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RESEARCH MEMORANDUM

PROBLEM OF SLIP FLOW IN AERODYNAMICS

By Robert E. Street

University of Washington

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SUMMARY

A survey of the present status of theory in the field of slip-flow aerodynamics leads to the conclusion that the Navier-Stokes equations of motion together with first-order velocity slip and temperature jump at any boundary are sufficient insofar as experimental confirmation is available. The use of the Burnett terms in the equations of motion as well as second-order terms in boundary conditions does not seem justified, at least for semirarefied gas flows. Further kinetic theoretical study and additional experimental data are urgently needed.

INTRODUCTION

The purpose of the present report is to survey the present status of theory in the field of slip-flow aerodynamics. No new problems will be set up and solved, but a fairly complete listing of books and papers which would seem to have some connection with this problem has been assembled and will be commented upon. The present report is therefore essentially preliminary, a small-scale map of the field, but with some suggestions as to areas which would seem to be the most desirable to explore in more detail. The realm of interest is the near-continuum flow domain (modern compressible viscous flow) in which the effects of slip in velocity and jump in temperature at the boundary of the flow begin to exhibit themselves.

The original survey of the field made for the aerodynamicist is the paper of Tsien (ref. 1). Although this paper was very fertile in that it led to the considerable activity in the investigation of flow of rarefied gases which has taken place since its publication, it is no longer a satisfactory introduction to the field. A better and more up-to-date survey of the whole problem of rarefied gas flow is the treatment by Schaaf and Chambré (ref. 2). The present report was outlined and planned just before a copy of the manuscript of this reference became available, so there is considerable overlapping, although much material which is included in the cited reference has been deleted from this report. A similar but less comprehensive survey was also published by Schaaf a few years ago (ref. 3).

The dimensionless parameter which is considered of paramount importance in the flow of semirarefied gases is the Knudsen number K, which is defined as the ratio of the mean free path l of the gas to some significant length L in the problem. The well-known connection between the Knudsen number and the Mach and Reynolds numbers which is taken as the governing factor determining the nature of the flow regime is

$$K = \frac{l}{L} = 1.253\gamma^{1/2} \frac{M}{Re} \tag{1}$$

where γ is the ratio of specific heats of the gas, $M = U/a$ is the Mach number, and $Re = \rho LU/\mu$ is the Reynolds number. The velocity U is some reference speed such as the mass velocity of the gas or the flight speed.

Some writers suppose that for large values of Re the characteristic length is the boundary-layer thickness δ and define the Knudsen number as $K = \frac{l}{\delta} \approx M/\sqrt{Re}$. This seems to be the proper parameter for boundary layers in slip flow, but it is not as yet known just how the boundary-layer thickness depends on Reynolds number except that it is certainly larger and hence may increase inversely as the square root of Re. The first-order slip on flat plates does depend upon the parameter $\sqrt{\gamma}M/\sqrt{Re}$ at low speeds, but in the hypersonic flow over flat plates the more significant parameter is a constant times $M^3/\sqrt{Re}$. Physically this seems to be due to the greater importance of M since it indicates compressibility in hypersonic flows while M occurs in conjunction with Re as an indication of rarefaction alone in equation (1). Since

$$\frac{M}{Re} = \frac{U}{a}\frac{\nu}{LU} = \frac{\nu}{aL} = \frac{l\bar{v}}{2aL} = \sqrt{\frac{2}{\pi\gamma}}\frac{l}{L}$$

independently of the magnitude of U, it follows that at any Mach number or Reynolds number the Knudsen number or the ratio M/Re remains as an indication of the amount of rarefaction of the gas. For $K \ll 1$ the laws of continuum mechanics must hold and for $K \gg 1$ the flow is highly rarefied or of free-molecule type. Nothing more can be said about the relevancy of one form of parameter as compared with that of another at the present time.

It is usually stated that the problem of skin friction and heat transfer for high-speed flight at extreme altitudes $K \gg 1$ must be based upon kinetic theory rather than on classical hydrodynamics (ref. 4). The question is, where does classical hydrodynamics as such break down and kinetic theory take over? It seems that modern phenomenological aerodynamics together with properly extended boundary conditions (that is, the compressible Navier-Stokes equations with first-order slip in the boundary conditions) is capable of extension to a broader domain of flow regimes than is sometimes assumed. In other words, it appears desirable, first, to see what the limitations are before abandoning hydrodynamics. For example, the problem of shock-wave structure appears to be contained within the classical equations (refs. 5 and 6) insofar as any experimental check has been made. As for other problems in semirarefied flow, the empirical theories based upon the phenomenological equations of motion have been surprisingly successful. Therefore, it seems of first importance to determine just how rarefied a gas must be or how fast a body must fly before this approach has to be abandoned. The free-molecular flow regime of highly rarefied gas is not considered here at all, except it is referred to as a limiting case in some places. The kinetic theory would be necessary in this flow regime. Also, the more theoretical problem of statistical mechanics of nonequilibrium and transport processes is not a proper subject. It has already been surveyed elsewhere (ref. 4), and the foundations of the subject are questions of extreme mathematical complication. The immediate question here is to exhibit those problems in semirarefied flow about which enough is now known to enable the design engineer to make some attempt at determining optimum aerodynamic shapes for aircraft and missiles of the immediate future.

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SYMBOLS

a	speed of sound
C_f	local skin friction, dimensionless
C_s	constant of proportionality in linear viscosity-temperature formula
$\vec{c}$	instrinsic velocity, $\vec{\xi} - \vec{u}$
c_i, c_j	components of instrinsic velocity

c_p	specific heat at constant pressure
c_v	specific heat at constant volume
d	distance between plates
f	distribution function depending on $\vec{x}$, $\vec{\xi}$, and t
G	coefficient of slip
g	function
E	energy
K	Knudsen number, l/L, dimensionless
L	reference length
l	mean free path of gas
M	Mach number, U/a, dimensionless
P	probability that molecule will collide with surface before it collides with another molecule
P_{ij}	components of complete stress
Pr	Prandtl number, $c_p\mu/\kappa$, dimensionless
p	pressure in gas
Q	any aerodynamic function of K such as drag coefficient or recovery factor
q_i	components of heat flux vector
R	gas constant
Re	Reynolds number
r	recovery factor on circular cylinders
T	absolute temperature
t	time

