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APPROXIMATE METHOD FOR CALCULATION OF LAMINAR BOUNDARY LAYER WITH HEAT TRANSFER ON A CONE AT LARGE ANGLE OF ATTACK IN SUPERSONIC FLOW

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SUMMARY

By the use of an integral technique, the laminar boundary-layer equations are reduced on the windward generator of the plane of symmetry to a set of simultaneous algebraic equations. The Chapman-Rubesin temperature-viscosity relation and a Prandtl number of 1 are assumed. The method enables the skin friction coefficients and Stanton number to be calculated in a much shorter time than was needed to obtain exact numerical solutions from the boundary-layer equations. The solutions obtained by this method are, for the most part, within 5 percentage points of the exact solutions.

INTRODUCTION

Some designs for supersonic aircraft and missiles indicate the use of a conical nose or forebody. For design purposes it is necessary to know the skin friction and heat transfer on the conical surface for all possible flight conditions. One important condition is flight at angle of attack, since the boundary-layer flow is complicated by the addition of a crossflow, which increases with increasing angle of attack.

Several analytical papers have been published dealing with the effect of angle of attack on the boundary layer of a cone. Laminar flow over cones at small angle of attack has been studied (ref. 1) by using a linearization of the nonlinear boundary-layer equations for a cone as derived in reference 2. In references 3 and 4, the results of reference 1 have been extended to cover the combined effect of spin and small angle of attack. In reference 5, the boundary-layer equations for large angles of attack were solved by a numerical method but the solutions were limited to the plane of symmetry. The preceding solutions were limited to the insulated case with a Prandtl number equal to 1.

Additional numerical solutions were computed in reference 6 for the equations for large angle of attack in the plane of symmetry. Heat-transfer solutions were also obtained in reference 6 by noting that for a Prandtl number of 1 the energy equation in the plane of symmetry could be written in the same form as the momentum equation along the generator and, therefore, the enthalpy profile is identical to the velocity profile in this direction. An approximate correction factor for heat transfer is given in reference 6 for the case of a Prandtl number not equal to 1.

Obtaining a numerical solution to the boundary-layer equations of reference 5 is an involved process and requires an electronic computor. 4854 It was therefore desirable to find an approximate method of solving the boundary-layer equations from which results could be obtained with only the use of a desk calculator. An integral method is presented herein, that results in a pair of simultaneous algebraic equations in place of the nonlinear ordinary differential equations of reference 5 for the plane of symmetry.

Interpolation of the exact solutions of reference 6 gives good results within the range of variables studied, but the method developed herein enables quick calculation of results outside the range of variables considered in reference 6. For this purpose, a computational procedure for the present method is given in appendix B. The technique given in this paper also indicates a method for studying the boundary layer on the cone other than in the plane of symmetry.

EQUATIONS OF MOTION

The boundary-layer equations are given in reference 2 for a general orthogonal coordinate system in which the body surface is defined by $y = 0$ and the coordinate x is defined so that the body cross sections are similar for various values of x. These equations can therefore be applied to a cone at angle of attack with the coordinate x along a generator, y perpendicular to the cone surface, and $x\beta\varphi$ on the circumference of the cone, where φ is the meridional angle measured from the windward generator on the plane of symmetry and β is the sine of the cone semivertex angle ϑ. Figure 1 shows this coordinate system. Since the flow past a cone at angle of attack is still conical, there is no pressure gradient along a generator of the cone. Therefore, for steady-state conditions the boundary-layer equations from reference 2 can be written as

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\rho w}{x\beta} \frac{\partial u}{\partial \varphi} - \frac{\rho w^2}{x} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{1a}$$

$$\rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \frac{\rho w}{x\beta}\frac{\partial w}{\partial \varphi} + \frac{\rho u w}{x} = -\frac{1}{x\beta}\frac{\partial p}{\partial \varphi} + \frac{\partial}{\partial y}\left(\mu \frac{\partial w}{\partial y}\right) \tag{1b}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{1}{x\beta}\frac{\partial(\rho w)}{\partial \varphi} + \frac{\rho u}{x} = 0 \tag{1c}$$

$$\rho u \frac{\partial E}{\partial x} + \rho v \frac{\partial E}{\partial y} + \frac{\rho w}{x\beta}\frac{\partial E}{\partial \varphi} = -p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{x\beta}\frac{\partial w}{\partial \varphi} + \frac{u}{x}\right) + \mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right) \tag{1d}$$

$$p = \rho R T \tag{1e}$$

Equations (1a) and (1b) are momentum equations in the x and φ directions, respectively, equation (1c) is the continuity equation, equation (1d) is the energy equation, and equation (1e) is the equation of state. All symbols are defined in appendix A.

The following physical assumptions, which are basically the same as in reference 5, are used in this paper:

(1) A thin laminar boundary layer across which the static pressure is constant and also c_p is constant

(2) Prandtl number equal to 1

(3) The Chapman-Rubesin temperature-viscosity relation (ref. 7) is

$$\frac{\mu}{\mu_e} = C\frac{T}{T_e} \tag{2}$$

where the constant C is evaluated to match the Sutherland value of viscosity at the cone surface

$$C = \left(\frac{T_s}{T_e}\right)^{1/2}\frac{T_e + 198.6}{T_s + 198.6} \tag{3}$$

(4) The cone surface temperature is constant. This differs from the assumption of zero heat transfer given in reference 5 in that heat transfer is considered.

For $Pr = 1$, equations (1) can be combined to give the new energy equation

$$\rho\left(u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} + \frac{w}{x\beta}\frac{\partial H}{\partial \varphi}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial H}{\partial y}\right) \tag{4}$$

where

$$H \equiv c_pT + \frac{1}{2}\left(u^2 + w^2\right) \tag{5}$$

If, in addition, the surface temperature is constant and a new quantity Θ is defined as

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$$\Theta \equiv \frac{H - H_s}{H_0 - H_s} \tag{6}$$

the energy equation (4) can be written as

$$\rho\left(u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y} + \frac{w}{x\beta}\frac{\partial \Theta}{\partial \varphi}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial \Theta}{\partial y}\right) \tag{7}$$

The boundary conditions on the cone at angle of attack are

$$u = v = w = \Theta = 0 \quad \text{at} \quad y = 0$$

and (8)

$$u = u_e,\ v = v_e = 0,\ w = w_e,\ \Theta = 1 \quad \text{at} \quad y = \delta$$

where $y = \delta$ at the outer edge of the boundary layer. It is assumed that both the velocity and thermal boundary layers have the same thickness δ.

