically for a fixed set of parametric values within certain limits for fully developed combined forced- and natural-convection flow of certain real fluids. The only restriction of the analysis which has not been evaluated is that of assuming the flow to be fully developed. Definitive experiments will have to be performed to evaluate this limitation in order to establish definitely the physical existance of the second state of flow and heat transfer indicated herein.

The analyses also show that a critical set of parameters exists beyond which no fully developed solutions can be found. Here, too, experimental observation of what occurs for parametric values beyond the critical would be of importance.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, March 20, 1958

APPENDIX A

SYMBOLS

A	longitudinal (axial) temperature gradient
C	constant, $1/\mu\left(\frac{\partial P}{\partial X} + \rho_R f_X\right)\sqrt{(Pr)d^4/c_p\theta_R K}$
c_p	specific heat at constant pressure
d	characteristic length (specifically, distance between plates)
f_X	negative of X-component of body force per unit mass
Gr	Grashof number, $\beta f_X \theta_R d^3/\nu^2$
h	step size
K	$(Pr)(Gr)\beta f_X d/c_p$
k	thermal-conductivity coefficient
m	constant defined by eq. (8)
P	pressure
Pr	Prandtl number, $c_p\mu/k$
Q	heat due to heat sources
Ra	Rayleigh number, $(c_p\mu/k)(\beta f_X A d^4/\nu^2)$
T	temperature function defined by eq. (1)
T^*	temperature
U	velocity
U_R	reference velocity
u	dimensionless velocity, $U\sqrt{(Pr)K/c_p\theta_R}$
V	perturbation function
X	longitudinal (axial) coordinate

Y	transverse coordinate
y	dimensionless transverse coordinate
α	dimensionless heat-source parameter, $Qd^2/K\theta_R$
β	volumetric expansion coefficient, $\rho\left[\frac{\partial(1/\rho)}{\partial T}\right]_P$
θ	temperature difference, $T - T_s$ for constant-wall-temperature case and $T - T_{w_0}$ for linear-wall-temperature case
θ_R	reference temperature, $T_{w_0} - T_s$ for constant-wall-temperature case and Ad for linear-wall-temperature case
μ	absolute viscosity coefficient
ν	kinematic viscosity coefficient
ρ	density
ρ_R	ρ_s for constant-wall-temperature case and ρ_{w_0} for linear-wall-temperature case
τ	dimensionless temperature differences, $K(\theta/\theta_R)$

Subscripts:

s	hydrostatic condition
w_0	wall conditions at $y = 0$
w_1	wall conditions at $y = 1$

Superscript:

Primes denote differentiation with respect to y

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APPENDIX B

ITERATION TECHNIQUE

Various integration methods were used: trapezoidal integration with end correction, three-point integration, five-point integration, Runge-Kutta integration, and finally Taylor's series integration. All methods gave the same results to six significant figures, but the last named gave good approximate results with a step size of 0.2 and, hence, was used in most of the calculations. When detailed profiles were desired, integrations were repeated with a step size of 0.1 or 0.05.

The Taylor series formulas were

$$u^{(j)}(y + h) = \sum_{n=0}^{10-j} u^{(n+j)}{}_y \frac{h^n}{n!} \qquad \text{for} \quad j = 0, 1, 2, 3$$

where the superscript on u refers to the order of the derivative. Derivatives of u of higher order than the fourth were obtained by differentiating equation (4).

The usual steps in a straightforward iteration were:

(1) Guess $u'''(0)$.

(2) Integrate from 0 to 1.

(3) Test whether $u(1)$ is sufficiently close to zero.

(a) If so, stop.

(b) If not, choose a new $u'''(0)$ and return to step (2).

