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APPROXIMATION METHOD FOR DETERMINING THE STATIC STABILITY OF A MONOPLANE GLIDER

By A. Lippisch

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Symbols (Figure 1)

F = wing area (m^2);

b^2/F = Λ_F = aspect ratio of wing;

t_m = F/b_F = mean chord of wing (m);

b_F = span of wing (m);

f = area of elevator (m^2);

b_H = span of elevator (m);

l = distance (m) from leading edge of wing to center of pressure of elevator, which may be assumed to be at 1/4 the mean chord of the elevator measured from its leading edge, though it is allowable to take the distance to the elevator hinge as an approximation;

e = $(l - \frac{t}{2})$ = distance (m) of elevator from lateral axis of wing.

${b_H}^2/f$ = Λ_H = aspect ratio of elevator;

σ^o = longitudinal dihedral angle, which is shown as positive in Figure 1;

${\alpha_F}^o$ = angle of attack of wing;

${\alpha_H}^o$ = effective angle of attack of elevator;

*"Näherungsverfahren zur Bestimmung der statischen Stabilität beim Eindecker," a communication from the Research Institute of the Rhön-Rossitten Society. From "Zeitschrift für Flugtechnik und Motorluftschiffahrt," June 14, 1927, pp. 251-256.

α_W° = downwash angle of elevator;

c_{a_F} = lift coefficient of wing;

c_{a_H} = lift coefficient of elevator;

$k_F = \frac{\partial \alpha_F}{\partial c_{a_F}} = \frac{\text{change in angle of attack}}{\text{change in lift coefficient of wing}}$;

$k_H = \frac{\partial \alpha_H}{\partial c_{a_F}} = \frac{\text{change in angle of attack}}{\text{change in lift coefficient of elevator}}$;

$\kappa = \frac{\partial \alpha_W}{\partial c_{a_F}} = \frac{\text{change in downwash angle}}{\text{lift coefficient of wing}}$;

$c_{m_{Fx}}$ = moment coefficient of wing with reference to location of center of gravity at the distance $x\ t$;

$c_{m_{Hx}}$ = moment coefficient of elevator with reference to location of center of gravity at the distance $x\ t$.

Since the following calculations afford an approximate solution of the static stability, c_a is put in place of c_n and, likewise, the lift is employed instead of the air-force component perpendicular to the longitudinal axis of the airplane.

Derivation of the Formulas

a) Moment Coefficient of Wing

The wing moment with reference to the rotation center at the forward end of the chord is represented by the well-known formula

$$M_F = c_{m_F}\ F\ t\ q \tag{1}$$

The course of c_m, as plotted against c_a in Figure 2, is obvious from the airfoil test in question. This course of the function $c_m = f(c_a)$, is known to be nearly independent of the aspect ratio of the wing ("Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen," Report I, Fig. 36) and is expressed approximately by

$$c_{m_F} = \alpha\ c_{a_F} + c_{m_0}, \tag{2}$$

in which the factor α represents an approximately constant quantity and can be assumed to have a mean value of 0.25. The moment coefficient of the wing with reference to the center of gravity at the distance $x\ t$ from the front end of the wing chord is then

$$c_{m_{F x}} = (\alpha' - x)\ c_{a_F} + c_{m_0} \tag{3}$$

$$c_{m_{F x}} = \frac{M_F}{F\ t\ q}$$

b) Moment Coefficient of Elevator

With a constant location of the center of pressure of the air forces on the elevator, the moment of the same, with reference to the center of gravity at the distance $x\ t$, is

$$M_H = f\ (l - x\ t)\ q\ c_{a_H} \tag{4}$$

In order to be able to refer the moment coefficient of the elevator also to the lift coefficient, we must, on the one hand, express c_{a_H} by c_{a_F}. On the other hand, the moment coefficient of the elevator must be developed as

$$c_{m_{Hx}} = \frac{M_H}{F\ t\ q}$$

$$c_{m_{Hx}} = \frac{f}{F}\left(\frac{l}{t} - x\right) c_{a_H} \tag{5}$$

The lift coefficient of the elevator is dependent on the effective angle of attack within the range of the same..

