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MULTIPLICITY OF SOLUTIONS IN AERODYNAMICS

By M. Dupont

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One of the most striking phenomena which accompany the flow of fluids such as air and water about bodies, is that of changes in character. This phenomenon is still very little understood. We will first discuss the nature of these changes and then show that pure theory leads to a multiplicity of characters of flow, among which we will endeavor to indicate those bearing some analogy to experimental results.

I. Historical Sketch

The first category of facts concerns bodies placed at the same angle of attack with reference to the air flow, for which it is found that the coefficient of resistance or drag changes suddenly in the neighborhood of a certain velocity corresponding, above and below this value, to two different appearances in the flow.

The discovery seems to have been made by Eiffel, who communicated it to the Academy of Sciences in 1912. Comparing his own results with those on spheres published by the Göttingen Laboratory in 1909, results differing from simple to triple,

* "Sur la multiplicite des solutions en aerodynamique." From "La Technique Aeronautique," Dec. 15, 1926, and Jan. 15, 1927.

Eiffel experimented with spheres of different diameters throughout the whole range of velocities, from 0 to 30 m (about 100 ft.) per second, attainable in his Auteuil Laboratory. He was surprised to find that the coefficient of drag varied with the velocity of the air and the size of the sphere. He was thus enabled to explain the considerable discrepancies with the Göttingen results. These discrepancies, however, have never been entirely eliminated, due to the fact that aerodynamically, the sphere is a very capricious obstacle. Thus originated the notion of critical velocity separating two different characters of flow, a notion which was soon to be generally adopted for all cylindrical and streamlined bodies.

There is another order of variation of the character of the flow, when there is a change in the angle of attack of certain bodies relative to the flow. The coefficients of drag and of lift generally increase regularly with this angle, except for a certain particular value, where a substantial change takes place in the flow. The first fact, in this sense, was noted by Rateau in 1909, for small oblique planes at an angle of attack of 25 to 35°. This was subsequently confirmed by Rateau, on carefully examining the polars of certain wing sections or profiles, and has since been generally adopted for all wing sections. For the latter, the critical angle of attack is particularly important. Below this angle, the high lifts, created by the large negative pressures on the upper surface of the leading edge, increase in direct proportion with the increase in the angle of attack, until the lift suddenly falls off, after a period of instability. This entails two consequences for the airplane: a sudden drop in altitude and loss of controllability. This constitutes the so-called "stalling" of the airplane. Knowledge of such an undesirable disturbance consequently has definite value.

It was very quickly recognized that the two types of changes in the character of the flow occur at the same values of the product of velocity × dimension or (more generally and for different fluids) at the same Reynolds Number VD/ν (ν being the kinetic coefficient of viscosity).

In Fig. 1, the variation in the coefficient of drag of cylinders according to Göttingen is plotted against VD/ν for a very wide range.* The logarithmic scale was adopted for both coordinates, the first consequence of which is to render the fall of the coefficient very noticeable. The intermediate horizontal range and the inception of the range at very high velocities are very clearly shown. The velocities and dimensions of importance in aerodynamics are precisely astride the critical value of VD. This value corresponds, for instance, to a velocity of 13 m (42.65 ft.) per second for a cylinder of 20 cm (7.87 in.) diameter.

*Extract from an article by Marchis on the laws of variation of the coefficient of drag of the air for a few bodies ("Bulletin technique du bureau Veritas," June, 1924.

Another advantage of the logarithmic scale is that it shows what takes place at very small VD. This zone is not important in practice. The point 1 farthest to the left corresponds to diameters of 0.05 mm (0.002 in.) and velocities of 1 m (3.28 ft.) per second. At most it would serve for piano wires of 1 mm (0.04 in.) with velocities of 5 cm (1.97 in.) per second. Aside from this utilization, the zone in question is of considerable theoretical interest by reason of the rapid and continuous variation in the coefficient of drag. Here there is a different kind of flow for each corresponding velocity. At this point, where the effect of the viscosity is predominant, the experimental multiplicity of the kinds of flow corresponds to the theoretical multiplicity of the solutions, as we shall see farther on.

The investigation of these kinds of flow has often led experimenters to make the filaments visible either in air or in water by smoke or colored fumes, or by liquids of different refractive powers, or by powders rendered brilliant by an electric spark. We cannot refer to them all, from Marey to Riabouchinsky and Colonel Lafay. We must, however, express our admiration for the remarkable results obtained by Camichel, of Toulouse, in water or viscous liquids. Very interesting experiments are now in progress under the direction of Toussaint at the Institute of Saint Cyr. Unfortunately the smoke or fumes become invisible when the velocity exceeds a few meters oer second, all the more because, in the small tubes or spaces, the flow may become turbulent at high velocities, thus confusing all observations.

It may be said that the upper range has never been materialized with the probable transfer of the separations toward the rear. It has been found possible to render visible the intermediate stage with separations near the main or midship section, a transition point with sudden changes of velocity on either side; and the turbulent zone within which are produced various motions which are quite gentle in immediate proximity to the body. When the velocity is diminished, one observes the vortices of Karman carried along by the current and constantly renewed. Camichel's most surprising results, however, concern very low velocities or what amounts to the same thing, the medium velocities with more or less viscous fluids.* They illustrate the continual variation in the character of the flow with the velocity. Each type of flow is characterized by two vortex cores, two points of bifurcation of the filaments on the surface of the body, and a neutral point downstream, all these characteristics changing gradually with the velocity. Some of the cores are even of spiral form, which is interesting, in view of what follows. Finally, at lower velocities or an exaggerated viscosity, the vortex cores disappear downstream and the fluid

*See the remarkable photographs by Camichel in the "Technique Aëronautique" of November 15, and December 15, 1925. The following descriptions especially concern Figs. 10, 17, 20, 21, 23, 25 and 26.

closely follows the contour of the body. Note that these figures have a gradually increasing resemblance to the theoretical flow of a perfect fluid, i.e., a fluid without viscosity. We have come to the conclusion that the characteristics of these flows can be studied in a perfect fluid and corrected by taking account of the viscosity, but without any substantial alteration.

In finishing this sketch, we will call attention to the work of Pawlowski at the University of Michigan (reported by Dzrewiecki), the results of which have not yet been published. This investigation has the advantage of having been made at velocities of 15-20 m (49-66 ft.) per second, which are quite near the velocities in current use. Since at these velocities, it was no longer a question of smoke or fumes which dissolved immediately, the motion was revealed, at least in the vicinity of the body, by the lines left on white lead by a current of H_2S liberated by a very fine tube, as close to the body as possible. Pavlowski found that the separations occurred at a point D on the front upper surface, even at the smallest angles of attack (Fig. 2), the separated filament then following a path very near the upper surface, but without its being possible to confound them, because in the angle thus formed, however small, the same condition occurs as for large angles of attack, i.e., the presence of a neutral point O separating a forward zone, where the fluid follows the surface in a direction contrary to that of the general flow, and a rearward zone where the fluid rejoins at the point P, the filament which has followed the lower surface. The point of separation would thus be fixed near the leading edge for the smallest angle of attack and would thereafter vary but very little. If the above results are confirmed, they are of such a nature as to modify the current conception of the nature of the flow about airfoils and to attribute to the phenomena of separation a very important role, even at the smallest angles of attack.

Whether with or without separation, we propose to show here that the mechanics of perfect fluids suggested to the imagination a multitude of solutions. To make an "a priori" choice presupposes a criterion which we do not yet possess. Lacking this, the best method will be comparison with experience and the elimination of solutions which do not agree with it. From this gradual refinement, there will perhaps be evolved, if not the true type, a type corresponding sufficiently to the reality to satisfy our need of knowledge and of application. This criterion will not be a comprehensive coefficient of drag ready for universal application, which can be sufficiently approximated even for a theory deficient in its details, but a local verification like the distribution of pressures or velocities on the surface of bodies sufficiently complicated so that the discordances are quickly revealed. Such, for example, are wing sections.

We will confine ourselves to the parallel flow, so that generalizations will be possible and easy. The surfaces tested will therefore be endless cylinders placed at right angles to the flow. These cylinders are reduced to their right sections in the motion diagrams, so that we will have only curves and the most convenient curve of all, namely, the circle. The circle can be studied either as an endless cylinder (symmetrical motion), or as transformed into a wing section (dissymmetrical problem). The result to be obtained on the circle will be the knowledge of the tangential velocities generating the pressures and, in particular, the location of the points of zero velocity. By a transformation of the Joukowski type with two poles P and P_1, it is in fact, possible to make a pointed profile correspond to a circle with the center O and the point P at a point of zero velocity (Fig. 3).

II. Solutions without Separation

These are the simplest to discuss and, in certain cases, useful to know approximately. The general conditions of the problem are too well known to require much attention. The fluid is supposed to be perfect and incompressible and the motion permanent on the whole. The problem is to determine a velocity potential which, combined with a general motion of translation, gives the complete flow around the object, here a circle. This condition of flow is a condition of nullity imposed on the normal velocity at the surface of the body in the resulting motion. The body is supposed to be stationary in a moving current. In certain parts of this discussion, it may be convenient, however, to suppose the body to be in motion and the fluid to be at rest to infinity.