The only potentially bothersome step is (b). After the first integration, $u'''(0)$ may be decreased if $u(1)$ is positive and increased if $u(1)$ is negative - but by how much is a matter of judgment and experience. After two iterations, linear interpolation or extrapolation usually gives a good new $u'''(0)$. Since an integration took 5 minutes, a machine operator did not have great difficulty in keeping ahead of several cases run in rotation. However, in situations were only one m-curve was being studied, it was desirable to use a scheme which would guess a new $u'''(0)$ automatically. Hence, the following perturbation problem was integrated along with the main problem:

Let u be the solution of the initial value problem:

$$u^{iv} = (u')^2 - (Ra)u + \alpha K$$

$$u(0) = 0,\ u'(0) = b,\ u''(0) = CK,\ u'''(0) = g \tag{B1}$$

Let W be the solution of the initial value problem:

$$W^{iv} = (W')^2 - (Ra)W + \alpha K$$

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$$W(0) = 0,\ W'(0) = b,\ W''(0) = CK,\ W'''(0) = g + e \tag{B2}$$

Let $S = W - u$. Then substituting in equation (B2) yields

$$u^{iv} + S^{iv} = u' + S')^2 - Ra(u + S) + \alpha K$$

$$u(0) + S(0) = 0,\ u'(0) + S'(0) = b \tag{B3}$$

$$u''(0) + S''(0) = CK,\ u'''(0) + S'''(0) = d + e$$

and simplifying by using equation (B1) yields

$$S^{iv} = 2S'u' + (S')^2 - (Ra)S$$

$$S(0) = 0,\ S'(0) = 0,\ S''(0) = 0,\ S'''(0) = e \tag{B4}$$

Now let $S = eZ$, so that $S' = eZ'$, and so forth. Then substituting in equation (B4) and dividing by e yield

$$Z^{iv} = 2Z'u' + e(Z')^2 - (Ra)Z$$

$$Z(0) = Z'(0) = Z''(0) = 0,\ Z'''(0) = 1 \tag{B5}$$

As e approaches zero, the function Z approaches the function V, which is the solution of

$$V^{iv} = 2V'u' - (Ra)V$$

$$V(0) = V'(0) = V''(0) = 0,\ V'''(0) = 1 \tag{B6}$$

From the preceding work, it can be observed that

$$\frac{du(1)}{du'''(0)} = \lim_{e \to 0} \frac{W(1) - u(1)}{e} = \lim_{e \to 0} \frac{S(1)}{e} = \lim_{e \to 0} Z(1) = V(1) \tag{B7}$$

Hence, a good approximation of a suggested change in g or $u'''(0)$ is

$$\Delta g = -u(1)/\left[du(1)/du'''(0)\right] = -u(1)/V(1) \tag{B8}$$

so that the formula after any trial integration for the new $u'''(0)$ is new $u'''(0)$ = old $u'''(0) - u(1)/V(1)$.

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Because of storage limitations of 38 ten-digit numbers, only four terms of the Taylor series for V were used, but even this yielded good convergence to a result in several trials. General-purpose floating point control panels were used for all calculations.

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TABLE I. - Ra = 0, α = 0

u'(0)	K = 3		K = 10		K = 35	
	u'''(0)	m	u'''(0)	m	u'''(0)	m
0	9	-2.0	30	-2.0	100	-2.2
10	-68	14.8	-44	3.3	34	-.4
20	-177	25.2	-152	6.6	-53	.8
30	-318	30.6	-290	8.4	-195	1.6
40	-487	31.7	-458	8.9	-356	1.8
50	-683	29.2	-652	8.3	-540	1.8
60	-905	23.5	-873	6.7	-768	1.5
70	-1152	15.0	-1118	4.2	-1014	.9
80	-1422	3.9	-1388	.9	-1284	.1
90	-1716	-9.6	-1680	-3.0	-1579	-1.0
100	-2031	-25.4	-1994	-7.7		

TABLE II. - K = 10, Ra = 0

u'(0)	α = 10		α = 100	
	u'''(0)	m	u'''(0)	m
0	4	-4.7	-230	-29.8
10	-69	.9	-293	-22.4
20	-175	4.3	-391	-17.6
30	-313	6.2	-522	-14.6
40	-478	6.8	-683	-13.3
50	-674	6.2	-873	-13.3
60	-894	4.7	-1089	-14.4
70	-1139	2.2	-1330	-16.6
80	-1408	-1.0	-1596	-19.7
90	-1700	-5.0	-1885	-23.6
100	-2014	-9.7	-2198	-28.4