U	reference velocity
$\vec{u}$	macroscopic flow velocity
u_i,u_j	components of flow velocity
v	mean molecular speed of gas
x_i,x_j	Cartesian coordinate components of position vector x
α	accommodation coefficient, dimensionless
γ	ratio of specific heats of gas, dimensionless, c_p/c_v
δ	boundary-layer thickness
κ	coefficient of heat conduction
λ	second coefficient of viscosity
ρ	density of gas
σ	reflection coefficient of Maxwell
μ	coefficient of viscosity of gas
ν	coefficient of kinematic viscosity, u/p
$\vec{\xi}$	molecular velocity vector
ξ_i	components of $\vec{\xi}$
τ	average time between molecular collisions

Subscripts:

c	value in continuum range
f	value in free-molecule range
o	free-stream conditions
s	value in slip flow
w	value at wall

THEORETICAL CONSIDERATIONS

The basis for the aerodynamics of rarefied gases is necessarily the kinetic theory of gases. Only the latter will be valid in highly rarefied flows where the mean free path is so large that the effects of all intermolecular collisions may be completely neglected as compared with those of collisions between the molecules and the surface of the body moving through the gas. At the other extreme of continuum flow, where the mean free path is so small that collisions between molecules themselves predominate, the Navier-Stokes equations for viscous compressible fluids are definitely established both theoretically and experimentally. However, the Navier-Stokes equations are also integrals of motion of the Boltzmann equation of kinetic theory. Thus it follows that some form of the Boltzmann equation is valid throughout the flow regime from a highly rarefied gas to the conventional gaseous continuum of phenomenological theory. A study of the assumptions underlying the derivation of the Boltzmann equation, such as stated clearly and succinctly by Grad (ref. 7), indicates that although restrictions upon the gas are necessary they are not unreasonable. In other words, real gases differ so inconsequentially from the gas described by the Boltzmann equation (or else the extraneous effects can be separated out) that the assumptions underlying the Boltzmann equation will be adhered to in the following discussion. Any really serious criticisms of the theory can be embodied within the range of this simplified gas theory.

The assumptions themselves are:

(1) The gas is monatomic. This statement means that each gas molecule behaves classically with only translational degrees of freedom, three in number. This is not a serious restriction since a monatomic gas, helium, is used in some wind tunnels, especially in hypersonic flows, and studies of shock-wave structure have included data on monatomic gases. The effects of internal degrees of freedom in more common gases, such as air, have been satisfactorily accounted for by the use of a second coefficient of viscosity in the Navier-Stokes equations. There are much more fundamental problems to be solved before the significance of the other degrees of freedom must be considered. The elimination of second viscosity from the Navier-Stokes equations by means of Stokes' relation (ref. 8)

$$3\lambda + 2\mu = 0$$

has in the past not seriously hampered the development of a fruitful boundary-layer theory of viscous and heat-conducting gases in the continuum flow regime. Only one paper so far has appeared in which an attempt has been made to develop a kinetic theory of polyatomic gases.

In this, Wang Chang and Uhlenbeck (ref. 9) have shown that internal degrees of freedom in polyatomic gases can be embodied in a second viscosity coefficient. Although this assumption restricts the discussion to gases at relatively low temperatures, this again is not serious, except for very strong shock waves and extremely high Mach number. Other considerations such as the deviation of the gas from a classical state would have to be taken into account in the case of high temperatures, or extremely low temperatures, anyway. Finally, the effects of relaxation times, which are also connected with the coefficient of second viscosity, are eliminated from the theory through this assumption.

(2) The gas is ideal. This statement means that the equation of state connecting pressure, temperature, and density is the simple relation

$$p = R\rho T$$

This is commonly assumed for gases in the continuum state and is certainly valid unless the gas is compressed to a pressure of many atmospheres. As the gas becomes more rarefied this equation of state is even better in the slip-flow and free-molecule flow regimes (ref. 10). There is no great restriction here, especially for monatomic gases.

(3) Point molecules, complete collisions (binary collisions only), a slowly varying distribution function f and, finally, molecular chaos. These assumptions are embodied in the derivation of the Boltzmann equation for f and are completely discussed by Grad (ref. 7). Later, another assumption made is that the molecules are either spherical (in that the collisions behave like those between perfectly elastic spheres) or Maxwellian (are repelled according to the inverse fifth power of the distance between them).

(4) The state of the gas is one of equilibrium or near equilibrium. This is implicit in the previous assumption, but is not commonly stressed. The strength of a good theory lies in its range of validity being so much broader than its basic assumptions would seem to imply. Such, for example, is true of the Prandtl-Glauert approximation in the subsonic flow of a compressible nonviscous fluid and the Prandtl boundary-layer theory in the flow of viscous compressible fluids. Since the Navier-Stokes equations work so well for shock-wave phenomena, it is not too much to expect this to be true of the kinetic theory based upon Boltzmann's equation. However, the amount of nonequilibrium allowed is neither known nor can as yet be estimated. The most significant approximation so far developed is that of Grad (ref. 7) and it is definitely limited to very near equilibrium conditions.

It must be emphasized that the difficulties with present theories of slip flow and the transition regime do not lie in any of the above assumptions but rather in the method of deduction of the suitable type of distribution function which determines the moments, the ordinary macroscopic variables: the density ρ, the velocity of the gas flow u_i, the stress components P_{ij}, the temperature T, and the heat flux vector components q_i. If the Boltzmann equation is multiplied by certain summational invariants and integrated over all the velocities, the collision integral vanishes, corresponding to the conservation of mass, momentum, and energy during collision, with the result that the following five conservation equations are obtained (refs. 2 and 7)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \tag{2}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho} \frac{\partial P_{ij}}{\partial x_j} = 0 \tag{3}$$