There is a solution known in principle since Laplace and D'Alembert and which has the characteristic of being regular, both outside the body and on its surface. The works of mathematicians of the nineteenth century have demonstrated that it is the only regular potential function giving the flow along the surface. Since Neumann, it has been known how to calculate it, as the potential of a simple layer spread over the surface. The resulting figure for a cylinder is well known (Fig. 4). This solution, so well determined that it seems to be the only possible one, has, nevertheless, two serious disadvantages. On making the integral of the pressures around the body, the result is a zero drag. Moreover, it has never been experimentally obtained in fluids such as air and water. By a paradox already mentioned, this pattern is obtained in its general appearance only by exaggerating the viscosity to the maximum, i.e., by keeping as far away as possible from the perfect fluid.

It is therefore necessary to discard this solution, which was, in fact, done long ago. Up to the end of the nineteenth century, the principal efforts crystallized around a solution with separation, the "wake" of Helmholtz, of which we will speak again. It was not until the beginning of the present century that an entirely new way was discovered. Its point of departure was Joukowski's theory, which has since been so greatly developed. Before discussing it, let us say that this solution is not the only one. This idea, actually so little diffused, was the object of an unpublished investigation by Mr. Girault of the S.T.Aé. The functions susceptible of representing the flow of a perfect fluid around a given body are as numerous as desired for choosing irregular ones according to certain lines or surfaces and for determining "a priori" the nature of these singularities. Physical motion is not generally possible in their vicinity (due, for example, to infinite velocities), but it will suffice to eliminate from the space utilized the neighboring lines of flow, in order to obtain, outside, a motion perfectly compatible with the physical data.

Joukowski's solution.- This is based on a dissymmetrical flow around a circle in view of its transformation into a profile. It is well known that an endless rectilinear vortex gives circular lines of flow (Fig. 5) in a perpendicular plane. The velocity along the circumference of a circle, of radius r, is equal to A/r, A being the intensity of the vortex. This number has a simple mechanical signification. The circulation around a circle, or product of the velocity at each point, multiplied by the element of arc, added for all the elements of the curve, is equal to $2\pi A$. It is independent of the circle considered and is generally the same for every closed circuit enclosing the point O which is the trace of the vortex. In the vicinity of this point, the velocity would theoretically be infinity, but this difficulty is avoided by not utilizing the part flowing inside a circle of finite radius. By superposing this motion on that of Laplace with uniform potential, we obtain a theoretical flow outside the circle, whose figure is well known (Fig. 6). The consequences deduced by Joukowski are as simple as they are remarkable.

1. The reaction of the flow on the circle develops a force perpendicular to the previous direction of translation of the air and proportional to the vortical circulation. The drag is zero.

2. A similar transformation of the circle on another curve changes neither the drag (zero), nor the lift, which remains proportional to the circulation, which is itself constant.

3. In particular, a judicious transformation discovered by Joukowski gives a well-streamlined profile, provided with a thin trailing edge, to every angle of attack of which there corresponds a flow transformed without discontinuity (Fig. 7). The resultant of the pressures would be a pure lift proportional to the primitive circulation around the circle.

The application will demonstrate that this transformation, chiefly arbitrary, would yield very good wing profiles and that the hypothesis of the circulation, no less arbitrary, would give a lift closely approximating reality. In reality, however, the drag is not zero. Beyond a certain critical angle of attack, the agreement would disappear completely. Nevertheless, the explanation seemed to be found for the great relative lift which was the essential fact in the properties of airplane wings at small angles of attack.

This agreement (of 20-25%) is unique in the history of aerodynamics, if we consider the pureness, without corrections, of the theory which leads to it. This was the foundation for the great success of Joukowski's idea. Generalizations have followed with Prandtl and his pupils, but its pureness has diminished with experience, as also the agreement in detail. It has also been sought to explain the origin of this circulation by reasoning based on the hypothesis of the "fluid wing" of Prandtl and which appeared to terminate naturally in it. This reasoning, however, does not eliminate the implicit postulate that things take place in reality the same as in theory.

The question rests entirely on a detailed comparison of experimental and theoretical results. The comparison of the theoretical and experimental pressures was made in the S.T.Aé. (Technical Section of Aeronautics - Bulletin No. 22 of the "Service Technique de l'Aéronautique," December, 1924) according to measurements made on a Joukowski wing (430) in the Eiffel Laboratory (Fig. 8). The value shown on the ordinate is the ratio k of the dynamic pressures to the generating pressure; if we adopt

the theorem of Bernoulli at the surface $k = \frac{V_0^2 - V^2}{V_0^2}$.
K = 0 corresponds to surface points where the velocity is equal to the velocity of the air stream in the wind tunnel. K = 1 corresponds to points of zero velocity (large front end). K < 0 corresponds to regions where the velocity exceeds that of the air stream, which is very clearly the case for a large part of the dorsal surface. The abscissas are the arcs of the profile traversed from point to point in passing through the upper and then the lower surface.

The curve is obviously quite uncertain. The first impression in the comparison is a certain remarkable conformity of appearance. A first difference appears on the upper rear, where the experimental pressure seems to become constant. It is thought that separations occur which disturb the phenomena.

We think, however, that the most important discrepancy, and the one the least remarked, occurs on the front upper surface in the zone of strong negative pressures, important for the lift, where the theoretical negative pressure attains 2 to 3 times the experimental pressure and the maximum theoretical velocity attains 1.45 times the experimental velocity, the net discrepancy at an angle of attack of a few degrees and which becomes very much greater above 15 or 20°. This difference, so great in detail, amounts in the aggregate, to only 20 to 25% for lifts below the critical angle of attack. This is why Joukowski's theory constitutes a satisfactory approximation at small angles of attack. We predict that it is not the last stage and that, independently of the separations to be considered, the distribution of the velocities at the sensitive points must be revised.

Solutions with sources and vortices.- The way being thus opened, we see that the problem can be approached in an infinite number of ways. The rectilinear vortex of Joukowski was suggested by the analogy of the equations of hydro (or aero) dynamics and electrodynamics, the field of velocity of the vortex being precisely the field of force of an endless rectilinear flow. Here are other examples, however, by comparison with static electricity.

In 1850, Rankin had combined two fields analogous to those of electric masses, in order to arrive at well-streamlined surfaces of revolution, which have produced good shapes for airships. Static electricity uniformly distributed on a straight line is represented by a point in a plane perpendicular to the straight line (Fig. 9). In this plane the lines of force are straight lines meeting at this point. Transposing in the domain of fluids, the velocities are centripetal or centrifugal and equal to B/r. B has a simple physical signification. $2\pi B$ represents the output of fluid appearing or disappearing per second at the point considered on the unit length of the straight line, which characterizes either a source or a "sink," to use an expression coined by Drzewiecki). B is the intensity of the source and is either positive or negative.

Certain precautions are necessary in order to apply to the motion of the fluids either an addition or subtraction of matter in permanent motion. Let us note, moreover, that the velocity is infinite in the vicinity of the individual points. Some writers, like Riabouchinsky, see in it the mathematical image of the cavitations which appear on the downstream side of the body.

By superposing a source on a vortex, we have another type of singularity characterized by spirals as lines of flow (Fig. 10). The source-vortices were systematically investigated by Riabouchinsky ("Bulletin du S.T.Aé.," April, 1926). In fact, the spiral form of the lines of flow is quite frequent in nature.

On applying these various singularities to the circle, it remains to find combinations of sources, vortices, or source-vortices, which yield circular lines of flow. By superposing them on the regular potential of Laplace, a very wide range of theoretical motions is obtained.

It should be mentioned that among the phenomena imaginable, some are found inside the body (like Joukowski's vortex) and are compatible with an external physical motion, while others may be internal and can be accepted only with reservations. Here are a few examples.

1. Two vortices of opposite intensities and conjugated with reference to a circle (more often a number of such pairs)

admit this circle as the line of flow. One application is known under the name of "Foppl vortices."* Behind a circle in a symmetrical flow (Fig. 11), there are two symmetrical pairs of opposite intensity. Foppl subjects them to the condition that each vortex center remain motionless under the action of the other centers and of Laplace's field of regular potential. The problem has one indetermination. We can choose, on a certain curve, one of the vortices, e.g., the upper one to the right. A well-determined intensity and a very distinct kind of flow correspond to each position, each figure being characterized by two rear points of zero velocity, which are normally the origins of two lines of flow separating the filaments continuing their downstream course from those which retrace their path by turning continuously on themselves. It is difficult not to attach to these theoretical figures the series of appearances obtained by Camichel behind a cylinder at a very low velocity. In this example we have acceptable internal vortices and other external ones, which must be subjected to the above reservations.

2. Circular lines of flow are produced by two sources of the same sign conjugate with reference to a circle and completed by a source of the contrary sign at the center of the circle (Riabouchinsky).

3. Such sources can be associated with the vortices of

*Foppl, "Münchene Sitzungsberichte," 1923. See also Howland, "The Journal of the Royal Aeronautical Society," April, 1925.

No. 1, in any number of pairs (Riabouchinsky).

4. Two opposite sources (or vortices) approaching each other indefinitely produce a double source (or double vortex) whose lines of flow are tangential circles (Fig. 12), the velocity y being of the form C/r^2. It may be considered either as a double source fixed on the circle in question, or two sources conjugate with reference to the circle. They may therefore be made to approach each other indefinitely and to have singularities of the order of D/r^3.

C. Witoszynski, in particular, utilized a fixed double source in a point of the circle combined with the regular potential of Laplace, in order to obtain a dissymmetrical flow, applicable to wing profiles. It was for the special purpose of verifying his conclusions that Pawlowski undertook the experiments already described.