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TABLE III. - $K = 10$, $\alpha = 0$

Ra = 10			Ra = 100			Ra = 1600		
$u'(0)$	$u'''(0)$	m	$u'(0)$	$u'''(0)$	m	$u'(0)$	$u'''(0)$	m
0	29	-2.2	0	24	-3.5	1.2	39	-4
10	-41	3.8	10	-5	10.2	1	52	6
40	-444	11.1	30	-181	27.7	0	125	62
60	-854	9.6	40	-318	33.2	0	1398	624
100	-1966	-3.2	50	-486	36.9	10	2053	775
			60	-681	39.4	20	2429	838
			100	-1713	39.4	30	2681	871
			150	-3501	24.6	40	2852	888
			200	-5777	-2.3	60	3015	893
						80	2999	876
						200	401	582
						300	-4140	238
						350	-7064	51
						400	-10,383	-144

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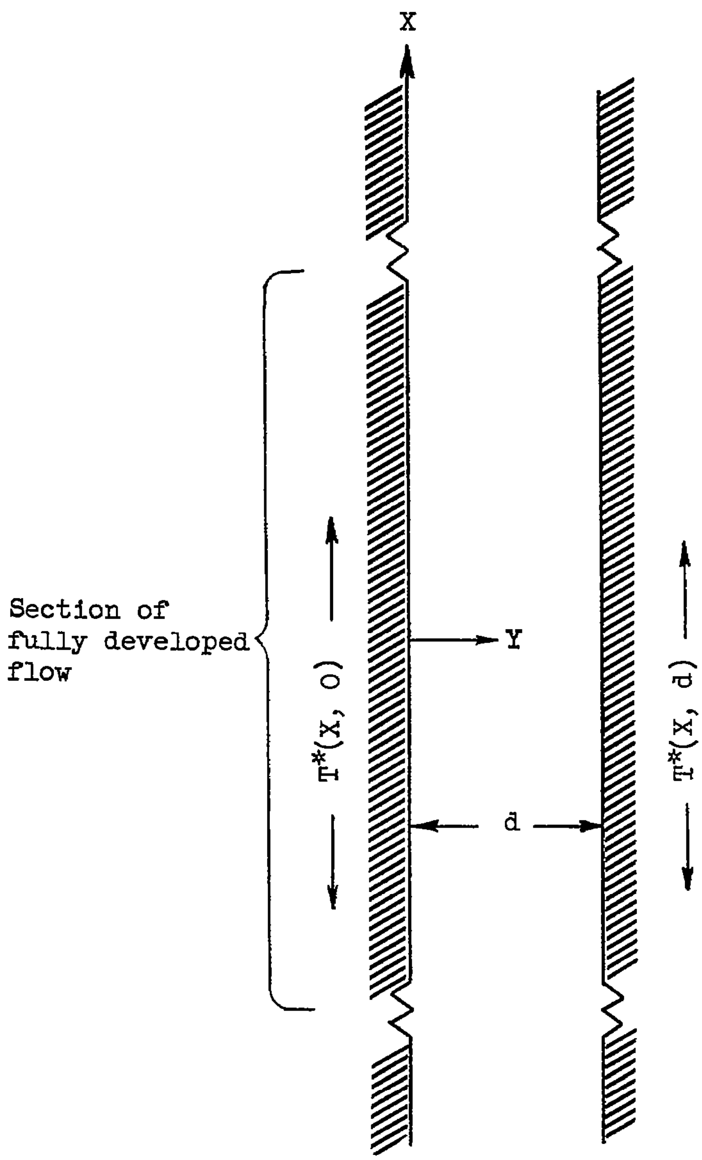


Figure 1. - Schematic sketch of configuration considered.