Combining each of the momentum and energy equations with the continuity equation and then integrating across the boundary layer from $y = 0$ to $y = \delta$ results in

$$\frac{\partial \zeta_{uu}}{\partial x} + \frac{1}{x\beta}\frac{\partial \zeta_{uw}}{\partial \varphi} + \frac{\zeta_{uw}}{\beta x \rho_e u_e^2}\frac{\partial\left(\rho_e u_e^2\right)}{\partial \varphi} + \frac{\zeta_{uu}}{x} -$$

$$\frac{1}{x\beta u_e}\frac{\partial u_e}{\partial \varphi}\int_0^\delta \rho^* w^* \, dy + \frac{1}{x}\int_0^\delta \rho^* w^{*2} \, dy = \frac{\mu_e}{\rho_e u_e}\left(\mu^*\frac{\partial u^*}{\partial y}\right)_{y=0} \tag{9a}$$

$$\frac{\partial \zeta_{wu}}{\partial x} + \frac{1}{x\beta}\frac{\partial \zeta_{ww}}{\partial \varphi} + \frac{\zeta_{ww}}{x\beta\rho_e u_e^2}\frac{\partial(\rho_e u_e^2)}{\partial \varphi} + \frac{\zeta_{wu}}{x} -$$

$$\frac{1}{x\beta u_e}\frac{\partial w_e}{\partial \varphi}\int_0^{\delta}\rho^* w^*\,dy - \frac{1}{x}\int_0^{\delta}\rho^* u^* w^*\,dy = \frac{\delta}{x\beta\rho_e u_e^2}\frac{\partial p}{\partial \varphi} + \frac{\mu_e}{\rho_e u_e}\left(\mu^*\frac{\partial w^*}{\partial y}\right)_{y=0} \qquad (9b)$$

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$$\frac{\partial \zeta_{\Theta u}}{\partial x} + \frac{1}{x\beta}\frac{\partial \zeta_{\Theta w}}{\partial \varphi} + \frac{\zeta_{\Theta w}}{x\beta\rho_e u_e}\frac{\partial(\rho_e u_e)}{\partial \varphi} + \frac{\zeta_{\Theta u}}{x} = \frac{\mu_e}{\rho_e u_e}\left(\mu^*\frac{\partial \Theta^*}{\partial y}\right)_{y=0} \qquad (9c)$$

where

$$\zeta_{uu} = \int_0^{\delta}\rho^* u^*(1 - u^*)dy \qquad (10a)$$

$$\zeta_{uw} = \int_0^{\delta}\rho^* w^*(1 - u^*)dy \qquad (10b)$$

$$\zeta_{ww} = \int_0^{\delta}\rho^* w^*(w_e^* - w^*)dy \qquad (10c)$$

$$\zeta_{wu} = \int_0^{\delta}\rho^* u^*(w_e^* - w^*)dy \qquad (10d)$$

$$\zeta_{\Theta u} = \int_0^{\delta}\rho^* u^*(1 - \Theta^*)dy \qquad (10e)$$

$$\zeta_{\Theta w} = \int_0^{\delta}\rho^* w^*(1 - \Theta^*)dy \qquad (10f)$$

The starred quantities introduced into equations (9) and (10) are dimensionless quantities formed as follows:

$$\left.\begin{array}{ll} u^* = u/u_e & \rho^* = \rho/\rho_e \\ w^* = w/u_e & \mu^* = \mu/\mu_e \\ \Theta^* = \Theta & T^* = T/T_e \end{array}\right\} \qquad (11)$$

Equations (9) and (10) are reduced to incompressible form by means of the transformation

$$\left.\begin{aligned} Y &\equiv \int_0^y \rho^* \, dy \\ X &\equiv x \\ \Phi &\equiv \varphi \end{aligned}\right\} \tag{12}$$

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The relation between the boundary-layer thickness in the physical coordinates δ and the thickness in the transformed coordinates Δ was obtained from equations (5), (6), and (12). The pressure gradient in the Φ direction was evaluated at the outer edge of the boundary layer. By using the Chapman-Rubesin temperature-viscosity relation (2) and the dimensionless variable

$$\eta \equiv \frac{Y}{\Delta} \tag{13}$$

the transformed boundary-layer equations become

$$\frac{\partial \theta_{uu}}{\partial X} + \frac{1}{X\beta} \frac{\partial \theta_{uw}}{\partial \Phi} + \frac{\theta_{uw}}{X\beta \rho_e u_e^2} \frac{\partial(\rho_e u_e^2)}{\partial \Phi} + \frac{\theta_{uu}}{X} - \frac{\Delta}{X\beta u_e} \frac{\partial u_e}{\partial \Phi} \int_0^1 w^* \, d\eta + \frac{\Delta}{X} \int_0^1 w^{*2} \, d\eta = \frac{\mu_e C}{\rho_e u_e \Delta} \left(\frac{\partial u^*}{\partial \eta}\right)_{\eta=0} \tag{14a}$$

$$\begin{aligned} &\frac{\partial \theta_{wu}}{\partial X} + \frac{1}{X\beta} \frac{\partial \theta_{ww}}{\partial \Phi} + \frac{\theta_{ww}}{X\beta \rho_e u_e^2} \frac{\partial(\rho_e u_e^2)}{\partial \Phi} + \frac{\theta_{wu}}{X} - \frac{\Delta}{X\beta u_e} \frac{\partial w_e}{\partial \Phi} \int_0^1 w^* \, d\eta - \frac{\Delta}{X} \int_0^1 u^* w^* \, d\eta = \\ &- \frac{\Delta w_e^*}{X} \left\{\frac{1}{\beta u_e} \frac{\partial w_e}{\partial \Phi} + 1\right\} \left\{1 + \frac{1}{T_1} \left(1 - \int_0^1 u^{*2} \, d\eta - \int_0^1 w^{*2} \, d\eta\right) + \frac{1}{T_1} w_e^{*2} \right. \\ &\left. + \left[\frac{T_s}{T_0} - 1\right] \left[1 + \frac{1}{T_1} \left(1 + w_e^{*2}\right)\right] \left[1 - \int_0^1 \Theta^* \, d\eta\right] \right\} + \frac{\mu_e C}{\rho_e u_e \Delta} \left(\frac{\partial w^*}{\partial \eta}\right)_{\eta=0} \end{aligned} \tag{14b}$$

$$\frac{\partial \theta_{\Theta u}}{\partial X} + \frac{1}{X\beta}\frac{\partial \theta_{\Theta w}}{\partial \Phi} + \frac{\theta_{\Theta w}}{X\beta \rho_e u_e}\frac{\partial(\rho_e u_e)}{\partial \Phi} + \frac{\theta_{\Theta u}}{X} = \frac{\mu_e C}{\rho_e u_e \Delta}\left(\frac{\partial \Theta^*}{\partial \eta}\right)_{\eta=0} \qquad (14c)$$

where

$$\theta_{uu} = \Delta \int_0^1 u^*(1 - u^*)d\eta \qquad (15a)$$

$$\theta_{uw} = \Delta \int_0^1 w^*(1 - u^*)d\eta \qquad (15b)$$

$$\theta_{ww} = \Delta \int_0^1 w^*(w_e^* - w^*)d\eta \qquad (15c)$$

$$\theta_{wu} = \Delta \int_0^1 u^*(w_e^* - w^*)d\eta \qquad (15d)$$

$$\theta_{\Theta u} = \Delta \int_0^1 u^*(1 - \Theta^*)d\eta \qquad (15e)$$

$$\theta_{\Theta w} = \Delta \int_0^1 w^*(1 - \Theta^*)d\eta \qquad (15f)$$

and

$$\frac{1}{T_1} \equiv \frac{u_e^2}{2c_pT_e} \qquad (16)$$

The boundary conditions are

$$u^* = w^* = \Theta^* = 0 \quad \text{at} \quad \eta = 0$$

and

$$\left.\begin{aligned} u^* = \Theta^* = 1,\ w^* = w_e^* \\ \frac{\partial u^*}{\partial \eta} = \frac{\partial w^*}{\partial \eta} = \frac{\partial \Theta^*}{\partial \eta} = 0 \end{aligned}\right\} \text{at} \quad \eta = 1 \qquad (17)$$

