NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1431

ON THE STATISTICAL THEORY OF TURBULENCE

By W. Heisenberg

Translation of "Zur statistischen Theorie der Turbulenz." Zeitschrift für Physik, vol. 124, 1948.

NACA

Washington
January 1958

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1431

ON THE STATISTICAL THEORY OF TURBULENCE*

By W. Heisenberg

The interpretation of turbulence presented in the preceding paper by v. Weizsaecker is treated mathematically with the aid of the customary method of Fourier analysis. The spectrum of the turbulent motion is derived to the smallest wave lengths, that is, into the laminar region; the mean pressure fluctuations and the correlation functions are calculated. Finally, an attempt is made to derive the constant which is characteristic for the energy dissipation in the statistical turbulent motion from the hydrodynamic equations.

In the statistical theory of turbulence developed by G. I. Taylor[1] and v. Kármán,[2] the irregular turbulent motion of a fluid is described by several characteristic functions between which simple mathematical relations exist: the "spectral" distribution of energy to waves of different wave length, the correlations between the velocities at points along a prescribed displacement in space or time, and the like. The reports of G. I. Taylor contain detailed empirical and theoretical data on these functions.

In his preceding report, v. Weizsaecker[3] derived the most important of these functions, the spectral distribution of energy for the limiting

---

*"Zur statistischen Theorie der Turbulenz." Zeitschrift für Physik, vol. 124, 1948, pp. 628-657.

[1]Taylor, G. I.: Proc. Roy. Soc. A 151, 421 (1935); 156, 307 (1936); 164, 15 (1938); 164, 476 (1938).

[2]Kármán, Th. v.: Journ. Aero. Sci. 4, 131 (1937).

[3]C. F. v. Weizsaecker: ZS f. Phys., being published. V. Weizsaecker's paper and the present treatise have been written in close collaboration during the time of our stay in England in 1945. Only after the articles had been finished, Mr. G. I. Taylor kindly told us (spring 1946) that essential ideas in these articles had been found and published already by Kolmogoroff: Compt. Rend. Acad. Sc. USSR 30, 301 (1941); 32, 16 (1941); and Onsager (Phys. Revue 68, 286, 1945). Compare a report by G. K. Batchelor at the VI. Internat. Kongress F. angew. Mechanik, Paris 1946. Approximately at the same time we learned, furthermore, about a paper by Prandtl and Wieghardt which contains similar concepts and has meanwhile been published in the Göttingen Academy reports (Nachrichten der Akademie der Wissenschaften in Göttingen, Math.-Physikal Klasse of the year 1945, p. 6). The present paper may therefore be regarded only as a supplement and completion of these earlier investigations.

case of large Reynolds numbers on the basis of similitude considerations. The following sections of this paper first will translate v. Weizsaecker's considerations into the accustomed terms of Fourier analysis and, with the aid of this translation, study the discontinuity of the spectrum for high frequencies due to molecular viscosity. Then conclusions will be drawn for the correlation functions and the pressure fluctuations, and finally, a derivation of the fundamental constant of the energy dissipation will be attempted.

1. Representation of v. Weizsaecker's Considerations in the Terms of Fourier Components

At sufficiently high Reynolds numbers the energy dissipation for the turbulent motion takes place in such a manner that the large turbulence elements lose energy due to the fact that, for them, the energy and momentum transfer by small turbulence elements has the effect of an additional viscosity (cf. for instance Prandtl[4]).

Under steady-state conditions, energy thus is continuously transferred from larger to smaller turbulence elements, with the spectral region of a certain wave length always receiving from larger waves as much energy as it gives off to smaller waves. For maintenance of this equilibrium a certain energy distribution is necessary which, when the molecular friction is neglected, is represented according to v. Weizsaecker by the law

$$\rho \frac{\overline{v^2}}{2} = \rho \frac{\overline{v_0}^2}{2} = \rho \int F(k)dk \tag{1}$$

$$F(k) \sim k^{-5/3}$$

$\left(k = \frac{2\pi}{\text{wave length}}\right.$ signifies the wave number, $v_0 = \sqrt{\overline{v^2}}$ or measure of the mean velocity.[5]$\left.\right)$ This spectrum $F(k) \sim k^{-5/3}$ is bounded on two sides. For small wave numbers, that is, large wave lengths, for one reason or another, it will no longer be possible to regard the flow as isotropically turbulent, for the largest turbulence elements are governed by the geometry of the devices which generate the turbulence. This end of the spectrum for small k therefore cannot in any case be the subject of a purely statistical theory. For large k, in contrast, the spectrum is bounded

[4] L. Prandtl, Strömungslehre (Flow theory). Vieweg, Braunschweig, 3rd edition 1942, p. 105 ff.

[5] Our v_0 differs from the v_0 in v. Weizsaecker's report by a numerical factor of the order of magnitude of 1.

by the molecular viscosity. For large k, finally, the molecular viscosity will become larger than the apparent turbulent viscosity, and the spectrum then will drop off very rapidly.

For the calculations, we shall use the following notation: the velocity $\underline{v}$ in a normalized volume V is to be expanded into a Fourier series

$$\underline{v} = \sum_{\underline{k}} \underline{v}_{\underline{k}} e^{i\underline{k}\underline{r}} \left(\underline{k}_x = \frac{2\pi}{L_x} n_x, \ldots ; n_x, \quad n_y, \quad n_z \quad \text{integers} \right) \tag{2}$$

Therein $\underline{v}_{\underline{k}} = \underline{v}^*_{-\underline{k}}$ and the number of the "natural vibrations" between k and $k + \Delta k$ is given[6] by $\frac{4\pi k^2 \Delta k V}{(2\pi)^3}$. Then one obtains

$$\frac{\overline{v^2}}{2} = \frac{1}{2} \sum_{\underline{k}} \left| \underline{v}_{\underline{k}}^2 \right| = \frac{1}{2} \int 4\pi k^2 dk \, \frac{V}{(2\pi)^3} \left| \underline{v}_{\underline{k}}^2 \right| = \int F(k) dk \tag{3}$$

thus

$$F(k) = (2\pi)^{-2} k^2 V \left| \underline{v}_{\underline{k}}^2 \right| \tag{4}$$

From div $\underline{v} = 0$ there follows

$$\left(\underline{v}_{\underline{k}} \underline{k} \right) = 0 \tag{5}$$

Let us call the coefficient of the molecular viscosity, μ. The mean energy loss then is, because of div $\underline{v} = 0$ with the assumption that the bounding surfaces are at rest:

---

[6]This method is somewhat less apparent, but mathematically more convenient than the customary expansion with respect to sin and cos. It amounts formally to the limiting condition that $\underline{v}, \partial\underline{v}/\partial x, \ldots$ at a bounding surface of the volume V are to have the same values as at the opposite bounding surface.

$$S = \mu\overline{(\mathrm{rot}\underline{v})^2} \tag{6}$$

$$= \mu \sum_{\underline{k}} \left|\left[\underline{v_k}\underline{k}\right]^2\right| = \mu \int F(k)2k^2 dk \tag{7}$$

If the spectrum obeys in a large region the law $F(k) \sim k^{-5/3}$, the total energy is determined by the largest turbulence elements. We may assume for instance that the law $k^{-5/3}$ is valid down to a minimum wave number k_0; for smaller k we shall assume $F_{(k)} = 0$. Then

$$v_0^2 = 2\int_0^\infty F(k)dk = 2\int_{k_0}^\infty \frac{C}{k^{5/3}} dk = 3Ck_0^{-2/3} \tag{8}$$

Thus

$$C = \frac{v_0^2}{3} k_0^{2/3}$$

and

$$F(k) = \frac{v_0^2}{3} \frac{k_0^{2/3}}{k^{5/3}} \quad \text{for} \quad k \geq k_0 \tag{9}$$

$$\overline{\left|v_k^2\right|} = \frac{v_0^2 k_0^{2/3}(2\pi)^3}{6\pi V k^{11/3}} \tag{10}$$

V. Weizsaecker considers the energy loss S_k of that portion of the total spectrum the wave numbers of which lie below k. For these turbulence elements, the turbulence elements of smaller wave length ($<2\pi/k$) have the effect of an additional viscosity. One may therefore generally write

$$S_k = \left(\mu + \eta_k\right) \int_0^k F(k')\ 2k'^2 dk' \tag{11}$$

where η_k is to designate the additional turbulent viscosity; it is produced by the cumulative action of all turbulence elements with wave lengths $<2\pi/k$. With respect to dimensions, η_k is according to Prandtl the product of density, mixing length, and velocity where the mixing length will be comparable to the diameter of the turbulence elements in question whereas the velocity of the turbulence elements is given for instance by $v_0(k_0/k)^{1/3}$. With reference to v. Weizsaecker's report one will therefore put

$$\eta_k = \kappa\rho \int_k^\infty dk' \sqrt{\frac{F(k')}{k'^3}} \tag{12}$$

(κ is a numerical factor)
The expression below the integration sign is essentially determined by dimensional considerations; but one could of course imagine that, for instance, the waves k' in the proximity of k enter into the integral with somewhat different weights than the waves with large k values; that is, the integrand could depend, in addition, on the dimensionless number k'/k. However, because of the homogeneous form of the spectrum $F(k) \sim k^{-5/3}$, one may include all these uncertainties in the numerical factor κ and give to the integral arbitrarily the exact form (12).

This method is unobjectionable in the region of the $k^{-5/3}$-law but becomes inaccurate at the ends of the region where the geometry or the molecular friction modifies the spectrum. But even at the latter limit (12) will still be a good approximation which at least qualitatively correctly represents the effect of friction.

The constant κ in equation (12) must be exactly determined by the hydrodynamic equations; it has the same numerical value in all cases where one may speak of statistical isotropic turbulence, and does not depend in any way on the geometry of the flow. The theoretical determination of this important number will be attempted in section 5. There, it will also be shown that the turbulent energy dissipation actually may be written as a double integral of the type

$\int_0^k dk' \ldots \int_k^\infty dk'' \ldots$, with the integrand signifying the energy transferred from k' into k'' per unit time (equation (89)). This integral is more complicated than the simplifying expression (13) which results from (11) and (12); however, for the following considerations (11) and (12) may be regarded as sufficient approximations. For S_k one thus obtains

$$S_k = \left(\mu + \rho\kappa \int_k^\infty dk'' \sqrt{\frac{F(k'')}{k''^3}}\right) \int_0^k F(k')\ 2k'^2 dk' \tag{13}$$

The decisive step of v. Weizsaecker's consideration is the statement that this expression for $k \gg k_0$ must be independent of k:

$$S_k = S = \text{const (for } k \gg k_0) \tag{14}$$

because the total energy lies, almost entirely, in the long-wave region of the spectrum, and the energy "transport" thus must become independent of k.

Equation (13) may therefore be interpreted as the determining equation of the turbulent spectrum $F(k)$ which must yield for the region of large Reynolds numbers the $k^{-5/3}$-law, and for still larger k values the fading of the spectrum due to the molecular viscosity.

2. Shape of the Spectrum in the Region of the Smallest Turbulence Elements

We put first $\mu/\rho = \nu$ and differentiate (13) with respect to k. Then there results

$$\left(\frac{\nu}{\kappa} + \int_k^\infty dk' \sqrt{\frac{F(k')}{k'^3}}\right) F(k)\ k^2 = \sqrt{\frac{F(k)}{k^3}} \int_0^k F(k')k'^2 dk' \tag{15}$$

Then we define new variables x and w by the equations

$$x = \lg \frac{k}{k_0}, \quad F(k) = F(k_0)e^{-w}; \quad w = w(x) \tag{16}$$

Therewith (15) is transformed into

$$e^{\frac{7}{2}x - \frac{w}{2}} \left(\frac{\nu}{\kappa}\sqrt{\frac{k_0}{F_0}} + \int_x^\infty e^{-\frac{w}{2} - \frac{x}{2}} dx\right) = \int_0^x e^{3x - w} dx \tag{17}$$

The constant $\frac{\nu}{\kappa}\sqrt{\frac{k_0}{F_0}} = \frac{\nu k_0 \sqrt{3}}{\kappa v_0}$ (cf. equation (9)) is essentially the reciprocal Reynolds number of the total flow and therefore always very small; if the Reynolds number itself were small, the flow could not be turbulent at all. By repeated differentiation there originates from (17)

$$\left(\frac{7}{2} - \frac{1}{2}\frac{dw}{dx}\right)\left(\frac{\nu}{\kappa}\sqrt{\frac{k_0}{F_0}} + \int_x^\infty e^{-\frac{w}{2} - \frac{x}{2}} dx\right) = 2e^{-\frac{w}{2} - \frac{x}{2}} \tag{18}$$

In this equation, one may approximately evaluate the integral $\int_x^\infty dxe^{-\frac{w+x}{2}}$ by expanding $w(x)$ in the proximity of x: $w(x_1) = w(x) + (x_1 - x)\frac{dw}{dx} + \ldots$ and breaking off after the second term. Since the exponential function rapidly decays, one thus obtains a good approximation

$$\int_x^\infty dxe^{-\frac{w+x}{2}} \approx 2\frac{e^{-\frac{w+x}{2}}}{1 + \frac{dw}{dx}} \tag{19}$$