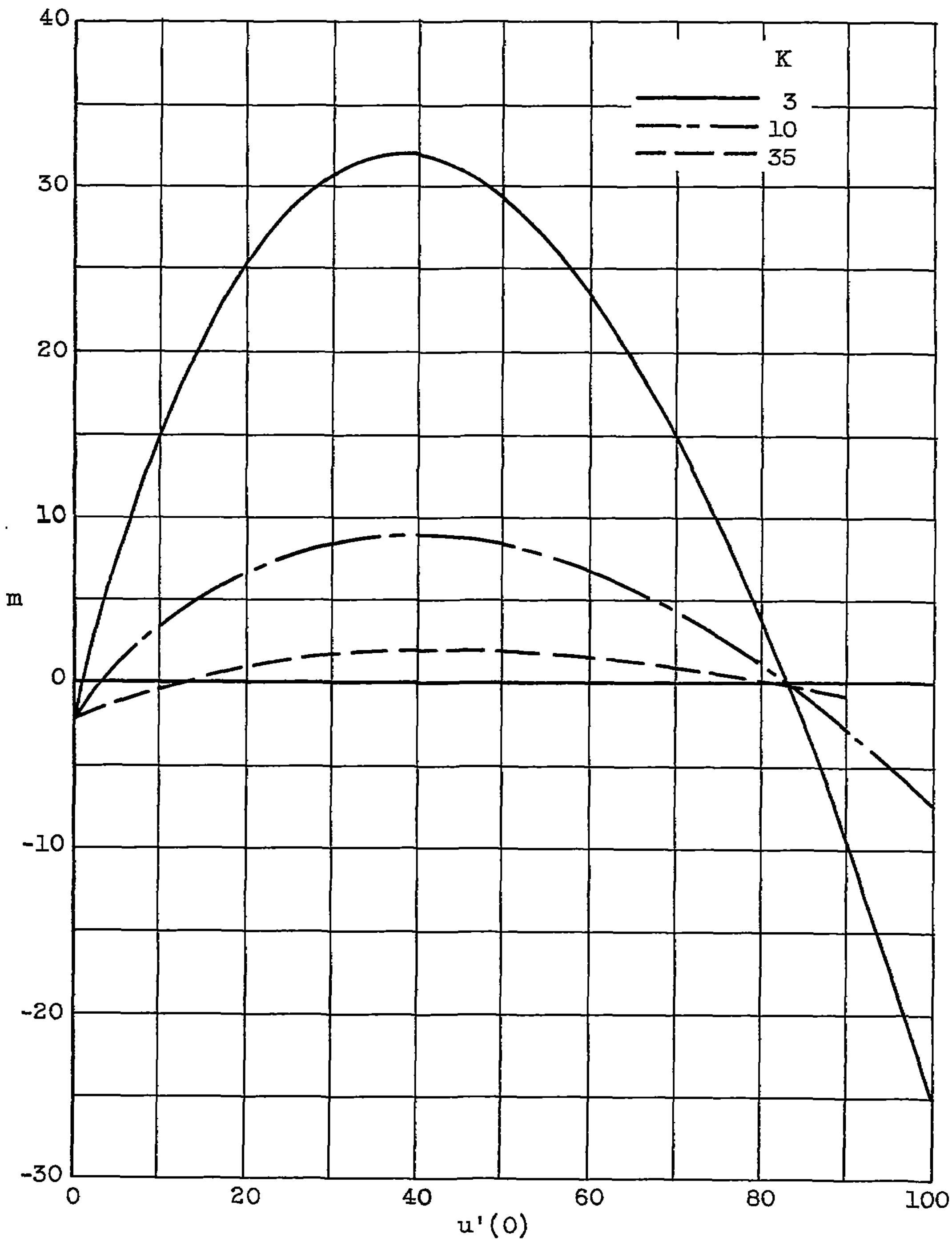


Figure 2. - The m-curves for various values of frictional heating parameter K. Ra = 0; α = 0.

