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By A. Betz and E. Petersohn

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In an attempt to treat theoretically the effect of ailerons, difficulty arises because an aileron may begin at any point of the wing. Since the deflection of an aileron has the same effect on the wing as increasing or decreasing the angle of attack, a wing with aileron in action behaves like a wing with irregularly varying angle of attack. From the wing theory it is known, however, that the lift at such a point with irregularly varying angle of attack does not vary irregularly. Hence the question arises as to how the transition of the lift distribution proceeds at such a point, since the effect of the aileron (i.e., the moment generated about the longitudinal axis) depends largely on this distribution.

In order to answer this question regarding the lift distribution during irregular variations in the angle of attack at first independently of other influences, especially those of the wing tips, we have taken as the basis of the following theoretical discussion a wing of infinite span and constant chord which exhibits at one point an irregular variation in the angle of

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*"Zur Theorie der Querruder" from Zeitschrift für angewandte Mathematik und Mechanik, Volume VIII, 1928, pp. 253-257.

attack.* As regards the mathematical treatment, we will first consider a wing with periodically recurring irregular angle of attack (upper part of Fig. 1). Ultimately we can let the period extend to infinity and then obtain the desired result for an infinitely long wing with a single point of irregular variation in the angle of attack. The treatment of a periodically variable wing offers the advantage that the functions involved can be expressed in a Fourier series, which gives especially simple relations in the present case.

In order to express the lift distribution, we will seek the circulation Γ in terms of the distance x from the point of disturbance. Between the circulation Γ and the lift per unit length $\frac{dA}{dx}$, there is known to be the relation

$$\frac{dA}{dx} = \rho \; v \; \Gamma \tag{1}$$

in which ρ is the air density and v the flight speed. Accordingly the lift coefficient at the given point is

*The application of the results to wings of finite span is discussed by E. Petersohn, "Theoretische und experimentelle Untersuchungen der unter Einwirkung von Querrudern an Tragflugeln auftretenden Momente," Luftfahrtforschung, Vol. II, No. 2.

Another treatment, based on an elliptical wing, was accorded the aileron problem by Dr. Max M. Munk (N.A.C.A. Technical Report No. 191: "Elements of the Wing Section Theory and of the Wing Theory," 1924).

While the present article was in press, another article, "Theoretische Untersuchungen über die Querruderwirkung beim Tragflugel," by C. Wieselsberger, appeared on this subject (Report No. 30 of the Aeronautical Research Institute, Tokyo Imperial University). In this article the lift distribution over a wing is approximately represented by a finite series of only eight terms.

$$c_a = \frac{d\ A}{\left(\frac{\rho}{2}\right) v^2\ t\ dx} = \frac{2\Gamma}{vt} \qquad (2)$$

in which t represents the wing chord.

The lift coefficient of a wing section or profile in an undisturbed two-dimensional flow, can, with sufficient accuracy, be assumed to be a linear function of the angle of attack α.*

$$c_a = c\ (\alpha - \alpha_0) \qquad (3).$$

Thereby

$$c = \frac{d\ c_a}{d\,\alpha}$$

a characteristic constant of the wing section. For flat plates the theoretical value is $c = 2\pi$; for thicker wing sections it is somewhat greater. The actual values are somewhat smaller than the theoretical.

From equations (2) and (3) we obtain the relation between Γ and α

$$\Gamma = c\,\frac{vt}{2}\,(\alpha - \alpha_0) \qquad (5)$$

where α_0 is the angle of attack at which $c_a = 0$. The angle of attack of the wing may vary irregularly from α_1 to α_2 (upper part of Fig. 1). The circulations corresponding to these angles of attack in undisturbed flow (i.e., for an infinitely long wing with constant angle of attack) are then

$$\Gamma_1 = c\,\frac{vt}{2}\,(\alpha_1 - \alpha_0) \qquad (6)$$

and

$$\Gamma_2 = c\,\frac{vt}{2}\,(\alpha_2 - \alpha_0) \qquad (7).$$

*Naturally this does not hold true in the vicinity of the burble point or after the flow has separated from the wing.