The angle of attack of the elevator is therefore

$$\alpha_H = \alpha_F - \alpha_w - \sigma \tag{6}$$

Between the angle of attack of a wing and the corresponding lift coefficient, we can now introduce, for the normal flight range, the approximate expression

$$\alpha_F = k_F\ c_a - \alpha_o \tag{7}$$

Hereby the factor k is a quantity essentially dependent only on the aspect ratio of the wing, so that the effect of the profile shape may be disregarded and be designated only by the constant term α_o, which is then the angle at which the lift becomes zero. The same result can likewise be obtained from the test result of the wing profile used, in which connection it may be remarked that, with tapered wings which change profile toward their tips, a mean value between the outer and inner profiles must be used.

The downwash angle α_w can likewise be brought into dependence on the lift coefficient of the wing. The formula derived from H. Helmbold ("Zeitschrift für Flugtechnik und Motorluftschiffahrt," 1927, p. 11) can be written

$$\alpha_W = \frac{57.3^\circ}{\pi \Lambda_F} c_a \left[u - \frac{c_a}{\sqrt{c_a^2 + 1}} (u - 0.5\ v)\right].$$

whereby, with the symbols there employed,

$$u = 1.36 + \frac{\sqrt{\epsilon^2 + 1}}{\epsilon} - \frac{0.45}{\epsilon} - 0.11\,\epsilon$$

$$v = \frac{1 + \frac{\epsilon}{\sqrt{\epsilon^2 + k^2}}}{k^2} + \frac{1}{\epsilon \sqrt{\epsilon^2 + 1}}.$$

Since $u - 0.5\ v$ is small in comparison with u, a mean value, independent of c_a, can be used for

$$\frac{c_a}{\sqrt{c_a^2 + 1}} (u - 0.5\ v)$$

so that the downwash angle can be expressed accurately enough by

$$\alpha_W = \kappa\, c_{a_F} \tag{8}$$

In Figure 4 the factor κ is plotted against the aspect ratio, the plain curves being for rectangular wings and the dotted curves for elliptical wings. For trapezoidal or semi-elliptical shapes, the corresponding mean values must be used.

The effective angle of attack of the elevator can then be written

$$\alpha_H = (k_F - \kappa)\ c_{a_F} - (\alpha_o + \sigma) \tag{9}$$

The lift coefficient is likewise

$$c_{a_H} = \frac{\alpha_H - \alpha_{o_H}}{k_H} \tag{10}$$

whereby, with the usual symmetrical profile shapes of the elevators $\alpha_{o_H} = 0$. The factor k_H depends, the same as for a wing, essentially on the aspect ratio of the elevator and may be obtained from Figure 3. Accordingly, we can represent the lift coefficient of the elevator by

$$c_{a_H} = \frac{k_F - \kappa}{k_H} c_{a_F} - \frac{\alpha_o + \sigma}{k_H} \tag{11}$$

Hence the moment coefficient of the elevator, with reference to the center of gravity, is

$$c_{m_{Hx}} = \frac{f}{F}\left(\frac{l}{t} - x\right)\left(\frac{k_F - \kappa}{k_H} c_{a_F} - \frac{\alpha_o + \sigma}{k_H}\right); \tag{12}$$

in which

$$\frac{k_F - \kappa}{k_H} = m \qquad \frac{\alpha_o + \sigma}{k_H} = n.$$

c) Total Moment of the Combined Wing and Elevator and the Moment Coefficient with Reference to the Center of Gravity