APPROXIMATE VELOCITY AND ENTHALPY PROFILES

Approximate solutions for equations (14) using the boundary conditions given by equation (17) can be obtained by assuming a polynomial to represent the boundary-layer profile for each of the velocity and thermal boundary layers. The parameters in the assumed profile are evaluated from the boundary conditions. Both a third- and fourth-degree polynomial were assumed. The results from the fourth-degree polynomial differed more from the exact solutions of reference 6 than those using the third-degree polynomial. Since it is desired to use an approximation that agrees as closely as possible to the exact solutions in the plane of symmetry, the method using the third-degree polynomial is presented.

Assume that

$$u^* = a_1\eta + a_2\eta^2 + a_3\eta^3 \tag{18a}$$

$$w^* = b_1\eta + b_2\eta^2 + b_3\eta^3 \tag{18b}$$

$$\Theta^* = c_1\eta + c_2\eta^2 + c_3\eta^3 \tag{18c}$$

Substituting boundary conditions (17) into equations (18), the assumed profiles become

$$u^* = (3 - 2\eta)\eta^2 + a\eta(1 - 2\eta + \eta^2) \tag{19a}$$

$$w^* = (3 - 2\eta)\eta^2 w_e^* + b\eta(1 - 2\eta + \eta^2) \tag{19b}$$

$$\Theta^* = (3 - 2\eta)\eta^2 + c\eta(1 - 2\eta + \eta^2) \tag{19c}$$

The expressions given in equations (14) and the integrals given in equations (15) become

$$\theta_{uu} = \frac{\Delta}{35}\left[\frac{9}{2} + \frac{3}{4}a - \frac{1}{3}a^2\right] \tag{20a}$$

$$\theta_{ww} = \frac{\Delta}{35}\left[\frac{9}{2}w_e^{*2} + \frac{3}{4}bw_e^* - \frac{1}{3}b^2\right] \tag{20b}$$

$$\theta_{wu} = \frac{\Delta}{35}\left[\frac{9}{2}\,w_e^* + \frac{11}{6}\,aw_e^* - \frac{13}{12}\,b - \frac{1}{3}\,ab\right] \tag{20c}$$

$$\theta_{uw} = \frac{\Delta}{35}\left[\frac{9}{2}\,w_e^* + \frac{11}{6}\,b - \frac{13}{12}\,aw_e^* - \frac{1}{3}\,ab\right] \tag{20d}$$

$$\theta_{\Theta u} = \frac{\Delta}{35}\left[\frac{9}{2} + \frac{11}{6}\,a - \frac{13}{12}\,c - \frac{1}{3}\,ac\right] \tag{20e}$$

$$\theta_{\Theta w} = \frac{\Delta}{35}\left[\frac{9}{2}\,w_e^* + \frac{11}{6}\,b - \frac{13}{12}\,cw_e^* - \frac{1}{3}\,bc\right] \tag{20f}$$

$$\int_0^1 w^*\,d\eta = \left[\frac{1}{2}\,w_e^* + \frac{1}{12}\,b\right] \tag{20g}$$

$$\int_0^1 \Theta^*\,d\eta = \left[\frac{1}{2} + \frac{1}{12}\,c\right] \tag{20h}$$

$$\int_0^1 u^{*2}\,d\eta = \frac{1}{35}\left[13 + \frac{13}{6}\,a + \frac{1}{3}\,a^2\right] \tag{20i}$$

$$\int_0^1 w^{*2}\,d\eta = \frac{1}{35}\left[13\,w_e^{*2} + \frac{13}{6}\,bw_e^* + \frac{1}{3}\,b^2\right] \tag{20j}$$

$$\int_0^1 u^*w^*\,d\eta = \frac{1}{35}\left[13\,w_e^* + \frac{13}{12}\,aw_e^* + \frac{13}{12}\,b + \frac{1}{3}\,ab\right] \tag{20k}$$

$$\left(\frac{\partial u^*}{\partial \eta}\right)_{\eta=0} = a \tag{20l}$$

$$\left(\frac{\partial w^*}{\partial \eta}\right)_{\eta=0} = b \tag{20m}$$

$$\left(\frac{\partial \Theta^*}{\partial \eta}\right)_{\eta=0} = c \tag{20n}$$

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The boundary-layer thickness is assumed to develop according to the expression

$$\Delta = K\sqrt{\frac{\mu_e}{\rho_e u_e}}\,X^{1/2} \tag{21}$$

where K is an unknown proportionality parameter. The quantities K, μ_e, ρ_e, and u_e are assumed to vary with Φ but not with X. The assumption of expression (21) for the laminar boundary-layer thickness is based on a statement in reference 5 that Blasius-type parabolic similarity exists in meridional planes. 4854

Substitution of equations (20) and (21) into equations (14) gives three ordinary differential equations in four unknowns: a, b, c, and C/K^2.

PLANE OF SYMMETRY

The equations of motion for the boundary layer can be greatly simplified by restricting them to the plane of symmetry. In the plane of symmetry Φ is equal to either 0 or π, however, both theory and experiment indicate that the boundary-layer assumptions do not hold for $\Phi = \pi$ at large angles of attack, since the boundary layer either becomes too thick or separation occurs. Therefore, the boundary-layer equations will be solved herein only for the case $\Phi = 0$ where, because of symmetry, $w = w_e = w^* = 0$ and therefore $b = 0$; also $p'(\Phi) = 0$.

Inspection of the Φ-momentum equation (14b) shows that each term vanishes on the plane of symmetry. In order to use this equation, it is necessary to differentiate it with respect to Φ before restricting it to the plane of symmetry. For $w = 0$, equations (14a) and (14c) are identical so that $u^* = \Theta^*$ and $a = c$. It is, therefore, only necessary to solve the X-momentum equation and the Φ derivative of the Φ-momentum equation.

