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ANALYSIS OF HARMONIC FORCES PRODUCED AT HUB BY IMBALANCES IN HELICOPTER ROTOR BLADES

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SUMMARY

General explicit expressions are derived for the harmonic forces produced at the hub by an n-bladed unbalanced helicopter rotor. Imbalances due to property differences among the blades and to nonuniform spacing between the blades are considered. It is shown mathematically that these two types of imbalances have equivalent effects. Forces applied to the hub both in and normal to the plane of rotation of the blades are analyzed. The additional first harmonic forces transmitted to the hub in the plane of rotation by an asymmetric rotor-blade system may be especially appreciable due to the influence of the high static centrifugal forces exerted by each blade. Simple expressions are derived for the amplitudes of these forces. The effect of small simultaneous property and spacing imbalances can be obtained by superposition of the separate effects of each imbalance. Numerical examples are given throughout to illustrate the order of magnitude of the results obtained herein. For convenient reference, an analysis of the forces transmitted to the hub by a balanced rotor of n blades is given.

INTRODUCTION

The purpose of this investigation is to present an analysis of the effect of imbalances in helicopter rotor blades on the net forces produced by n blades at the hub in flight. Two types of imbalances are distinguished in this analysis. One type, to be denoted as a "property imbalance," occurs when one or more blades differ slightly in construction from that of a standard or reference blade in the rotor. For example, a blade may have a slightly different blade angle, blade twist, or chord distribution from that of the reference blade. The other type of imbalance, to be denoted herein as "eccentricity imbalance," is due to slightly unequal angular (azimuth) spacings between the blades. Both types of imbalances, which result in an azimuthal lack of perfect symmetry in the rotor-blade system, are usually unintentional, and hence small, and are due to imperfections or tolerances in manufacturing. Basically, the effect of a property imbalance is to cause the unbalanced blade, at any given azimuth position, to transmit to the hub a dynamic or aerodynamic load different from that of the reference blade when it is at the same azimuth. It will be proved, moreover, that the effect of an eccentricity imbalance is analogous to that of a property imbalance.

A study of the effect of rotor imbalances is of practical interest, since even a comparatively small imbalance may be capable of producing a significant effect on the harmonic loads transmitted to the hub simultaneously by all of the blades. The reason for this is that a balanced rotor with n blades will produce at the hub only harmonic forces with frequencies which are multiples of n (e.g., refs. 1 to 3). Consequently, only the higher harmonics acting on a single blade will be transmitted to the hub by a balanced rotor of more than two blades. For a rotor of three blades, for example, only the second and higher harmonic loads acting on a single blade will contribute to the lowest (third) harmonic forces acting at the hub. An imbalance in the rotor, however, will transmit additional harmonic loads which otherwise would not appear. In particular, appreciable first harmonic loads at the hub may now appear.

A systematic analysis of the effect of rotor imbalance on the net forces transmitted to the hub by all of the blades does not appear to have been given thus far in the literature. The aim of the present report is to furnish such an analysis. A comparatively brief discussion of the effect of rotor imbalance is given in reference 3.

The analysis in the present investigation is sufficiently general to take account of simultaneous property and eccentricity imbalances in any number of the blades of a rotor. Moreover, this analysis is also sufficiently general to take account of property imbalances due to any cause. Restriction to a particular type of property imbalance is made only in the numerical examples. Finally, forces applied to the hub both in and normal to the plane of rotation of the blades are analyzed.

This investigation consists of three main sections. First, an analysis of loads transmitted to the hub by balanced blades will be given. Although a limited number of references on this subject exist (refs. 1 to 3), it appears worthwhile to include a separate section on balanced blades for convenient reference and for a clearer appreciation of the analysis and effect of unbalanced blades. In the second section, the additional loads transmitted to the hub in a direction normal to the plane of rotation of the blades by imbalances in a rotor are derived. The effects of property imbalances, eccentricity imbalances, and simultaneous property and eccentricity imbalances in the blades are analyzed in general terms, and general conclusions of physical interest are drawn. Numerical examples are then given to illustrate the results. The third section deals similarly with the loads produced at the hub in the plane of rotation by unbalanced blades. In the entire analysis, the results are given in terms of the forces transmitted to the hub by a single rotating helicopter blade in flight, and these are regarded as known or given.

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SYMBOLS

A_{e1} amplitude of increment of first harmonic load component produced at hub by small eccentricity imbalances

A_{t1} amplitude of increment of first harmonic load component produced at hub by small simultaneous property and eccentricity imbalances

$\left.\begin{array}{l}\Delta a_{x'm}, \Delta a_{y'm}, \Delta a_{zm}, \\ \Delta b_{x'm}, \Delta b_{y'm}, \Delta b_{zm}\end{array}\right\}$ coefficients of increments in mth harmonic forces in x'-, y'-, and z-directions, respectively, produced at hub by property imbalances in a rotor (eqs. (18a) and (33))

$\left.\begin{array}{l}\Delta a'_{x'm}, \Delta a'_{y'm}, \Delta a'_{zm}, \\ \Delta b'_{x'm}, \Delta b'_{y'm}, \Delta b'_{zm}\end{array}\right\}$ coefficients of increments in mth harmonic forces in x'-, y'-, and z-directions, respectively, produced at hub by eccentricity imbalances in a rotor (eqs. (25a) and (39a))

$\left.\begin{array}{l}\Delta a''_{x'm}, \Delta a''_{y'm}, \Delta a''_{zm}, \\ \Delta b''_{x'm}, \Delta b''_{y'm}, \Delta b''_{zm}\end{array}\right\}$ coefficients of increments in mth harmonic forces in x'-, y'-, and z-directions, respectively, produced at hub by simultaneous property and eccentricity imbalances in a rotor (eqs. (29a) and (44a))

e distance between blade hinge and axis of rotation (fig. 1)

F_{xm}, F_{ym}, F_{zm} net mth harmonic forces in x-, y-, and z-directions, respectively, transmitted by rotor to hub

$F_{xms}, F_{yms}, F_{zms}$ mth harmonic forces in x-, y-, and z-directions, respectively, produced at hub by a single blade of a balanced rotor

$\overline{F}_{xms}, \overline{F}_{yms}, \overline{F}_{zms}$ mth harmonic forces in x-, y-, and z-directions, respectively, produced at hub by a single blade of an unbalanced rotor

$F_{x'}, F_{y'}, F_z$	total forces in x'-, y'-, and z-directions, respectively, produced at hub by a balanced rotor
$\Delta F_{x'}, \Delta F_{y'}, \Delta F_z$	total additional forces in x'-, y'-, and z-directions, respectively, produced at hub by rotor due to imbalances
$f_{xm}, f_{ym}, f_{zm}, g_{xm}, g_{ym}, g_{zm}$	coefficients of mth harmonic forces in x-, y-, and z-directions, respectively, transmitted to hub by a single blade (eqs. (1) and (8))
$\Delta f_{xmj}, \Delta f_{ymj}, \Delta f_{zmj}, \Delta g_{xmj}, \Delta g_{ymj}, \Delta g_{zmj}$	increments in values of f_{xm}, f_{ym}, f_{zm}, g_{xm}, g_{ym}, and g_{zm} for jth blade at azimuth position ψ_j due to property imbalance of this blade
$\Delta f_{zmj}', \Delta g_{zmj}'$	increments in the values of f_{zm} and g_{zm} for jth blade due to eccentricity imbalance of this blade (eq. (23a))
$(\Delta f_{xmj})_{eq}, (\Delta f_{ymj})_{eq}, (\Delta f_{zmj})_{eq}, (\Delta g_{xmj})_{eq}, (\Delta g_{ymj})_{eq}, (\Delta g_{zmj})_{eq}$	values of Δf_{xmj}, Δf_{ymj}, Δf_{zmj}, Δg_{xmj}, Δg_{ymj}, and Δg_{zmj} which will produce same effect as eccentricity imbalance or combined property and eccentricity imbalance
j	integer denoting azimuth position of a blade of rotor, $j = 0,1,2,. . .,n - 1$
m	integer denoting frequency of harmonic forces
n	number of rotor blades
r	distance of blade element from axis of rotation (fig. 1)
s	distance along a blade, measured from root
t	time
x,y,z	Cartesian coordinate system of a rotating blade (z coincides with axis of rotation, and x coincides with arm of blade; see fig. 1)
x',y'	fixed Cartesian coordinate system in plane of rotation of rotor

ϵ_j eccentricity imbalance of jth blade (eq. (21))

η magnitude of imbalance in blade angle of a blade

ϑ blade angle

ρ mass per unit length of a blade

ψ azimuth angle

ψ_j azimuth position of jth blade in a balanced rotor (eq. (10))

ψ_j' azimuth position of jth blade in an unbalanced rotor (eq. (21))

Ω angular speed of rotor

Subscripts:

e denotes eccentricity imbalances

p denotes property imbalances

t denotes combined property and eccentricity imbalances

FORCES TRANSMITTED TO HUB BY BALANCED BLADES

A concise analysis of harmonic forces transmitted to the hub by n balanced blades of a helicopter rotor is given in this section.

Forces Normal to Plane of Rotation of Blades

Let the mth harmonic force transmitted to the hub normal to the plane of rotation by a single rotating blade of a helicopter rotor of n blades be

$$F_{zms}(\psi) = f_{zm} \sin m\psi + g_{zm} \cos m\psi \tag{1}$$

where ψ is the azimuth position of the blade and f_{zm} and g_{zm} are independent of ψ. Then, at any instant, the total mth harmonic force in the z direction due to all of the uniformly spaced blades will be

$$F_{zm} = \sum_{j=0}^{n-1} \left[f_{zm} \sin m\left(\psi + \frac{2\pi j}{n}\right) + g_{zm} \cos m\left(\psi + \frac{2\pi j}{n}\right)\right] \tag{2}$$

If the blades are all exactly alike, as is assumed in this section, then f_{zm} and g_{zm} are the same for all blades, that is, these coefficients are independent of j.