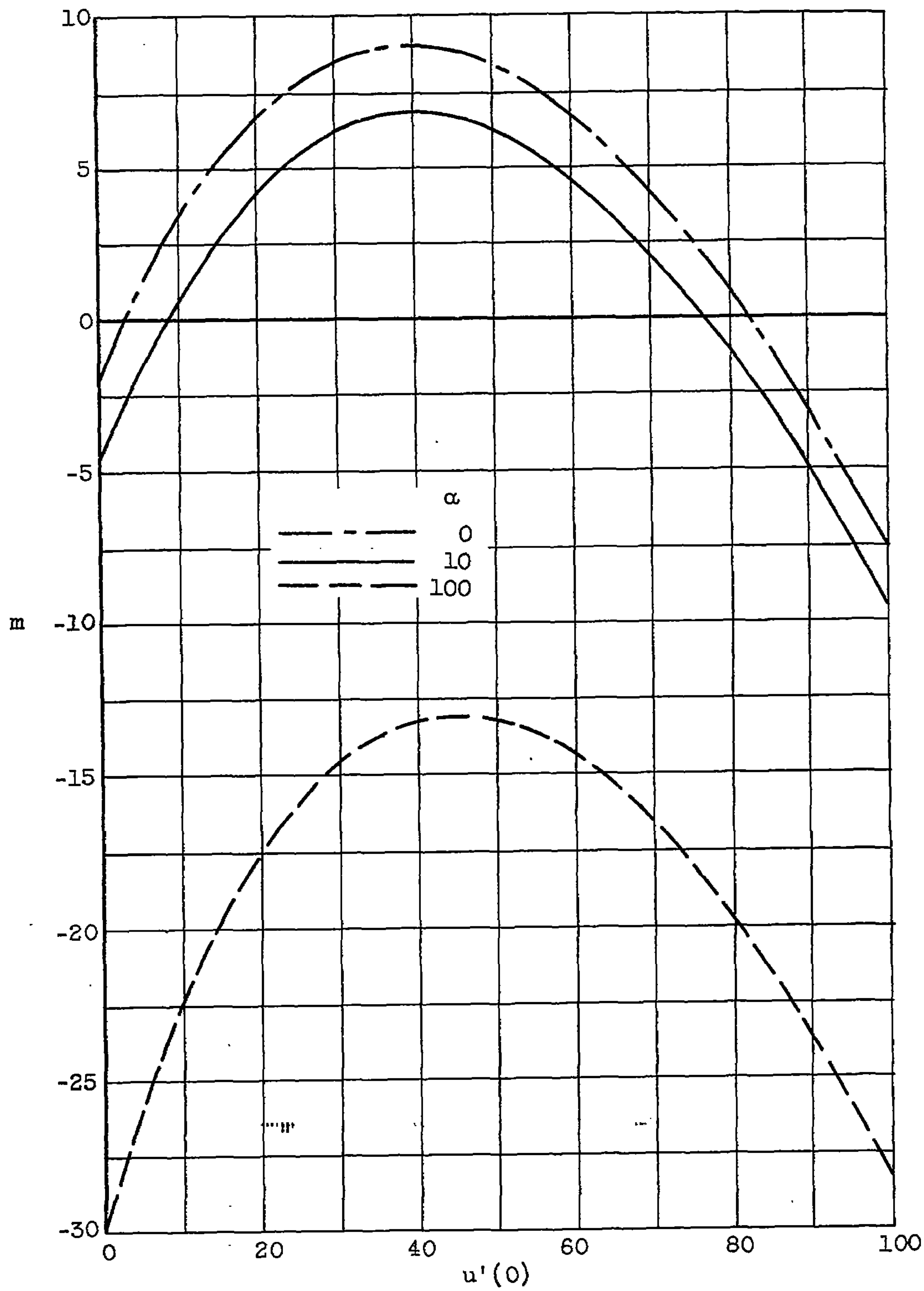


Figure 3. - The m-curves for various values of heat-source parameter α. Ra = 0; K = 10.