The resulting equations in the plane of symmetry are

$$\frac{27}{4} + \frac{9}{8}a - \frac{1}{2}a^2 + \left(\frac{9}{2} - \frac{13}{12}a\right)\left(\frac{1}{\beta}\frac{dw_e^*}{d\Phi}\right) + \left(\frac{11}{6} - \frac{1}{3}a\right)\left(\frac{1}{\beta}\frac{db}{d\Phi}\right) = 35a\frac{C}{K^2} \tag{22a}$$

$$\left(\frac{25}{4} - \frac{5}{3} a\right)\left(\frac{dw_e^*}{d\Phi}\right) + \left(\frac{65}{24} + \frac{5}{6} a\right) \frac{db}{d\Phi} + \frac{17}{2\beta} \left(\frac{dw_e^*}{d\Phi}\right)^2 + \frac{17}{12\beta} \frac{db}{d\Phi} \frac{dw_e^*}{d\Phi} + \frac{2}{3\beta} \left(\frac{db}{d\Phi}\right)^2 - \frac{dw_e^*}{d\Phi}\left[\frac{1}{\beta} \frac{dw_e^*}{d\Phi} + 1\right]\left[35 + \frac{1}{T_1}\left(22 - \frac{13}{6} a - \frac{1}{3} a^2\right) + \left(\frac{T_s}{T_0} - 1\right)\left(1 + \frac{1}{T_1}\right)\left(\frac{35}{2} - \frac{35}{12} a\right)\right] = -35 \frac{C}{K^2} \frac{db}{d\Phi} \tag{22b}$$

These equations are simultaneous algebraic equations in the three unknowns: $db/d\Phi$, C/K^2, and a.

Examination of the exact boundary-layer profiles as given in reference 6 indicated similarity in the x-direction that was practically independent of angle of attack, temperature ratio, and cone angle. Therefore, a is assumed constant for all values of the variables and its value can be determined by comparison with the exact solutions.

The quantities $\frac{dw_e^*}{d\Phi}$ and $\frac{1}{T_1}$ are functions of the flow at the outer edge of the boundary layer and can be evaluated from data tabulated in the M.I.T. tables of flow over a cone (refs. 8 and 9). As given in reference 5, these quantities are

$$\frac{dw_e^*}{d\Phi} = \alpha \frac{z}{\bar{u}} + 2\alpha^2\left(\frac{w_2}{\bar{u}} - \frac{1}{\beta} - \frac{\sqrt{1 - \beta^2}}{\beta} \frac{z}{\bar{u}} - \frac{1}{2} \frac{xz}{\bar{u}^2}\right) \tag{23}$$

$$\frac{1}{T_1} = \frac{\bar{u}^2}{1 - \bar{u}^2}\left\{1 + \alpha\left(2 \frac{x}{\bar{u}} + \frac{\xi}{\bar{\rho}} - \frac{\eta}{\bar{p}}\right) + \alpha^2\left[2\left(1 + \frac{u_0}{\bar{u}} + \frac{u_2}{\bar{u}}\right) + \frac{\rho_0}{\bar{\rho}} + \frac{\rho_2}{\bar{\rho}} - \frac{p_0}{\bar{p}} - \frac{p_2}{\bar{p}} - \frac{2\bar{u}^2}{1 - \bar{u}^2} + \frac{2x}{\bar{u}}\left(\frac{\xi}{\bar{\rho}} - \frac{\eta}{\bar{p}}\right) + \frac{x^2}{\bar{u}^2} + \frac{\eta^2}{\bar{p}^2} - \frac{\eta}{\bar{p}} \frac{\xi}{\bar{\rho}}\right]\right\} \tag{24}$$

where α is the angle of attack, β is the sine of the cone semivertex angle, and all other quantities on the right side of the equations are in the notation of references 8 and 9. The barred quantities refer to zero angle of attack.

The quantities given in equations (23) and (24) are exact up to the order of α^2 but terms of higher order of α are considered negligible. This assumption sets an upper limit on the angle of attack. This limit is also a function of the other variables of the problem.

COEFFICIENTS OF FRICTION AND HEAT TRANSFER

The expressions for the components of viscous shear stress for the direction along the most windward generator and the circumferential direction are

$$\left(C_{f,x}\right)_{\varphi=0} = \frac{1}{\frac{1}{2}\rho_e u_e^2}\left(\mu\frac{\partial u}{\partial y}\right)_{\substack{y=0\\ \varphi=0}} \tag{25a}$$

$$\left(C_{f,\varphi}\right)_{\varphi=0} = 0 \tag{25b}$$

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$$\left(\frac{\partial C_{f,\varphi}}{\partial \varphi}\right)_{\varphi=0} = \frac{1}{\frac{1}{2}\rho_e u_e^2}\left[\mu\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial \varphi}\right)\right]_{\substack{y=0\\ \varphi=0}} \tag{25c}$$

In terms of the quantities obtained from this report, equations (25) can be written as

$$\left(C_{f,x}\right)_{\varphi=0} = 2a\frac{C}{K}\left(Re_x\right)^{-1/2} \tag{26a}$$

$$\left(C_{f,\varphi}\right)_{\varphi=0} = 0 \tag{26b}$$

$$\left(\frac{\partial C_{f,\varphi}}{\partial \varphi}\right)_{\varphi=0} = 2\frac{C}{K}\frac{db}{d\Phi}\left(Re_x\right)^{-1/2} \tag{26c}$$

Similarly, the expression for the Stanton number is

$$St = \frac{q}{\rho_e u_e\left(H_{as} - H_s\right)}$$

$$= \frac{\varkappa_s\left(\frac{\partial T}{\partial y}\right)_{y=0}}{\rho_e u_e\left(H_{as} - H_s\right)} \tag{27}$$

which, in terms of the presented quantities becomes

$$St = a\frac{C}{K}\left(Re_x\right)^{-1/2} \tag{28}$$

SOLUTIONS OF EQUATIONS

The exact solutions for the boundary-layer equations as given in reference 6 were used to evaluate the constant a. The expressions given in reference 6 for skin-friction coefficients and Stanton number are

$$(C_{f,x})_{\varphi=0} = 2f_w'' \sqrt{\frac{3C}{Re_x}} \tag{29a}$$

$$\left(\frac{\partial C_{f,\varphi}}{\partial \varphi}\right)_{\varphi=0} = 3\beta k \psi_w'' \sqrt{\frac{3C}{Re_x}} \tag{29b}$$

$$(St)_{\varphi=0} = \Theta_w' \sqrt{\frac{3C}{Re_x}} \tag{29c}$$

where f_w'', ψ_w'', and Θ_w' are defined in reference 6 and

$$k \equiv \frac{2}{3\beta}\left(\frac{dw_e^*}{d\Phi}\right)_{\Phi=0} \tag{30}$$

It should be noted here that in reference 6, as in the present paper, the energy equation was similar to one of the momentum equations and therefore $f_w'' = \Theta_w'$.

The skin-friction coefficients as given in equations (26) were set equal to the exact skin-friction coefficients as given in equations (29) for the particular case of $k = 0$, which is for zero angle of attack. Combining these coefficients with the X-momentum equation, the resulting value for a is

$$a = 1.5921$$

Using this value of a, the boundary-layer equations for the windward generator (eqs. (22a) and (22b)) become

$$7.2737 + \frac{2.7753}{\beta}\frac{dw_e^*}{d\Phi} + \frac{1.3026}{\beta}\frac{db}{d\Phi} = 55.724\frac{C}{K^2} \tag{31a}$$

$$3.5964 \frac{dw_e^*}{d\Phi} + 4.0350 \frac{db}{d\Phi} + \frac{1}{\beta}\left[8.5000\left(\frac{dw_e^*}{d\Phi}\right)^2 + 1.4167 \frac{dw_e^*}{d\Phi}\frac{db}{d\Phi} + 0.6667\left(\frac{db}{d\Phi}\right)^2\right] -$$

$$\left[\frac{1}{\beta}\left(\frac{dw_e^*}{d\Phi}\right)^2 + \left(\frac{dw_e^*}{d\Phi}\right)\right]\left[35 + 17.706 \frac{1}{T_1} + 12.856\left(\frac{T_s}{T_0} - 1\right)\left(1 + \frac{1}{T_1}\right)\right] = -35 \frac{C}{K^2}\frac{db}{d\Phi} \tag{31b}$$

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These boundary-layer equations can be solved simultaneously to give $db/d\Phi$ and C/K^2 once the conditions of the problem are given.