For reasons of symmetry a mean circulation $\frac{\Gamma_1 + \Gamma_2}{2}$ will prevail at the point of irregularity. The circulation from there on will approach asymptotically the value Γ_1 on one side and Γ_2 on the other side. We can therefore write

$$\Gamma = \frac{\Gamma_1 + \Gamma_2}{2} + \frac{\Gamma_1 - \Gamma_2}{2} \epsilon \tag{8}$$

in which ϵ is a temporarily unknown function of x. Our task is to determine the function $\epsilon (x)$.

The process of calculation is as follows. We develop α in a Fourier series and put Γ likewise in the form of a Fourier series with temporarily unknown coefficients. From this distribution of Γ we can calculate, by the well-known wing theory method, the vertical induced velocities w on the wing, which alter the effective angle of attack by the amount

$$\Delta \alpha = - \frac{w}{V} \tag{9}$$

so that the effective angle of attack is

$$\alpha' = \alpha - \frac{w}{V} .$$

The circulation at every point x of the wing is calculated from this effective angle of attack according to equation (5). Since all functions are represented in the form of Fourier series, the circulation distribution thus calculated is in the form of a Fourier series. The still undetermined coefficients of this series can be found by comparing the calculated circulation distribution with that originally assumed.

The series for an irregularly varying angle of attack is

$$\alpha = \frac{\alpha_1+\alpha_2}{2} + \frac{\alpha_1-\alpha_2}{2}\,\frac{4}{\pi}\left(\sin\frac{2\pi x}{l} + \frac{1}{3}\sin 3\frac{2\pi x}{l} + \frac{1}{5}\sin 5\frac{2\pi x}{l} + \ldots\right) \tag{10}$$

(Cf. Hütte, 25th edition, Volume I, p.169.) For the distribution of Γ we write

$$\Gamma = \frac{\Gamma_1+\Gamma_2}{2} + \frac{\Gamma_1-\Gamma_2}{2}\left(\alpha_1 \sin\frac{2\pi x}{l} + \alpha_3 \sin 3\frac{2\pi x}{l} + \alpha_5 \sin 5\frac{2\pi x}{l} + \ldots\right) \tag{11}$$

From the distribution of Γ and according to the well-known calculation method of the wing theory the induced velocity w becomes

$$w = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{\partial\Gamma}{\partial x}\,\frac{1}{x - x_1}\,dx \tag{12}$$

at a point on the wing x_1 distant from the point of disturbance. The summation of Γ according to equation (11)* gives

$$w = \frac{\Gamma_1 - \Gamma_2}{2}\,\alpha(2n + 1)\quad \frac{\pi}{2l}\,(2n + 1)\,\sin\,(2n + 1)\frac{2\pi x}{l} \tag{13}$$

Since this calculation naturally holds good for any distance x_1 and not simply for a certain fixed distance, the subscript 1 may be omitted and equation (13) would then represent in general the relation between the induced velocity w and the distance x from the point of disturbance.

We may express the effective angle of attack $\alpha_1 = \alpha - w/v$ as a function of x and from it calculate the circulation Γ

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*L. Prandtl, "Tragflugeltheorie" Part I, Vier Abhandlungen zur Hydrodynamik und Aerodynamik, Göttingen, 1927, published by J. Springer, Berlin. Under No. 14 it is shown that a circulation distribution $\Gamma = \bar{\Gamma}\cos\mu x$ gives an induced velocity $w = \frac{\mu}{4}\bar{\Gamma}\cos\mu x = \frac{\mu}{4}\Gamma$.

$$\Gamma = c\,\frac{vt}{2}\left(\alpha - \alpha_0 - \frac{w}{v}\right) = c\,\frac{vt}{2}\left[\frac{\alpha_1 + \alpha_2}{2} - \alpha_0 + \right.$$

$$\left. + \frac{\alpha_1 - \alpha_2}{2}\,\frac{4}{\pi}\sum_0^\infty \frac{1}{2n+1}\sin(2n+1)\frac{2\pi x}{l}\right]$$

$$- c\,\frac{t}{2}\,\frac{\Gamma_1 - \Gamma_2}{2}\sum_0^\infty \alpha_{(2n+1)}\,\frac{\pi}{2l}(2n+1)\sin(2n+1)\frac{2\pi x}{l} \qquad (14)$$