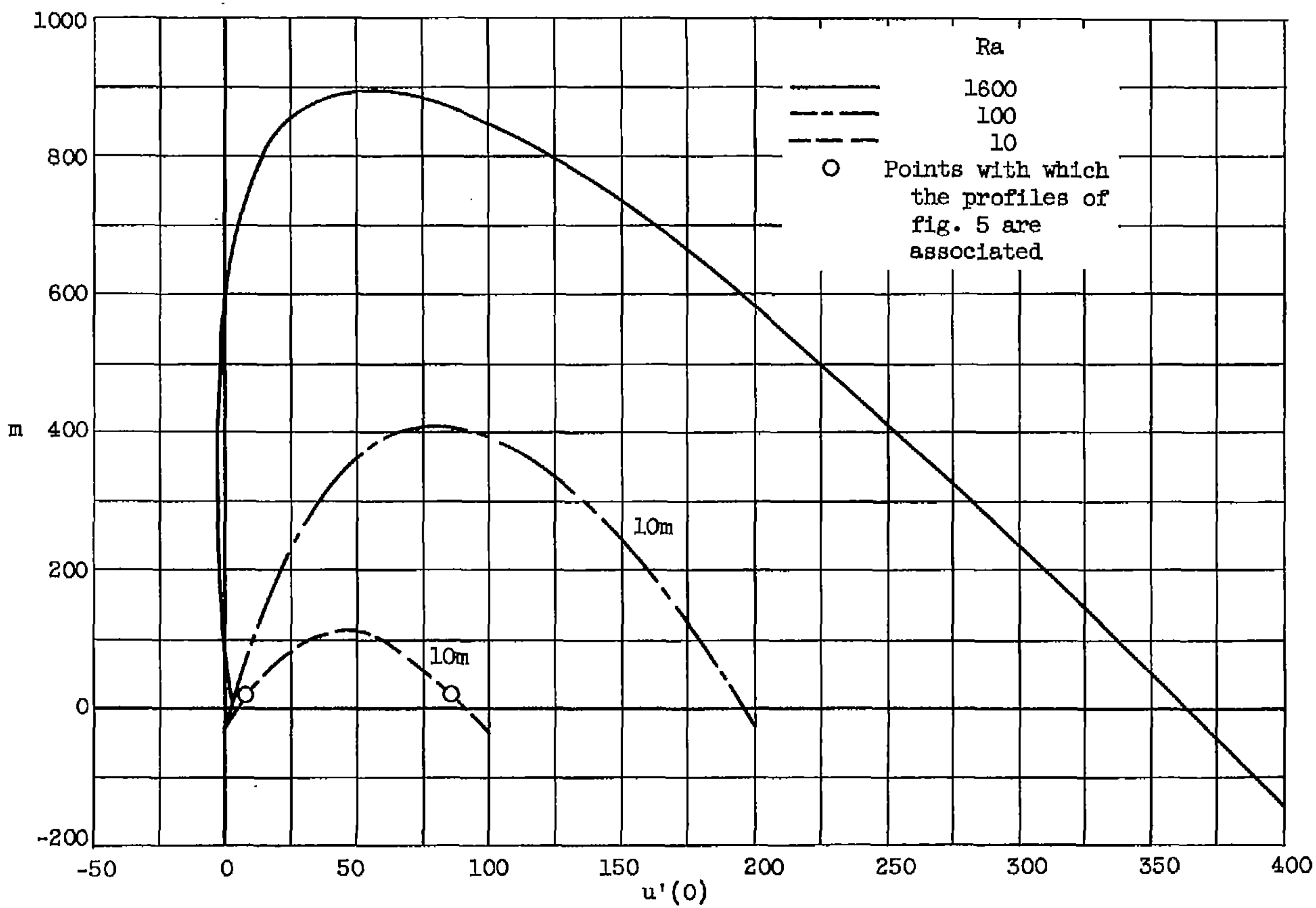