The total moment is the sum of the wing and elevator moments. It is therefore

$$M_{total} = M_F + M_H$$

$$M_{G_x} = c_{m_{F\,x}}\,F\,t\,q + c_{m_{Hx}}\,F\,t\,q$$

$$\frac{M_{G_x}}{F\,t\,q} = c_{m_{gx}} = c_{m_{F\,x}} + c_{m_{Hx}} \tag{13}$$

By using équations (3) and (12), we obtain

$$c_{m_{gx}} = \left[(\alpha - x) + m\,\frac{f}{F}\left(\frac{l}{t} - x\right)\right] c_{a_F} +$$

$$+ \left[c_{m_o} - \frac{f}{F}\left(\frac{l}{t} - x\right) n\right].$$

If we combine the terms which vary with the displacement of the center of gravity, the moment coefficient of the system becomes

$$c_{m_{gx}} = \left[\left(\alpha + n\,\frac{f}{F}\,\frac{l}{t}\right) - x\left(1 + n\,\frac{f}{F}\right)\right] c_{a_F} -$$

$$-\left(\frac{f}{F}\,\frac{l}{t}\,n - c_{m_o}\right) + \frac{f}{F}\,n\,x \tag{14}$$

After introducing the abbreviations

$$\alpha + m\frac{f}{F}\frac{l}{t} = A \qquad \left(\frac{f}{F}\frac{l}{t}\,n - c_{m_0}\right) = C$$

$$1 + m\frac{f}{F} = B \qquad \frac{f}{F}\,n = D,$$

we can write

$$c_{m_{gx}} = (A - Bx)\; c_{a_F} - (C - Dx) \tag{15}$$

The function for $c_{m_{gx}}$ is therefore restored to an equation of the first degree, from whose course the following deduction can be made with reference to the static stability of the wing-and-elevator system in question.

If $(A - Bx) > 0$ and $(C - Dx) > 0$, then the course is the same as represented in Figure 5. The intersection with the c_a axis determines the lift coefficient at which the airplane flies with the elevator in its normal position. Since this lift coefficient usually lies between $c_{a_F} = 0.7$ and 1, the condition

$$\frac{C - Dx}{A - Bx} = 0.70 \div 1.00.$$

must be fulfilled. The dash lines then indicate, respectively, tail heaviness and nose heaviness. The positive direction of the lines indicates that the airplane is stable, i.e., that any change in the flying attitude, due to external causes, releases forces which tend to restore the previous attitude.

The stability limit is reached when $(C - Dx) = 0$, i.e., when the line passes through the origin of the coordinates (Fig. 6). The extreme limit, however, at which the airplane can be regarded as still manageable, is reached when the inclination of the lines is likewise zero, and the function $c_{m_{gx}} = f\,(c_{a_F})$ coincides with the c_a axis. In this extreme case, any steering maneuver in either the positive or negative direction produces instability, while in the equalization, as shown in Figure 5, the elevator produces stability.

If the picture of the function is as represented by Figure 7, the airplane is unstable and requires, in individual cases, a change in the location of the center of gravity, of the longitudinal dihedral angle, or of the dimensions of the elevator or its distance. The longitudinal dihedral angle for the extreme rear limit of the location of the center of gravity can be determined by the formulas

$$\left.\begin{aligned} C - Dx &= 0 = \frac{f}{F}\frac{l}{t}\,n - \frac{f}{F}\,n\,x - c_{m_0} \\ n_0 &= \frac{c_{m_0}}{\frac{f}{F}\frac{l}{t} - \frac{f}{F}\,x} \end{aligned}\right\} \qquad (16)$$

The longitudinal dihedral angle is then

$$\sigma_0 = k_H\,n_0 - \alpha_0 \qquad (17)$$

If, therefore, it is desired to test the correctness of a finished airplane, the longitudinal dihedral angle (σ) or the location of the center of gravity (x) is introduced as a variable.