It is observed that (e.g., as shown in ref. 2)

$$\left.\begin{aligned} \sum_{j=0}^{n-1} \sin m\left(\psi + \frac{2\pi j}{n}\right) &= \varphi(m,n) \sin m\psi \\ \sum_{j=0}^{n-1} \cos m\left(\psi + \frac{2\pi j}{n}\right) &= \varphi(m,n) \cos m\psi \end{aligned}\right\} \tag{3}$$

where the function $\varphi(m,n)$ is defined by[1]

$$\left.\begin{aligned} \varphi(m,n) &= 0 \quad \text{if } m \text{ is not a multiple of } n \\ \varphi(m,n) &= n \quad \text{if } m \text{ is a multiple of } n \end{aligned}\right\} \tag{4}$$

It then follows from equation (2) that for balanced blades

$$F_{zm} = \varphi(m,n)\left(f_{zm} \sin m\psi + g_{zm} \cos m\psi\right) \tag{5}$$

The total forces transmitted to the hub by n balanced blades over all harmonics will be

$$F_z = \sum_{m=0}^{\infty} F_{zm} \tag{6}$$

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[1]The function $\varphi(m,n)$ can, if desired, be expressed in terms of the comparatively familiar Kronecker delta symbol δ_{ij} defined by $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. Thus, $\varphi(m,n) = n\delta_{m,kn}(k = 0, 1, 2, \ldots)$.

Hence, from equation (5), it follows that

$$F_z = n \sum_{k=0} \left(f_{z,kn} \sin kn\psi + g_{z,kn} \cos kn\psi\right) \tag{7}$$

Forces in Rotor Plane of Rotation

Let (x,y) be Cartesian coordinate axes in the plane of rotation of the blades fixed to a typical blade of an n-bladed rotor and hence rotating with the blade. Moreover, let (x',y') be fixed Cartesian coordinate axes in the rotor plane of rotation and let the angle between the x' and x axes be ψ. Finally, let the mth harmonic forces transmitted to the hub in the x and y directions by a single rotating blade at instantaneous azimuth position ψ be

$$\left.\begin{aligned} F_{xms}(\psi) &= f_{xm} \sin m\psi + g_{xm} \cos m\psi \\ F_{yms}(\psi) &= f_{ym} \sin m\psi + g_{ym} \cos m\psi \end{aligned}\right\} \tag{8}$$

Then the forces in the x'- and y'-directions, due to F_{xms} and F_{yms}, transmitted to the hub at any instant by the n uniformly spaced blades of the rotor are

$$\left.\begin{aligned} F_{x'm} &= \sum_{j=0}^{n-1} \left[F_{xms}(\psi_j)\cos \psi_j - F_{yms}(\psi_j)\sin \psi_j\right] \\ F_{y'm} &= \sum_{j=0}^{n-1} \left[F_{xms}(\psi_j)\sin \psi_j + F_{yms}(\psi_j)\cos \psi_j\right] \end{aligned}\right\} \tag{9}$$

where

$$\psi_j \equiv \psi + \frac{2\pi j}{n} \tag{10}$$

and $F_{xms}(\psi_j)$ and $F_{yms}(\psi_j)$ are obtained from equations (8) by replacing ψ by ψ_j there. For identical blades, f_{xm}, f_{ym}, g_{xm}, and g_{ym} are all independent of j.

The following well-known trigonometric relations are noted:

$$\left.\begin{aligned}
\sin A \cos B &= \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right] \\
\cos A \cos B &= \frac{1}{2}\left[\cos(A + B) + \cos(A - B)\right] \\
\sin A \sin B &= \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]
\end{aligned}\right\} \tag{11}$$

By using equations (11) and (3) in conjunction with equations (9) and then summing over all harmonics m, the following expressions are obtained for the total loads in the x'- and y'-directions transmitted by n balanced blades to the hub:

$$\left.\begin{aligned}
F_{x'} = \frac{1}{2}\sum_{m=0} \Big\{ &\varphi(m+1,n)\Big[(f_{xm} - g_{ym})\sin(m+1)\psi + (g_{xm} + f_{ym})\cos(m+1)\psi\Big] + \\
&\varphi(m-1,n)\Big[(f_{xm} + g_{ym})\sin(m-1)\psi + (g_{xm} - f_{ym})\cos(m-1)\psi\Big]\Big\} \\
F_{y'} = \frac{1}{2}\sum_{m=0} \Big\{ &\varphi(m+1,n)\Big[(g_{ym} - f_{xm})\cos(m+1)\psi + (g_{xm} + f_{ym})\sin(m+1)\psi\Big] + \\
&\varphi(m-1,n)\Big[(f_{xm} + g_{ym})\cos(m-1)\psi + (f_{ym} - g_{xm})\sin(m-1)\psi\Big]\Big\}
\end{aligned}\right\} \tag{12}$$

Equations (12) can be written in a slightly more explicit form as

$$\left.\begin{aligned}
F_{x'} = \frac{1}{2}n\sum_{k=0} \Big[&\left(f_{x,kn-1} - g_{y,kn-1} + f_{x,kn+1} + g_{y,kn+1}\right)\sin kn\psi + \\
&\left(g_{x,kn-1} + f_{y,kn-1} + g_{x,kn+1} - f_{y,kn+1}\right)\cos kn\psi\Big] \\
F_{y'} = \frac{1}{2}n\sum_{k=0} \Big[&\left(g_{y,kn-1} - f_{x,kn-1} + f_{x,kn+1} + g_{y,kn+1}\right)\cos kn\psi + \\
&\left(g_{x,kn-1} + f_{y,kn-1} + f_{y,kn+1} - g_{x,kn+1}\right)\sin kn\psi\Big]
\end{aligned}\right\} \tag{13}$$

where

$$f_{x,-1} = f_{y,-1} = g_{x,-1} = g_{y,-1} = 0$$

In all the preceding equations, the explicit dependence of the forces on time follows from the relation $\psi = \Omega t$ where Ω is the angular speed of the rotor and t is the time.

General Implications

From equation (7), it is seen that the only frequencies produced at the hub in a direction normal to the plane of rotation of the blades by all of the n balanced blades of a rotor will be harmonics which are multiples of n. Moreover, only those harmonics transmitted by a single blade which are multiples of n will contribute to the net forces at the hub produced by the n blades.

From equations (12) and (13), it is seen that, as in the case of forces normal to the plane of rotation, the net forces produced at the hub in the plane of rotation by all of the n blades of a balanced rotor will be harmonics which are multiples of n. From equations (12) and (13), it also follows, however, that the forces transmitted by a single blade which contribute to the net forces transmitted by n blades will be those harmonics which differ by 1 from the multiples of n. For example, for a balanced rotor of three blades, the net forces acting at the hub, in and normal to the rotor plane of rotation, due to the n blades will be the zeroth (static), third, sixth, and so forth harmonics. These same harmonics acting on a single blade normal to the plane of rotation will contribute to the net normal forces acting at the hub. On the other hand, the first, second, fourth, fifth, seventh, and so forth harmonic forces acting on a single blade in the plane of rotation will contribute to the net forces acting in the plane of rotation at the hub. A tabular summary, without formal derivation, illustrating these results is given in reference 1.

From equations (13), it can be shown directly, by determining $\left(F_{x'}^{2} + F_{y'}^{2}\right)$, that the magnitude of the resultant force in the plane of rotation transmitted to the hub by a balanced rotor will be constant with time, or independent of ψ, if and only if a single net harmonic force acts at the hub and only a single harmonic force acting on an individual blade has contributed to this net harmonic force at the hub. In such a case, this net force at the hub will rotate with constant magnitude in the plane of rotation at angular speed equal to a multiple of $n\Omega$ (usually $n\Omega$ itself).

FORCES TRANSMITTED TO HUB NORMAL TO PLANE OF ROTATION BY UNBALANCED BLADES

In this section, the general effect of imbalances in a rotor of n blades on the net forces transmitted to the hub normal to the plane of rotation of the blades is derived. This effect, in general, can be expressed in the form of increments in the harmonic loads transmitted by a balanced rotor. General expressions for these increments will be developed here. Property imbalances, eccentricity imbalances, and combined property and eccentricity imbalances will be considered.

Property Imbalances

Consider a rotor with n uniformly spaced blades. Then, at any instant, the blades may be considered to be situated at azimuth angles $\psi_j (j = 0, 1, 2, \ldots, n - 1)$ in accordance with equation (10). Considering the blade at azimuth ψ_0 as a reference blade, suppose that any other blade at azimuth $\psi_j (j \neq 0)$ has properties different from those of the reference blade. Then for a given azimuth position, this blade will as a result transmit dynamic or aerodynamic forces to the hub different from those transmitted by the reference blade when the latter is at this position. Let the total mth harmonic-force component transmitted to the hub by the reference blade normal to the plane of rotation be given by the right side of equation (1). Then the total mth harmonic-force component transmitted to the hub in the z-direction by an unbalanced blade when it is at any azimuth position, will be

$$F_{zmj}(\psi) = \left(f_{zm} + \Delta f_{zmj}\right)\sin m\psi + \left(g_{zm} + \Delta g_{zmj}\right)\cos m\psi \qquad (14)$$

Here, Δf_{zmj} and Δg_{zmj} denote the increments in the amplitudes of the mth harmonic forces transmitted to the hub by the jth blade due to its property imbalance. If several blades have imbalances, then equation (14) holds for each of these blades, with a different value of Δf_{zmj} and Δg_{zmj} for each blade.

The total mth harmonic force transmitted to the hub in the z-direction by the uniformly spaced, but different, blades is

$$F_{zm} = \sum_{j=0}^{n-1} F_{zmj}(\psi_j) \tag{15}$$

where $F_{zmj}(\psi_j)$ is obtained from equation (14) by replacing ψ by ψ_j. Equation (15) yields

$$F_{zm} = \left(F_{zm}\right)_b + \left(\Delta F_{zm}\right)_p \tag{16}$$

where $\left(F_{zm}\right)_b$ is given by the right side of equation (2) while

$$\left(\Delta F_{zm}\right)_p = \sum_{j=1}^{n-1} \left(\Delta f_{zmj} \sin m\psi_j + \Delta g_{zmj} \cos m\psi_j\right) \tag{17}$$

According to equation (16), $\left(\Delta F_{zm}\right)_p$ represents the increment in the net mth harmonic load transmitted to the hub by the n blades due to their imbalances. Summing over all harmonics, it is found that the total of these increments is

$$\left(\Delta F_z\right)_p = \sum_{m=0} \left(\Delta a_{zm} \sin m\psi + \Delta b_{zm} \cos m\psi\right) \tag{18a}$$

where

$$\left.\begin{aligned} \Delta a_{zm} &= \sum_{j=1}^{n-1} \left(\Delta f_{zmj} \cos \frac{2\pi mj}{n} - \Delta g_{zmj} \sin \frac{2\pi mj}{n}\right) \\ \Delta b_{zm} &= \sum_{j=1}^{n-1} \left(\Delta f_{zmj} \sin \frac{2\pi mj}{n} + \Delta g_{zmj} \cos \frac{2\pi mj}{n}\right) \end{aligned}\right\} \tag{18b}$$

Equations (18a) and (18b) can be used to calculate directly the effect of property imbalances in the rotor blades on the net forces transmitted to the hub, if the effect of each property imbalance on the forces acting on a single blade is known, that is, if the values of Δf_{zmj} and Δg_{zmj} are known.