By substitution into (18) there finally results

$$\left(7 - \frac{dw}{dx}\right)\left(\frac{\nu}{2\kappa}\sqrt{\frac{k_0}{F_0}}\, e^{\frac{w+x}{2}} + \frac{1}{1 + \frac{dw}{dx}}\right) = 2 \tag{20}$$

One recognizes from (20) immediately the variation of the spectrum. For not too large x and w the first term in the sum may be neglected and one obtains

$$\left(7 - \frac{dw}{dx}\right) = 2\left(1 + \frac{dw}{dx}\right), \qquad \text{i.e. } \frac{dw}{dx} = \frac{5}{3};\ F(k) = F_0\left(\frac{k}{k_0}\right)^{-5/3} \tag{21}$$

as must be the case according to v. Weizsaecker's theory. For large x and w, in contrast, the first term predominates; therefore one must then have

$$\frac{dw}{dx} = 7 \qquad F \sim \text{const } k^{-7} \tag{22}$$

In the region of the smallest turbulence elements the spectrum therefore decays very rapidly, namely with the seventh power of the wave number.

Only in the transitional region from (21) to (22) are numerical calculations necessary in order to determine the solution of (20). Since one may put for smaller x, that is, for the region

$$\left.\begin{aligned} 1 \ll x \ll \frac{3}{4}\,\lg\frac{2\kappa}{\nu}\sqrt{\frac{F_0}{k_0}} \\ w = \frac{5}{3}\,x \end{aligned}\right\} \tag{23}$$

(therewith not only (18), but also (17) then is satisfied with sufficient accuracy), one may calculate, progressing from point to point, dw/dx according to (20) from w, and therewith derive w for higher x. It is sufficient to perform the numerical calculation for a particular large value of the constant, for instance $\frac{2\kappa}{\nu}\sqrt{\frac{F_0}{k_0}} = a$. For another value b one may then obtain w by a simple similitude transformation:

$$w_b(x) = w_a\left(x + \frac{3}{4}\lg\frac{a}{b}\right) - \frac{5}{4}\lg\frac{a}{b} \tag{24}$$

as one recognizes by substitution into (20) and (23).

Figure 1 shows the result of the numerical calculation for $\frac{2\kappa}{\nu}\sqrt{\frac{F_0}{k_0}}$ = about 1000. The numerical integration shows that for this value in the region

$$\left.\begin{aligned} x &> 5 \\ w(x) &\approx 7x - 21.85 \end{aligned}\right\} \tag{25}$$

More generally, one obtains therefore in this region of the k^{-7} - law:

$$w(x) = 7x + 3.0 - 4\lg\left(\frac{\kappa}{\nu}\sqrt{\frac{F_0}{k_0}}\right) \tag{26}$$

that is,

$$F(k) = 0.0496\,\frac{F_0^{\,3}}{k_0^{\,2}}\left(\frac{\kappa}{\nu}\right)^4\left(\frac{k_0}{k}\right)^7 \tag{27}$$

A serviceable interpolation formula which is correct in the two limiting cases and does not result in any large errors in the transitional region, either, reads:

$$F(k) = F_0\left(\frac{k_0}{k}\right)^{5/3}\left[1 + \left(\frac{k}{k_s}\right)^{8/3}\right]^{-2} \tag{28}$$

If one defines $L_0 = \pi/k_0$ as the "diameter of the largest turbulence elements" and introduces as the Reynolds number of the total flow

$$R_0 = \frac{\rho v_0 L_0}{\mu} \tag{29}$$

one obtains according to (9), (27), and (29)

$$k_s = 0.16k_0\left(R_{0\kappa}\right)^{3/4} \tag{30}$$

One may denote $L_s = \pi/k_s$ as the "diameter of the smallest turbulence elements" and obtains

$$L_s = 6.25L_0\left(R_{0\kappa}\right)^{-3/4} \tag{31}$$

By (9), (28), and (30) the form of the spectrum is determined for the entire k-region. For the actual flows, of course, the shape of the spectrum will be different for small k-values $(k \sim k_0)$ since there the geometry of the tests plays a role, for instance, the shape of the grids by means of which the turbulence is produced. In order to be able to sensibly carry out the comparison with experiments, one will then introduce a quantity k_0 in such a manner that, for instance in the domain of the $k^{-5/3}$-law (thus for $k_s \gg k \gg k_0$), the formula

$$F(k) = \frac{{v_0}^2}{3k_0}\left(\frac{k_0}{k}\right)^{5/3} \tag{32}$$

becomes correct. The quantity k_0 thus defined then does not give any direct statement regarding the variation of the spectrum for the smallest k-values. Generally, though, the spectrum will greatly deviate from the $k^{-5/3}$-law in the region $k \sim k_0$.

For $k \gg k_s$, too, the spectrum will not unlimitedly retain the form k^{-7}. The well-known investigations of Burgers[7] make it very probable that at sufficiently small Reynolds numbers finally no turbulent motions whatever exist. On the other hand, the k^{-7}-law shows such rapid decay that the region $k \gg k_s$ is practically insignificant. A somewhat larger error will arise, particularly in the transitional region, due to the inaccuracy of equation (13) itself; but it is probably not worth while to apply already at this point the much more complicated equation of section 5 to the problem stated here. The correct equations would at any rate lead to somewhat different numerical factors in (27), (30), and (31).

---

[7]Burgers, J. M.: Verh. d. Kgl. Nied. Akad. d Wiss. 17, Nr. 2, 1 (1939); 18, Nr. 1, 1 (1940).