In general, these two factors suffice for obtaining a perfect trim. For designing, however, the formulas are available for determining the magnitude and distance of the control surfaces. If, for example, the condition is made that, in normal flight (always to be sought with $(c_a/c_w)_{max}$ or $(c_w^3/c_w^2)_{max}$ according to the structural viewpoint), the elevator is not to be subjected to pressure, either up or down, there are obtained for the dimensions certain fiducial lines, which are deduced from the following formulas.

If the elevator exerts no lift, its moment is zero. It must therefore be

$$c_{m_{Hx}} = 0 = \frac{f}{F}\left(\frac{l}{t} - x\right)(m c_{a_F} - n); \tag{19}$$

In order that there shall be stability in this case, the wing moment must likewise disappear and

$$c_{m_{Fx}} = 0 = (\alpha - x)\ c_{a_F} + c_{m_o} \tag{20}$$

c_{a_F} is defined by the attitude of normal flight, so that x may be computed as

$$x = \alpha + \frac{c_{m_o}}{c_{a_F}} \tag{21}$$

With this value of x we then obtain, by equation (19), an expression for the possible characteristics of the airplane (wing area, length of tail, longitudinal dihedral and aspect ratio). For the further calculation, one can utilize the stability limits, which are given by the formulas already explained.

$$(A - Bx) \geqq 0$$

$$(C - Dx) \geqq 0$$

d) Illustration of the Computation Formulary by an Example for Testing the Static Stability

In order to enable a simple evaluation of the derived formula, a formulary of the computation process was published by the Aviation Section of the Research Institute of the Rhön-Rossitten Society on the Wasserkuppe. The application of this formulary may be illustrated by an example (See Formulary and Figs. 8-9).

Formulary (referring to Figure 8).

1. Wing area	$F_{\sim}$	18 m²
2. Wing span	b_F	12.0 m
3. Mean wing chord	$t_m = \frac{F}{b_F}$	1.5 m
4. Wing aspect ratio	$\frac{b_F^2}{F} = \Lambda_F$	$\frac{144}{18} = 8$
5. Elevator area	f	2.4 m²
6. Elevator span	b_H	2.5 m
7. Elevator aspect ratio	$\frac{b_H^2}{f} = \Lambda_H$	$\frac{6.25}{2.4} = 2.6$
8. Distance of elevator from leading edge of wing.	l	4.0 m

Formulary (referring to Figure 8) Cont.

9. Distance of elevator from lateral axis of wing.	e	3.25 m
10. Longitudinal dihedral angle.	σ	1°
11. Distance between c.g. and leading edge of wing.	$S = x\ t_m$	0.53 m
12. $\epsilon = \frac{e}{b_F/2} = \frac{3.25}{6.0} = 0.54$	Table II κ	4.74°
13.	Table I. k_F	13.1°
14.	Table I. k_H	17.8°