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i}(pu_i) + \frac{2}{3} P_{ij} \frac{\partial u_i}{\partial x_j} + \frac{2}{3} \frac{\partial q_i}{\partial x_i} = 0 \tag{4}$$

where $i,j = 1,2,3 \ldots$ and the double summational convention is used. To these must be added the equation of state

$$p = R\rho T \tag{5}$$

These equations can be deduced by elementary means and form the continuum basis of fluid dynamics, for which see any standard reference such as Lamb, Goldstein, Howarth or Kuethe, and Schetzer. As they stand there are 6 equations, but many more unknowns (15 in all). For complete equilibrium the solution of Boltzmann's equation is the well-known distribution function of Maxwell

$$f^{(0)} = \frac{\rho}{(2\pi RT)^{3/2}} e^{-c^2/2RT} \tag{6}$$

where c^2 is the square of the intrinsic velocity whose components are the difference between the components of molecular velocity ξ_i and mass velocity u_i. All odd moments of $f^{(0)}$ vanish as do the second-order ones, leading to $P_{ij} = p\delta_{ij}$ and $q_i = 0$, and equations (2) to (5) reduce to the completely determined set for a nonviscous non-heat-conducting ideal fluid, the Euler equations. In macroscopic theory the same result is obtained by assuming that the coefficients of viscosity and heat conduction are zero.

From the phenomenological point of view the assumption that departure from equilibrium is slight, such that the stress and heat flux are linear functions of the rate of deformation of the fluid and the temperature gradient, respectively, leads to the relations

$$\left.\begin{aligned} P_{ij} &= p\delta_{ij} + p_{ij} \\ p_{ij} &= -\mu(u_{i,j} + u_{j,i}) - \lambda\delta_{ij}u_{k,k} \end{aligned}\right\} \tag{7}$$

$$q_i = -\kappa T_i \tag{8}$$

Then equations (2) to (5) reduce to the Navier-Stokes equations of a compressible viscous fluid, which are also completely determined. The coefficients μ, λ, and κ are then left to experiment for determination. A determination of higher order, nonlinear terms in the relations between stress and heat flux with velocity and temperature gradients based upon the theory of invariants in tensor analysis has been carried out by Truesdell (ref. 11) but leads to an enormous number of undetermined coefficients. To date no corresponding variety of solutions and experiments has been devised which would determine these additional coefficients.

As mentioned earlier, the fact that the Navier-Stokes equations lead to correct solutions in the continuum range substantiates the correctness of equations (2) to (5) together with relations (7) and (8). Any kinetic theory must therefore begin with the first approximation (6) and as second approximation give equations (7) and (8). The Chapman-Enskog-Burnett method does essentially do so by taking the complete distribution function to be

$$f = f^{(0)}\left(1 + f^{(1)} + f^{(2)} + . . .\right) \tag{9}$$

Substitution of this expression into Boltzmann's integral equation and the determination of $f^{(0)}$, $f^{(1)}$, $f^{(2)}$, . . . by a complicated scheme of iteration leads in succession to the Euler, Navier-Stokes, and Burnett equations of motion.

The Burnett expressions are not written down here for several reasons. First, they are long and can be found in many places, for example, references 1 to 3 and 11. Secondly, they can be deduced from Grad's distribution function

$$f = f^{(0)}\left[1 + \frac{p_{ij}}{2pRT} c_i c_j - \frac{q_i c_i}{pRT}\left(1 - \frac{c^2}{5RT}\right)\right] \tag{10}$$

as was first shown by Schaaf and Chambré (refs. 2 and 3). Thirdly, no solution of the Burnett equations has yet been found which gives any better results than the Navier-Stokes equations, if as good. Finally, it has been pointed out by Grad, Schaaf, Truesdell, and others that only a very limited number of possible solutions of the Boltzmann equation can be found by an iteration such as equation (9). Since the iteration is made upon the collision integral and only collisions between molecules are contained therein, it is strange that the Burnett terms are considered valid in the transition regime between slip flow and free-molecule flow where collisions between the surface and the molecules are as important as those between the molecules themselves.

For this last reason, it is to be expected that the Burnett terms would be more significant in the continuum regime, where collisions between molecules predominate and large deviations from equilibrium take place, such as in shock-wave structure or in hypersonic flow. Yet in these two cases in particular the Burnett terms, when applied, have not yet led to any significant results (refs. 5 and 12) different from the Navier-Stokes terms. In hypersonic flow Von Kryzwoblocki has deduced equations to boundary-layer order of magnitude (ref. 13) and has indicated means of solving these but has not worked out a solution. Yang (ref. 14) has found a solution to Rayleigh's problem of the impulsive motion of a flat plate in low-speed flow which agrees with both limiting cases of free-molecule and continuum flow. This solution was obtained from Grad's equations and could not be deduced from the Burnett equations.

The Couette flow between parallel plates at low Mach number and the heat transport between the plates with small temperature difference have been calculated by Wang Chang and Uhlenbeck (refs. 15 and 16) for arbitrary Knudsen numbers starting with the Boltzmann equation. As expansions in powers of the reciprocal of the Knudsen number $K = l/d$, where d is the distance between the plates, they obtain solutions in the far transition regime approaching free-molecule flow. In the terminology used by Wang Chang and Uhlenbeck, $K = d/l$, or the reciprocal of the Knudsen number defined in this report, and the continuum and free-molecule flow regimes are called a Clausius gas and Knudsen gas, respectively. Since their expansions in powers of l/d do not converge while those in powers of d/l do converge, except for a minor difficulty due to the special geometry of parallel plates, the results imply, although they do not prove, that an expansion such as equation (9) is not valid at all. Thus, while a solution of the Boltzmann equation in a power series may be quite possible in highly rarefied flows, there is more doubt of such a solution in the continuum or slip-flow regimes. This is the solution which Yang showed to be useless in the Rayleigh problem.