Witoszynski's solution.- In addition to the long list derived from electrical analogies, other functions can be found. A remarkable model was recently made by Witoszynski ("Mécanique des profiles d'aviation," translated from Polish into French by Ziembinski, Cheron, 1924) according to an original principle which which we will now consider. The task is to produce a dissymmetrical motion around a circle for the purpose of transforming it into a profile (Fig. 13). In contrast with the preceding examples, in which one complete revolution around a body brings the same velocity of the fluid back to the same point, the function here adopted requires two revolutions.

Let us imagine that the leaf of paper is split from a point of the contour clear to the left, and let us follow a closed circuit from a certain point on the verso (front or upper side of the paper). Encountering the slit at a certain moment, we pass to the recto (back of the paper) and return to the initial point with a different velocity. Continuing the revolution, we return to the slit with a velocity different from the preceding revolution and finish the second revolution on the verso, in order to recover the elements of departure at the initial point. By this device, each side of the leaf represents a different motion of the fluid. Witoszynski places the cusp (singular point of separation or discontinuity where the velocity becomes infinite, since it varies as $1 : r^{1/2}$) on the back of the paper and uses the front of the paper only where the velocities are finite. A lift and a drag are also found.

One difficulty remains. Along the slit the velocity differs from point to point, which fact requires a sudden change of pressure. Physically a troubled zone must be created, of which the slit is a sort of skeleton. In the transformation into profile, it is arranged for the beginning of the slit to be a point of zero velocity which will correspond to the point, and the transformed slit will be the wake of the profile.

The comparison of the pressures obtained by this method with those obtained experimentally was made on the same profile 430, which has already served as a criterion for the results obtained by Joukowski ("Bulletin du S.T.Aé.," December, 1924). Aside from a few differences (Fig. 14) due to separation at the rear, the zone of great negative pressures shows an agreement much more complete than with the first theory. For such a complicated curve, this can hardly be the result of chance. The tentative method, obligatory in the present state of aerodynamics, here gave very good results. The above comparison is due to Girault. At angles of attack above 10 or 15°, the agreement rapidly disappears.

We have shown what a diversity of solutions is already offered when one seeks a flow surrounding the whole body, where the region behind the body does not oppose the region in front, from the continuity viewpoint.

By adding certain singularities to the regular potential, one can obtain images of flow about a cylinder, of striking analogy to the reality, though only at very low velocities. For wing profiles, it is at small angles of attack that certain theories give a satisfactory agreement for practical utilization. The effect of the separations is encountered, however, at the highest velocities and the largest angles of attack. The surface is divided into two very distinct regions: the front region, which the fluid washes in a very regular and permanent flow; the rear region, in contact with a fluid more or less dead and possessing various motions. This brings us to the complete problem in the latter hypothesis. The multiple solutions, which have already appeared, will increase still further. We will endeavor to give an idea of them.

III. Solutions with Separation

The separation hypothesis has long been offered to explain the drag of a body. For this, Helmholtz invented the theory of the wake and attached to the rear of the body a mass of relatively immobile fluid at constant pressure. It is conceived that one thus breaks the equilibrium of the pressures before and behind, characterizing the solution of Laplace, and that a drag is created (Fig. 15). In considering the problem, it is seen, however, that the two surfaces bounding the wake must separate indefinitely, and that the rear pressure must be the atmospheric pressure, a double consequence far from experience.

Witoszynski's conception is nearer the reality. It reduces the phenomena behind the body to the diagram shown in Fig. 16. The fluid separates at two points of the contour and follows two lines parallel to the general direction of the relative flow. Between these two lines, there is a space which has the following properties. In contact with the body there is a negative pressure p_a and the velocity V of the whole. At infinity the dynamic pressure is 0 and the velocity 0. Everything takes place as if, in this domain, each second, a certain mass of air took the velocity V in passing from the dynamic pressure 0 to the negative pressure p_a. These two facts require the creation of a certain amount of motion, at the expense of drag. Witoszynski has drawn from this idea very interesting conclusions, but this scheme does not take into account the details of the phenomena on the downstream side of the body.

Separation.- We do not consider as separation proper a case already encountered in which the fluid leaves the wall normally by fusion of the two contrary currents at a point of zero velocity (Fig. 17). The pressure would there be a maximum and equal to the generative pressure. Nothing similar has even been confirmed as positive pressure in aerodynamics, the zone of the separations being always under negative pressure. From this we conclude that the velocity at the point of separation is not zero and that the fluid leaves the contour tangentially (on excluding, at this point, any discontinuity of velocity (Fig. 18)). This conclusion involves an aspect of the problem very different from the complete contouring of the body. After separation, the course of the fluid, which has previously followed the contour, becomes the new boundary of the current coming from upstream and will result from the solution itself. Within these limits there is a surface comprising, in part, the body itself and, in part, a certain downstream region. In this latter domain, there is a second motion subordinated to the first by the conditions at the limits. We will here consider only the first motion. The second current can be at the velocity potential with a different Bernoulli constant (the limit of the wake being then a discontinuity of velocity), or more generally vortical in all its points and still more generally "non-permanent," although the first motion may be. In brief, the conditions at the limits are not simply a condition of zero normal velocity (tangent to the downstream wall of the body and to the wake itself), but a condition of equality of pressure on both sides of the wake. The latter entails a vortical non-permanent flow. These conclusions from the theory would moreover agree with the current investigation of a turbulent downstream zone. Its theoretical determination is difficult and an analysis would doubtless be unable to give it. It would here be rather a question of dragging by friction, along the surface of discontinuity and of physical mixture, to the points where the equality of pressure could not be maintained.

It is these motions which would absorb the energy necessary to produce the drag. If their analysis is difficult, it must be remembered that they are manifested by a negative pressure, very nearly constant on the downstream wall. The knowledge of this negative pressure, combined with the pressures of the general motion, would give by integration the drag sought, but we have not yet proceeded that far.

For the time being, we are trying to treat the problem of the first motion. It admits as unknowns, the point of separation and the shape of the wake. The latter will depend on the solution itself, but we need to know in advance at least how this wake behaves quite far behind the body. The existence of a drag necessitates a residue of turbulence downstream. This residue is slight and, since on the other hand it is natural to think that the fluid, separated by the body, tends to combine behind it, we will attach to the wake the condition of closing completely at infinity a limiting case susceptible of being modified by the result, which will make it possible to seek the different types of solution.

They are here more numerous than in Section II, not only because of ignorance of the separation point, but also because the regular solution of the potential of the velocities, so well determined for a closed contour or surface, is so no longer for a portion of the contour or surface. In addition to the normal velocity on the wall (the only precise condition at our disposal), it would be necessary to give the value of the potential at every point, which nothing renders possible to calculate. If we add to this the fact that special solutions can be superposed on the regular ones, we will have an idea of the complexity of the problem.

A very important observation must be made on the subject of these singularities. Thus far, for locating them, we had at our disposal the interior of the body (example of Joukowski's vortex). Here we have also the interior of the wake, since the motion considered is expressly outside this domain. The problem of separation, thus stated, therefore enables the introduction of certain singularities outside the body, without including the reservations required in Section II. We will find both cases in succession and for the same body. Perhaps it will be possible to see in them the image of two distinct kinds of flow analogous to those obtained experimentally.

Lastly, in order to terminate with the principles, we can generalize the well-known conception which consists, for a given body, in envisaging on the inside, a sort of prolongation of the outside flow, a prolongation not materialized in the fluid, but which is explained by the existence of a motion of the whole, in which the partial contour of the body, prolonged by the wake, is a line of flow. All the internal lines of flow cut the downstream wall and must be taken away from the real motion. Nevertheless, their conception can be of service. In fact, the theoretical solutions, which will be presented later, comprise these two parts: the real motion and the extended fictitious or imaginary motion.

These statements were necessary in order to comprehend the character of the problem. They are all based on the fact that the fluid separates tangentially to the surface with a velocity not zero.

Solution with simple layer and source-vortices.- We will now give a solution of the problem for a circle in a symmetrical flow, with a view to its application to a cylinder, the generalization being immediate for any contour whatsoever. Since the points of separation are not known in advance, it will be necessary to study all the possible cases. Since the solutions, whether regular or irregular, are all multiples, we will begin by choosing one of them according to the only definite condition we have, namely, the value of the normal velocity at every point of the washed portion of the body.

We have a motion with a preliminary potential which is that of the translation of the whole. This translation has a component normal to the washed surface, known at evry point. The complementary potential to be sought will have at every point a normal velocity equal and opposite to the first. We know how to calculate such a function, since it is the potential of a certain simple layer on the surface or partial contour.

For the physics of liquids, a simple layer is a source, but a source distributed on a contour or surface, the density μ at every point being such that the output of a small element ds is $+2 \pi \mu\, ds$ toward the outside and $-2 \pi \mu\, ds$ toward the inside. The normal being directed toward the concavity, μ is negative for a source and positive for a sink (Fig. 19).

$2 \int \mu\, ds = M$ is equivalent to the total intensity of a source, the total output being $2 \pi M$. According to the sign of M, the simple layer is either repulsive (source) or attractive (sink) in the whole.

This solution is only the first step. Had it been necessary to hold to it, it would have suggested to us "a posteriori" a condition for the point of separation. In fact, according to the point chosen, the density on the boundaries of the layer has a given value, which entails a discontinuity of velocities. By annulling the density, we obtain a continuous separation, as to velocities, which enjoys a sort of privilege. Fig. 20 shows an example of a cylinder with the optimum angle of $77^{\circ} 28'$, to be compared with Figs. 10 and 17 of Camichel's article previously referred to.