If we consider that, according to equations (6) and (7),

$$c\,\frac{vt}{2}\left(\frac{\alpha_1 + \alpha_2}{2} - \alpha_0\right) = \frac{\Gamma_1 + \Gamma_2}{2} \quad \text{and} \quad c\,\frac{vt}{2}\,\frac{\alpha_1 - \alpha_2}{2} = \frac{\Gamma_1 - \Gamma_2}{2},$$

we obtain, by comparing the last equation with the original equation for Γ (equation 11), the following relation

$$\frac{\Gamma_1 - \Gamma_2}{2}\sum_0^\infty \alpha_{(2n+1)}\sin(2n+1)\frac{2\pi x}{l}$$

$$= \frac{\Gamma_1 - \Gamma_2}{2}\sum_0^\infty\left[\frac{4}{\pi(2n+1)} - \frac{ct}{2}\,\alpha_{(2n+1)}\,\frac{\pi}{2l}(2n+1)\right]\sin(2n+1)\frac{2\pi x}{l} \quad (15).$$

Since the coefficients of the corresponding terms of the two Fourier series must be the same, we obtain, for the coefficients $\alpha_{(2n+1)}$ of the original summation for Γ, the expression

$$\alpha_{(2n+1)} = \frac{4}{\pi(2n+1)} - \frac{ct\pi}{4l}\,\alpha_{(2n+1)}(2n+1),$$

$$\alpha_{(2n+1)} = \frac{4}{\pi(2n+1)}\,\frac{1}{1 + \frac{ct\pi}{4l}(2n+1)} \qquad (16)$$

For the desired function ϵ in equation (8) we therefore obtain

$$\epsilon = \sum_0^\infty \frac{4}{\pi(2n+1)\left(1 + \frac{ct\pi}{4l}(2n+1)\right)}\sin(2n+1)\frac{2\pi x}{l} \quad (17).$$

Thus we have solved the problem for periodical alternations in the angle of attack. In order to adapt the results to the case of a single point of disturbance, we must let the period l extend to infinity. For very large values of l and small values of x all the terms in the above series having small values of n approach zero as a limit. For large values of n, however, since n and $n + 1$ differ but little, we can replace Σ by an integral by introducing a uniformly varying quantity λ in place of the whole numbers n, so that $2n + 1 = 2\lambda$. The series (equation 17) then becomes

$$\epsilon = \frac{4}{\pi} \int_0^\infty \frac{\sin 2\lambda\mu}{2\lambda(1 + 2\nu\lambda)} \tag{18}$$

where, for brevity, we put

$$\mu = \frac{2\pi x}{l} \tag{19}$$

and

$$\nu = \frac{c\,t\,\pi}{4\,l} \tag{20}$$

This integral can be reduced to the well-known functions* sine integral

$$\mathrm{Si}\,\xi = \int_0^\xi \frac{\sin z}{z}\,dz \tag{21}$$

and cosine integral

$$\mathrm{Ci}\,\xi = \int_\xi^\infty \frac{\cos z}{z}\,dz \tag{22}$$

By partial fractional resolution the integral of equation (18) can be transformed into

*E. Jahnke and F. Emde, "Funktionentafeln mit Formeln und Kurven," Leipzig, B. G. Teubner, 1923.

$$\int_0^\infty \frac{\sin 2\lambda\mu}{2\lambda(1+2\nu\lambda)}\,d\lambda = \int_0^\infty \frac{\sin 2\lambda\mu}{2\lambda}\,d\lambda - \int_0^\infty \frac{\nu\sin 2\lambda\mu}{1+2\nu\lambda}\,d\lambda$$