15. $c_{m_o} = 0.090\ (100\ c_m = 9.0)$

16. $\alpha_o = -\ 6.5^\circ$

17. $x = \frac{s}{t_m} = \frac{0.53}{1.50} = 0.35$

Registration No.	Group School Glider III
Acceptance No.	Stability

18 $m=$	$\frac{k_F - \kappa}{k_H}$	$\frac{13.1 - 4.7}{17.8}$	$= \frac{8.4}{17.8} = 0.47$
19a $\sigma=1^\circ$ $n=$	$\frac{\alpha_o + \sigma}{k_H}$	$\frac{6.5 + 1}{17.8}$	$= \frac{7.5}{17.8} = 0.42$
19b $\sigma= -1^\circ$ (n)		$\frac{6.5 - 1}{17.8}$	$= \frac{5.5}{17.8} = 0.31$
19c $\sigma=3^\circ$ (n)		$\frac{6.5 + 3}{17.8}$	$= \frac{9.5}{17.8} = 0.53$
20 $A=$	$0.25+m\,\frac{f}{F}\,\frac{l}{t_m}$	$0.25+0.47\,\frac{2.4\times4.0}{18.0\times1.5}$	$= 0.25+0.47\,\frac{9.6}{27} = 0.42$
21 $B=$	$1 + m\,\frac{f}{F}$	$1 + 0.47\,\frac{2.4}{18}$	$= 1 + 0.063 = 1.063$
22a $\sigma=1^\circ$ $C=$	$\frac{f}{F}\,\frac{l}{t_m}\,n-c_{m_o}$	$\frac{2.4\times4.0}{18\times1.5}\,0.42-0.09$	$= 0.149-0.09 = 0.059$
22b $\sigma= -1^\circ$ (C)		$\frac{2.4\times4.0}{18\times1.5}\,0.31-0.09$	$= 0.110-0.09 = 0.020$
22c $\sigma=3^\circ$ (C)		$\frac{2.4\times4.0}{18\times1.5}\,0.53-0.09$	$= 0.188-0.09 = 0.098$
23a $\sigma=1^\circ$ $D=$	$\frac{f}{F}\,n$	$\frac{2.4}{18}\,0.42$	$= 0.056$
23b $\sigma= -1^\circ$ (D)		$\frac{2.4}{18}\,0.31$	$= 0.041$
23c $\sigma=3^\circ$ (D)		$\frac{2.4}{18}\,0.53$	$= 0.071$

Let us suppose that an airplane is to be tested, whose dimensions are shown in Figure 8. Lines 1-12 of the formulary are then filled out by inscribing the pertinent values. The c_{a_F}, α_F, and c_m values of the given wing profile are plotted against one another in Figure 9. These values can be obtained directly from the results of the normal tests, with the aspect ratio 5, at the Göttingen Aerodynamic Institute. Attention is again called to the fact that, for wings with variable angle of attack, variable profile, or variable chord, the corresponding mean values are to be found. The measuring points are joined by two straight lines, deviations from which, especially at small and large c_a values, are disregarded.

The values c_{m_0} and α_0 given in lines 15-16, are then obtained for $c_a = 0$. The values for k_F, k_H, and κ are taken from Figures 3-4. The values of k_F and k_H, for the aspect-ratio coefficients of the wing and elevator, can be read directly from Figure 8. The value of κ, however, is interpolated from Figure 4, whereby, if the aspect ratio in question is not represented by a curve, we must interpolate between the adjacent curves, remembering that the distance between the curves diminishes as the aspect ratio increases. For elliptical outlines the dotted curves are used.

For the aspect ratio of a rectangular wing $(\Lambda_F = 8)$ we thus obtain the value $\kappa = 4.7$. This number is to be found as the intersection of the curve

$$\frac{b_F^2}{F} = \Lambda_F = 8 \quad \text{with} \quad \epsilon = \frac{e}{b_F/2} = 0.54$$

After lines 12-14 have thus been filled out, we calculate line 17 and can now determine the factors of the stability equations, by inserting the corresponding values and calculations in the formulary. If we also desire to obtain information concerning the changed positions of the elevator and their effects, we must find, in addition to the value of n in the attitude of normal flight, two other values, corresponding respectively, to a raised and a depressed position of the elevator, which are characterized by a slight increase or decrease in the longitudinal dihedral angle σ ($\pm 2^\circ$ in our example). The values of A and B can be found directly by the determination of m. For C and D we then calculate three pairs of values, corresponding to the three elevator positions.

Lines 24-27 (a, b, c) of the formulary give the factors of the stability equations for different positions of the center of gravity. The portions enclosed by heavy lines distinguish, as also for n, C and D, the normal flight attitude for the neutral position of the elevator, which corresponds to the length determined by the attitude of the manned airplane and entered in line 11. The effect of the shifting of the center of gravity is shown by the other values, which are calculated for the different distances (x) to the center of gravity.

The values in these four lines serve for plotting the course of the stability equations in Figure 10, by entering the hundredfold values of $C - Dx$ on the c_m axis in the negative direction, because this value has a negative sign corresponding to equation (15). The pertinent hundredfold value of $\frac{C - Dx}{A - Bx}$ is sought on the positive c_a axis, and the corresponding points on the c_m and c_a axes are joined by a straight line. For each line of x (24-27) there is a separate diagram, since the lines corresponding to the different elevator positions must run parallel to one another.