It is worthwhile to take a closer look at Yang's thesis, which is the only attempt so far to solve the flat-plate problem by purely kinetic theory. Yang uses Grad's 13-moment approximation to the Boltzmann equation which assumes the distribution function (10) above and deduces therefrom a system of nine additional equations for the tensors P_{ij} and q_i, which together with equations (2) to (5) above give a determined system with unknowns p, ρ, T, u_i, q_i, and P_{ij}. Thus Grad's equations contain Euler's, Navier-Stokes', and Burnett's equations as successive approximations, for which Schaaf and Chambré (refs. 2 and 3) and Yang give the method of iteration. In his thesis Yang summarizes Grad's theory and gives a clearer and more physical derivation of Grad's boundary conditions, about which more will be stated later in the present report (cf. section entitled "Boundary Conditions"). Applying the equations of Grad to a flat plate starting suddenly from rest with velocity U in its own plane, Yang finds it possible to linearize these and the boundary conditions by assuming that U is small and the temperature difference between the plate and undisturbed gas is also small. Solving the linearized equations and boundary conditions for the adiabatic plate by an approximate method Yang obtains series expansions for small and large times t. Considering only the dominant terms in these two limiting expressions he deduces a closed expression which is approximately valid for all time. This is done for shear stress, tangential heat flux, and slip velocity on a plate where the interaction between plate surface and molecules is completely diffuse or $\alpha = 0$. For example, the result for local skin friction C_f is

$$M_oC_f \approx 0.683 \exp\left(0.342 \frac{\sqrt{Re}}{M_o}\right)^2 \operatorname{erfc}\left(0.342 \frac{\sqrt{Re}}{M_o}\right) \tag{11}$$

Here $M_o = U/a_o$ and $Re = \frac{\rho_o}{\mu_o} U^2 t$, the subscript o denoting free-stream conditions. The ratio of the speed of sound to the rate of diffusion of vorticity is $\sqrt{Re}/M_o = a_o\sqrt{\dfrac{\rho_o t}{\mu_o}} = \sqrt{\dfrac{a_o{}^2 t}{\nu_o}}$ or, alternatively, $Re/M_o{}^2 = a_o{}^2 t/\nu_o$ is proportional to the ratio of elapsed time to the average time between molecular collisions τ, that is, $Re/M_o{}^2 = \dfrac{4}{3}\dfrac{t}{\tau}$.

Schaaf and Sherman (ref. 17) obtained a similar expression for C_f as did Mirels (ref. 18) by considering Rayleigh's problem using Navier-Stokes equations and first-order slip at the surface. Their result for M_oC_f is the same as equation (11) in functional form but differs in the magnitude of the constants. Because of this, both formulas give twice the correct value of C_f for free-molecule flow and Schaaf's equation fails to give the correct value for C_f in continuum flow. Many papers have been published lately in which Rayleigh's problem has been extended to compressible flow, but in most cases only with the no-slip boundary conditions. Besides the two mentioned papers by Schaaf and Mirels, a report by Martino (ref. 19) has just appeared which repeats the calculations of Mirels but considers in addition the case of heat transfer.

Although Martino surveys the kinetic-theory approach and includes another method of solving Boltzmann's equation, attributed to Chapman, which is to appear in a forthcoming book by Dr. G. N. Patterson entitled "The Molecular Flow of Gases," he deduces the usual boundary-layer equations of motion and first-order slip from the Chapman-Patterson equations. The parameter used is $\zeta = 1.30\sqrt{\dfrac{C_s}{M_oK}}$ where $C_s = \left(\dfrac{T_w}{T_o}\right)^{1/2} \dfrac{T_o + S}{T_w + S}$ is the constant of proportionality in the linear viscosity-temperature formula. But using equation (1) $M_oK = 1.256\gamma^{1/2}\dfrac{M^2}{Re}$ and since $C_s = 1$ for the adiabatic no-heat-transfer case $\zeta \approx \sqrt{Re}/M$, or ζ is essentially the same parameter used by the others, including Yang.

The results of Martino's analysis are essentially the same as Mirel's insofar as the skin friction is concerned, although no explicit form of the relation is given. Again the limiting value of skin friction in free-molecule flow is twice its correct value. Martino is primarily interested in heat transfer expressed by means of the Stanton number and finds the ratio of the wall temperature to the free-stream gas temperature a significant parameter. This agrees with the result of Maslen (ref. 20), who expanded the boundary-layer equations and slip-boundary conditions in powers of the parameter $\epsilon = \sqrt{\gamma}\, M/Re^{1/2}$, retaining only terms to the first power in ϵ.

Maslen's paper is the only one to treat the flat-plate boundary-layer problem with first-order slip by the usual boundary-layer analysis instead of by Rayleigh's method. His expressions for skin friction and heat transfer show a different functional form. However, the recent measurements by Kendall (ref. 21) tend to confirm Maslen, and Martino's independent determination of the Nusselt number also confirms Maslen's heat-transfer relation. On the other hand, Martino's heat-transfer characteristics expressed as the Stanton number differ from the value for free-molecule flow by the factor of two, as does the skin-friction coefficient. Also, the recovery factor, deduced by Martino, exhibits exactly the opposite characteristics observed with increasing rarefaction. Instead of increasing to a value greater than 1 as observed by Stalder, Goodwin, and Creager (ref. 22), the calculated recovery factor actually decreases with decreasing values of the Chapman-Patterson parameter ζ and becomes zero in free-molecule flow. On the other hand, the value of Nusselt number obtained from the Rayleigh solution by Martino does check experimental data of reference 22 according to the authors, although no graph of the curve is presented.

Thus, on the whole, the attempt to solve the flat-plate boundary layer with slip boundary conditions is hardly to be considered a complete success. The significant parameter does seem to be $(M_oK)^{1/2}$ or $M/\sqrt{Re}$ for relatively low-speed flow and for conditions in the near slip flow regime where the flow is not too far from continuum conditions. A study of experimental results, discussed below, tends to lead to the belief that effects of interaction between thickening boundary layer and external flow are more significant than slip in the observed decrease in heat transfer and, for supersonic flow, increase in skin friction with increasing Knudsen number. A particularly simple analysis by Schaaf and Chambré (ref. 2) shows why there is no first-order effect of slip upon the skin friction of a flat plate. If the stream function is expanded in a series of terms multiplied by increasing powers of the mean free path l, it turns out that the term in C_f containing l to the first power also contains the pressure gradient as a factor. Hence, slip is effective only when an induced pressure gradient is present or else flow over surfaces other than flat plates is considered.

Schaaf and Lin suggest the flow over a wedge as a case for which the pressure gradient is not zero, or the stagnation point flow (ref. 23), both of which have been analyzed only in incompressible flow. However, the effect of slip is to reduce skin friction in both cases. The experimental data of Schaaf and Sherman (ref. 17) show the effects of both slip and boundary-layer interaction, the first tending to lower the drag coefficient while the latter tends to increase it. This interaction effect is expected to be larger at higher Mach numbers and near to the leading edge of a flat plate as borne out by the experiments of Kendall (ref. 21) performed at a Mach number of 5.8.