By the introduction of $2\lambda\mu = z$ the first integral of the right-hand member of the above equation becomes

$$\frac{1}{2}\int_0^\infty \frac{\sin z}{z}\,dz = \pm\frac{\pi}{4},^{*}$$

and by the introduction of $\frac{\mu}{\nu}(1 + 2\nu\lambda) = y$ the second integral becomes

$$\frac{1}{2}\int_{\mu/\nu}^\infty \left(\cos\frac{\mu}{\nu}\sin y - \sin\frac{\mu}{\nu}\cos y\right)\frac{dy}{y} = \frac{1}{2}\cos\frac{\mu}{\nu}\left(\pm\frac{\pi}{2} - \mathrm{Si}\,\frac{\mu}{\nu}\right) + $$
$$+ \frac{1}{2}\sin\frac{\mu}{\nu}\,\mathrm{Ci}\,\frac{\mu}{\nu}\,.^{*}$$

By the further introduction of $\frac{\mu}{\nu} = \frac{8x}{ct}$ (equations 19 and 20) we obtain

$$\epsilon = \frac{2}{\pi}\left[\pm\frac{\pi}{2} \mp \frac{\pi}{2}\cos\frac{8x}{ct} + \cos\frac{8x}{ct}\,\mathrm{Si}\,\frac{8x}{ct} - \sin\frac{8x}{ct}\,\mathrm{Ci}\,\frac{8x}{ct}\right]$$

$$= \left(\pm 1 - \frac{2}{\pi}\sin\frac{8x}{ct}\,\mathrm{Ci}\,\frac{8x}{ct}\right) - \cos\frac{8x}{ct}\left(\pm 1 - \frac{2}{\pi}\,\mathrm{Si}\,\frac{8x}{ct}\right).$$

The behavior of the function ϵ for positive values of x is shown in Figure 2. Negative values of x give the same curve but with the opposite sign. For large values of x the function $\epsilon(x)$ can be represented by the semiconvergent series

$$\epsilon = \pm 1 - \frac{2}{\pi z}\left(1 - \frac{2!}{z} + \frac{4!}{z} - - \ldots\right)^{**}$$

into which $z = \frac{8x}{ct}$ has been introduced for brevity. For small values of x the function is represented by the expression

*The positive sign corresponds to positive μ and x; the negative sign to negative μ and x.

**The series can be used only so long as the terms decrease.

$$\epsilon = \frac{2}{\pi} (1 - C - \ln z) z$$

(for $z \ll 1$), in which C is the Euler constant $= 0.577$.

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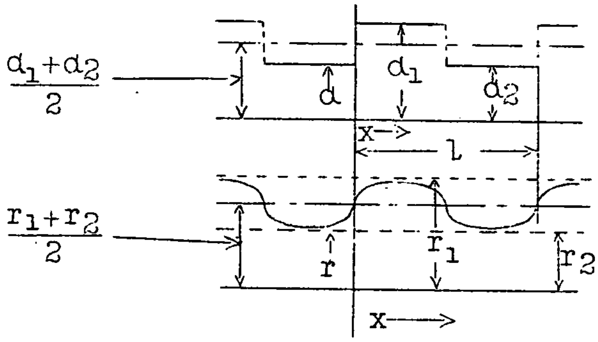
$\frac{d_1+d_2}{2}$
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Fig.1

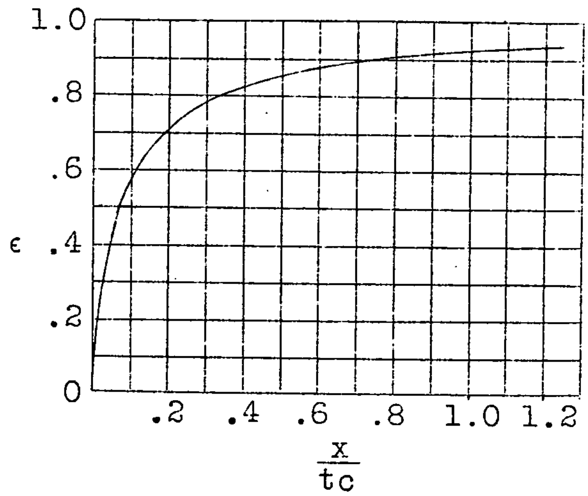
1.0
.8
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Fig.2