For the comparison with experiment one needs the energy distribution with wave number for a certain direction, for instance with k_x, since the spectra have been measured experimentally by Simmons[8] and Dryden[9] by means of the fluctuations of the velocity with time in an airstream which is guided past the measuring point at a constant velocity U which is large relative to v_0. Also, this spectrum has a different form according to whether the Fourier expansion of $\underline{v}_x$ or $\underline{v}_y$ is concerned. Experimentally, first the spectrum for $\underline{v}_x$ is required; however, we shall also derive the spectrum for $\underline{v}_y$ since it will be necessary later on, in the calculation of the correlation functions. Since according to equation (5) $(\underline{v}_k\underline{k}) = 0$, one obtains

$$\overline{v_{kx}^2} = \frac{v_k^2}{2}\left(1 - \frac{k_x^2}{k^2}\right) \tag{33}$$

The spectrum of $\underline{v}_x$ in k_x which we shall designate by $F_x(k_x)$ becomes therefore

$$\begin{aligned} F_x\left(k_x\right) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{dk_y dk_z}{4\pi k^2}\,\frac{1}{2}\left(1 - \frac{k_x^2}{k^2}\right)F(k) \\ &= \frac{1}{4}\int_{k_x}^{\infty} \frac{dk}{k^3}\left(k^2 - k_x^2\right)F(k) \end{aligned} \tag{34}$$

In a similar manner, the spectrum for $\underline{v}_y$ becomes

$$\begin{aligned} F_y\left(k_y\right) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{dk_y dk_z}{4\pi k^2}\,\frac{1}{2}\left(1 - \frac{k_y^2}{k^2}\right)F(k) \\ &= \frac{1}{8}\int_{k_x}^{\infty} \frac{dk}{k^3}\left(k^2 + k_x^2\right)F(k) \end{aligned} \tag{35}$$

From (34) and (35) there follows for

(a) $k_0 \ll k \ll k_s$:

$$F(k) = F_0\left(\frac{k_0}{k}\right)^{5/3} \quad \text{thus} \quad F_x(k_x) = \frac{9F_0}{110}\left(\frac{k_0}{k_x}\right)^{5/3} \quad \text{and} \quad F_y(k_x) = \frac{6F_0}{55}\left(\frac{k_0}{k_x}\right)^{5/3} \tag{36}$$

---

[8] Simmons, U. Salter: Proc. Roy. Soc. A 165, 73, (1938).

[9] Dryden, Schubauer, Mock u. Skramstad: Nation. Adv. Comm. Aero., Nr. 581, 1938; Dryden, H. L.: Proc. V. Intern. Congr. f. Applied. Mech. Cambridge, Mass., p. 362, 1938.

(b) $k \gg k_s$:

$$F(k) = F_0 \frac{k_0^{5/3} k_s^{16/3}}{k^7}$$

$$F_x(k_x) = \frac{F_0}{126} k_0^{5/3} k_s^{16/3} k_x^{-7}$$

$$F_y(k_x) = \frac{2F_0}{63} k_0^{5/3} k_s^{16/3} k_x^{-7} \tag{37}$$

As a serviceable interpolation formula (which, however, is somewhat less accurate than (28) in the transitional region), one may again put

$$F_x(k_x) = \frac{9F_0}{110}\left(\frac{k_0}{k_x}\right)^{5/3}\left[1 + \left(\frac{k_x}{k_s^{xx}}\right)^{8/3}\right]^{-2} \tag{38}$$

with

$$k_s^{xx} = 0.645 \ k_s \tag{39}$$

and

$$F_y(k_x) = \frac{6F_0}{55}\left(\frac{k_0}{k_x}\right)^{5/3}\left[1 + \left(\frac{k_x}{k_s^{yx}}\right)^{8/3}\right]^{-2} \tag{40}$$

with

$$k_s^{yx} = 0.793 \ k_s \tag{41}$$

Before the comparison with experiment is carried out in detail, we want to raise the question at what critical Reynolds numbers the transition from the $k^{-5/3}$-law to the k^{-7}-law takes place, that is, - if one wants to express it in this manner - the transition from the really turbulent motion proper to the laminar motion. One may regard as the critical Reynolds number for this, perhaps, the expression

$$R_s = \frac{\rho v_s L_s}{\mu} \tag{42}$$

wherein, according to v. Weizsaecker, $v_s = v_0\left(\frac{k_0}{k_s}\right)^{1/3}$. From (29), (30), and (31) one then obtains

$$R_s = \frac{10.2}{\kappa} \tag{43}$$

The numerical value of κ will be discussed later on. At any rate, the transition therefore takes place at a certain numerical value of the Reynolds number, as was to be expected from general similitude considerations.

In figure 2, the measurements of the spectrum $F_x(k_x)$ by Simmons[10] are compared with the theory. The measurements in question are intensity measurements on an airstream which flows past the measuring point at the velocities $U = 456$ cm/sec (O), 608 cm/sec (X), 1060 cm/sec (□), and which has been made turbulent by a grid of 7.6 cm mesh width; the measurement was made 2.1 m behind the grid. The measured points of Simmons are plotted individually only in the right part of the figure, in the left half the approximate scatter of the measured points is indicated by a vertical line. The abscissa is k in cm^{-1} (in logarithmical scale), the ordinate $F_x(k_x)$, likewise logarithmically, in arbitrary units. If one assumes U/v_0 to have the same value in all three measuring series, which is confirmed by other measurements by Taylor, one obtains, in the case of a suitable selection of this ratio, the three curves plotted in the figure. Qualitatively, the Simmons data are well represented by the curves, particularly also the divergence of the three test series in the short-wave part of the spectrum. In details, however, there exist considerable discrepancies; one recognizes from the figure that the range of validity of the $k^{-5/3}$-law is here so small that a reliable check is not possible. The reason for this is the smallness of the Reynolds number R_0. For $k = 1\ cm^{-1}$ the diameter of the turbulence elements is 3 cm, thus about half as large as the mesh width of the grid; in this region, the turbulence is not yet fully isotropic, therefore the $k^{-5/3}$ -law cannot yet be valid. However, already for $k = 4\ cm^{-1}$ the influence of the molecular viscosity becomes noticeable, and the intensity drops off markedly. The related measurements by Dryden quoted before which extend over a large spectral region have been made at Reynolds numbers so small that the validity of the $k^{-5/3}$-law can hardly be checked. Therefore it would be desirable that similar measurements be carried out at very much larger Reynolds numbers. For the ratio U/v_0, one obtains from the adjustment of the theoretical curves to the measuring points of $U/v_0 = 53\kappa$ if one identifies L_0 with the mesh width of the grid. This value agrees well with measurements[11] of this ratio in similar tests if one assumes κ to be about 0.5.

---

[10] Simmons, U. Salter: Proc. Roy. Soc. A 165, 73 (1938).
[11] Cf. G. I. Taylor, Proc. Royal Soc. A, 164, 486, (1938).