It remains to satisfy our last condition, which is the closing of the wake. It is easy to see that it is not realized by the simple layer. Here the fiction of the prolonged motion will be of service. Considered as a fissure in the sheet of paper, on its upstream portion, the contour leaves other than $2 r M$, which is the output of all the fictitious or imaginary filaments within the wake. Far enough in the rear, their residual velocity is only one of translation, if h is the distance of the two asymptotes from the wake. $h V = 2 r M$, which gives h a value other than 0. (In the abovementioned example, $h = 1.25 D$.) There is no closure for any simple layer of output not zero. (Let us note that $M = 0$ for the total contour, as is suitable.)

In order to remedy this, it is always possible to add any function to the tentative potential, thus far represented only by the translation. The new potential can be calculated from the simple layer which, combined with the first two, will render it possible to pass around a predetermined portion of the object.

If this additive initial function is regular, especially within the wake, the output of the internal fictitious filaments is not altered up to infinity. A singularity is therefore required in the interior, which is capable of absorbing the output of the source, i.e., a total assembly of sinks of capacity, equal and of opposite sign. One can thus be assured that, beyond the first sink, the total output of the filaments within the wake is zero and that, sufficiently far to the rear, the two boundary lines of the wake will reunite.

Let us note that the additive function no longer needs, as in Section II, to have a circular line of flow. Therefore the number and disposition of the sinks remain arbitrary within the body or the wake, but the conception is there. The body throwing off the fluid acts as a source spread over a portion of its surface. The fluid separates. At the rear there is a vacuum, which is closed by the effect of the sinks situated in this zone or even inside the body. We thus return, by a general and deductive method, to the former conception of Rankine when he sought good shapes for airships. It is always understood that the real motion will be limited to the fluid which has followed the contour of the body, and that the theoretical existence of sinks entails no physical contradiction.

A secondary condition arises here, although natural in permanent motion. Every singular center in the domain of the fictitious motion is subjected to a field composed of the translation, of the simple layer and of the other sinks. Let us take, for example (Fig. 19), two symmetrical downstream sinks of the cylinder. The attraction of one of the sinks for the other can equilibrate the vertical component of the field of the simple layer. Nothing counterbalances the component of translation which is preponderant. The sinks therefore have a motion which is incompatible with the permanent motion. We will try to realize the equilibrium even for the fictitious motion. It will be necessary to add to each sink a suitable vortex which can offset the translation and whose direction is indicated on the figure.

In summing up, we are led to consider a general solution based on a simple layer and on source-vortices, in order to realize the three conditions: partial circumferential flow, closing at the rear, and equilibrium of the singularities.

It is obvious, from the simplest example, that these accumulated conditions do not suffice to determine completely the problem of a circle with two symmetrical source-vortices. The unknowns are reduced by the symmetry to the angle of separation, to two coordinates and to the respective densities of the sources and of the vortex constituting one of the singular points, or five in all.

To recapitulate the conditions, the horizontal and vertical equilibrium furnish two, and the closure a third. It is necessary to add one more, the annulment of the residual density at the point of separation (continuity of the velocities). In spite of everything, there still remains one indeterminate. The point of separation remains arbitrary.

The only positions of equilibrium found are inside the circle, one corresponding to each point of separation. It can likewise be demonstrated that the sheet separated at angles above 120° does not again cut the circle (Fig. 21). This kind of solution tends to unite the flow at high velocities, at less important separations, without external vortical appearances.

We have found no positions of equilibrium outside the circle. If we consider two source-vortices in this domain, they cannot remain stationary and this opens up the much more complicated question of their evolution. These mobile singularities cannot be rejected, because they remind us of the vortex trails investigated and confirmed by Karman. They correspond rather to the flow with large separations and to a temporary wake.

We have not completed the study of the analogous problem in a dissymmetrical flow, for the purpose of transforming the circle into a profile. It is certain, however, that for a slight dissymmetry (small angle of attack) the singularities will be internal and that, on exaggerating the dissymmetry (large angles of attack), they will come out at a certain moment, creating a second appearance to the flow about the profile.

This is only one example, because it is obvious that we can complicate the problem by varying the number and disposition of the source-vortices. The field is therefore still vast for this type of solution based on the simple layer and the source-vortex. We may add that its generalization is immediate in the case of the three dimensions.

Solution with double layer and simple vortices.- In closing, we wish to show how an entirely different method likewise leads to a general type very distinct from the first, by utilizing the resources of conform representation. We will first examine the case of a symmetrical flow. We can arrive at a flow separating from a circle tangentially with a finite velocity, by transforming a continuous solution without separation. Starting with the vortices of Foppl already mentioned, there are two pairs, one internal and one external, which are in equilibrium. There are two points of zero velocity in the direction of the flow, and two other symmetrical ones on the downstream contour. The two latter points give off filaments which reunite on the axis quite far to the rear. They form the separation between the upstream fluid which flows to infinity and that which flows indefinitely around the vortex cores. They constitute, without any discontinuity, a sort of wake which is closed at the rear.

Let us transform the circle according to Joukowski (Fig. 22) by taking as the foci of the transformation, the two symmetrical points of zero velocity situated on the flanks of the circle. We will obtain an arc, ending at these two points, and two filaments which will reunite at the rear, forming the new wake. From the four initial vortices, there will remain in the new system only one pair which will, moreover, be in equilibrium. (It should be noted that the space inside the original circle is transformed into a complete system communicating with the former by a fissure which is the arc of the transformed circle.) Foppl found an infinity of these systems in equilibrium, each one corresponding, in the transformation, to a point of separation.

Hence we have a second figure of flow, where the point of separation remains arbitrary, and where the singularities consist simply of two symmetrical vortices within the circle. Their position is shown on the same figure (21) as the source-vortices of the first type presented. The points differ a little, but there is, on the whole, quite a decided resemblance between the two results.

Nevertheless, the type is very different. This is obvious on seeing that, at the upstream wall of the transformed arc, the velocities of the motion of the whole arc tangent to the two faces and that the only thing which differs is the value of the potential on one side and on the other. We recognize here, without being able to proceed farther, the action of a double layer spread over the surface. In brief, the type thus discovered restores a system of double layer and of vortices. It required an artifice to arrive at it in a simple manner, because the direct problem of the determinations of double layer by normal velocities is very complicated.

The case of the dissymmetrical flow is interesting. We will first try to generalize the vortices of Foppl in the dissymmetrical hypothesis. It gives, in general, four points of zero velocity without symmetry (Fig. 23). A first transformation, with the two of zero velocity above the rear wall, as foci, realizes a dissymmetrical flow around a circle (Fig. 24). Then, on taking, as the foci, the third point of zero velocity at the rear and another within the circle, the arc is transformed into a profile about which there is obtained a flow with separation (Fig. 25).

The solution involves very complicated calculations. At a predetermined angle of attack of the profile, one has the choice of the separation between certain limits, as in the first type. There exist, after all, two vortices which, according to the case, are either within or without or astride the profile. Our investigation is not yet complete.

Conclusion.- The essential thing in this discussion is a new class of possible solutions added to the other and causing us to anticipate, between these two extreme types, a whole series of intermediate types. In this great multiplicity there are included, as special cases, all those of non-separation.

It should be noted, moreover, that all the fixed conditions which, strictly speaking, are not equally indispensable, still leave a variable parameter, which is the choice of the separation. Therefore, we still have to select one condition, but which? Will it enable us to make a choice in their infinite diversity? We are far from being able to answer this question. This is the reason why, for the time being, comparison with experience will be our best guide.

It may be asked in what direction to search. Is it the correction required by the viscosity? Is it a more general condition, analogous to the natural laws of minimum work, of minimum action? It is impossible to say yet. In the latter direction, we can indicate only one result, in the incomplete solution constituted by the pure simple layer.

In this hypothesis, we have seen that the separation can be determined by the condition of annulling the density of the simple layer at its limit. This condition entails, among others, one remarkable consequence. For this particular separation, the potential, in every point of the space, passes through a minimum or a maximum, so that there would thus be a position of equilibrium relative to the value of the potential. The condition of equilibrium imposed on the singularities of the motion is based on the same idea. Without our being able to draw any further precise conclusions, it is, perhaps, an indication to be borne in mind in studying problems relating to the perfect fluid.

Translation by Dwight M. Miner,
National Advisory Committee
for Aeronautics.