Equations (18a) and (18b) indicate that the effect of property imbalances in a rotor is to transmit to the hub in the z-direction additional forces in all harmonics. This is in contrast to the case of balanced blades, in which only harmonics which are multiples of the number of blades are transmitted to the hub normal to the plane of rotation. The first harmonics, in particular, in equations (18a) and (18b) may have appreciable amplitudes.

As a numerical example, to illustrate order of magnitude, consider a rotor of three blades with the following values of the load coefficients[2] in pounds:

$$\left.\begin{array}{lll} g_{z0} = 1516 & f_{z1} = -1561 & g_{z1} = -1078 \\ f_{z2} = 162.2 & g_{z2} = -76.3 & f_{z3} = 78.7 \\ g_{z3} = -64.3 & f_{z4} = 12.2 & g_{z4} = 1.8 \end{array}\right\} \qquad (19)$$

Then, according to equation (7), the net nonstatic force transmitted by the three balanced blades will be a third harmonic force with an amplitude of $3\sqrt{(78.7)^2 + (64.3)^2} = 305$ pounds. Suppose now that one of the blades, corresponding to $j = 1$, for example, has a slightly different blade angle, namely, $\vartheta = \vartheta_0(1 + \eta)$, from that of the other two $(\vartheta = \vartheta_0)$. For purposes of a simple order-of-magnitude calculation, it may be assumed that the forces transmitted to the hub in the z-direction by a single blade are proportional to the blade angle. Then for the blade at $j = 1$, $\Delta f_{zm1} = \eta f_{zm}$ and $\Delta g_{zm1} = \eta g_{zm}$. Hence equations (18b) yield

$$\Delta a_{zm} = \eta\left(f_{zm} \cos \frac{2\pi m}{3} - g_{zm} \sin \frac{2\pi m}{3}\right)$$

$$\Delta b_{zm} = \eta\left(f_{zm} \sin \frac{2\pi m}{3} + g_{zm} \cos \frac{2\pi m}{3}\right)$$

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[2]These values, and all other values given in succeeding numerical examples in this report, are based on theoretical calculations which were made in ref. 4 for a three-bladed helicopter of gross weight 4,660 pounds in forward flight at an advance ratio μ of 0.3. Ref. 4 is essentially a slight modification of the analysis and calculations made in ref. 2. The rotating (x,y,z) coordinate system pertinent to these calculations is shown in fig. 1.

With the data in equations (19), equation (18a) thus yields

$$(\Delta F_z)_p = \eta(1516 + 1713 \sin \psi - 811 \cos \psi - 147.1 \sin 2\psi - 102.3 \cos 2\psi + 78.7 \sin 3\psi - 64.3 \cos 3\psi - 7.66 \sin 4\psi + 9.7 \cos 4\psi) \tag{20}$$

The most important additional nonstatic harmonic-load component which occurs here is the first harmonic load, which has an amplitude of $1,900\eta$ pounds. Thus, if $\eta = 0.02$, the imbalances here produce a first harmonic-load component at the hub with an amplitude which is 0.82 percent of the gross helicopter weight, but which is 12.5 percent of the amplitude of the net (third) harmonic load transmitted by balanced blades.

Eccentricity Imbalances

To determine the effect of unequal azimuth spacings of the blades, it will first be assumed that all the blades of the rotor are exactly alike.

Let the azimuth angle of each of the n blades of a rotor, at any instant, be

$$\psi_j' = \psi + \frac{2\pi j}{n} + \epsilon_j \tag{21}$$

where $j = 0, 1, 2, \ldots n - 1$ and $\epsilon_0 = 0$, so that the blade corresponding to $j = 0$ will be considered as the reference blade. For uniform azimuth spacing, $\epsilon_j = 0$ for all values of j. The mth harmonic forces transmitted to the hub in the z-direction by all n blades are

$$F_{zm} = \sum_{j=0}^{n-1} \left(f_{zm} \sin m\psi_j' + g_{zm} \cos m\psi_j'\right) \tag{22}$$

Subtracting the result for uniformly spaced blades, equation (22) leads to the following expression for the increments in these forces:

$$\left(\Delta F_{zm}\right)_e = \sum_{j=0}^{n-1}\left(\Delta f_{zmj}' \sin m\psi_j + \Delta g_{zmj}' \cos m\psi_j\right) \tag{23a}$$

where ψ_j is given by equation (10) and

$$\left.\begin{aligned} \Delta f_{zmj}' &= f_{zm}\left(\cos m\epsilon_j - 1\right) - g_{zm} \sin m\epsilon_j \\ \Delta g_{zmj}' &= f_{zm} \sin m\epsilon_j + g_{zm}\left(\cos m\epsilon_j - 1\right) \end{aligned}\right\} \tag{23b}$$

For small values of ϵ_j, equations (23b) to first powers of ϵ_j become:

$$\left.\begin{aligned} \Delta f_{zmj}' &= -mg_{zm}\epsilon_j \\ \Delta g_{zmj}' &= mf_{zm}\epsilon_j \end{aligned}\right\} \tag{23c}$$

A comparison of equations (23a) with equation (17) shows that the effect of eccentricity imbalances is equivalent to property imbalances. For example, the effect of eccentricity imbalances can be produced if each blade has property imbalances such that

$$\left.\begin{aligned} \left(\Delta f_{zmj}\right)_{eq} &= \Delta f_{zmj}' \\ \left(\Delta g_{zmj}\right)_{eq} &= \Delta g_{zmj}' \end{aligned}\right\} \tag{24a}$$

where $\Delta f_{zmj}'$ and $\Delta g_{zmj}'$ are given by equations (23b) or (23c) and the subscript eq denotes property imbalances which will produce the same results as the given eccentricity imbalances. For small spacing differences, these equivalence relationships become[3]

$$\left.\begin{aligned} \left(\Delta f_{zmj}\right)_{eq} &= -mg_{zm}\epsilon_j \\ \left(\Delta g_{zmj}\right)_{eq} &= mf_{zm}\epsilon_j \end{aligned}\right\} \tag{24b}$$

From equations (23a) and (23b) it follows that the total increments, over all harmonics, in the loads transmitted to the hub normal to the plane of rotation which are due to nonuniform azimuth spacings of the blades are

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[3]The most general equivalence relations can be obtained by writing $\left(\Delta a_{zm}\right)_{eq} = \Delta a'_{zm}$ and $\left(\Delta b_{zm}\right)_{eq} = \Delta b'_{zm}$ and using equations (18b) and (25b) (or (25c)).

$$\left(\Delta F_z\right)_e = \sum_m \left(\Delta a'_{zm} \sin m\psi + \Delta b'_{zm} \cos m\psi\right) \tag{25a}$$

where

$$\left.\begin{aligned} \Delta a'_{zm} &= \sum_{j=1}^{n-1} \left(\Delta f_{zmj}' \cos \frac{2\pi mj}{n} - \Delta g_{zmj}' \sin \frac{2\pi mj}{n}\right) \\ \Delta b'_{zm} &= \sum_{j=1}^{n-1} \left(\Delta f_{zmj}' \sin \frac{2\pi mj}{n} + \Delta g_{zmj}' \cos \frac{2\pi mj}{n}\right) \end{aligned}\right\} \tag{25b}$$

and $\Delta f_{zmj}'$ and $\Delta g_{zmj}'$ are given explicitly by equations (23b) and (23c). For small eccentricities in the blade spacings, equations (25b) become

$$\left.\begin{aligned} \Delta a'_{zm} &= -m\left(g_{zm}\sum_{j=1}^{n-1} \epsilon_j \cos \frac{2\pi mj}{n} + f_{zm}\sum_{j=1}^{n-1} \epsilon_j \sin \frac{2\pi mj}{n}\right) \\ \Delta b'_{zm} &= m\left(f_{zm}\sum_{j=1}^{n-1} \epsilon_j \cos \frac{2\pi mj}{n} - g_{zm}\sum_{j=1}^{n-1} \epsilon_j \sin \frac{2\pi mj}{n}\right) \end{aligned}\right\} \tag{25c}$$

Equations (23) and (25) are convenient general expressions permitting a direct calculation of the effect of eccentricity imbalances.

As a numerical example, consider a rotor with three identical blades with the load coefficients given in equations (19). If the blades were equally spaced, the angles between two adjacent blades would be $\frac{2\pi}{3}$ radians. Suppose that the angle between the reference blade ($j = 0$) and the succeeding blade ($j = 1$) is $\frac{2\pi}{3} + \epsilon_1$ and that the third blade is separated from the reference blade by an angle $\frac{4\pi}{3}$ radians. If it is assumed that $|\epsilon_1| \ll 1$ then equations (25c) yield

$$\Delta a'_{zm} = -m\epsilon_1\left(g_{zm} \cos \frac{2\pi m}{3} + f_{zm} \sin \frac{2\pi m}{3}\right)$$

$$\Delta b'_{zm} = m\epsilon_1\left(f_{zm} \cos \frac{2\pi m}{3} - g_{zm} \sin \frac{2\pi m}{3}\right)$$

Evaluating $\Delta a'_{zm}$ and $\Delta b'_{zm}$ from the data in equations (19) and inserting them into equation (25a) yield

$$\left(\Delta F_z\right)_e = \epsilon_1(813 \sin \psi + 1715 \cos \psi + 206 \sin 2\psi - 294 \cos 2\psi + \ldots) \tag{26}$$

The magnitude of the first harmonic-load component in $\left(\Delta F_z\right)_e$ is thus $\epsilon_1\sqrt{(813)^2 + (1715)^2} = 1900\epsilon_1$ pounds. If $\epsilon_1 = 0.02$ radian (which is 1.15°), then this produces a first harmonic force at the hub with an amplitude of 38 pounds, which is 12.5 percent of the amplitude of the net third harmonic load transmitted to the hub by balanced blades.