In our example we see that the location of the center of gravity at 0.35 of the chord gives good stability and maneuverability. The latter is assured by the fact that the elevator deflection of $\pm 2^\circ$ covers the whole usual range of the lift coefficients.

The extreme position of the center of gravity is calculated from lines 28-29, whereby the smaller value of x_o applies, namely x_{o_1} in our case.

This formulary is therefore intended to furnish, in addition to a stability check for the builder, a practical aid in calculating and dimensioning gliders.

The Aviation Section of the Rhön-Rossitten Society, Wasserkuppe, will give special information at any time to any one who encounters difficulties in using the formulary.

Translation by Dwight M. Miner,
National Advisory Committee
for Aeronautics.

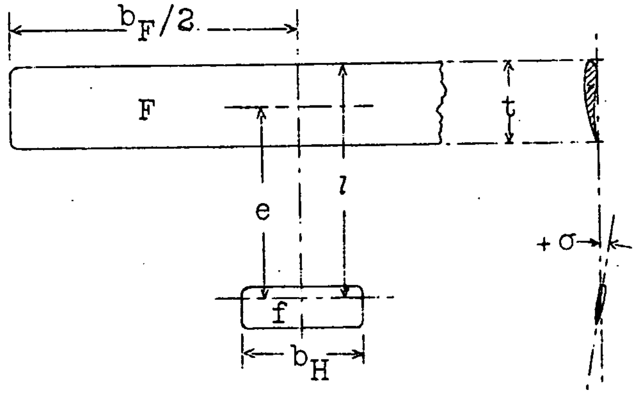


Fig.1.

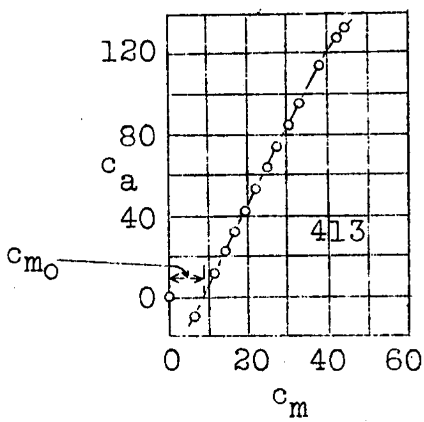
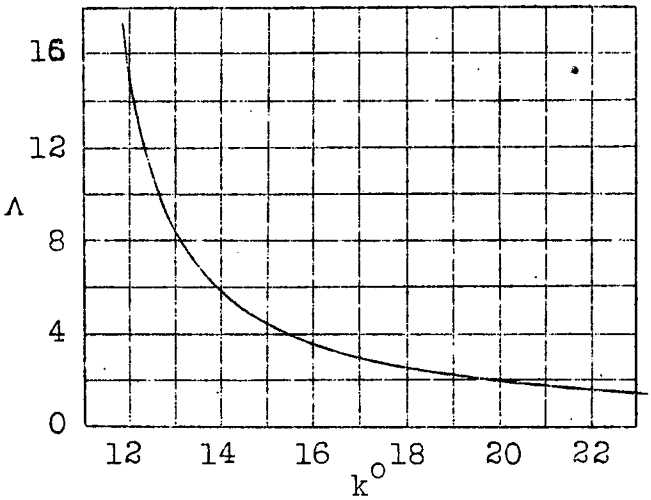


Fig.2.



$$k = \frac{\partial \alpha}{\partial c_a} \cong 10.8 + \frac{57.3}{\pi \Lambda}$$

Fig.3.