Solutions of the boundary-layer equations for the windward generator are given in table I along with the exact solutions of reference 6 for those conditions for which exact solutions have been obtained. The solutions are given in terms of the quantities obtained in reference 6. The relations between these quantities and the quantities presented herein are

$$f_w'' \equiv \frac{(C_{f,x})_{\varphi=0}}{2}\sqrt{\frac{Re_x}{3C}} = 0.9192\sqrt{\frac{C}{K^2}} \tag{32a}$$

$$\psi_w'' \equiv \frac{1}{3\beta k}\left(\frac{\partial C_{f,\varphi}}{\partial \varphi}\right)_{\varphi=0}\sqrt{\frac{Re_x}{3C}} = \frac{0.5774}{\left(\frac{dw_e^*}{d\Phi}\right)_{\Phi=0}}\frac{db}{d\Phi}\sqrt{\frac{C}{K^2}} \tag{32b}$$

$$\Theta_w' \equiv (St)_{\varphi=0}\sqrt{\frac{Re_x}{3C}} = 0.9192\sqrt{\frac{C}{K^2}} \tag{32c}$$

The data presented in table I are plotted in figure 2.

The errors between the approximate solutions obtained by the present method and the exact numerical solutions of reference 6 are less than 1 percent for both Stanton number and skin-friction coefficient along the most windward generator. With the exception of the extreme cooling case where $T_s/T_0 = 0$, the errors are less than 5 percent for the derivative of the circumferential skin-friction coefficient. In the range of conditions covered in reference 6, interpolation of the exact solutions gives better results than the use of the present method. However, in regions not covered by the exact solutions the present method will give useable results. A computational procedure is given in appendix B.

The integral method used reduces the boundary-layer equations to ordinary differential equations over those parts of the cone where the boundary-layer assumptions hold. Figure 2 shows that the solutions of these equations, when reduced to the plane of symmetry, agree closely with exact solutions of the original equations. Therefore, solutions of the ordinary differential equations away from the plane of symmetry can be obtained and will give some idea of the boundary-layer flow off the plane of symmetry.

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CONCLUSIONS

By means of an integral technique, the nonlinear partial differential equations for the laminar boundary layer over a cone at angle of attack are reduced in the plane of symmetry to a set of simultaneous algebraic equations. Third-degree polynomials are assumed for both velocity and enthalpy profiles. The approximate solutions agree with exact solutions for a Prandtl number of 1 within 1 percent for Stanton number and skin friction coefficient along the most windward generator and, for the most part, within 5 percent for the derivative of the circumferential skin-friction coefficient.

The method presented gives an easy method of determining skin-friction coefficients and Stanton numbers outside the range for which exact solutions are available.

Due to the close agreement between the approximate solutions and the exact solutions on the plane of symmetry, the same technique can be used to study the boundary-layer flow off the plane of symmetry.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, August 4, 1958

APPENDIX A

SYMBOLS

a	parameter in generator velocity profile
a_1,a_2,a_3	coefficients in assumed generator velocity profile
b	parameter in circumferential velocity profile
b_1,b_2,b_3	coefficients in assumed circumferential velocity profile
C	constant in Chapman-Rubesin temperature-viscosity relation
C_f	skin-friction coefficient
c	parameter in enthalpy profile
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
c_1,c_2,c_3	coefficients in assumed enthalpy profile
E	internal energy, c_vT
f_w''	generator shear parameter (defined in ref. 6)
H	enthalpy as defined in equation (5)
K	proportionality parameter in transformed boundary-layer thickness
k	circumferential velocity-gradient parameter
Pr	Prandtl number
p	static pressure
q	heat-transfer coefficient
R	gas constant
Re_x	Reynolds number based on distance x
St	Stanton number

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T	temperature
T_1	dimensionless external temperature
u	velocity in x-direction
v	velocity in y-direction
w	velocity in circumferential direction
X	transformed coordinate along generator
x	coordinate along generator
Y	transformed coordinate perpendicular to surface of cone
y	coordinate perpendicular to surface of cone
α	angle of attack
β	sine of cone semivertex angle
Δ	boundary-layer thickness in transformed coordinates
δ	boundary-layer thickness
ζ_{ij}	defined by equation (10)
η	dimensionless coordinate perpendicular to cone
Θ	enthalpy difference ratio
ϑ	semivertex angle of cone
θ_{ij}	defined by equation (15)
Θ_w'	heat-transfer parameter (defined in ref. 6)
$\varkappa$	coefficient of thermal conductivity
μ	viscosity
ρ	density
Φ	transformed angular coordinate
φ	angular coordinate measured from most windward generator

ψ_w'' circumferential shear-gradient parameter (defined in ref. 6)

Subscripts:

as adiabatic surface

e outer edge of boundary layer

s surface

x at distance x along generator

φ at angular distance φ

0 free-stream stagnation

Superscripts:

$*$ dimensionless quantity

$'$ differentiation with respect to independent variable

APPENDIX B

COMPUTATIONAL PROCEDURE

The analysis given in the present paper enables one to calculate skin-friction coefficients and Stanton numbers on the windward generator of a cone at angle of attack. The analysis is, however, limited to the case of Prandtl number equal to 1. It is necessary to know the following conditions:

(1) Mach number upstream of shock wave

(2) Cone surface to free-stream stagnation temperature ratio

(3) Semivertex angle of cone

(4) Angle of attack

The following quantities are to be evaluated:

$$\left(\frac{dw_e^*}{d\Phi}\right)_{\Phi=0} = \alpha \frac{z}{\bar{u}} + 2\alpha^2 \left(\frac{w_2}{\bar{u}} - \frac{1}{\beta} - \frac{\sqrt{1 - \beta^2}}{\beta} \frac{z}{\bar{u}} - \frac{1}{2} \frac{xz}{\bar{u}^2}\right)$$

$$\frac{1}{T_1} = \frac{\bar{u}^2}{1 - \bar{u}^2} \left\{1 + \alpha\left(2 \frac{x}{\bar{u}} + \frac{\xi}{\bar{\rho}} - \frac{\eta}{\bar{p}}\right) + \alpha^2 \left[2\left(1 + \frac{u_0}{\bar{u}} + \frac{u_2}{\bar{u}}\right) + \frac{\rho_0}{\bar{\rho}} + \frac{\rho_2}{\bar{\rho}} - \frac{p_0}{\bar{p}} - \frac{p_2}{\bar{p}} - 2 \frac{\bar{u}^2}{1 - \bar{u}^2} + 2 \frac{x}{\bar{u}} \left(\frac{\xi}{\bar{\rho}} - \frac{\eta}{\bar{p}}\right) + \frac{x^2}{\bar{u}^2} + \frac{\eta^2}{\bar{p}^2} - \frac{\eta}{\bar{p}} \frac{\xi}{\bar{\rho}}\right]\right\}$$

where α and $\beta = \sin \vartheta$ are given and the other quantities on the right side of the equations are given in references 8 and 9. The barred quantities refer to zero angle of attack.