Simultaneous Property and Eccentricity Imbalances

If the n blades of a rotor have both property and eccentricity imbalances, then equations (15) or (22) must be replaced by

$$F_{zm} = \sum_{j=0}^{n-1}\left[\left(f_{zm} + \Delta f_{zmj}\right)\sin m\psi_j' + \left(g_{zm} + \Delta g_{zmj}\right)\cos m\psi_j'\right] \tag{27}$$

It is assumed that $\Delta f_{zmo} = \Delta g_{zmo} = \epsilon_o = 0$; that is, the blade at $j = 0$ is taken as a reference blade. Subtracting from equation (27) the net load transmitted by balanced blades, the following general expression is obtained for the increments in the mth harmonic forces transmitted to the hub by a rotor with both property and eccentricity imbalances:

$$\begin{aligned}\left(\Delta F_{zm}\right)_t = \sum_{j=0}^{n-1}\Big\{&\Big[f_{zm}\left(\cos m\epsilon_j - 1\right) + \Delta f_{zmj} \cos m\epsilon_j - \\ &\left(g_{zm} + \Delta g_{zmj}\right)\sin m\epsilon_j\Big]\sin m\psi_j + \Big[\left(f_{zm} + \Delta f_{zmj}\right)\sin m\epsilon_j + \\ &g_{zm}\left(\cos m\epsilon_j - 1\right) + \Delta g_{zmj} \cos m\epsilon_j\Big]\cos m\psi_j\Big\}\end{aligned} \tag{28}$$

Summing over all harmonics, it is found that the total of these increments is

$$(\Delta F_z)_t = \sum_m \left(\Delta a''_{zm} \sin m\psi + \Delta b''_{zm} \cos m\psi\right) \tag{29a}$$

where

$$\left.\begin{aligned}
\Delta a''_{zm} &= \sum_{j=1}^{n-1} \left\{\left[f_{zm}(\cos m\epsilon_j - 1) + \Delta f_{zmj} \cos m\epsilon_j - \right.\right. \\
&\quad \left.(g_{zm} + \Delta g_{zmj})\sin m\epsilon_j\right]\cos \frac{2\pi j}{n} - \left[(f_{zm} + \Delta f_{zmj})\sin m\epsilon_j + \right. \\
&\quad \left.\left. g_{zm}(\cos m\epsilon_j - 1) + \Delta g_{zmj} \cos m\epsilon_j\right]\sin \frac{2\pi j}{n}\right\} \\
\Delta b''_{zm} &= \sum_{j=1}^{n-1} \left\{\left[f_{zm}(\cos m\epsilon_j - 1) + \Delta f_{zm} \cos m\epsilon_j - \right.\right. \\
&\quad \left.(g_{zm} + \Delta g_{zmj})\sin m\epsilon_j\right]\sin \frac{2\pi j}{n} + \left[(f_{zm} + \Delta f_{zmj})\sin m\epsilon_j + \right. \\
&\quad \left.\left. g_{zm}(\cos m\epsilon_j - 1) + \Delta g_{zmj} \cos m\epsilon_j\right]\cos \frac{2\pi j}{n}\right\}
\end{aligned}\right\} \tag{29b}$$

For small imbalances, equations (29b) to first powers of Δf_{zmj}, Δg_{zmj}, and ϵ_j reduce to

$$\left.\begin{aligned}
\Delta a''_{zm} &= \sum_{j=1}^{n-1} \left[(\Delta f_{zmj} - g_{zm}m\epsilon_j)\cos \frac{2\pi j}{n} - (f_{zm}m\epsilon_j + \Delta g_{zmj})\sin \frac{2\pi j}{n}\right] \\
\Delta b''_{zm} &= \sum_{j=1}^{n-1} \left[(\Delta f_{zmj} - g_{zm}m\epsilon_j)\sin \frac{2\pi j}{n} + (f_{zm}m\epsilon_j + \Delta g_{zmj})\cos \frac{2\pi j}{n}\right]
\end{aligned}\right\} \tag{29c}$$

A comparison of equations (29a) and (29c) with equations (18a), (18b), (25a), and (25c) shows that, for small imbalances, the effect of simultaneous property and eccentricity imbalances can be obtained simply by superposition, that is, by adding the load increments due to property imbalances alone to the load increments due to eccentricity imbalances alone. In this regard, it may also be noted that the effect of small simultaneous property and eccentricity imbalances is equivalent to that of small property imbalances alone. This equivalence relation may, for example, be given by

$$\left.\begin{aligned}\left(\Delta f_{zmj}\right)_{eq} &= \Delta f_{zmj} - mg_{zm}\epsilon_j \\ \left(\Delta g_{zmj}\right)_{eq} &= \Delta g_{zmj} + mf_{zm}\epsilon_j\end{aligned}\right\} \tag{30}$$

From this equation it is seen that the effect of property imbalances may either be diminished or increased in any harmonic by a simultaneous eccentricity imbalance, depending on the signs of Δf_{zmj}, Δg_{zmj}, and ϵ_j.

As a numerical example, consider a rotor of three blades having the property imbalances discussed in the first example above and the eccentricity imbalances discussed in the second example. Since these imbalances are small, the principle of superposition justified above is valid, and the increments in the loads due to each imbalance alone may be added. Thus the additional first-harmonic load produced by the simultaneous imbalances is

$$\Delta F_{z1} = \left(1713\eta + 813\epsilon_1\right)\sin\psi + \left(-813\eta + 1713\epsilon_1\right)\cos\psi$$

If $\eta = 0.02$ and $\epsilon_1 = 0.02$, the amplitude of this harmonic is $\sqrt{(50.5)^2 + (18.0)^2} = 53.5$ pounds.

FORCES TRANSMITTED TO HUB IN PLANE OF ROTATION BY UNBALANCED BLADES

The effect of imbalances in a rotor of n blades on the forces transmitted to the hub in the rotor plane of rotation will be derived herein. The effects of property imbalances in one or more of the blades, nonuniform spacing between the blades, and the combination of these imbalances will be considered. As in the section on balanced blades, a fixed Cartesian coordinate system (x',y') in the plane of rotation and a system (x,y) rotating with a blade will be used.

Effects of Property Imbalances

At any instant, let the n blades of a rotor have azimuth angles ψ_j. Moreover, let any blade at $\psi_j(j \neq 0)$ have properties different from the corresponding properties of the reference blade at ψ_0. Then, if the mth harmonic forces transmitted to the hub at any instant in the x- and y-directions by the reference blade are given by the right sides of equations (8), the corresponding forces transmitted to the hub at this instant by the jth blade are

$$\left.\begin{aligned} \overline{F}_{xms}(\psi_j) &= \left(f_{xm} + \Delta f_{xmj}\right)\sin m\psi_j + \left(g_{xm} + \Delta g_{xmj}\right)\cos m\psi_j \\ \overline{F}_{yms}(\psi_j) &= \left(f_{ym} + \Delta f_{ymj}\right)\sin m\psi_j + \left(g_{ym} + \Delta g_{ymj}\right)\cos m\psi_j \end{aligned}\right\} \tag{31}$$

where Δf_{xmj}, Δg_{xmj}, Δf_{ymj}, and Δg_{ymj} are the increments in the blade force coefficients f_{xm}, g_{xm}, f_{ym}, and g_{ym}, respectively, due to the property imbalance. The total forces transmitted to the hub by all n blades can be obtained from equations (9) and (10) by replacing $F_{xms}(\psi_j)$ and $F_{yms}(\psi_j)$ in equation (9) by $\overline{F}_{xms}(\psi_j)$ and $\overline{F}_{yms}(\psi_j)$ in accordance with equations (31). The following expressions for the additional loads transmitted to the hub because of these property imbalances are thus obtained:

$$\begin{aligned} \left(\Delta F_{x'}\right)_p = \frac{1}{2} \sum_m \sum_{j=0}^{n-1} \Big[&\left(\Delta f_{xmj} - \Delta g_{ymj}\right)\sin(m + 1)\psi_j + \\ &\left(\Delta f_{xmj} + \Delta g_{ymj}\right)\sin(m - 1)\psi_j + \\ &\left(\Delta g_{xmj} + \Delta f_{ymj}\right)\cos(m + 1)\psi_j + \\ &\left(\Delta g_{xmj} - \Delta f_{ymj}\right)\cos(m - 1)\psi_j \Big] \end{aligned} \tag{32a}$$

$$\begin{aligned}(\Delta F_{y'})_p = \frac{1}{2} \sum_m \sum_{j=0}^{n-1} \Big[&(\Delta f_{ymj} + \Delta g_{xmj}) \sin(m + 1)\psi_j + \\ &(\Delta f_{ymj} - \Delta g_{xmj}) \sin(m - 1)\psi_j + \\ &(\Delta g_{ymj} - \Delta f_{xmj}) \cos(m + 1)\psi_j + \\ &(\Delta g_{ymj} + \Delta f_{xmj}) \cos(m - 1)\psi_j \Big]\end{aligned} \tag{32b}$$