Yang's attempt to solve the flat-plate problem by applying Grad's equations does not seem sufficiently more fruitful than the attempts of others who have used the Navier-Stokes equations with only first-order slip and like them is definitely limited to extremely low Mach numbers. He exposes one of the characteristics of Grad's equations in that his solution for the shear stress away from the surface contains a term showing the presence of waves in the flow which disappear only after an infinite time lapse, the limit of continuum flow. This is not at all surprising since Grad's equations are hyperbolic, not parabolic like the Navier-Stokes equations, and, as Yang points out, the shear stress satisfies the wave equation at the start of the motion rather than the heat-conduction equation.

The most serious objection to the application of Rayleigh's technique to the flat-plate problem is its neglect of the leading-edge conditions, however. In high-speed flow the leading-edge region is where the effects of mean free path are most important; rarefaction is greatest just at the leading edge and becomes less as the flow goes farther downstream. The Knudsen number, if based upon the distance from the leading edge, the same as Re_x, decreases rapidly from infinity to any finite value as x increases. The rarefaction parameter $M/\sqrt{Re}$ exhibits the same behavior for any fixed value of M. At high Mach numbers approaching the hypersonic range, the interaction between the shock wave and boundary layer becomes more serious as well. Attempts to develop a second-order boundary-layer theory valid for small values of Re have been made by various investigators, but so far without much success. The Low Pressures Research Group at the University of California, Berkeley, are supposedly working on this problem as is Sidney Goldstein at Harvard. Until their studies are complete little can be said about the best theoretical approach to the slip flow over flat plates.

The same statement probably holds true for blunt bodies or bodies of revolution. For the time being all that theory can contribute to the problem is to apply the usual boundary-layer equations or the Navier-Stokes equations themselves, but with first-order slip and temperature jump in the boundary conditions. Thus far the results so obtained do not seem to be too far from agreement with experiment where available, especially when interaction between the boundary layer and external flow is included in the analysis. Otherwise, definitely empirical relations such as equation (11) or some of those suggested in experimental literature should be used for engineering design and analysis. A more serious criticism of the 13-moment equation as well as of the Burnett equations has been made by a purely theoretical but rigorous analysis of kinetic theory by Ikenberry and Truesdell in two papers which appeared in print just as this was written (ref. 24). A brief summary of this new theory is presented in the appendix. A study of this theory, either applying it to the shock-wave problem or determining some compatible set of boundary conditions which can lead to a simple problem such as Couette flow might be carried out with some hope of success, at least to obtaining working results.

BOUNDARY CONDITIONS

A thorough analysis of the proper macroscopic boundary conditions would have to be based upon an analysis of the molecular forces between molecules in the gas and the atoms of the solid wall which forms the boundary of the flow. It is clear that such an interaction would be statistical in nature and that the conditions sought would be some sort of integrated average over an interval of time or of velocity and space. The problem is essentially the same as that known as the physical adsorption of gases on solid surfaces. Various models have been proposed for the latter phenomenon, but so far the intuitive picture is the best available. That is, the molecules approaching the wall are in an equilibrium state which is Maxwellian. At the wall this state is disturbed to a considerable extent and probably destroyed as the molecules become attached temporarily. The time of adsorption is really so very short from the macroscopic point of view (about 10^{-10} second) that incidence and reemission can be considered almost instantaneous. The reemission is called diffuse since the speed and direction of the reemitted molecules is random and has no connection with the same values they had at incidence.

From the newer point of view the molecules have wave properties and can be diffracted by the potential field of the surface. Thus, diffuse reflection can occur without any adsorption actually taking place. The crystals of the solid surface are randomly orientated with the result that diffracted molecules are reflected in all directions. Some may reflect in the right direction and with no loss of tangential momentum, whence the term specular reflection is appropriately applied to them. Upon reemission, whether diffuse or specular, the molecules rapidly approach another state of equilibrium which is also Maxwellian. Actually, collisions between incident and reemitted molecules tend to smooth out the temperature difference between them, but this effect is a second-order one and not considered likely to take place right at the wall. Also, some molecules may be bounced back and forth between the gas and wall many times except when the flow velocity is extremely high.

The original theoretical approach to the boundary conditions was made by Maxwell (ref. 25) who assumed that a fraction σ of the incident molecules is temporarily adsorbed by the surface and then reemitted in arbitrary directions, but with an energy corresponding to the temperature of the wall, while a fraction $1 - \sigma$ is reflected perfectly like light rays at a plane mirror. The fraction σ thus reflects diffusely while the fraction $1 - \sigma$ reflects specularly. If x is the coordinate normal to the surface and v is the velocity of the gas parallel to the surface, the original calculations of Maxwell led to the velocity of slip

$$v(0) = G\left(\frac{\partial v}{\partial x} - \frac{3}{2}\frac{\mu}{\rho T}\frac{\partial^2 T}{\partial x\,\partial y}\right)_{x=0} + \left(\frac{3}{4}\frac{\mu}{\rho T}\frac{\partial T}{\partial y}\right)_{x=0}$$

where $G = \frac{2}{3}\left(\frac{2}{\sigma} - 1\right)l$ and l is the mean free path of the molecules. If there is no temperature gradient then

$$v(0) = G\left(\frac{\partial v}{\partial x}\right)_{x=0}$$

The coefficient of slip G is defined by Maxwell to be

$$G = \frac{1}{2}\,\mu(2\pi)^{1/2}(p\rho)^{-1/2}\left(\frac{2}{\sigma} - 1\right)$$

with $\mu = 4c\rho l(2\pi RT)^{-1/2}$ and $c = 0.499$ (ref. 26, pp. 292 to 297)

$$G = 2c\left(\frac{2}{\sigma} - 1\right)l \approx \left(\frac{2}{\sigma} - 1\right)l$$

Modern derivations omit the temperature gradient normal to the wall (ref. 26) and obtain as the first-order slip velocity at the wall

$$v(0) = \frac{2 - \sigma}{\sigma}\,l\left(\frac{\partial v}{\partial x}\right)_{x=0} + \frac{3}{4}\frac{\mu}{\rho T}\left(\frac{\partial T}{\partial y}\right)_0 = \frac{\mu}{p}\left[\sqrt{\frac{\pi}{2}}\,\frac{2 - \sigma}{\sigma}\sqrt{RT}\left(\frac{\partial v}{\partial x}\right)_0 + \frac{3}{4}R\left(\frac{\partial T}{\partial y}\right)_0\right] \quad (12)$$

The last term containing the temperature gradient in the direction of flow is called "thermal creep;" it is a flow along the surface induced by the temperature gradient.