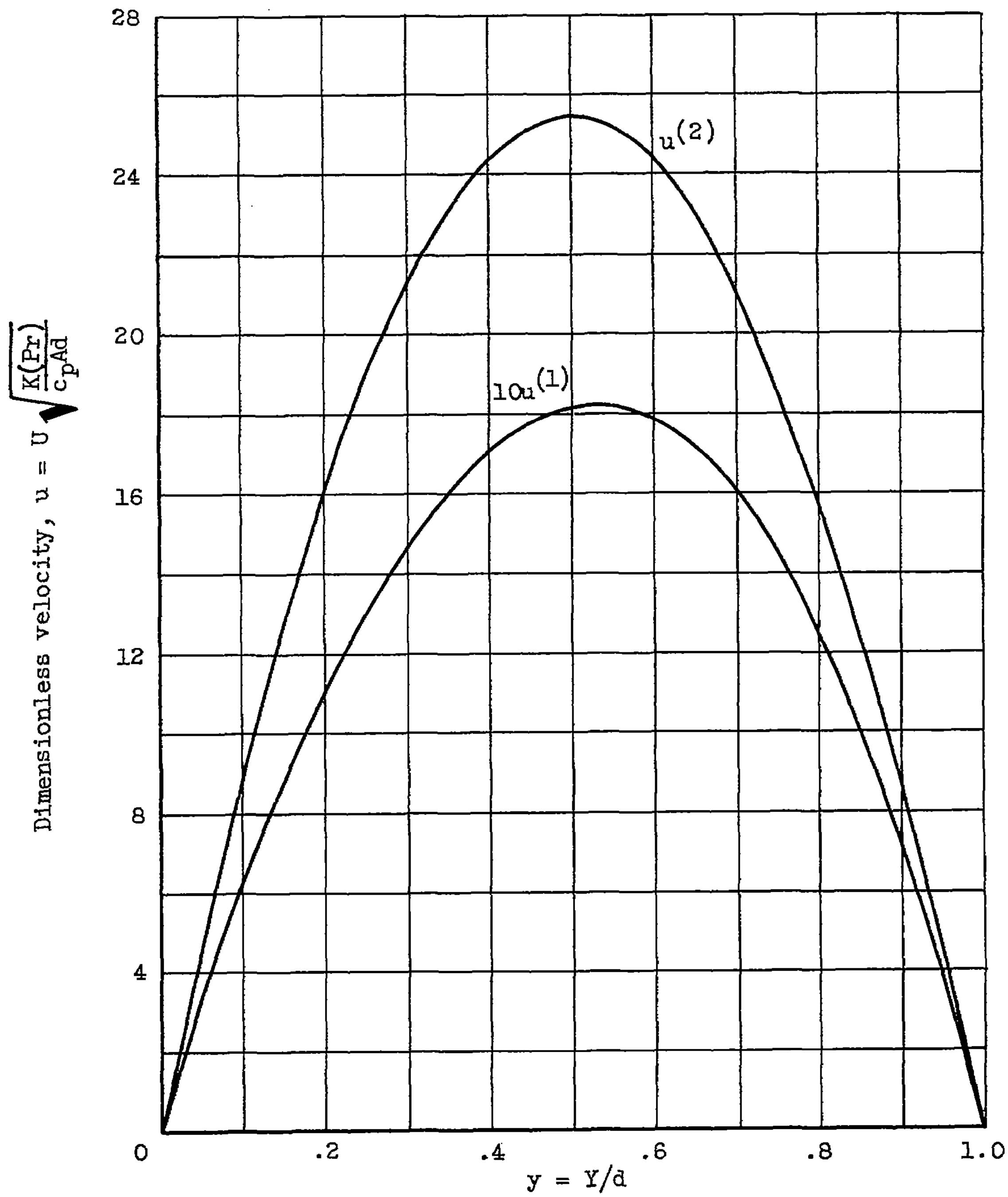