Equations (32) can be written more explicitly in the following form:

$$\left.\begin{aligned}(\Delta F_{x'})_p &= \sum_m (\Delta a_{x'm} \sin m\psi + \Delta b_{x'm} \cos m\psi) \\ (\Delta F_{y'})_p &= \sum_m (\Delta a_{y'm} \sin m\psi + \Delta b_{y'm} \cos m\psi)\end{aligned}\right\} \tag{33}$$

where

$$\left.\begin{aligned}\Delta b_{x'0} &= \frac{1}{2} \sum_{j=1}^{n-1} (\Delta g_{x1j} - \Delta f_{y1j}) \\ \Delta b_{y'0} &= \frac{1}{2} \sum_{j=1}^{n-1} (\Delta f_{x1j} + \Delta g_{y1j}) \\ \Delta a_{x'1} &= \frac{1}{2} \sum_{j=1}^{n-1} \left[(-2\Delta g_{y0j} + \Delta f_{x2j} + \Delta g_{y2j}) \cos \frac{2\pi j}{n} + (-2\Delta g_{x0j} + \Delta f_{y2j} - \Delta g_{x2j}) \sin \frac{2\pi j}{n}\right] \\ \Delta b_{x'1} &= \frac{1}{2} \sum_{j=1}^{n-1} \left[(2\Delta g_{x0j} + \Delta g_{x2j} - \Delta f_{y2j}) \cos \frac{2\pi j}{n} + (-2\Delta g_{y0j} + \Delta g_{y2j} + \Delta f_{x2j}) \sin \frac{2\pi j}{n}\right] \\ \Delta a_{y'1} &= \frac{1}{2} \sum_{j=1}^{n-1} \left[(2\Delta g_{x0j} - \Delta g_{x2j} + \Delta f_{y2j}) \cos \frac{2\pi j}{n} + (-2\Delta g_{y0j} - \Delta f_{x2j} - \Delta g_{y2j}) \sin \frac{2\pi j}{n}\right] \\ \Delta b_{y'1} &= \frac{1}{2} \sum_{j=1}^{n-1} \left[(2\Delta g_{y0j} + \Delta f_{x2j} + \Delta g_{y2j}) \cos \frac{2\pi j}{n} + (2\Delta g_{x0j} - \Delta g_{x2j} + \Delta f_{y2j}) \sin \frac{2\pi j}{n}\right]\end{aligned}\right\} \tag{33a}$$

For $m \geqq 2$:

$$
\left.
\begin{aligned}
\Delta a_{x'm} &= \frac{1}{2}\sum_{j=1}^{n-1}\left[\left(\Delta f_{x,m-1,j} - \Delta g_{y,m-1,j} + \Delta f_{x,m+1,j} + \Delta g_{y,m+1,j}\right)\cos\frac{2\pi jm}{n} + \right. \\
&\qquad \left.\left(-\Delta g_{x,m-1,j} - \Delta f_{y,m-1,j} - \Delta g_{x,m+1,j} + \Delta f_{y,m+1,j}\right)\sin\frac{2\pi jm}{n}\right] \\
\Delta b_{x'm} &= \frac{1}{2}\sum_{j=1}^{n-1}\left[\left(\Delta g_{x,m-1,j} + \Delta f_{y,m-1,j} + \Delta g_{x,m+1,j} - \Delta f_{y,m+1,j}\right)\cos\frac{2\pi jm}{n} + \right. \\
&\qquad \left.\left(\Delta f_{x,m-1,j} - \Delta g_{y,m-1,j} + \Delta f_{x,m+1,j} + \Delta g_{y,m+1,j}\right)\sin\frac{2\pi jm}{n}\right] \\
\Delta a_{y'm} &= \frac{1}{2}\sum_{j=1}^{n-1}\left[\left(\Delta f_{y,m-1,j} + \Delta g_{x,m-1,j} + \Delta f_{y,m+1,j} - \Delta g_{x,m+1,j}\right)\cos\frac{2\pi jm}{n} + \right. \\
&\qquad \left.\left(\Delta f_{x,m-1,j} - \Delta g_{y,m-1,j} - \Delta f_{x,m+1,j} - \Delta g_{y,m+1,j}\right)\sin\frac{2\pi jm}{n}\right] \\
\Delta b_{y'm} &= \frac{1}{2}\sum_{j=1}^{n-1}\left[\left(-\Delta f_{x,m-1,j} + \Delta g_{y,m-1,j} + \Delta f_{x,m+1,j} + \Delta g_{y,m+1,j}\right)\cos\frac{2\pi jm}{n} + \right. \\
&\qquad \left.\left(\Delta f_{y,m-1,j} + \Delta g_{x,m-1,j} + \Delta f_{y,m+1,j} - \Delta g_{x,m+1,j}\right)\sin\frac{2\pi jm}{n}\right]
\end{aligned}
\right\} \tag{33b}
$$

From equations (33), it is seen that the additional forces transmitted to the hub due to property imbalances in the blades are composed of all harmonics. This is in contrast to the net forces transmitted by balanced blades, which consist only of those harmonics which are multiples of the number of rotor blades. This was also the case in the previous section which dealt with forces applied along the axis of rotation. On the other hand, in that section it was seen that only the mth harmonics acting on a single blade contribute to the net mth harmonic forces at the hub (whether the blades are balanced or unbalanced), whereas here the harmonics acting on a single blade which contribute to the net mth harmonic forces at the hub are the $(m + 1)$th and the $(m - 1)$th harmonics. This conclusion may be particularly significant for the additional first and second harmonic forces produced at the hub, since these will depend on the relatively large zeroth (static) and first harmonic forces acting on a single rotating blade.

The instantaneous magnitude of the total force increment in equations (33) is $\sqrt{\left(\Delta F_{x'}\right)^2 + \left(\Delta F_{y'}\right)^2}$. In general, this varies with ψ and hence with time. It is independent of time under conditions similar to those for balanced blades. Of the various harmonic components in equations (33), the first harmonic will usually be of greatest practical interest since the zeroth harmonic is a static load, while the higher harmonics have smaller amplitudes. If the contribution of the second-harmonic forces acting on a single blade are neglected in comparison with the static forces, while the y-components of the static forces are also neglected, as will be justified in the succeeding paragraph, the following expression is obtained for the square of the magnitude of the additional first harmonic forces transmitted to the hub:

$$\left(\Delta a_{x'1} \sin \psi + \Delta b_{x'1} \cos \psi\right)^2 + \left(\Delta a_{y'1} \sin \psi + \Delta b_{y'1} \cos \psi\right)^2 \cong$$

$$\left(\sum_{j=1}^{n-1} \Delta g_{x0j} \cos \frac{2\pi j}{n}\right)^2 + \left(\sum_{j=1}^{n-1} \Delta g_{x0j} \sin \frac{2\pi j}{n}\right)^2 \tag{34}$$

The magnitude of this additional first harmonic is thus seen to be approximately constant with time and hence this additional force component will rotate with constant magnitude, given by equation (34), at an angular velocity equal to that of the rotor angular velocity Ω.

If the x-axis is chosen to coincide with the arm of a blade, then the static load transmitted to the hub by a single blade will be almost entirely the centrifugal force component in the x-direction[4]. Thus,

$$g_{x0} \cong \Omega^2 \int_0^l \rho r \, ds \tag{35}$$

where $\rho(s)$ is the mass per unit length of a blade, r is the distance of a blade mass element $\rho \, ds$ from the rotor axis of rotation, l is the length of the blade, and s is the distance along the blade measured from its root. From equation (35), the value of Δg_{x0j} for a given

[4]There will be a relatively small centrifugal force component in the y-direction due to blade lagging. There may also be some slight aerodynamic static force components in the x-direction. (See, e.g., ref. 2.)

property imbalance (e.g., a change in $\rho(s)$) can be readily calculated if the slight effects of blade flapping and lagging (cf., e.g., ref. 2) are neglected, and r is therefore replaced by $(s + s_1)$, where s_1 is the distance of the root of an undeflected blade from the axis of rotation.

As a numerical example, a three-bladed rotor with the following data for the blade force coefficients (in pounds) is considered:

$$\left.\begin{array}{lllll} g_{x0} = 12944 & g_{y0} = -1878 & f_{x1} = -245.8 & g_{x1} = 774.6 & f_{y1} = 41.3 \\ g_{y1} = 344.2 & f_{x2} = -535.9 & g_{x2} = -60.9 & f_{y2} = -65.5 & g_{y2} = -141.4 \\ f_{x3} = 9.3 & g_{x3} = -26.2 & f_{y3} = -87.1 & g_{y3} = -55.5 & \end{array}\right\} \quad (36)$$

If the blades are perfectly balanced, then according to equations (12) or (13), the following net forces will be transmitted to the hub in the plane of rotation:[5]

$$F_{x'} = \frac{3}{2}\left(733.3 - 394.5 \sin 3\psi - 126.4 \cos 3\psi + \ldots\right)$$
$$F_{y'} = \frac{3}{2}\left(98.4 - 126.4 \sin 3\psi + 394.5 \cos 3\psi + \ldots\right)$$

Consequently, the nonstatic force component transmitted to the hub in the plane of rotation will have a constant magnitude of 413 pounds and will rotate with an angular speed of 3Ω. Suppose now that one of the blades, corresponding to $j = 1$, for example, has a slightly different mass distribution from that of the other two, causing an increment in g_{x01} of Δg_{x01} for this blade. Then according to equation (34), the main effect of this increment is to transmit to the hub an additional force in the first harmonic (i.e., of frequency Ω) with an amplitude of $\left|\Delta g_{x01}\right| = \alpha g_{x0} = 12{,}944\alpha$ pounds, where $\alpha = \left|\frac{\Delta g_{x01}}{g_{x0}}\right|$. If the imbalance is such that $\alpha = 0.02$, then the amplitude of the additional first harmonic will be 259 pounds or 62.7 percent of the amplitude of the third-harmonic force transmitted to the hub by balanced blades.