Both Kennard and Knudsen (refs. 26 and 27) found experimentally a temperature jump at any surface in contact with a rarefied gas as was originally predicted by Poisson

$$T_k - T_w = g\left(\frac{\partial T}{\partial x}\right)_0$$

where T_w is the wall temperature and T_k is the temperature which the gas would have if the temperature curve near the wall is linearly extrapolated up to the wall. Knudsen introduced an accommodation coefficient α to account for the fact that adsorbed molecules are not reemitted from the wall with energies corresponding to the wall temperature T_w but corresponding to some temperature T_r which is intermediate between T_w and the temperature T_i of the incident molecules. The definition is given by $E_i - E_r = \alpha(E_i - E_w)$ in terms of the energies, or in terms of the temperatures it becomes

$$T_i - T_r = \alpha(T_i - T_w)$$

It is seen that for $\alpha = 1$ the accommodation is complete so that the emitted molecules do leave at the temperature of the wall. Although time of attachment of molecules is very short, there can be considerable accommodation and α is only slightly less than 1 for aerodynamical surfaces. For a complete discussion see the book by Kennard or Loeb (ref. 26 or 28) or the monograph of Schaaf and Chambré (ref. 2).

The elementary kinetic-theory derivation of the temperature jump (ref. 26) gives g in terms of α, whence the temperature jump at a wall is to the first order

$$T(0) - T_w = \frac{2}{\gamma + 1}\,\frac{\kappa}{\mu c_v}\,\frac{2 - \alpha}{\alpha}\,l\left(\frac{\partial T}{\partial x}\right)_0 \tag{13}$$

or, in terms of the Prandtl number of the gas $Pr = \mu c_p/\kappa$,

$$T(0) - T_w = \frac{2 - \alpha}{\alpha}\,\frac{2\gamma}{\gamma + 1}\,\frac{l}{Pr}\left(\frac{\partial T}{\partial x}\right)_0 = \frac{\mu}{p}\,c_1\sqrt{RT}\left(\frac{\partial T}{\partial x}\right)_0 \tag{14}$$

For polyatomic gases

$$c_1 = \sqrt{\frac{\pi}{2}}\,\frac{2 - \alpha}{\alpha}\,\frac{2\gamma}{\gamma + 1}\,\frac{1}{Pr}$$

and for monatomic gases

$$c_1 = \frac{15}{8}\,\frac{2 - \alpha}{\alpha}\sqrt{\frac{\pi}{2}} \qquad (14a)$$

because $\frac{\kappa}{\mu c_v} = \frac{1}{4}(9\gamma - 5)$ and $\gamma = \frac{5}{3}$.

This expression for the temperature jump in a polyatomic gas is carefully derived by Weber (ref. 29) but by intuitive means which are ascribed to Knudsen. He arrives at the above expression for a monatomic gas as long as the accommodation coefficient is the same for translational and inner energies of a molecule. Then Weber concludes from the experimental evidence that equation (13) together with equation (14) gives the correct temperature jump for all gases, polyatomic as well as monatomic. The Weber formula (identical with Smoluchowski's) is then

$$\Delta T = T(0) - T_w = \frac{15}{8}\,\frac{2 - \alpha}{\alpha}\,l\left(\frac{\partial T}{\partial x}\right)_0 \qquad (15)$$

which is independent of γ and Pr.

Welander (ref. 30) has recently redetermined the first-order temperature jump for Maxwellian molecules by finding a solution of the linearized Boltzmann equation (i.e., small disturbances from a Maxwellian distribution) under the assumption that the reflected molecules have a Maxwellian distribution. His method is similar to the Chapman-Enskog expansion with one additional term and his result simply replaces the factor $(2 - \alpha)/\alpha$ by $(2 - k\alpha)/\alpha$ where $k = 0.827$. Welander also gives an excellent survey of the literature up to 1954. His results imply that except for the lightest gases, hydrogen and helium, accommodation is perfect. In that case equation (15) reduces for air to

$$\Delta T = 2.2l\left(\frac{\partial T}{\partial x}\right)_0$$

The assumption that the distribution function for the gas molecules is Maxwellian in the neighborhood of the wall is the one most often criticized. Payne (ref. 31) attempted to remove the assumption that the molecules approaching the wall have the same distribution in velocity and temperature over the last short distance of the order of magnitude of 1 mean free path but had to make additional assumptions as to the order of magnitude of velocities and temperatures. His result is that the lower limit on the velocity of slip is the first term in equation (12) and the upper limit is 1.27 times the lower limit for Maxwellian molecules. The temperature jump which Payne derives has the same minimum value as equation (14), but with the factor $2\gamma/(\gamma + 1)$ in c_1 replaced by $\gamma/2(\gamma - 1)$. For monatomic molecules $\gamma = 5/3$ and both of these factors are $5/4$, so Payne's result is equation (14) with c_1 given by equation (14a). He obtains 1.32 times the minimum value as the upper bound to ΔT. The result obtained by Welander would therefore be about the average of Payne's maximum and minimum values.

It thus appears that all attempts to derive the first-order boundary conditions lead to the same result, equation (12) for the slip velocity and equation (13) or (14) for the temperature jump, but with different values for the coefficients c_1 and $a_1 = \sqrt{\frac{\pi}{2}}\frac{2 - \sigma}{\sigma}$. Since the parameters α and σ are not predicted by theory anyway, but must be determined from experiment, it would seem that the boundary conditions (12) and (14) are perfectly adequate to handle any problem in the slip-flow regime. Parameters α and σ are apparently a macroscopic expression of unknown but intuitively understood adjustment of state and equilibrium of gas molecules near solid boundaries.