Substitute the values of $(dw_e^*/d\Phi)_{\Phi=0}$ and $1/T_1$ into the following equations:

$$7.2737 + \frac{2.7753}{\beta} \frac{dw_e^*}{d\Phi} + \frac{1.3026}{\beta} \frac{db}{d\Phi} = 55.724 \frac{C}{K^2}$$

$$3.5964 \frac{dw_e^*}{d\Phi} + 4.0350 \frac{db}{d\Phi} + \frac{1}{\beta}\left[8.5000 \left(\frac{dw_e^*}{d\Phi}\right)^2 + 1.4167 \frac{dw_e^*}{d\Phi} \frac{db}{d\Phi} + 0.6667 \left(\frac{db}{d\Phi}\right)^2\right] -$$

$$\left[\frac{1}{\beta}\left(\frac{dw_e^*}{d\Phi}\right)^2 + \left(\frac{dw_e^*}{d\Phi}\right)\right]\left[35 + 17.706 \frac{1}{T_1} + 12.856 \left(\frac{T_s}{T_0} - 1\right)\left(1 + \frac{1}{T_1}\right)\right] = -35 \frac{C}{K^2} \frac{db}{d\Phi}$$

These equations can be solved simultaneously to obtain $db/d\Phi$ and C/K^2. Since the second equation is quadratic in $db/d\Phi$, two solutions will occur. The desired solution is that which makes C/K^2 positive. The skin-friction coefficients and Stanton numbers are then obtained directly from

$$\sqrt{\frac{Re_x}{3C}} \left(C_{f,x}\right)_{\varphi=0} = 1.8384 \sqrt{\frac{C}{K^2}}$$

$$\sqrt{\frac{Re_x}{3C}} \left(\frac{\partial C_{f,\varphi}}{\partial \varphi}\right)_{\varphi=0} = 1.1547 \frac{db}{d\Phi} \sqrt{\frac{C}{K^2}}$$

$$\sqrt{\frac{Re_x}{3C}} (St)_{\varphi=0} = 0.9192 \sqrt{\frac{C}{K^2}}$$

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TABLE I. - COMPARISON OF APPROXIMATE AND EXACT SKIN-FRICTION AND HEAT-TRANSFER PARAMETERS FOR MOST WINDWARD GENERATOR OF A CONE AT ANGLE OF ATTACK FOR PRANDTL NUMBER OF 1

$\frac{T_s}{T_0}$	$\frac{1}{T_1}$	k	f''_w or θ'_w		ψ''_w	
			Exact	Approx-imate	Exact	Approx-imate
0	0	0	0.3321	0.3321	0.4238	0.4514
		.6	.4330	.4318	.5527	.5837
		1.2	.5143	.5124	.6570	.6923
	2.5	0	0.3321	0.3321	0.6532	0.7464
		.6	.4598	.4563	.8596	.9673
		1.2	.5569	.5516	1.0281	1.1518
	5.0	0	0.3321	0.3321	0.8826	1.0407
		.6	.4815	.4764	1.1422	1.3249
		1.2	.5898	.5823	1.3634	1.5739
0.5	0	0	0.3321	0.3321	0.5923	0.6079
		.6	.4468	.4455	.7888	.7910
		1.2	.5367	.5346	.9460	.9417
	2.5	0	0.3321	0.3321	1.2430	1.2920
		.6	.4944	.4912	1.5922	1.6159
		1.2	.6092	.6045	1.8998	1.9147
	5.0	0	0.3321	0.3321	1.8937	1.9726
		.6	.5291	.5247	2.3034	2.3565
		1.2	.6594	.6532	2.7287	2.7752
1.0	0	0	0.3321	0.3321	0.7609	0.7642
		.4	.4215	.4205	.9358	.9187
		.8	.4935	.4918	1.0850	1.0560
		1.2	.5559	.5537	1.2165	1.1783
	1.0	0	0.3321	0.3321	1.1897	1.1937
		.4	.4436	.4420	1.4266	1.4004
		.8	.5278	.5253	1.6445	1.6014
		1.2	.5995	.5962	1.8393	1.7833
	2.5	0	0.3321	0.3321	1.8329	1.8360
		.4	.4704	.4680	2.1030	2.0701
		.8	.5675	.5639	2.4026	2.3492
		1.2	.6487	.6443	2.6767	2.6077
	5.0	0	0.3321	0.3321	2.9049	2.8980
		.4	.5051	.5019	3.1300	3.0950
		.8	.6172	.6126	3.5389	3.4809
		1.2	.7096	.7040	3.9250	3.8496