Effects of Eccentricity Imbalances

If the n blades of a rotor are identical but have a nonuniform azimuth spacing ψ_j' given by equation (21), then in equations (9) ψ_j

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[5]The contributions of the fourth harmonic coefficients f_{x4}, g_{x4}, f_{y4}, and g_{y4} have been neglected here.

must be replaced by $\psi_j{}'$. In this manner, the following expressions are obtained for the additional forces transmitted to the hub by all n blades:

$$\begin{aligned}(\Delta F_{x'})_e = \frac{1}{2} \sum_m \sum_{j=1}^{n-1} \Bigg(& \sin(m + 1)\psi_j \left\{ (f_{xm} - g_{ym})\left[\cos(m + 1)\epsilon_j - 1\right] - \right. \\ & \left. (g_{xm} + f_{ym})\sin(m + 1)\epsilon_j \right\} + \\ & \cos(m + 1)\psi_j \left\{ (g_{xm} + f_{ym})\left[\cos(m + 1)\epsilon_j - 1\right] + \right. \\ & \left. (f_{xm} - g_{ym})\sin(m + 1)\epsilon_j \right\} + \\ & \sin(m - 1)\psi_j \left\{ (f_{xm} + g_{ym})\left[\cos(m - 1)\epsilon_j - 1\right] - \right. \\ & \left. (g_{xm} - f_{ym})\sin(m - 1)\epsilon_j \right\} + \\ & \cos(m - 1)\psi_j \left\{ (g_{xm} - f_{ym})\left[\cos(m - 1)\epsilon_j - 1\right] + \right. \\ & \left. (f_{xm} + g_{ym})\sin(m - 1)\epsilon_j \right\} \Bigg)\end{aligned} \tag{37a}$$

$$\begin{aligned}(\Delta F_{y'})_e = \frac{1}{2} \sum_m \sum_{j=1}^{n-1} \Bigg(& \sin(m + 1)\psi_j \left\{ (f_{ym} + g_{xm})\left[\cos(m + 1)\epsilon_j - 1\right] - \right. \\ & \left. (g_{ym} - f_{xm})\sin(m + 1)\epsilon_j \right\} + \\ & \cos(m + 1)\psi_j \left\{ (g_{ym} - f_{xm})\left[\cos(m + 1)\epsilon_j - 1\right] + \right. \\ & \left. (f_{ym} + g_{xm})\sin(m + 1)\epsilon_j \right\} + \\ & \sin(m - 1)\psi_j \left\{ (f_{ym} - g_{xm})\left[\cos(m - 1)\epsilon_j - 1\right] - \right. \\ & \left. (f_{xm} + g_{ym})\sin(m - 1)\epsilon_j \right\} + \\ & \cos(m - 1)\psi_j \left\{ (f_{xm} + g_{ym})\left[\cos(m - 1)\epsilon_j - 1\right] + \right. \\ & \left. (f_{ym} - g_{xm})\sin(m - 1)\epsilon_j \right\} \Bigg)\end{aligned} \tag{37b}$$

A comparison of these equations with equations (32) shows that the effect of eccentricity imbalances in this case is equivalent to property imbalances, as in the case of forces normal to the plane of rotation. For example, the effect of eccentricity imbalances can be obtained by property imbalances which are characterized by increments $(\Delta f_{xmj})_{eq}$, $(\Delta g_{xmj})_{eq}$, and so forth in the forces on a single blade, such that

$$\left.\begin{aligned}
&(\Delta f_{xmj})_{eq} - (\Delta g_{ymj})_{eq} = \\
&(f_{xm} - g_{ym})\left[\cos(m + 1)\epsilon_j - 1\right] - (g_{xm} + f_{ym})\sin(m + 1)\epsilon_j \\
&(\Delta g_{xmj})_{eq} + (\Delta f_{ymj})_{eq} = \\
&(g_{xm} + f_{ym})\left[\cos(m + 1)\epsilon_j - 1\right] + (f_{xm} - g_{ym})\sin(m + 1)\epsilon_j \\
&(\Delta f_{xmj})_{eq} + (\Delta g_{ymj})_{eq} = \\
&(f_{xm} + g_{ym})\left[\cos(m - 1)\epsilon_j - 1\right] - (g_{xm} - f_{ym})\sin(m - 1)\epsilon_j \\
&(\Delta g_{xmj})_{eq} - (\Delta f_{ymj})_{eq} = \\
&(g_{xm} - f_{ym})\left[\cos(m - 1)\epsilon_j - 1\right] + (f_{xm} + g_{ym})\sin(m - 1)\epsilon_j
\end{aligned}\right\} \tag{38a}$$

Solving these equations for the equivalent property-imbalance increments yields

$$\left.\begin{aligned}
(\Delta f_{xmj})_{eq} &= f_{xm}(\cos m\epsilon_j \cos \epsilon_j - 1) + g_{ym} \sin m\epsilon_j \sin \epsilon_j + \\
&\qquad f_{ym} \sin \epsilon_j \cos m\epsilon_j - g_{xm} \sin m\epsilon_j \cos \epsilon_j \\
(\Delta g_{xmj})_{eq} &= g_{xm}(\cos m\epsilon_j \cos \epsilon_j - 1) - f_{ym} \sin m\epsilon_j \sin \epsilon_j - \\
&\qquad g_{ym} \sin \epsilon_j \cos m\epsilon_j + f_{xm} \sin m\epsilon_j \cos \epsilon_j \\
(\Delta f_{ymj})_{eq} &= -g_{xm} \sin m\epsilon_j \sin \epsilon_j + f_{ym}(\cos m\epsilon_j \cos \epsilon_j - 1) + \\
&\qquad f_{xm} \sin \epsilon_j \cos m\epsilon_j - g_{ym} \sin m\epsilon_j \cos \epsilon_j \\
(\Delta g_{ymj})_{eq} &= f_{xm} \sin m\epsilon_j \sin \epsilon_j + g_{ym}(\cos m\epsilon_j \cos \epsilon_j - 1) + \\
&\qquad g_{xm} \sin \epsilon_j \cos m\epsilon_j + f_{ym} \sin m\epsilon_j \cos \epsilon_j
\end{aligned}\right\} \tag{38b}$$

For small eccentricity imbalances, these relations, to first powers of ϵ_j become[6]

$$\left.\begin{aligned}(\Delta f_{xmj})_{eq} &= -(mg_{xm} + f_{ym})\epsilon_j \\ (\Delta g_{xmj})_{eq} &= (mf_{xm} - g_{ym})\epsilon_j \\ (\Delta f_{ymj})_{eq} &= (f_{xm} - mg_{ym})\epsilon_j \\ (\Delta g_{ymj})_{eq} &= (g_{xm} + mf_{ym})\epsilon_j\end{aligned}\right\} \tag{38c}$$

Equations (37) may be written in a more explicit manner. The following expressions give the additional loads, due to eccentricity imbalances, transmitted to the hub in the plane of rotation of a rotor of n identical blades.

$$\left.\begin{aligned}(\Delta F_{x'})_e &= \sum_m \left(\Delta a'_{x'm} \sin m\psi + \Delta b'_{x'm} \cos m\psi\right) \\ (\Delta F_{y'})_e &= \sum_m \left(\Delta a'_{y'm} \sin m\psi + \Delta b'_{y'm} \cos m\psi\right)\end{aligned}\right\} \tag{39}$$

where

$$\left.\begin{aligned}&\Delta b'_{x'0} = 0 \\ &\Delta b'_{y'0} = 0 \\ &\Delta a'_{x'1} = \frac{1}{2} \sum_{j=1}^{n-1} \Bigg\{\cos \frac{2\pi j}{n}\Big[(-2g_{y0} + f_{x2} + g_{y2})(\cos \epsilon_j - 1) + \\ &\qquad (-2g_{x0} - g_{x2} + f_{y2})\sin \epsilon_j\Big] + \\ &\qquad \sin \frac{2\pi j}{n}\Big[(2g_{y0} - f_{x2} - g_{y2})\sin \epsilon_j + \\ &\qquad (-2g_{x0} - g_{x2} + f_{y2})(\cos \epsilon_j - 1)\Big]\Bigg\}\end{aligned}\right\} \tag{39a}$$

(Eqs. (39a) continued on next page.)

---

[6]The most general equivalence relations can be obtained by writing $(\Delta a_{x'm})_{eq} = \Delta a'_{x'm}$, $(\Delta b_{x'm})_{eq} = \Delta b'_{x'm}$, and so forth and by using equations (33a), (33b), (39a), and (39b).

$$\left.\begin{aligned}
\Delta b'_{x'1} &= \frac{1}{2}\sum_{j=1}^{n-1}\left\{\cos\frac{2\pi j}{n}\left[\left(-2g_{y0} + f_{x2} + g_{x2}\right)\sin\epsilon_j + \left(2g_{x0} + g_{x2} - f_{y2}\right)\left(\cos\epsilon_j - 1\right)\right] + \sin\frac{2\pi j}{n}\left[\left(-2g_{y0} + f_{x2} + g_{y2}\right)\left(\cos\epsilon_j - 1\right) + \left(-2g_{x0} - g_{x2} + f_{y2}\right)\sin\epsilon_j\right]\right\} \\
\Delta a'_{y'1} &= \frac{1}{2}\sum_{j=1}^{n-1}\left\{\cos\frac{2\pi j}{n}\left[\left(2g_{x0} + f_{y2} - g_{x2}\right)\left(\cos\epsilon_j - 1\right) + \left(-2g_{y0} - f_{x2} - g_{y2}\right)\sin\epsilon_j\right] + \sin\frac{2\pi j}{n}\left[\left(-2g_{x0} - f_{y2} + g_{x2}\right)\sin\epsilon_j + \left(2g_{y0} - f_{x2} - g_{y2}\right)\left(\cos\epsilon_j - 1\right)\right]\right\} \\
\Delta b'_{y'1} &= \frac{1}{2}\sum_{j=1}^{n-1}\left\{\cos\frac{2\pi j}{n}\left[\left(2g_{x0} + f_{y2} - g_{x2}\right)\sin\epsilon_j + \left(2g_{y0} + f_{x2} + g_{y2}\right)\left(\cos\epsilon_j - 1\right)\right] + \sin\frac{2\pi j}{n}\left[\left(2g_{x0} + f_{y2} - g_{x2}\right)\left(\cos\epsilon_j - 1\right) + \left(-2g_{y0} - f_{x2} - g_{y2}\right)\sin\epsilon_j\right]\right\}
\end{aligned}\right\} \quad (39a)$$