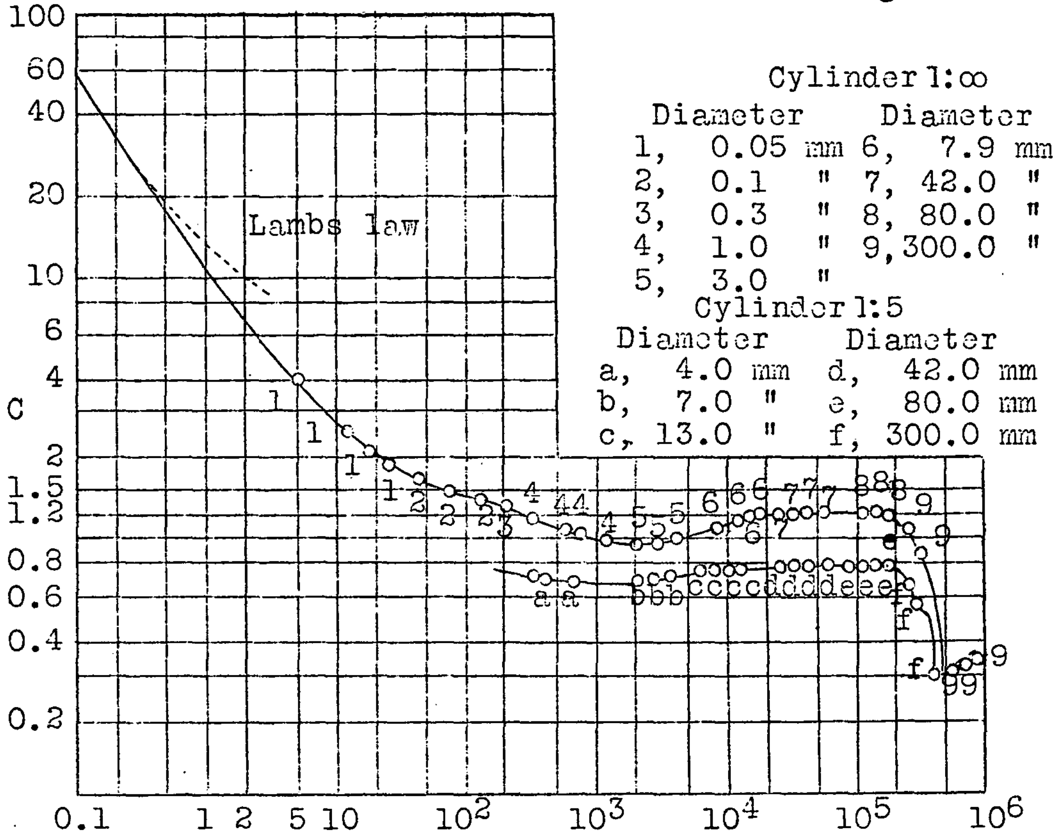

Fig.1

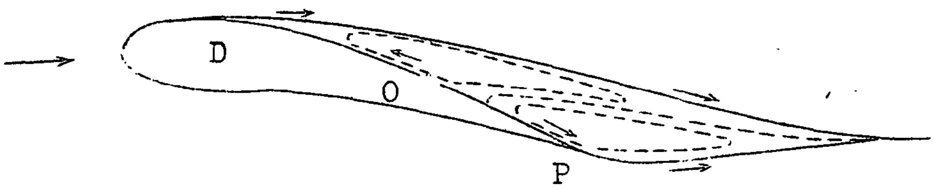

Fig.2 Separation according to Prof. Pavlowski.

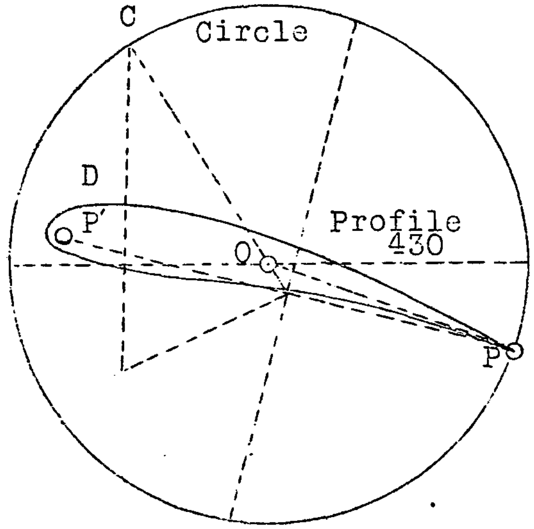

Fig.3 Transformation of a circle into profile 430 according to Joukowski.

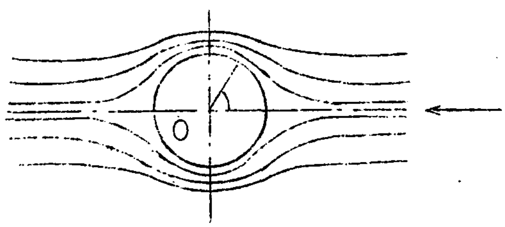

Fig.4

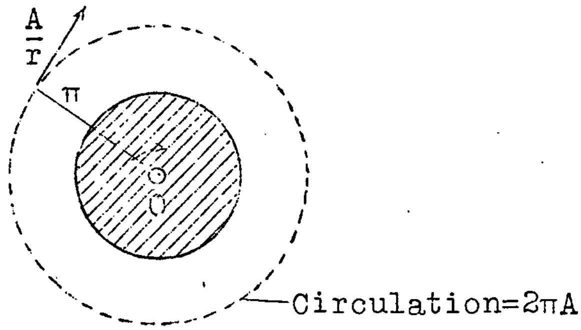

Fig.5 Indefinite rectilinear vortex.

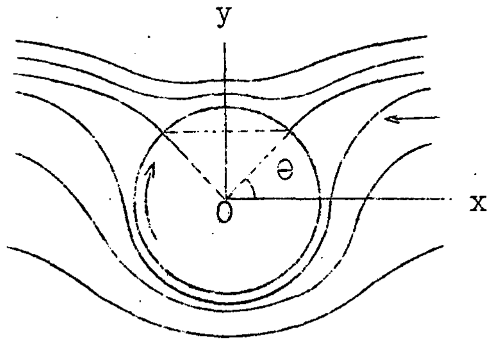

Fig.6

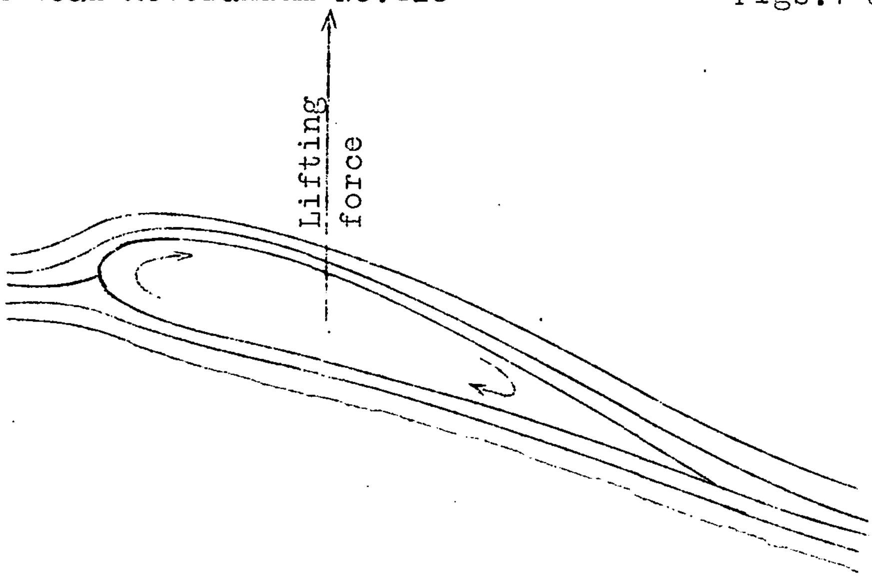

Fig.7 Flow according to Joukowski.

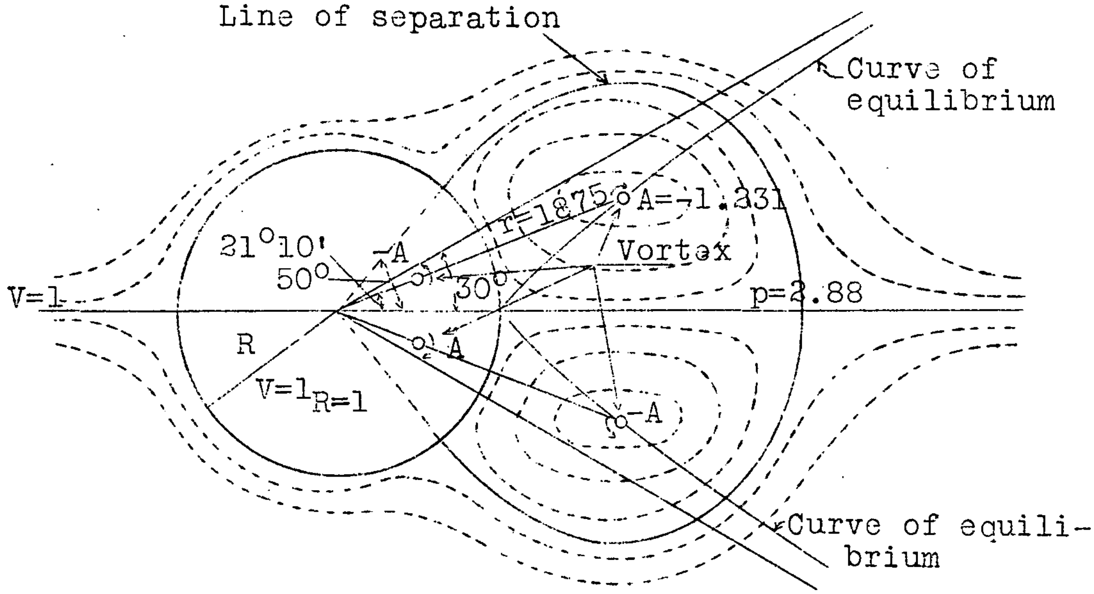

Fig.11 Vortices of Foppl.