Another and probably more accurate determination of κ is obtained from the damping with time of the turbulence, already theoretically treated completely by Taylor (cited before). For the total energy loss per cm^3 and second: S, there results from (9), (13), and (14)

$$S = \rho\kappa \frac{\sqrt{3}}{8} v_0{}^3 k_0 \tag{44}$$

For the damping with time of v_0 there must thus apply

$$\frac{d}{dt}\left(\frac{v_0{}^2}{2}\right) = -\kappa \frac{\sqrt{3}}{8} v_0{}^3 k_0 \tag{45}$$

with the solution[12]

$$v_0(t) = \frac{v_0(0)}{1 + \frac{\sqrt{3}}{8}\kappa\ k_0 v_0(0)t} \tag{46}$$

Taylor who essentially derived this equation reports on measurements by Simmons in which $U/v_0(t)$ was ascertained as a function of $t = x/U$ (x equals distance of the measuring point from the grid). From (46) follows

$$\frac{U}{v_0(t)} = \frac{U}{v_0(0)} + \frac{\sqrt{3}}{8}\kappa\pi \frac{x}{L_0} = \frac{U}{v_0(0)} + 0.68\kappa \frac{x}{L_0} \tag{47}$$

If one puts $v_0 = u'\sqrt{3}$ $\left(u' = \sqrt{\overline{v_x{}^2}}\right.$ according to Taylor) and identifies L_0 with the mesh width, there follows from Taylor's measurements $\kappa = 0.85$, from the corresponding measurements by Dryden a somewhat smaller value. However, because of the uncertainty regarding the value to be inserted for L_0, this determination is probably still uncertain by about 50 percent.

3. The Correlation Functions

Taylor and von Kármán (cited before) studied the correlations which exist between the velocities at two points at a given distance. The two

[12]Footnote at the time of proof correction: For this solution, k_0 = const. is presupposed which certainly is not the case for larger times. The problem of damping is investigated more closely in a paper of the author about to be published (Proc. Roy. Soc. A.).

correlation functions $R_1(x)$ and $R_2(x)$ which therein play the main role, are defined as

$$R_1(x) = \frac{\overline{v_x(P_1)v_x(P_2)}}{\overline{v_x^2}} \qquad R_2(x) = \frac{\overline{v_y(P_1)v_y(P_2)}}{\overline{v_y^2}} \tag{48}$$

with the point P_2 displaced with respect to the point P_1 by the distance x in the x-direction.

These functions are, according to Taylor, in a simple relationship with the spectra:

$$\left.\begin{aligned} R_1(x) &= \frac{\int_0^\infty dk_x F_x(k_x)\cos k_x x}{\int_0^\infty dk_x F_x(k_x)} \\ R_2(x) &= \frac{\int_0^\infty dk_x F_y(k_x)\cos k_x x}{\int_0^\infty dk_x F_y(k_x)} \end{aligned}\right\} \tag{49}$$

With the aid of equations (34) and (35), (49) is transformed into

$$\left.\begin{aligned} R_1(x) &= \frac{3\int_0^\infty dkF(k)(\sin kx - kx\cos kx)k^{-3}x^{-3}}{\int_0^\infty dkF(k)} \\ R_2(x) &= \frac{\frac{3}{2}\int_0^\infty dkF(k)(k^2x^2\sin k x + kx\cos kx - \sin kx)k^{-3}x^{-3}}{\int_0^\infty dkF(k)} \end{aligned}\right\} \tag{50}$$

From these expressions one recognizes immediately the correctness of von Kármán's relationship

$$R_2 = R_1 + \frac{x}{2}\frac{dR_1}{dx} \tag{51}$$

The formulas (50) may be approximately evaluated in the two limiting cases $x \ll 1/k_s$ and $1/k_s \ll x \ll 1/k_0$. When $x \ll \frac{1}{k_s}$, it is advisable to expand the integrands with respect to powers of x. The first terms of the expansion then lead to the quantity

$$\overline{k^2} = \frac{\int_0^\infty dkF(k)k^2}{\int_0^\infty dkF(k)} \tag{52}$$

which may easily be calculated from (13), (44), and (29):

$$\overline{k^2} = \frac{\sqrt{3}}{8\pi} \kappa R_0 k_0^2 \tag{53}$$

Thus one obtains:

$$R_1(x) = 1 - \frac{x^2\overline{k^2}}{10} + \ldots$$

For $x \ll 1/k_s$ (54)

$$R_2(x) = 1 - \frac{x^2\overline{k^2}}{5} + \ldots$$

Taylor defined a length λ by the equation

$$\lambda^2 = \frac{5}{\overline{k^2}} \tag{55}$$

and designated it as a measure for the magnitude of the smallest turbulence elements. According to (53), λ becomes

$$\lambda = 2.71 \frac{L_0}{\sqrt{R_0\kappa}} = 0.434 L_s (R_0\kappa)^{\frac{1}{4}} \tag{56}$$

It must be emphasized that λ is not identical with the quantity L_s (equation (31)) which we have denoted as "diameter of the smallest turbulence elements" and that λ also depends on L_0 and v_0 in a manner different from that of L_s. A comparison of (31) and (56) shows that for sufficiently large Reynolds numbers L_s becomes $\ll\lambda$.

In the opposite limiting case $1/k_s \ll x \ll 1/k_0$ one obtains from (50)

$$R_1(x) = 1 - 0.643(k_0x)^{2/3} + \ldots$$
$$R_2(x) = 1 - 0.858(k_0x)^{2/3} + \ldots \qquad (57)$$

Here these first terms of the expansion do not depend on the special form of the spectrum in the proximity of k_0; only for $x \sim 1/k_0$ does the form of the spectrum in the proximity of k_0 becomes important; there, however, the problem may no longer be treated with purely statistical methods. The formulas (54) and (57) thus give essentially a complete description of the correlations in so far as they may be regarded as a consequence of statistical isotropic turbulence.

The formulas (54) and (57) also show clearly that the correlation function does not have the same form in all flows but that rather, in the case of variations of the parameters, the inner and outer parts of the function undergo different similitude transformations. This point has been stressed particularly by Taylor[13] in contrast to a different conjecture of von Kármán (cited before).

For comparison with experiment, the measurements of $R_1(x)$ and $R_2(x)$ made by Simmons have been plotted (circles and dots, respectively) in figure 3; furthermore, the theoretical curves calculated according to the exact formula (50) are shown. Here again L_0 has been identified with the mesh width of the grid and λ has been calculated from the spectrum for $U = 1060$ cm/sec. The experimental values agree, in fact, with the theoretical ones very exactly at the smaller values of x, actually more exactly than could have been expected in view of the uncertainty of L_0. Beginning from $xk_0 \sim 1$ the deviation of the experimental points from the theoretical curves becomes noticeable which was to be expected from the derivation. The variation for larger x-values depends on the behavior of the spectrum in the proximity of k_0 which cannot, in principle, be represented by our formulas. But even for larger x-values the deviations from the theoretical curves remain small.