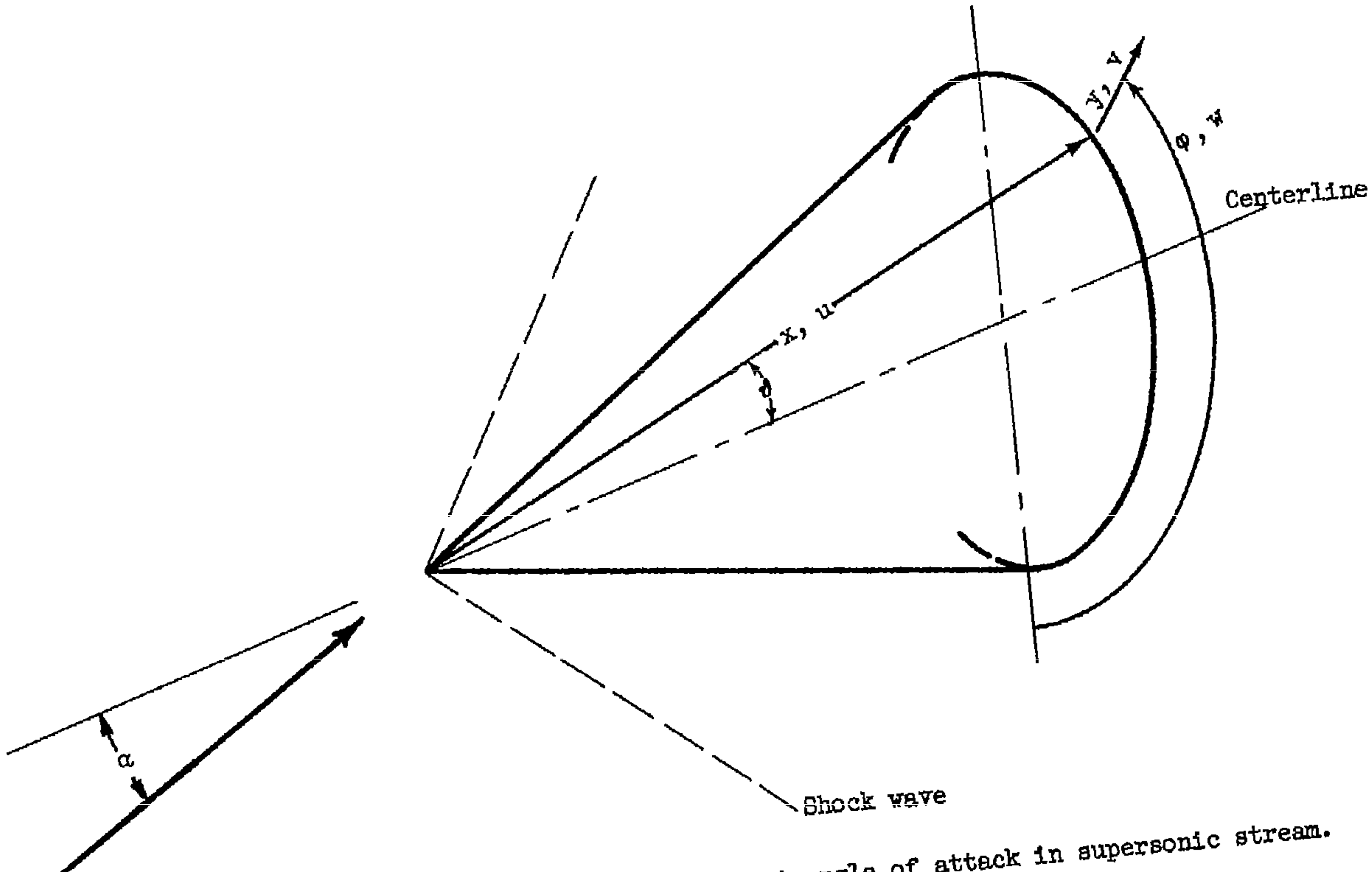
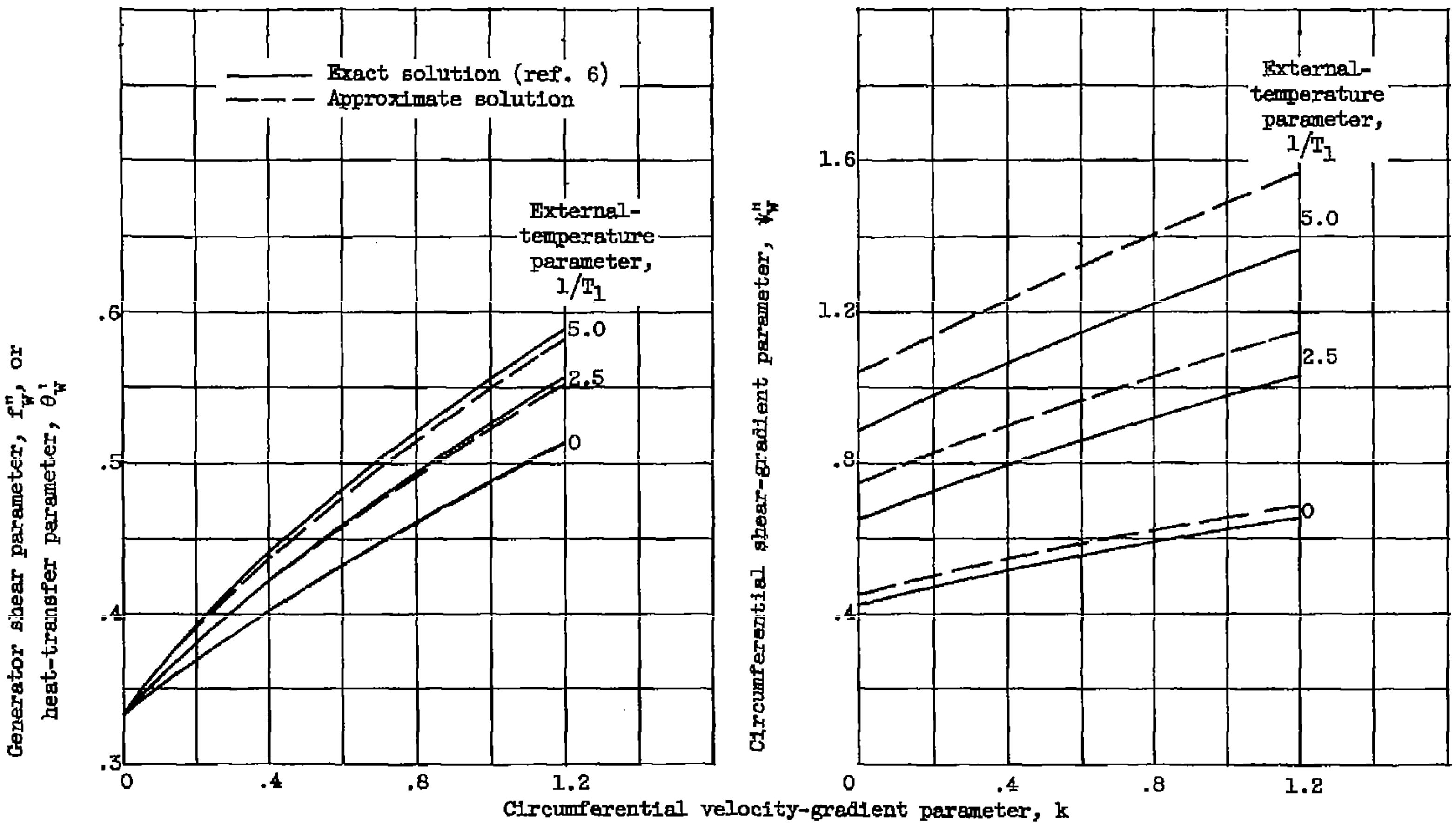
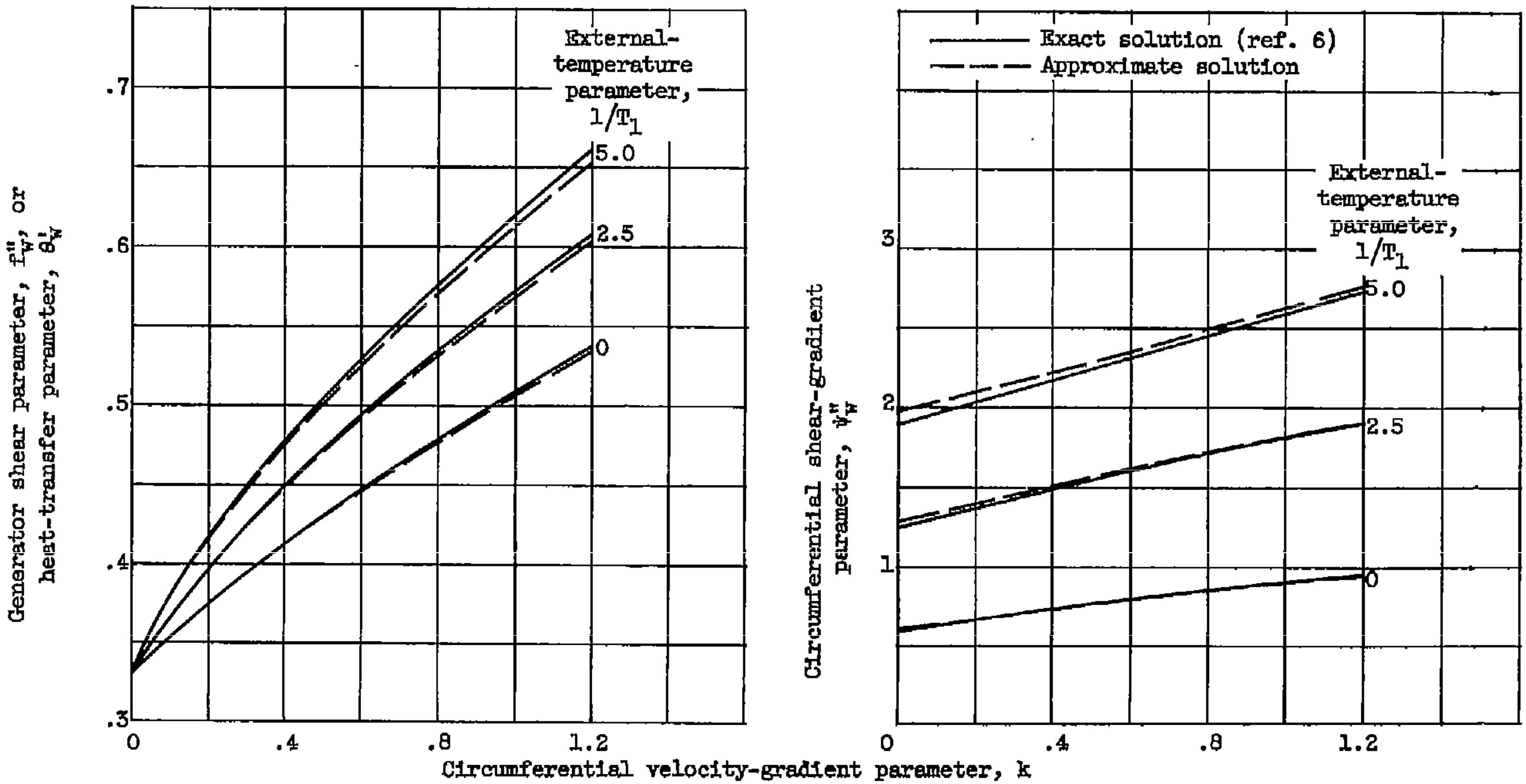