Pressure distribution
----- Experimental
——— Theoretical according to Joukowsky.
I Corresponds to the incidence of -5.1°
II " " " " " 4.7°
III " " " " " 9.6°

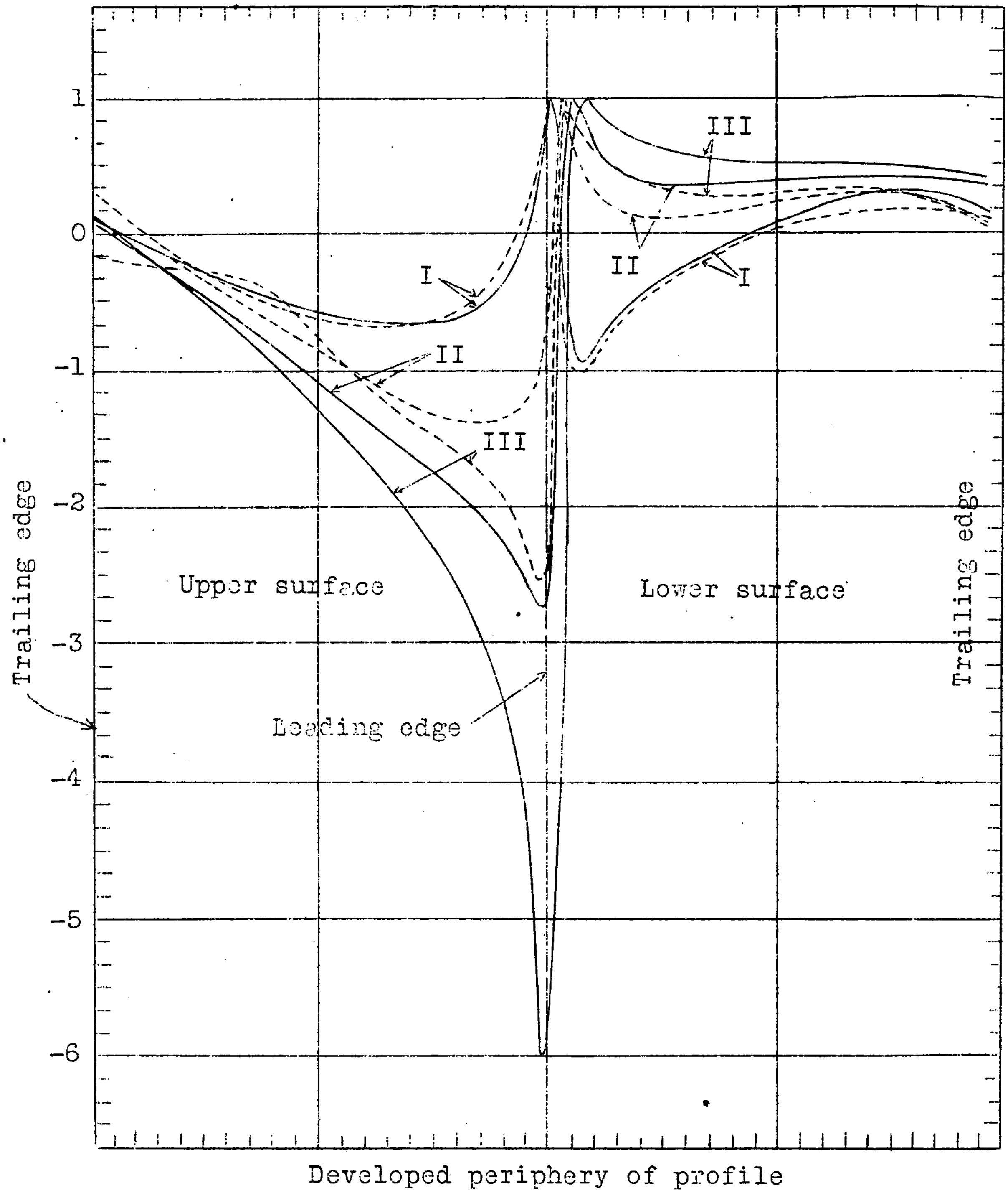

Fig.8 Comparative curves showing experimental and theoretical values of ratio k.

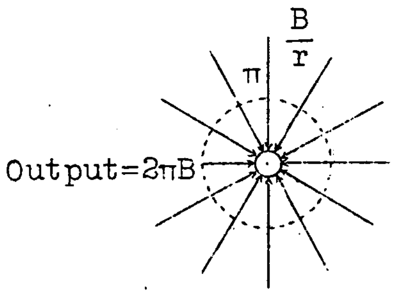

Fig.9 Negative source or sink.

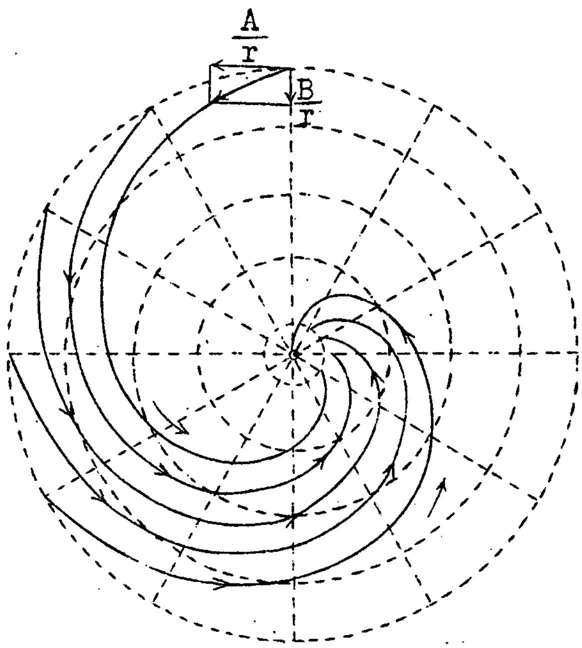

Fig.10 Source vortex.

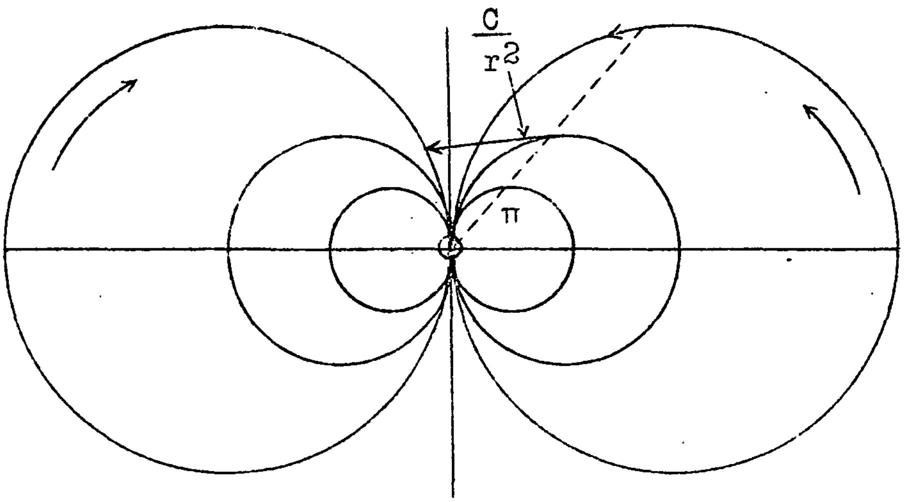

Fig.12 Double source.

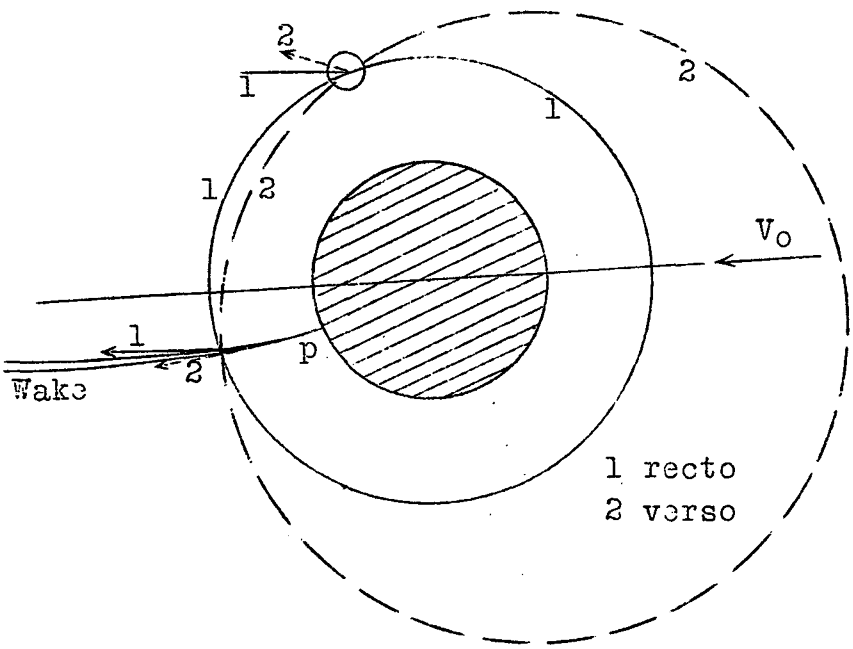

Fig.13 Solution of Prof. Witoszynski.