All the hypersonic viscous flow theory has so far been based upon the well-known Prandtl boundary-layer equations which originally were derived for low-speed flow from the Navier-Stokes equations. No kinetic theory has been proposed which has special qualities that would make it more correct for high Mach numbers; in fact, both Grad's theory and the Burnett equations have been shown to break down at a relatively low Mach number. The study of shock-wave structure is excellent for the determination of the range of validity of a theory since it is independent of boundary conditions, as also is the absorption and dispersion of high-frequency sound waves. The Navier-Stokes equations have been completely confirmed by the known data in both of these fields (refs. 5, 6, and 32) so at the present time there seems to be no theory more adquate than the Navier-Stokes equations together with the first-order slip boundary conditions. This is not to imply that the Navier-Stokes equations are proven to be valid for all densities and all Mach numbers but only that present experimental evidence agrees best with them for Mach numbers less than 2 and mean-free-path lengths of 0.005 to 0.03 inch (ref. 6). Theoretical work does show (refs. 5 and 12) that nonlinear terms in the stress-deformation relation do lead to thicker shock waves at Mach numbers above 2 but experimental data are as yet too meager to determine the significance of these nonlinear terms. The one measurement of Sherman at $M = 3.70$ has not apparently been compared with either of these two theories.

Second-order-slip boundary conditions are implicitly contained in the formalization of Epstein (ref. 33) but were not explicitly written out until Schamberg used Burnett's theory to obtain terms to order $(\mu/p)^2$ in the nonlinear pressure and temperature gradients. Since Grad (ref. 7) obtains more rigorously and simply a set of boundary conditions by iteration in the same manner in which an iteration on the 13-moment equations give the Burnett terms (ref. 2), there is no reason to consider Schamberg's results at all. Yang (ref. 14) has rederived Grad's boundary conditions from a more physical point of view while Kryzwoblocki (ref. 34) has reconsidered and generalized Epstein's derivation. Since Von Kryzwoblocki does not work out the second-order terms, showing only that the boundary conditions are consistent with those for the Navier-Stokes equations with no slip, there is need for an analysis of his integrals, which look like those of Grad, to determine whether or not they are the same as Grad's. A superficial examination indicates that they are, but both Yang's and Von Kryzwoblocki's papers arrived too late to determine if they are the same for the present report.

EXPERIMENTAL DATA

Experimental checks of theory or disagreement therewith have been touched upon in the earlier section on theory. While there is not so much experimental data available as would be desired in order definitely to prefer one theory above all others, there is a definite tendency for the data to agree best in the slip flow regime with the predictions of the Navier-Stokes order of approximation. Furthermore, there is a smooth transition of data all the way from the continuum flow domain through slip flow and the transition regime into free-molecule flow. No theory yet proposed is capable of predicting when the Navier-Stokes equations break down and kinetic theory considerations take over. Experiment can help in this respect but is not a complete answer, since the conditions which distinguish one theory from another are too fine to be caught in the coarse-grained experiments so far performed. For example, measurements of drag in the rotating-cylinder apparatus do not determine the pressure and temperature distribution between the cylinders which is necessary to distinguish one theory from another. The Navier-Stokes equations with first-order-slip boundary conditions are sufficient to predict the measured drag.

Since Schaaf and Chambré (refs. 2 and 3) have presented excellent surveys of all the experimental data, no attempt will be made here to refer to all the significant papers. All data so far obtained have come from the Ames Laboratory of the NACA or the Low Pressures Research Project of the University of California at Berkeley.

Some of the most interesting experimental results are the definite increase in drag coefficient with decreasing Reynolds numbers and increasing Mach number, the same for the recovery factor, and the decrease in heat transfer as expressed in the form of a Nusselt number for both spheres and cylinders. The changes apparently begin in the slip flow regime and the experimental curves fair into the free-molecule flow values as predicted by the theory for the latter. Hence, it is possible to obtain a rough indication of the Knudsen number at which slip begins, which it is not yet possible to do theoretically. However, no precise value of K can be said to be that at which slip flow starts, only that it seems to be of the order of 0.1.

Experimental data on flat plates and cones have so far shown negligible effects of slip, the effects of interaction between the thick boundary layer and the external flow (as well as the transverse curvature correction for cones) tending to overshadow any slip which might be present. For cones there is also the decrease in heat transfer and increase in the recovery factor with increasing rarefaction.

The considerable increase in the temperature recovery factor from the value $\sqrt{Pr} \approx 0.9$ for continuum flow to values of 1.3 or so in free-molecule flow is a significant result. Martino (ref. 19) has proposed a simple method of expressing the recovery factor for circular cylinders, as measured by Stalder, Goodwin, and Creager (ref. 22), which is definitely empirical but is suggested here as the answer to those who need simple relations for immediate design problems.

If P is the probability that a molecule will collide with the surface before colliding with another molecule then

$$\begin{aligned}(\text{Slip system} &= P(\text{Free-molecule flow system})\\ &\neq (1 - P)(\text{Continuum system})\end{aligned}$$

Evidently P is a function of the Knudsen number K such that

$$\lim_{K \to 0} P = 0$$

and

$$\lim_{K \to \infty} P = 1$$

since Knudsen interpreted what is now called the Knudsen number as the ratio of the number of collisions of molecules with the wall to the number of collisions between molecules themselves. Martino then takes P as equal to the fraction of molecules colliding with the surface and obtains

$$P = \frac{K}{1 + K}$$

$$1 - P = \frac{1}{1 + K}$$

If Q is any aerodynamic function of K such as the drag coefficient or recovery factor, Q_o is its value in the continuum range, and Q_f is its value in the free-molecule flow, the value Q_s in slip flow is

$$Q_s = \frac{1}{1 + K} Q_c + \frac{K}{1 + K} Q_f$$

For the recovery factor on circular cylinders

$$r_s = \frac{r_c}{1 + K} + \frac{Kr_f}{1 + K}$$

and with $r_o = 0.95$ and $r_f = 1.35$ this formula agrees very closely with the data of figure 6 in reference 22. On the other hand, an attempt to use this method for the skin-friction coefficient and Nusselt number in heat transfer did not work. The reason lay in the dependence of both of these quantities upon the Reynolds number and through the latter on the Knudsen number. Thus, Martino's suggestion would appear to be valid for cases in which the quantity Q is independent of Knudsen number in both continuum and free-molecule flow. Yang's empirical approach appears better for quantities like skin friction, heat transfer, and velocity of slip.