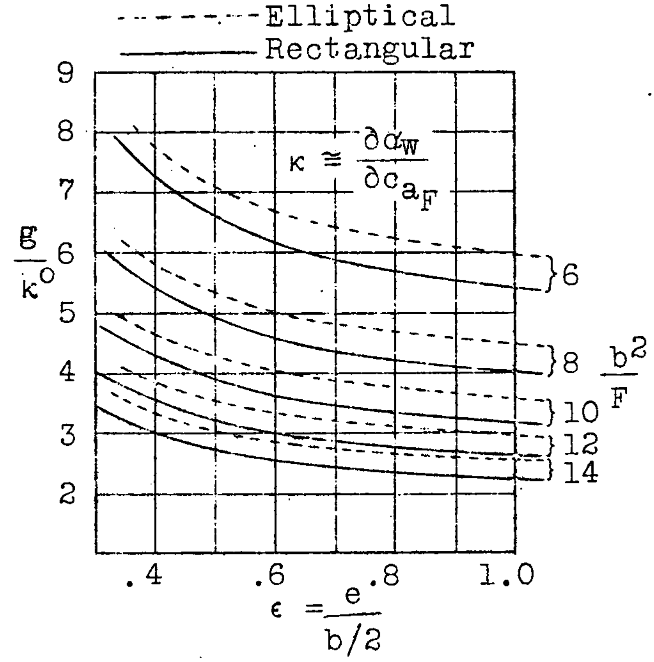


Fig.4.

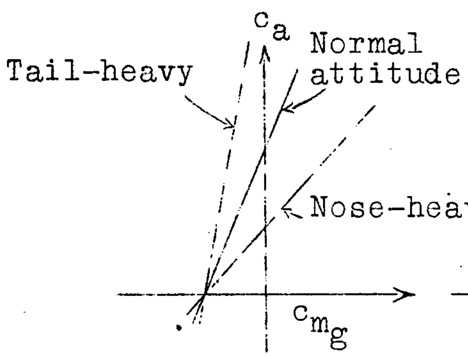


Fig.5.

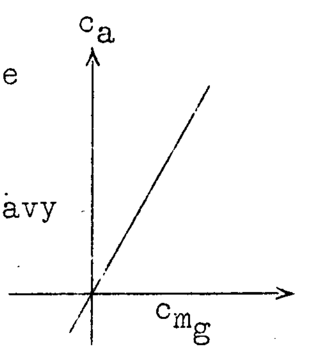


Fig.6.

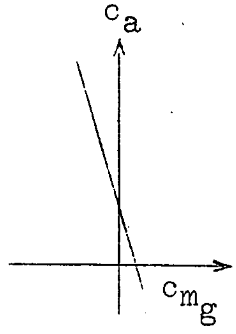


Fig.7.

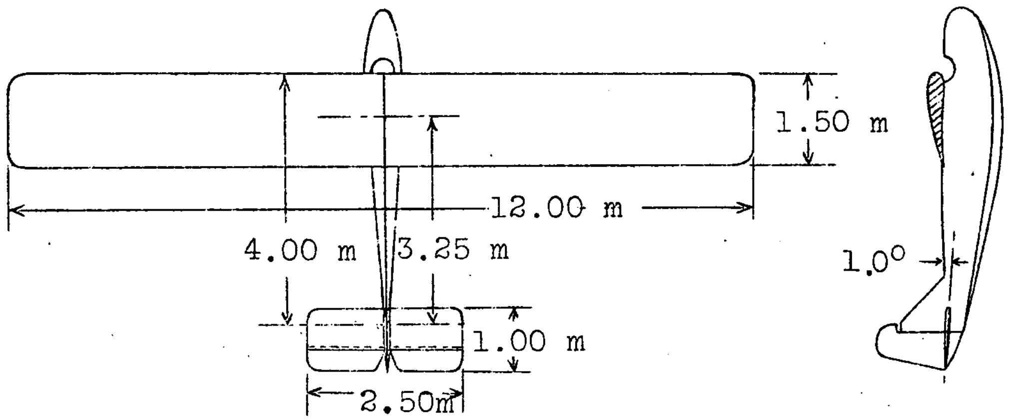


Fig.8.