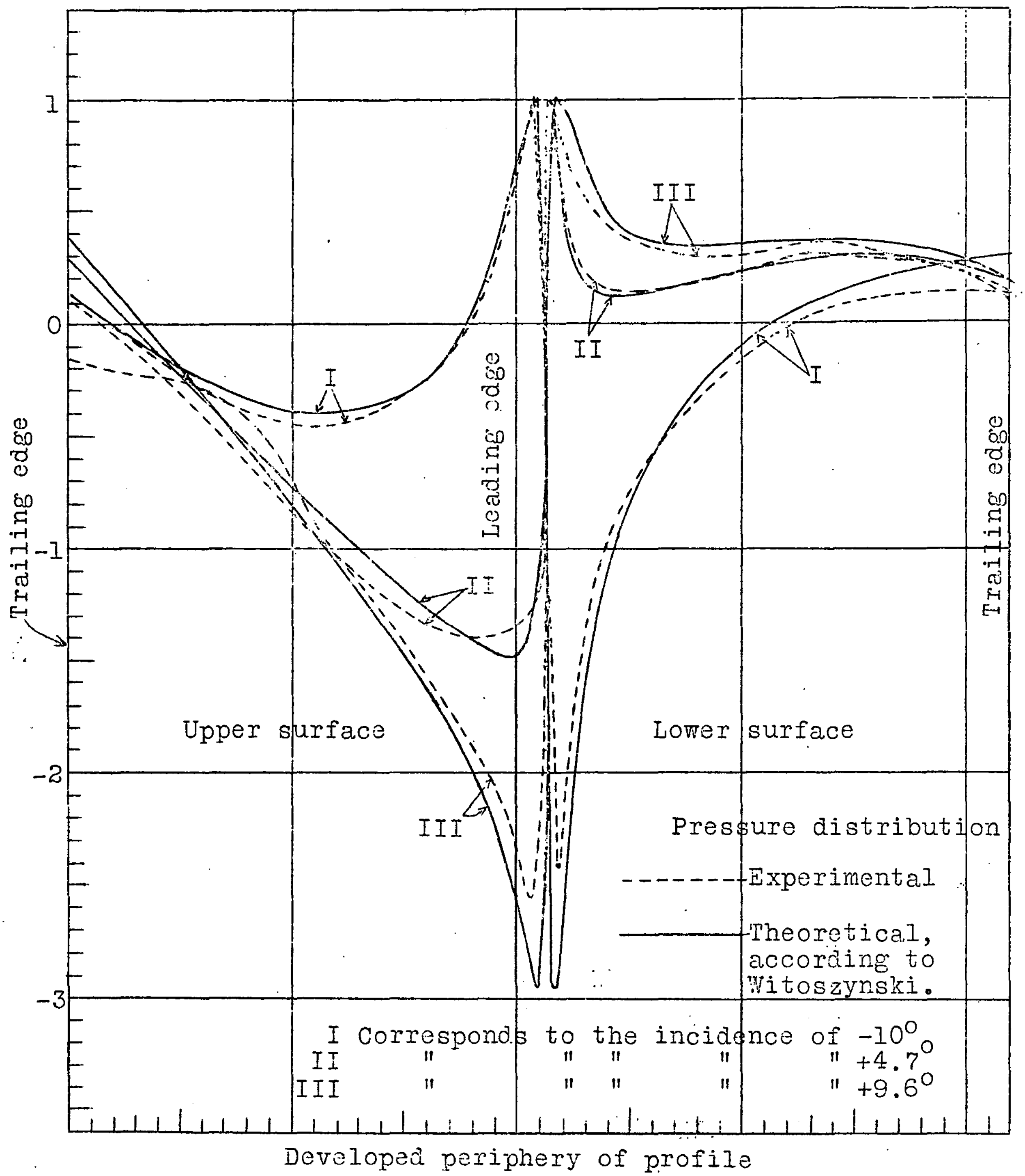

Fig.14 Comparative curves showing experimental and theoretical values of ratio k.

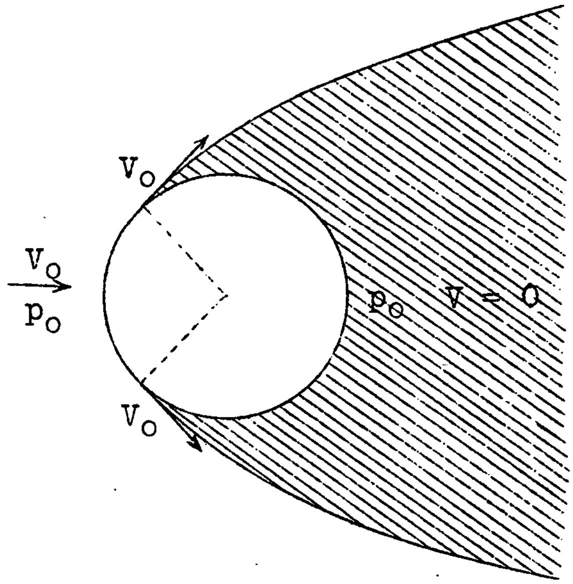

Fig.15 Wake according to Helmholtz.

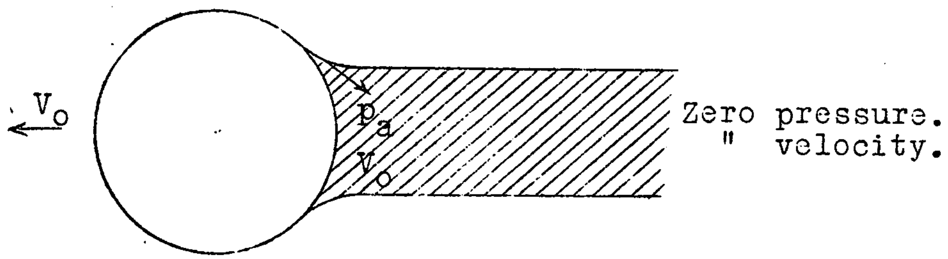

Fig.16 Wake according to Prof. Witoszynski.

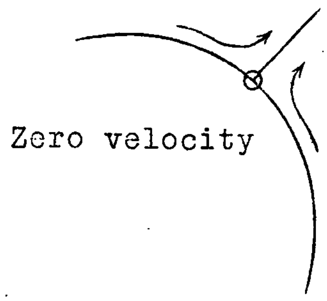

Fig.17

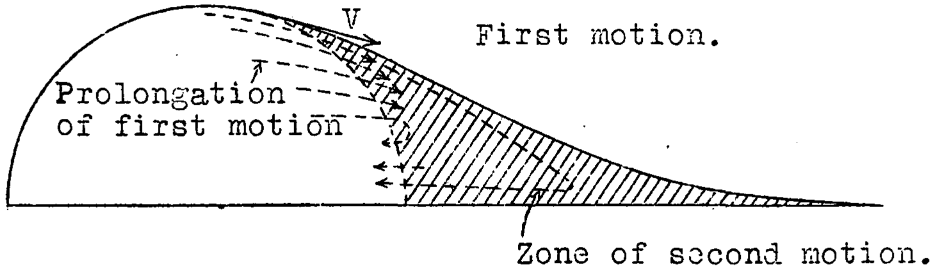
V
First motion.
Prolongation
of first motion
Zone of second motion.

Fig.18

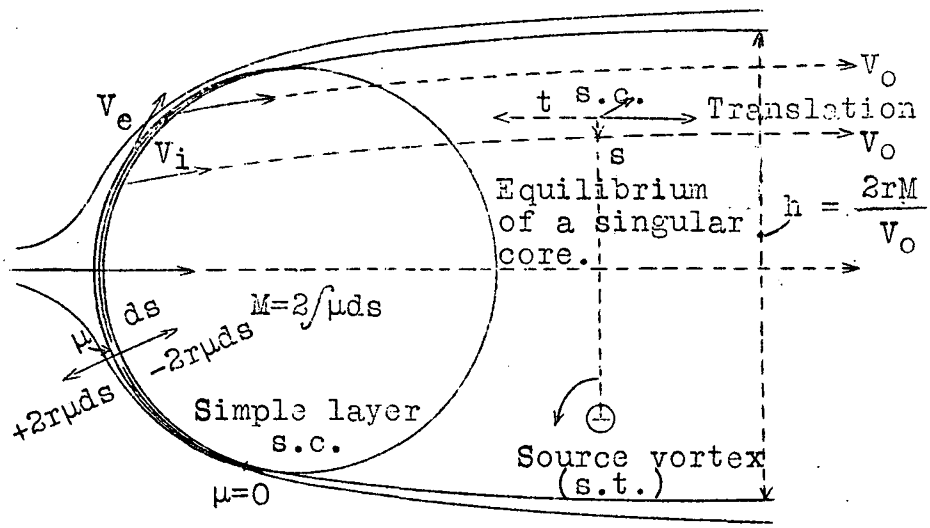
V
o
V
e
t
s.c.
Translation
V
i
V
o
s
Equilibrium
of a singular
core.
h = 2rM / V
o
ds
M=2∫μds
μ
-2rμds
+2rμds
Simple layer
s.c.
Source vortex
(s.t.)
μ=0