---

[13] G. I. Taylor, Jour. Aero. Sci. 4, No. 8, 311, 1937.

4. The Pressure Fluctuations

While studying the diffusion in a turbulent airstream, Taylor (cited before) has derived a relationship between the correlation function for the diffusion and the root mean square value of the pressure gradient. We shall investigate therefore also the root mean square values of the pressure fluctuations from the viewpoint of the theory here described.

With reference to equation (2) one may expand the pressure into a Fourier series

$$p = \sum_{\underline{k}} p_{\underline{k}} e^{i\underline{k}\underline{r}}, p_{-\underline{k}} = p_{\underline{k}}^* \tag{58}$$

and the fundamental hydrodynamic equation

$$\dot{\underline{v}} = -(\underline{v}\nabla)\underline{v} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\Delta\underline{v}$$

is transformed into

$$\dot{\underline{v}}_{\underline{k}} = -\sum_{\underline{k}'} i\left(\underline{v}_{\underline{k}'}\underline{k}\right)\underline{v}_{\underline{k}-\underline{k}'} - \frac{i\underline{k}}{\rho} p_{\underline{k}} - \frac{\mu}{\rho} k^2 \underline{v}_{\underline{k}} \tag{59}$$

Because of $\left(\underline{v}_{\underline{k}}\underline{k}\right) = 0$ it follows

$$\dot{\underline{v}}_{\underline{k}} = -\sum_{\underline{k}'} i(\underline{v}_{\underline{k}'}\underline{k})\left[\underline{v}_{\underline{k}-\underline{k}'} - \frac{\underline{k}}{k^2}\, \underline{v}_{\underline{k}-\underline{k}'},\underline{k}\right] - \frac{\mu}{\rho} k^2 \underline{v}_{k}$$

$$p_{\underline{k}} = -\frac{\rho}{k^2}\sum_{\underline{k}'} (\underline{v}_{\underline{k}'}\underline{k})(\underline{v}_{\underline{k}-\underline{k}'},\underline{k}) \tag{60}$$

For the root mean square values of pressure and pressure gradient there results

$$\overline{p^2} = \sum_{\underline{k}} \left|\overline{p_{\underline{k}}^2}\right| \qquad \overline{\text{grad}^2 p} = \sum_{\underline{k}} k^2 \left|\overline{p_{\underline{k}}^2}\right| \tag{61}$$

We are interested, first, in this latter mean value:

$$\overline{\text{grad}^2\ p} = \sum_{\underline{k}} \sum_{\underline{k}'} \sum_{\underline{k}''} \frac{\rho^2}{k^2} \overline{(\underline{v}_{\underline{k}}\cdot\underline{k})(\underline{v}_{\underline{k}-\underline{k}'}\cdot\underline{k})(\underline{v}_{\underline{k}''}\cdot\underline{k})(\underline{v}_{\underline{k}-\underline{k}''}\cdot\underline{k})} \qquad (62)$$

The superscript bar indicating approximation to the mean signifies here simply approximation to the mean in time. If one wants to calculate mean values of the type (62), one must take as a basis some kind of "assumption of disorder" regarding the turbulent motion. One may start from the fact that the amplitudes $\underline{v}_{\underline{k}}$ in the course of time fluctuate by a value given by (10) or (28), respectively, so that the time average $\overline{\underline{v}_{\underline{k}}\underline{v}_{-\underline{k}}}$ is given simply by the spectrum (28). The phases of the $\underline{v}_{\underline{k}}$, however, will in the course of time pass through all possible values; all values of the phase will occur, on the average, with the same frequency. If one could regard the phases pertaining to different wave numbers as completely independent statistically, there would, in taking the mean of such products of four factors $\overline{\underline{v}_{\underline{k}_1}\underline{v}_{\underline{k}_2}\underline{v}_{\underline{k}_3}\underline{v}_{\underline{k}_4}}$, be left only the terms in which every two wave numbers are equal and opposite; thus, terms of the type $\overline{\underline{v}_{\underline{k}_1}\underline{v}_{-\underline{k}_1}\underline{v}_{\underline{k}_2}\underline{v}_{-\underline{k}_2}}$, and these mean values could be replaced by the products of the mean values of the squares of the amplitudes:

$$\overline{\underline{v}_{\underline{k}_1}\underline{v}_{-\underline{k}_1}\underline{v}_{\underline{k}_2}\underline{v}_{-\underline{k}_2}} = \overline{\underline{v}_{\underline{k}_1}\underline{v}_{-\underline{k}_1}}\ \overline{\underline{v}_{\underline{k}_2}\underline{v}_{-\underline{k}_2}} \qquad (63)$$

Actually, however, statistical correlations will exist between the phases pertaining to different wave numbers since the waves mutually influence one another. In section 5 we shall attempt to estimate such correlations in a simple case. In spite of the presence of the correlations, however, we are inclined to assume that in a sum of the kind (62) the terms of the type (63) make the largest contribution since their mean value is different from zero even in the first approximation, without any assumption regarding the behavior of the waves with respect to time whereas the other mean values attain a value different from zero only because of the finer fluctuations of various waves. We believe therefore that one obtains a serviceable approximation if one takes only the terms of the type (63) in (62) into consideration. Then there results

$$\overline{\text{grad}^2 p} = 2\rho^2 \sum_{\underline{k}'\underline{k}''}^{\bullet} \frac{\left(\underline{v}_{\underline{k}'}\cdot\underline{k}''\right)\left(\underline{v}_{-\underline{k}'}\cdot\underline{k}''\right)\left(\underline{v}_{\underline{k}''}\cdot\underline{k}'\right)\left(\underline{v}_{-\underline{k}''}\cdot\underline{k}'\right)}{(\underline{k}' - \underline{k}'')^2} \qquad (64)$$

In taking the mean with respect to the directions of the $\underline{v}_{\underline{k}}$ - one assumes again all directions perpendicular to $\underline{k}$ for $\underline{v}_{\underline{k}}$ to be equally probable - it is expedient to use the relation

$$\overline{(\underline{v}_{\underline{k}}\underline{a})(\underline{v}\text{-}_{\underline{k}}\underline{b})} = \frac{\overline{\left|\underline{v}_{\underline{k}}{}^2\right|}}{2}\left[\underline{a}\underline{b} - \frac{(\underline{a}\underline{k})(\underline{b}\underline{k})}{k^2}\right] \tag{65}$$

Thus one obtains

$$\overline{\text{grad}^2 p} = \frac{\rho^2}{2}\sum_{\underline{k}'\underline{k}''}\overline{\left|v_{\underline{k}'}{}^2\right|}\,\overline{\left|v_{\underline{k}''}{}^2\right|}\,\frac{\left[k'^2k''^2 - (\underline{k}'\underline{k}'')^2\right]^2}{(\underline{k}' - \underline{k}'')^2} \tag{66}$$