Figure 1. - Coordinate system for circular cone at angle of attack in supersonic stream.



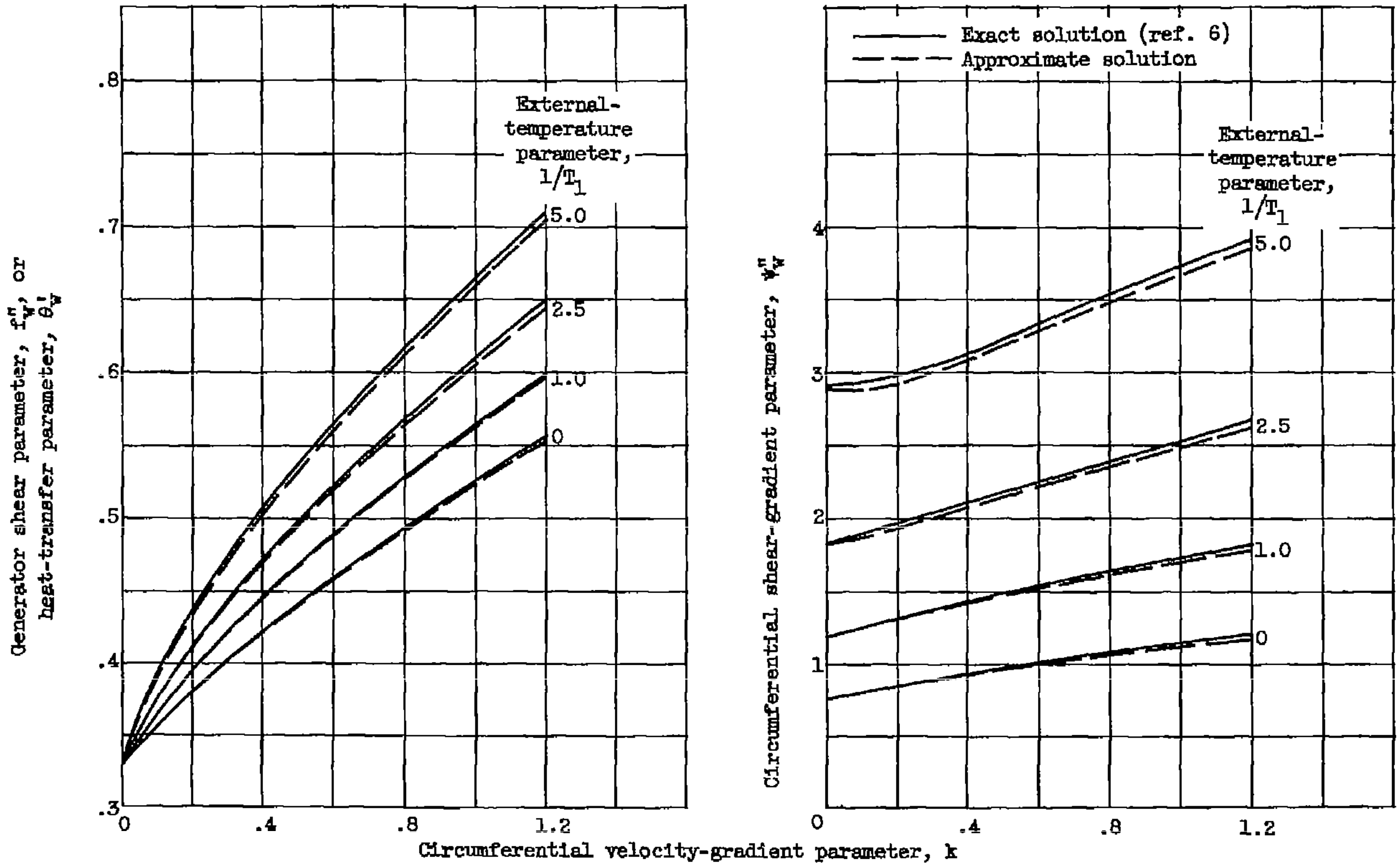
(a) Ratio of surface temperature to free-stream total temperature, $T_s/T_0 = 0$.

Figure 2. - Shear and heat-transfer parameters from exact and approximate solutions in plane of symmetry.



(b) Ratio of surface temperature to free-stream total temperature, $T_s/T_0 = 0.5$.

Figure 2. - Continued. Shear and heat-transfer parameters from exact and approximate solutions in plane of symmetry.



(c) Ratio of surface temperature to free-stream total temperature, $T_s/T_0 = 1.0$.

Figure 2. - Concluded. Shear and heat-transfer parameters from exact and approximate solutions in plane of symmetry.

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