Most of the experimental papers contain empirical or semiempirical formulas which correlate the data given. In general, some of the best fitting curves are derived from low-speed incompressible analysis, but with first-order slip in the boundary conditions. This is the real justification for using the Navier-Stokes equations with first-order slip at the present time. No theoretical justification has been found, but the agreement with experiment is all the designer has to go on.

Besides reference 22, reference 35 also gives data on heat transfer from circular cylinders and correlates the data by an analytical expression. Data on skin friction on flat plates are given in reference 17. For spheres, drag data are to be found in reference 36, while the measured heat transfer from spheres is presented and analyzed in references 37 and 38.

CONCLUSIONS

As a result of this study of the problem of slip flow in aerodynamics, it is possible to make several significant conclusions:

1. Two extreme cases in the flow regime are well known and established both theoretically and experimentally: (a) Continuum flow (Knudsen number much less than 1). Here the Navier-Stokes equations with no slip and no temperature jump at the boundaries are perfectly valid and seem to hold for laminar flow to extremely large Mach number well into the hypersonic range (turbulent flow included); (b) Free-molecule flow (Knudsen number much greater than 1). The gas is in equilibrium with a Maxwellian distribution of velocities and collisions between molecules can be neglected; the collisions between the body and the molecules can be accounted for by means of the accommodation coefficient and reflection coefficient.

2. The slip-flow regime near to continuum flow can best be described by the Navier-Stokes equations together with first-order slip and temperature jump at any bounding surfaces. In this region, rarefaction is slight, and collisions between molecules predominate, but some effect of collisions between surfaces and the molecules must be taken into account.

3. The transition regime between first-order slip flow and the free-molecule regime has no valid theoretical basis. The collisions between molecules themselves and between the molecules and surfaces are apparently of equal importance so any correct theory must embody both. Since the new theory of Ikenberry and Truesdell contains no treatment of boundary conditions, it is not yet considered suitable to describe flow about bodies or between surfaces. For problems not involving solid boundaries, such as the propagation of sound or the structure of shock waves, the Ikenberry-Truesdell theory holds the greatest promise.

4. Experimental data to some extent exist for both subsonic and supersonic flows which span the region of rarefaction or the complete Knudsen number range. Empirical or semiempirical relations can be devised to agree with these data and these should be used in engineering analyses at the present time.

5. Finally, some problems of interest which could be attacked with some hope of success, at least for obtaining working results, are: (1) A first-order slip investigation of a flow involving pressure gradients such as the flow over a wedge or biconvex profile, and (2) a study of the Ikenberry-Truesdell kinetic theory, either applying it to the shock-wave problem or determining some compatible set of boundary conditions which can lead to a solution of a simple problem such as Couette flow.

University of Washington,
Seattle, Wash., March 20, 1956.

APPENDIX

THEORY OF TRUESDELL AND IKENBERRY

Thanks to Professor Truesdell, copies of the manuscripts of the two papers of reference 24 were obtained before the present paper was completed. A hurried reading indicated that severe criticism of present kinetic theory was contained therein. On the other hand, the work of Truesdell and Ikenberry included a new approach, figuratively speaking, to the kinetic-theory problem. However, their approach is not exactly new, since the authors go back to a method which was used by Maxwell himself and which they call Maxwellian iteration. All developments in kinetic theory since Maxwell are criticized as wrong in principle, if not in results.

The new iteration method does lead directly from a few basic assumptions to nine exact equations of transfer. No knowledge of any distribution function is needed, and it is precisely in this respect that the new scheme seems to be the most powerful approach so far advanced. It is restricted to Maxwellian molecules, but, as pointed out earlier in this report, that is not considered serious when the question considered is the one of the foundations of kinetic theory.

While reference 24 is critical of some aspects of Grad's theory there remains an impression of a great debt to some of the ideas and methods which Grad proposed in his examination of kinetic theory. The new theory is still based upon Boltzmann's integro-differential equation, although it nowhere explicitly appears in the papers. The foundations of kinetic theory are still the same; only the manner of constructing the edifice thereupon is new. By using spherical harmonics as originally suggested by Maxwell, the technique is straightforward, although just as tedious as Maxwell's in other approaches. The great advantage is in the exactness due to the fact that any iteration depends only upon iterates of lower order.

The treatment is thorough. Other methods of iteration are touched upon and comparisons are made with other approaches, those of Hilbert, Enskog, Chapman, and Grad. One exact solution is found, that of shearing flow, which is the first ever to be obtained from kinetic theory for dissipative flows.

On the other hand, there are new difficulties as well as successes. The solutions are not general but special and may or may not turn out to be correct representations of physical phenomena. The complete validity of the method is shown only for a contrived mathematical model, not for the kinetic theory itself. Despite the author's claim for simplicity, the techniques developed are no more elementary than in the methods of Grad or Enskog. Similar integrals have to be evaluated and systems of equations have to be solved. However, this treatise in 120 pages of text certainly contains a more thorough and less forbidding introduction for aerodynamicists to the foundations of advanced kinetic theory than the thicker treatise by Chapman and Cowling. Actually, there is no real comparison, since the latter book was really concerned with the derivation of the physical coefficients of diffusion, viscosity, and heat conduction in transport processes. Given the latter, this new treatment is more suitable for the setting up of the equations to be solved in a physical problem.

There are two questions which arise from this work and they are: (1) Are there any more exact solutions than the one found or even any valid approximate solutions? (2) What happens if boundary conditions are added? There is no mention at all of boundary conditions, particularly of slip, and hence no indication as to whether the method is compatible with the addition of conditions at the walls. In the concluding remarks some indications of incompatibility with continuum mechanics is mentioned with the remark that a new theory of the latter nature is urgently needed. It is not mentioned that, no matter how elaborate a correct continuum theory must be, the Navier-Stokes equations will be contained therein and will reduce the the latter when classical conditions are valid.

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