If one sets $(\underline{k}'\underline{k}'') = k'k''\zeta$, one can perform the integration with respect to ζ and finds

$$\overline{\text{grad}^2 p} = \frac{\rho^2}{2}\sum_{\underline{k}'\underline{k}''}\overline{\left|v_{\underline{k}'}{}^2\right|}\,\overline{\left|v_{\underline{k}''}{}^2\right|}k'k''\psi\left(\frac{k'}{k''}\right) \tag{67}$$

where

$$\psi(s) = \psi\,\frac{1}{s} = \frac{1}{16s^3}\left[-1 + \frac{11}{3}s^2 + \frac{11}{3}s^4 - s^6 + \frac{(1 - s^2)^4}{2s}\lg\frac{1 + s}{|1 - s|}\right] \tag{68}$$

For $0 \leq s \ll 1$ there applies approximately

$$\psi(s) \approx \frac{8s}{15}\left(1 - \frac{3}{7}s^2 + \frac{s^4}{21}\right) \tag{69}$$

If one transforms the sums into integrals and substitutes the spectrum (28) into (67), there follows finally

$$\overline{\text{grad}^2 p} = \frac{2}{9}\rho^2 v_0{}^4 k_0{}^{\frac{4}{3}}\int_{k_0}^{\infty}\frac{dk'}{k'^{\frac{2}{3}}}\int_{k_0}^{\infty}\frac{dk''}{k''^{\frac{2}{3}}}\frac{\psi\left(\frac{k'}{k''}\right)}{\left[1 + \left(\frac{k'}{k_s}\right)^{\frac{8}{3}}\right]^2\left[1 + \left(\frac{k''}{k_s}\right)^{\frac{8}{3}}\right]^2} \tag{70}$$

One recognizes from (70) that the integrals converge at small k-values, and that one may therefore perform the integration from $k = 0$ without a considerable error. This shows that $\overline{\mathrm{grad}^2 p}$ is actually determined by the behavior of the spectrum at large k, that is, by the smallest turbulence elements. Had we calculated $\overline{p^2}$, we would have found, on the contrary, that the integral diverges at small values of k, thus that its value is determined entirely by the largest turbulence elements. Therefore, the value of $\overline{p^2}$ cannot at all be calculated according to the method used here; for, first, in the case of small k, the spectrum has a form dependent on the geometry; and second, it would surely by quite unjustified to consider, for the largest turbulence elements, only the mean values of the type (63) since the geometry certainly impresses definite phase relations upon the system of the largest eddies.

Equation (70) now becomes

$$\overline{\mathrm{grad}^2 p} = \frac{2}{9}\rho^2 v_0{}^4 k_0{}^{4/3} k_s{}^{2/3} \int_0^\infty \frac{d\zeta}{\zeta^{2/3}} \int_0^\infty \frac{d\eta}{\eta^{2/3}} \frac{\psi\left(\frac{\zeta}{\eta}\right)}{\left[1 + \zeta^{8/3}\right]^2 \left[1 + \eta^{8/3}\right]^2} \tag{71}$$

The double integral at the right was estimated, according to a graphical method, to be 0.763; thus there follows finally (cf. (30))

$$\overline{\mathrm{grad}^2 p} = 0.17\rho^2 v_0{}^4 k_0{}^{4/3} k_s{}^{2/3} \tag{72}$$

$$= 0.05\rho^2 v_0{}^4 k_0{}^2 \sqrt{R_0 \kappa} \tag{73}$$

Taylor (cited before) had expressed the conjecture that $\overline{\mathrm{grad}^2 p}$ should have the same order of magnitude as $\overline{\rho^2 v_0{}^2 (\partial v/\partial x)^2}$ - thus the order of magnitude $\rho^2 v_0{}^4 k_0{}^{2/3} k_s{}^{4/3}$. One recognizes now from (72) that $\overline{\mathrm{grad}^2 p}$ must be considerably smaller, the more so, the larger the ratio k_s/k_0. The length λ_η defined by Taylor:

$$\overline{\left(\frac{\partial p}{\partial y}\right)^2} = 2\rho^2 \frac{\left(\overline{v_y{}^2}\right)^2}{\lambda_\eta{}^2} \tag{74}$$

that is,

$$\overline{\text{grad}^2 p} = \frac{2}{3} \rho^2 \frac{v_0^4}{\lambda_\eta^2} \tag{75}$$

must, therefore, for large Reynolds numbers, become considerably larger than the length λ of equation (55). From (75), (73), and (56) follows

$$\lambda_\eta = \lambda 0.42 \sqrt[4]{R_0 \kappa} \tag{76}$$

It is true that this result does not agree with the experimental findings. Taylor indicates, for a test which Simmons had performed adjoining similar experiments by Schubauer[14], $\lambda_\eta/\lambda = 0.5$ where one must assume approximately $\sqrt[4]{R_0 \kappa} \approx 3.9$ according to the test conditions. Thus one must raise the question whether the result (72) has perhaps been falsified by the fact that only the terms of the type (63) were taken into consideration in taking the mean. However, one can easily see that this may perhaps affect the numerical factor in (72) but that the dependence of k_0 and k_s, that is, the dependence of λ_η on $(R_0 \kappa)$ is in no way connected with this approximation. For already equation (62) shows that on the right side the normalization factor $v_0^4 k_0^{4/3}$ must appear, because of equation (10). After the mean has been taken, this factor is supplemented by a factor of the dimension $k^{2/3}$ which obviously can be at most of the order $k_s^{2/3}$; it must be, because the pertaining integral with respect to k would diverge like $k^{2/3}$ if the decay of the spectrum would not set in for $k \sim k_s$ with k^{-7}. There would remain the possibility that only the numerical factor in (72) has been estimated as too low due to the consideration solely of the terms of the type (63). But it is hard to imagine that the correct expression would increase by more than a tenfold - which would be necessary for interpretation of the experiments.

Perhaps the contradiction may be cleared up in the following manner: The main contribution to $\overline{\text{grad}^2 p}$ stems from wave numbers of the order k_s, thus, from turbulence elements whose diameter measures a few millimeters. In Simmons' test, the airstream is heated by a hot wire of 20 cm length stretched across the wind tunnel, and then the distribution of the heated air is measured at a certain distance behind the hot wire. Precisely the smaller distances (5 to 15 cm) are decisive for the determination of λ_η.