Fig.19

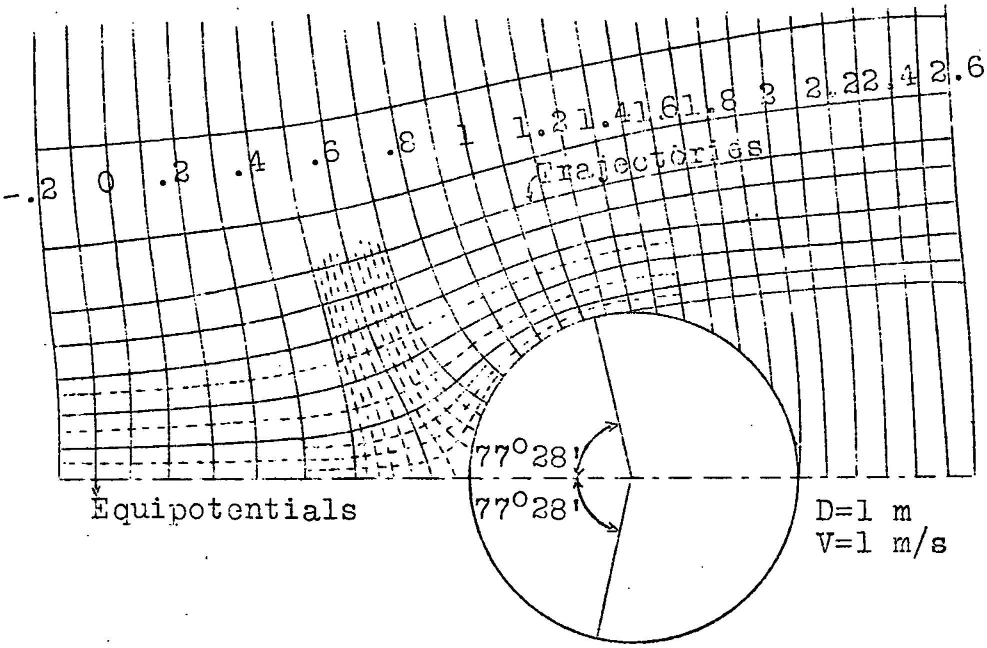

Fig.20 Indefinite cylinder separation at 77°28'.

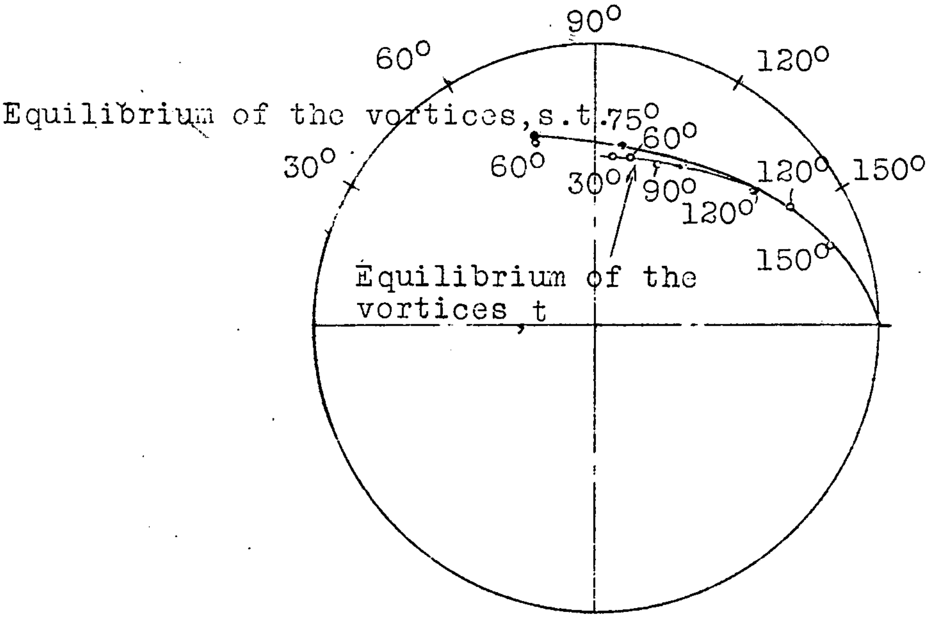

Fig.21 Interior position of equilibrium of the source-vortices and vortices.

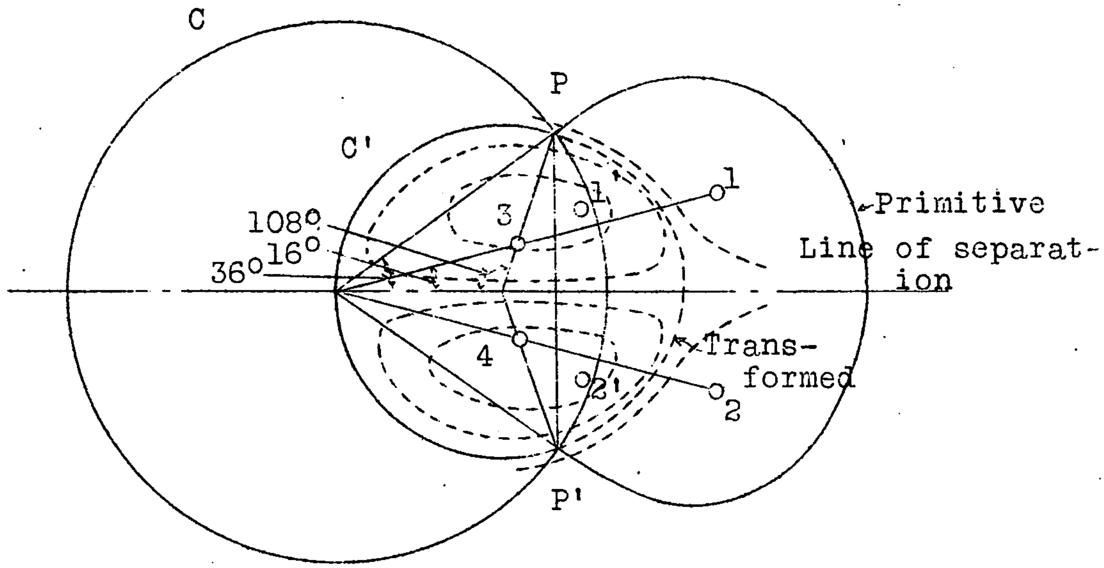

Fig.22 Transformation of the vortices of Foppl into a flow around and arc.

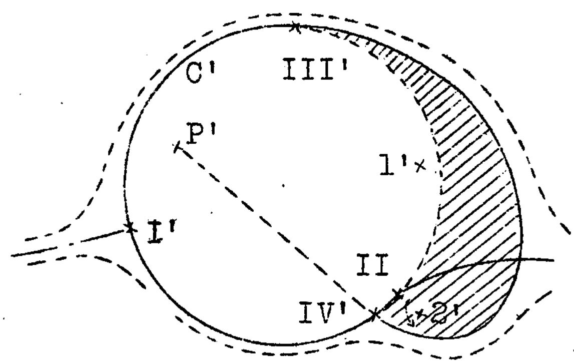

Fig.24 Dissymmetrical flow.(Transformation of Fig.23 with II and III as poles) 1' and 2' vortices transformed.

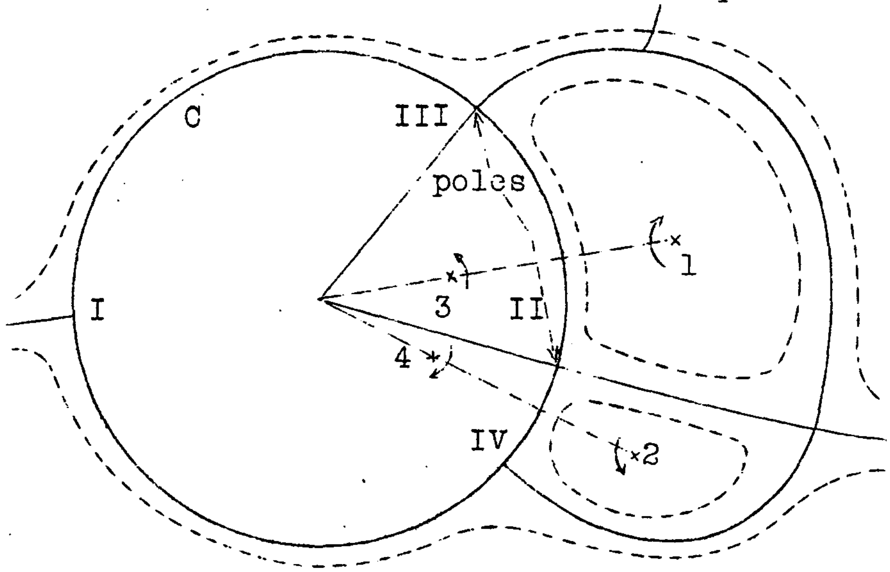

Fig.23 Dissymmetrical vortices of Foppl.

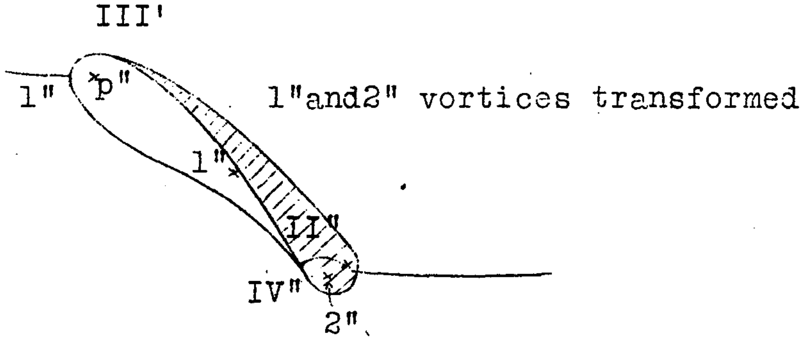

Fig.25 Separation on a profile(Transformation of Fig. 24 with IV' and p' as poles.