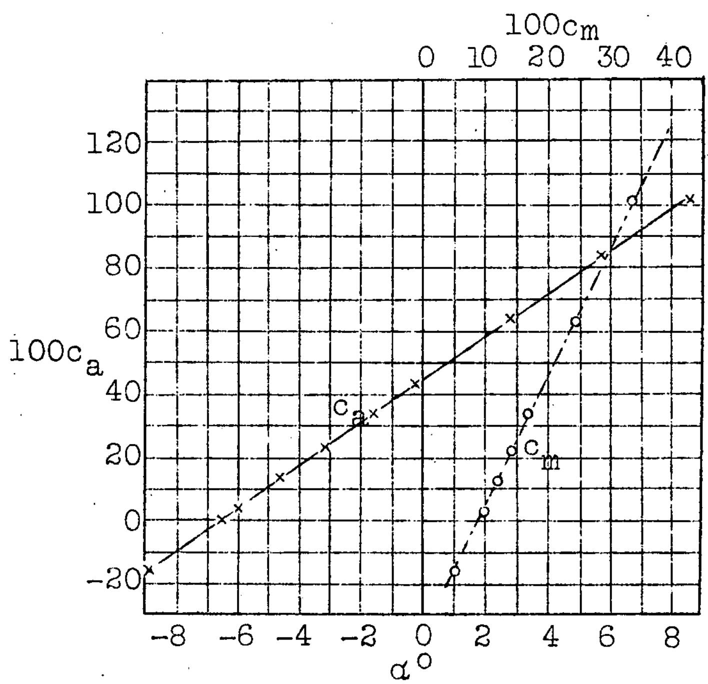
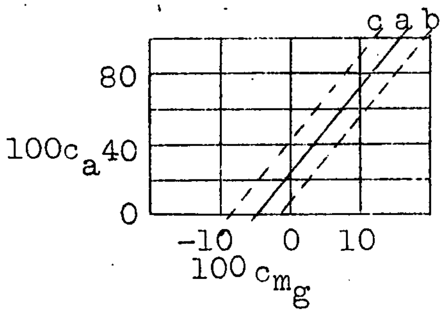
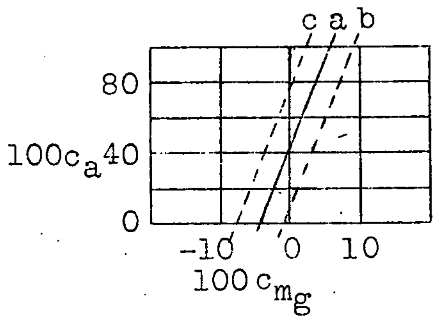


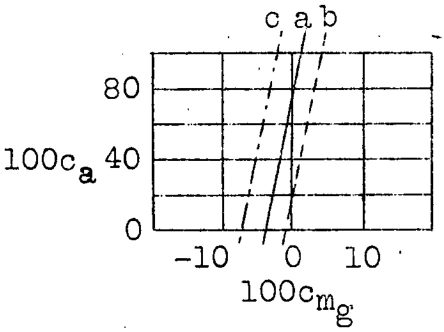
Fig.9 398 Göttingen airfoil.



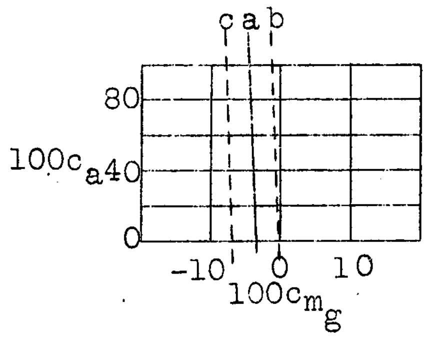
x = 0.20



x = 0.30



x = 0.35



x = 0.40

Fig.10.