For $m \geqq 2$:

$$
\left.
\begin{aligned}
\Delta a'_{x'm} = \frac{1}{2}\sum_{j=1}^{n-1}\Bigg\{ & \cos\frac{2\pi jm}{n}\Big[\left(f_{x,m-1} - g_{y,m-1} + f_{x,m+1} + g_{y,m+1}\right)\left(\cos m\epsilon_j - 1\right) + \\
& \left(g_{x,m-1} + f_{y,m-1} + g_{x,m+1} - f_{y,m+1}\right)\left(-\sin m\epsilon_j\right)\Big] + \\
& \sin\frac{2\pi jm}{n}\Big[\left(-f_{x,m-1} + g_{y,m-1} - f_{x,m+1} - g_{y,m+1}\right)\sin m\epsilon_j - \\
& \left(g_{x,m-1} + f_{y,m-1} + g_{x,m+1} - f_{y,m+1}\right)\left(\cos m\epsilon_j - 1\right)\Big]\Bigg\} \\
\Delta b'_{x'm} = \frac{1}{2}\sum_{j=1}^{n-1}\Bigg\{ & \cos\frac{2\pi jm}{n}\Big[\left(f_{x,m-1} - g_{y,m-1} + f_{x,m+1} + g_{y,m+1}\right)\sin m\epsilon_j + \\
& \left(g_{x,m-1} + f_{y,m-1} + g_{x,m+1} - f_{y,m+1}\right)\left(\cos m\epsilon_j - 1\right)\Big] + \\
& \sin\frac{2\pi jm}{n}\Big[\left(f_{x,m-1} - g_{y,m-1} + f_{x,m+1} + g_{y,m+1}\right)\left(\cos m\epsilon_j - 1\right) - \\
& \left(g_{x,m-1} + f_{y,m-1} + g_{x,m+1} - f_{y,m+1}\right)\left(\sin m\epsilon_j\right)\Big]\Bigg\} \\
\Delta a'_{y'm} = \frac{1}{2}\sum_{j=1}^{n-1}\Bigg\{ & \cos\frac{2\pi jm}{n}\Big[\left(f_{y,m-1} + g_{x,m-1} + f_{y,m+1} - g_{x,m+1}\right)\left(\cos m\epsilon_j - 1\right) + \\
& \left(g_{y,m-1} - f_{x,m-1} + f_{x,m+1} + g_{y,m+1}\right)\left(-\sin m\epsilon_j\right)\Big] + \\
& \sin\frac{2\pi jm}{n}\Big[\left(f_{y,m-1} + g_{x,m-1} + f_{y,m+1} - g_{x,m+1}\right)\left(-\sin m\epsilon_j\right) + \\
& \left(f_{x,m-1} - g_{y,m-1} - f_{x,m+1} - g_{y,m+1}\right)\left(\cos m\epsilon_j - 1\right)\Big]\Bigg\} \\
\Delta b'_{y'm} = \frac{1}{2}\sum_{j=1}^{n-1}\Bigg\{ & \cos\frac{2\pi jm}{n}\Big[\left(g_{y,m-1} - f_{x,m-1} + f_{x,m+1} + g_{y,m+1}\right)\left(\cos m\epsilon_j - 1\right) + \\
& \left(f_{y,m-1} + g_{x,m-1} + f_{y,m+1} - g_{x,m+1}\right)\left(\sin m\epsilon_j\right)\Big] + \\
& \sin\frac{2\pi jm}{n}\Big[\left(g_{y,m-1} - f_{x,m-1} + f_{x,m+1} + g_{y,m+1}\right)\left(-\sin m\epsilon_j\right) + \\
& \left(f_{y,m-1} + g_{x,m-1} + f_{y,m+1} - g_{x,m+1}\right)\left(\cos m\epsilon_j - 1\right)\Big]\Bigg\}
\end{aligned}
\right\}(39b)
$$

For small eccentricity imbalances, equations (39a) and (39b) to first powers of ϵ_j become

$$
\left.
\begin{aligned}
&\Delta b'_{x'0} = 0 \\
&\Delta b'_{y'0} = 0 \\
&\Delta a'_{x'1} = \frac{1}{2}\left[\left(-2g_{x0} - g_{x2} + f_{y2}\right)\sum_{j=1}^{n-1}\left(\epsilon_j \cos\frac{2\pi j}{n}\right) + \right. \\
&\qquad\qquad \left. \left(2g_{y0} - f_{x2} - g_{y2}\right)\sum_{j=1}^{n-1}\left(\epsilon_j \sin\frac{2\pi j}{n}\right)\right] \\
&\Delta b'_{x'1} = \frac{1}{2}\left[\left(-2g_{y0} + f_{x2} + g_{y2}\right)\sum_{j=1}^{n-1}\left(\epsilon_j \cos\frac{2\pi j}{n}\right) + \right. \\
&\qquad\qquad \left. \left(-2g_{x0} - g_{x2} + f_{y2}\right)\sum_{j=1}^{n-1}\left(\epsilon_j \sin\frac{2\pi j}{n}\right)\right] \\
&\Delta a'_{y'1} = \frac{1}{2}\left[\left(-2g_{y0} - f_{x2} - g_{y2}\right)\sum_{j=1}^{n-1}\left(\epsilon_j \cos\frac{2\pi j}{n}\right) + \right. \\
&\qquad\qquad \left. \left(-2g_{x0} - f_{y2} + g_{x2}\right)\sum_{j=1}^{n-1}\left(\epsilon_j \sin\frac{2\pi j}{n}\right)\right] \\
&\Delta b'_{y'1} = \frac{1}{2}\left[\left(2g_{x0} + f_{y2} - g_{x2}\right)\sum_{j=1}^{n-1}\left(\epsilon_j \cos\frac{2\pi j}{n}\right) + \right. \\
&\qquad\qquad \left. \left(-2g_{y0} - f_{x2} - g_{y2}\right)\sum_{j=1}^{n-1}\left(\epsilon_j \sin\frac{2\pi j}{n}\right)\right]
\end{aligned}
\right\} \qquad (39c)
$$

For $m \geqq 2$:

$$
\left.
\begin{aligned}
\Delta a'_{x'm} &= \frac{1}{2} m \Bigg[\left(-g_{x,m-1} - f_{y,m-1} - g_{x,m+1} + f_{y,m+1}\right) \sum_{j=1}^{n-1} \left(\epsilon_j \cos \frac{2\pi jm}{n}\right) + \\
&\qquad \left(-f_{x,m-1} + g_{y,m-1} - f_{x,m+1} - g_{y,m+1}\right) \sum_{j=1}^{n-1} \left(\epsilon_j \sin \frac{2\pi jm}{n}\right)\Bigg] \\
\Delta b'_{x'm} &= \frac{1}{2} m \Bigg[\left(f_{x,m-1} - g_{y,m-1} + f_{x,m+1} + g_{y,m+1}\right) \sum_{j=1}^{n-1} \left(\epsilon_j \cos \frac{2\pi jm}{n}\right) + \\
&\qquad \left(-g_{x,m-1} - f_{y,m-1} - g_{x,m+1} + f_{y,m+1}\right) \sum_{j=1}^{n-1} \left(\epsilon_j \sin \frac{2\pi jm}{n}\right)\Bigg] \\
\Delta a'_{y'm} &= \frac{1}{2} m \Bigg[\left(-g_{y,m-1} + f_{x,m-1} - f_{x,m+1} - g_{y,m+1}\right) \sum_{j=1}^{n-1} \left(\epsilon_j \cos \frac{2\pi jm}{n}\right) + \\
&\qquad \left(-f_{y,m-1} - g_{x,m-1} - f_{y,m+1} + g_{x,m+1}\right) \sum_{j=1}^{n-1} \left(\epsilon_j \sin \frac{2\pi jm}{n}\right)\Bigg] \\
\Delta b'_{y'm} &= \frac{1}{2} m \Bigg[\left(f_{y,m-1} + g_{x,m-1} + f_{y,m+1} - g_{x,m+1}\right) \sum_{j=1}^{n-1} \left(\epsilon_j \cos \frac{2\pi jm}{n}\right) + \\
&\qquad \left(-g_{y,m-1} + f_{x,m-1} - f_{x,m+1} - g_{y,m+1}\right) \sum_{j=1}^{n-1} \left(\epsilon_j \sin \frac{2\pi jm}{n}\right)\Bigg]
\end{aligned}
\right\} \quad (39d)
$$

From equations (39), it is seen that eccentricity imbalances in a rotor, as in all of the previous cases of imbalances, will produce at the hub additional loads of all harmonics and not only those which are multiples of the number of blades. Moreover, the (m + 1)th and the (m - 1)th harmonic forces acting on a single rotating blade will contribute to the mth harmonic force transmitted to the hub.

Of particular practical interest in ordinary cases will be the increment in the first harmonic load transmitted to the hub. This may be especially large because of the contribution of the static blade forces to its magnitude. The contribution of only the blade static (zeroth harmonic) load in the x-direction to this additional first harmonic load will now be considered. This has already been justified in the derivation of equation (34). For small eccentricity imbalances, the magnitude of the first harmonic load component according to equations (39) is then found to be

$$A_{e1} = g_{x0}\left[\left(\sum_{j=1}^{n-1}\epsilon_j \cos\frac{2\pi j}{n}\right)^2 + \left(\sum_{j=1}^{n-1}\epsilon_j \sin\frac{2\pi j}{n}\right)^2\right]^{1/2} \tag{40a}$$

where g_{x0} is the centrifugal load given by equation (35). For the special cases of rotors with two, three, or four blades, this equation becomes

for $n = 2$,

$$A_{e1} = g_{x0}\epsilon_1 \tag{40b}$$

for $n = 3$,

$$A_{e1} = g_{x0}\sqrt{\epsilon_1{}^2 + \epsilon_2{}^2 - \epsilon_1\epsilon_2} \tag{40c}$$

and for $n = 4$,

$$A_{e1} = g_{x0}\sqrt{\epsilon_2{}^2 + (\epsilon_1 - \epsilon_3)^2} \tag{40d}$$

These equations show that, for a rotor of three blades, eccentricity imbalances of opposite signs will enhance each other appreciably. For a rotor of four blades, the effect of eccentricity imbalances is seen to depend on the azimuth angles between alternate blades.

As a numerical example, in order to indicate order of magnitude, consider a three-bladed rotor with $g_{x0} = 12{,}944$ pounds. If the angles between the reference blade ($j = 0$) and the succeeding blades at $j = 1$ and $j = 2$ are $\left(\frac{2}{3}\pi + \epsilon_1\right)$ radians and $\frac{4}{3}\pi$ radians, respectively, then according to equation (40c) the magnitude of the additional first harmonic force transmitted to the hub in the plane of rotation by this imbalance will be $A_{e1} = 12{,}944\epsilon_1$ pounds. If $\epsilon_1 = 0.02$ radian (1.15°), then $A_{e1} = 259$ pounds, which is 62.7 percent of the amplitude of the third-harmonic force transmitted by balanced blades.

Simultaneous Property and Eccentricity Imbalances

If the n blades of a rotor have simultaneous property and eccentricity imbalances, then equations (9) and (10) must be modified to the form

$$\left.\begin{aligned} F_{x'm} &= \sum_{j=0}^{n-1}\left[\overline{F}_{xms}(\psi_j')\cos\psi_j' - \overline{F}_{yms}(\psi_j')\sin\psi_j'\right] \\ F_{y'm} &= \sum_{j=0}^{n-1}\left[\overline{F}_{xms}(\psi_j')\sin\psi_j' + \overline{F}_{yms}(\psi_j')\cos\psi_j'\right] \end{aligned}\right\} \tag{41}$$

where $\overline{F}_{xms}$ and $\overline{F}_{yms}$ are the mth harmonic individual blade force coefficients given by equations (31) with ψ_j replaced by ψ_j'.