Figure 4. - The m-curves for various values of Rayleigh number. K = 10; α = 0.



(a) Velocity.

Figure 5. - A representative pair of profiles for Ra = 10, K = 10, m = 2, α = 0, and C = -1.

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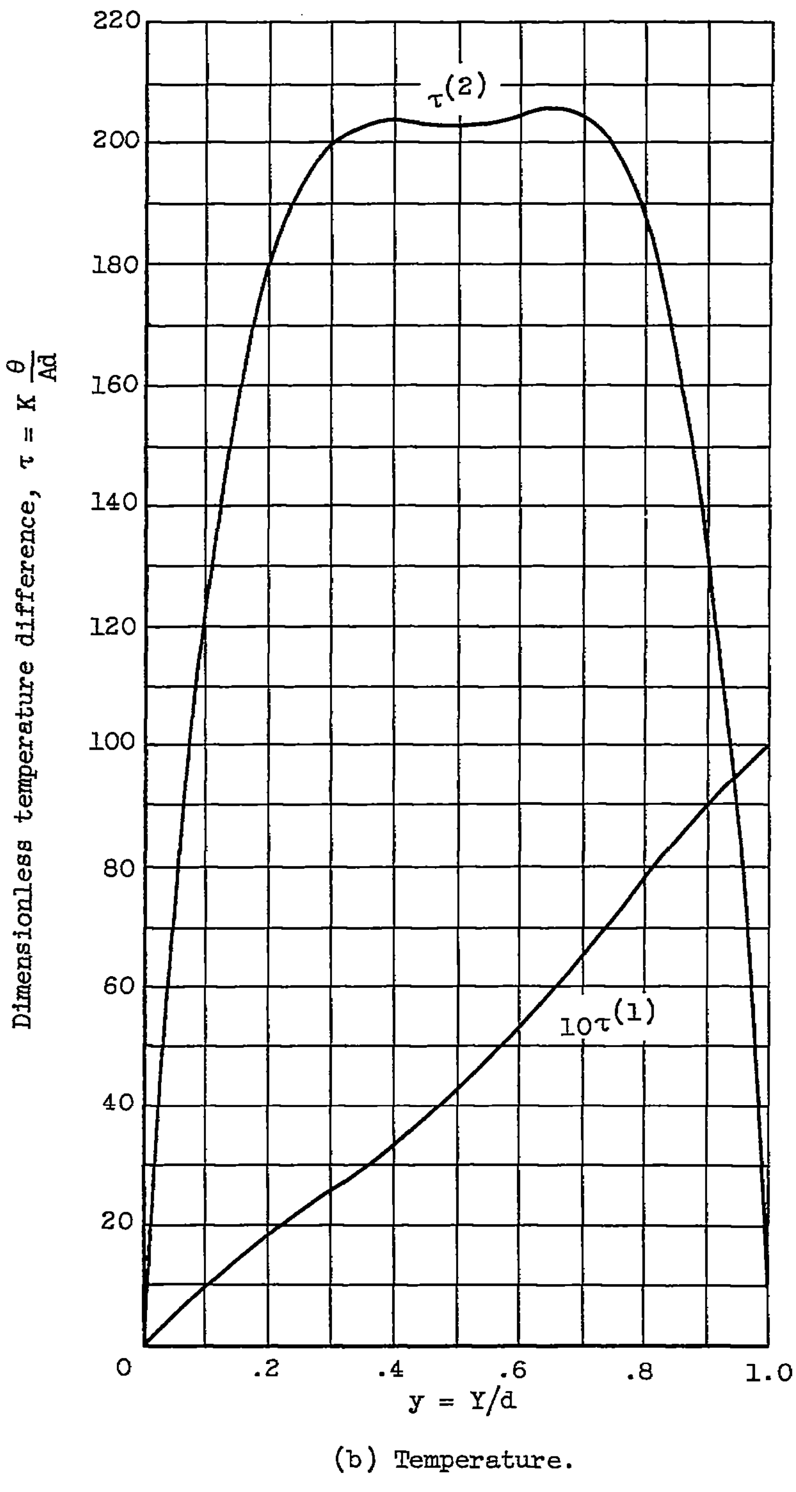

(b) Temperature.

Figure 5. - Concluded. A representative pair of profiles for Ra = 10, K = 10, m = 2, α = 0, and C = -1.