By using straightforward trigonometric manipulations and summing over all harmonics m, it is possible to derive from equations (41) general explicit expressions for the additional harmonic forces transmitted to the hub by simultaneous property and eccentricity imbalances. For the case in which both types of imbalances are sufficiently small so that powers of Δf_{xmj}, Δf_{ymj}, Δg_{xmj}, Δg_{ymj}, and ϵ_j above the first can be neglected, it can be shown that the effect of the simultaneous imbalances can be obtained by superposition of the effects of property imbalances alone and the effects of eccentricity imbalances alone. Thus the increments in the forces produced at the hub by the simultaneous blade imbalances can be expressed as

$$\left.\begin{aligned} \left(\Delta F_{x'}\right)_t &= \sum_m\left(\Delta a''_{x'm}\sin m\psi + \Delta b''_{x'm}\cos m\psi\right) \\ \left(\Delta F_{y'}\right)_t &= \sum_m\left(\Delta a''_{y'm}\sin m\psi + \Delta b''_{y'm}\cos m\psi\right) \end{aligned}\right\} \tag{42a}$$

where

$$\left.\begin{aligned} \Delta a''_{x'm} &= \Delta a_{x'm} + \Delta a'_{x'm} \\ \Delta b''_{x'm} &= \Delta b_{x'm} + \Delta b'_{x'm} \\ \Delta a''_{y'm} &= \Delta a_{y'm} + \Delta a'_{y'm} \\ \Delta b''_{y'm} &= \Delta b_{y'm} + \Delta b'_{y'm} \end{aligned}\right\} \tag{42b}$$

Explicit expressions for the right sides of equations (42b) can be obtained directly from equations (33b), (33c), (39d), and (39e).

It may be further noted that the effect of simultaneous property and eccentricity imbalances can be shown to be equivalent to property imbalances alone. For example, the effect of small simultaneous property and eccentricity imbalances can be obtained by property imbalances characterized by the following equations:

$$\left.\begin{aligned} (\Delta f_{xmj})_{eq} &= \Delta f_{xmj} - (mg_{xm} + f_{ym})\epsilon_j \\ (\Delta g_{xmj})_{eq} &= \Delta g_{xmj} + (mf_{xm} - g_{ym})\epsilon_j \\ (\Delta f_{ymj})_{eq} &= \Delta f_{ymj} - (mg_{ym} - f_{xm})\epsilon_j \\ (\Delta g_{ymj})_{eq} &= \Delta g_{ymj} + (mf_{ym} + g_{xm})\epsilon_j \end{aligned}\right\} \tag{43}$$

A comparison of equations (43) with equations (38c) illustrates further the principle of superposition for small imbalances.

As has been previously explained, the additional first harmonics, due almost entirely to the static centrifugal forces exerted by each blade, will ordinarily be of particular practical interest. If the effect of the centrifugal forces in the x-direction only is considered, then for small simultaneous property and eccentricity imbalances, the additional first harmonic force components, according to equations (42), (33), and (39), will be

$$\left.\begin{aligned} \left[(\Delta F_{x'})_t\right]_1 &= \Delta a''_{x'1} \sin \psi + \Delta b''_{x'1} \cos \psi \\ \left[(\Delta F_{y'})_t\right]_1 &= \Delta a''_{y'1} \sin \psi + \Delta b''_{y'1} \cos \psi \end{aligned}\right\} \tag{44a}$$

where

$$\left.\begin{aligned}
\Delta a''_{x'1} &= \sum_{j=1}^{n-1}\left(-\Delta g_{x0j}\sin\frac{2\pi j}{n} - \epsilon_j g_{x0}\cos\frac{2\pi j}{n}\right) \\
\Delta b''_{x'1} &= \sum_{j=1}^{n-1}\left(\Delta g_{x0j}\cos\frac{2\pi j}{n} - \epsilon_j g_{x0}\sin\frac{2\pi j}{n}\right) \\
\Delta a''_{y'1} &= \Delta b''_{x'1} \\
\Delta b''_{y'1} &= -\Delta a''_{x'1}
\end{aligned}\right\} \tag{44b}$$

From these equations, the amplitude of the additional first harmonic force in the plane of rotation is constant with time and has the value

$$A_{t1} = \left[\left(\Delta a''_{x'1}\right)^2 + \left(\Delta b''_{x'1}\right)^2\right]^{1/2} \tag{44c}$$

For the special case in which the simultaneous property and eccentricity imbalances occur in a single blade, corresponding to $j = k$, for example, equation (44c) yields

$$A_{t1} = \left[\left(\Delta g_{x0k}\right)^2 + \left(g_{x0}\epsilon_k\right)^2\right]^{1/2} = g_{x0}\left(\alpha^2 + \epsilon_k{}^2\right)^{1/2} \tag{44d}$$

where $\alpha \equiv \Delta g_{x0k}/g_{x0}$.

As a numerical example, consider an n-bladed rotor for which $g_{x0} = 12{,}944$ pounds (as in the previous numerical examples) and in which the blade corresponding to $j = k$ has a property imbalance characterized by $\alpha = 0.02$ and an eccentricity imbalance $\epsilon_k = 0.02$. Then $A_{t1} = 366$ pounds, which is 88.5 percent of the amplitude of the third harmonic force transmitted by the balanced blades of the three-bladed rotor of the preceding examples. It may be noted that this result is independent of the signs of α and ϵ_k.

CONCLUDING REMARKS

From the present analysis of the net forces transmitted to the hub by a rotor of n blades with property and eccentricity imbalances, the following concluding remarks may be made:

The effect of imbalances in the rotor blades is to produce at the hub, both in and normal to the rotor plane of rotation, additional forces in all of the harmonics. The magnitudes of the additional lower harmonics may be especially appreciable. The effect of eccentricity imbalances, or nonuniform spacing between the blades, is equivalent to the effect of property imbalances, or property differences among the blades. The magnitude of the first harmonic forces transmitted to the hub in the plane of rotation of the blades by property or eccentricity imbalances may be especially high because of the influence of the static centrifugal forces exerted by the blades. The effect of small simultaneous property and eccentricity imbalances in a rotor is equivalent to the superposition of the effects of property imbalances alone and the effects of eccentricity imbalances alone. Explicit general expressions have been derived for the calculation of all of the additional harmonic forces transmitted to the hub both in and normal to the plane of rotation by property and eccentricity imbalances in an n-bladed rotor. These expressions are given in terms of the forces transmitted to the hub by a single rotating blade. For convenient reference, a concise analysis of forces transmitted to the hub by a balanced rotor of n blades is given. Numerical examples are given to indicate the order of magnitude of the results obtained herein.

Polytechnic Institute of Brooklyn,
Brooklyn, N. Y., January 29, 1957.

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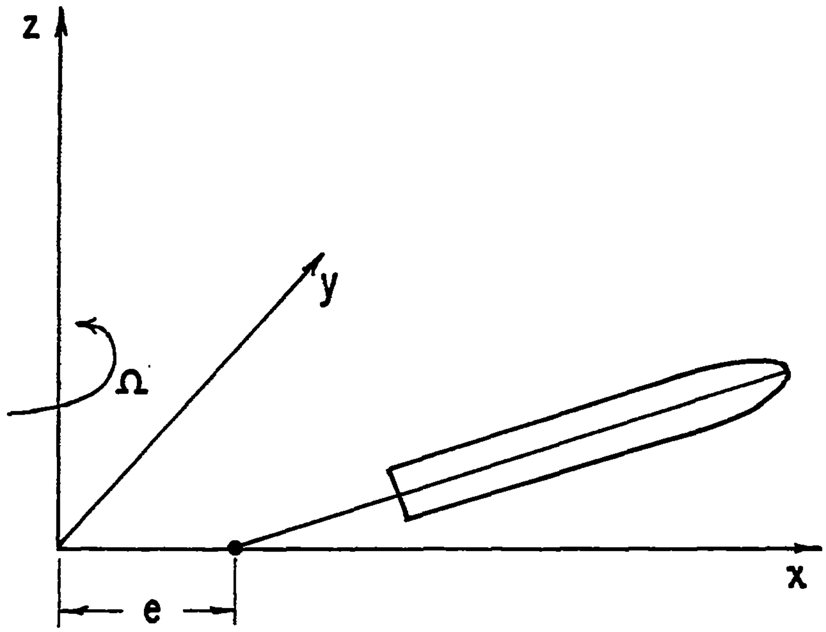
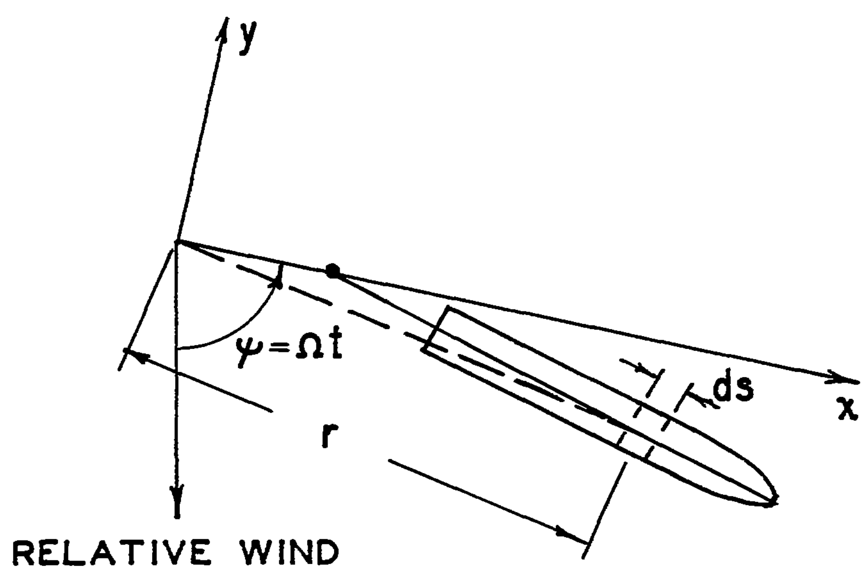


(a) Space view.



(b) Plan view.

Figure 1.- Coordinate system (x,y,z) rotating with a blade.