---

[14]Schubauer: Rep. Nat. Adv. Comm. Aero. Nachr. Nr. 524, 1935.

It suggests itself to assume that the hot wire itself produces in the airstream a small vortex street and additional turbulence, with the turbulence elements probably having a length of a few millimeters - that is, the wire increases precisely the intensity of the turbulence in the spectral region which exerts the strongest influence on $\overline{\text{grad}^2 p}$. The additional turbulence in the immediate proximity of the wire is probably much more intense than the original turbulence of the same wave range. However, this additional turbulence is rapidly damped, of course, and it is surely difficult to estimate whether this turbulence alone can explain the discrepancy between (76) and the empirical λ_η-value.

5. Energy Dissipation for Normal Isotropic Turbulence

The investigations of the preceding section are already closely connected with a basic problem of the statistical turbulence theory: namely, with the determination of the energy dissipation in the case of normal energy distribution, that is, the determination of the constant κ in equation (12). In this problem the molecular friction may be neglected entirely. The fundamental hydrodynamical equations could therefore be presupposed in the form

$$\dot{\underline{v}} = -(\underline{v}\nabla)\underline{v} - \frac{1}{\rho}\nabla p \qquad \nabla\underline{v} = 0$$

Furthermore, a suitably singled-out partial volume of the fluid is to be selected as normalization volume which under certain circumstances is moved simultaneously with the fluid, corresponding, for instance, to the mean value of the velocity with respect to the volume. We assume therefore that the volume moves with the velocity $\underline{u}$. Then (60) is transformed into

$$\dot{\underline{v}}_{\underline{k}} = -i\sum_{\underline{k}'}(\underline{v}_{\underline{k}'}\underline{k})\left[\underline{v}_{\underline{k}-\underline{k}'} - \frac{\underline{k}}{k^2}(\underline{v}_{\underline{k}-\underline{k}'},\underline{k})\right] + i(\underline{u}\underline{k})\underline{v}_{\underline{k}} \tag{77}$$

For the calculation of the energy dissipation, one has to ascertain how the intensity $\left|\underline{v}_{\underline{k}}^2\right|$ of a certain natural vibration $\Big($or perhaps better: the sum of such squares of amplitudes with respect to a small spectral region Δk: $\sum_{k}^{k+\Delta k}\left|\underline{v}_{\underline{k}}^2\right|\Big)$ varies in the course of time. One recognizes from (77) that one needs for this purpose time averages with respect to products of the type

$$\overline{\underline{v}_{\underline{k}_1}\,\underline{v}_{\underline{k}_2}\,\underline{v}_{\underline{k}_3}} \tag{78}$$

wherein $\underline{k}_3 = -\underline{k}_1 - \underline{k}_2$. Because of the statistically uniform distribution of the phases, these mean values would disappear if there would not exist statistical correlations between the phases pertaining to different $\underline{k}$ which stem from the mutual influence of the various waves, as has been explained already in section 4. In order to ascertain these correlations, one must somehow express in the equations the influence exerted upon a wave with given $\underline{k}$ by waves with a different $\underline{k}$; one may do this for instance by representing one of the three amplitudes in (78) as a time integral over $\dot{\underline{v}}_{\underline{k}}$ and expressing $\dot{\underline{v}}_{\underline{k}}$ in turn by a sum over two other $\underline{v}_{\underline{k}}$ according to (77). Then one obtains products of four amplitudes $\underline{v}_{\underline{k}}$ each of which, however, must partly be taken at different times. For such products the considerations of section 4 are valid according to which one obtains a first approximation by taking only products of the type (63) into consideration. Of course, one could continue the procedure in principle and attempt to calculate the other mean values of quadruple products by tracing them back to six-fold ones, etc. However, such calculations would probably become much too complicated; the higher terms probably also would make a lesser contribution, and we shall thus be content with the first step.

For these calculations, one will obviously need mean values of the type

$$\overline{\underline{v}_{\underline{k}}(t)\underline{v}_{-\underline{k}}(t + \tau)}$$

and we define therefore

$$R_k(t,\tau) = \frac{\sum_{k}^{k+\Delta k} \overline{\underline{v}_{\underline{k}}\left(t + \frac{\tau}{2}\right)\underline{v}_{-\underline{k}}\left(t - \frac{\tau}{2}\right)}}{\sum_{k}^{k+\Delta k} \overline{\underline{v}_{\underline{k}}\underline{v}_{-\underline{k}}}} \tag{79}$$

The summation over a small spectral region Δk has been included into the definition of $R_k(t,\tau)$ so that the magnitude of the normalization volume does not directly enter into $R_k(t,\tau)$ and that the mean is taken equally over all directions of $\underline{k}$. Evidently the spectral region Δk must be selected wide enough that many natural vibrations of the normalization volume still may be accommodated in it (that is, $k^2 \Delta k V \gg 1$), yet very small compared to k itself. These requirements are for the turbulence elements of the order of magnitude V themselves no longer compatible but for those, the statistical methods cannot be applied anyway. The whole procedure thus can be carried out only if it is found that the large turbulence elements practically do not any more contribute to the mean values to be investigated.

In order to obtain from the hydrodynamical equations information about the quantities $R_k(t,\tau)$, it suggests itself to examine the following expression:

$$\dot{\underline{v}}_{\underline{k}}\left(t + \frac{\tau}{2}\right)\underline{v}_{-\underline{k}}\left(t - \frac{\tau}{2}\right) = -i\sum_{\underline{k}'}\left(\underline{v}_{\underline{k}'}^{t+\frac{\tau}{2}},\underline{k}\right)\left(\underline{v}_{\underline{k}-\underline{k}'}^{t+\frac{\tau}{2}}\underline{v}_{-\underline{k}}^{t-\frac{\tau}{2}}\right) + i(\underline{u}\underline{k})\underline{v}_{\underline{k}}^{t+\frac{\tau}{2}}\,\underline{v}_{-k}^{t-\frac{\tau}{2}} \qquad (80)$$

In this expression one can replace $\underline{v}_{\underline{k}'}^{t+\frac{\tau}{2}}$ by a time integral with respect to $\dot{\underline{v}}_{\underline{k}'}^{t+\frac{\tau}{2}}$:

$$\underline{v}_{\underline{k}'}^{t+\frac{\tau}{2}} = \int_{-T}^{t+\frac{\tau}{2}} \dot{\underline{v}}_{\underline{k}'}^{t'}\, dt' + \underline{v}_{\underline{k}'}(-T) = \int_{0}^{t+\frac{\tau}{2}+T} d\tau'\,\dot{\underline{v}}_{\underline{k}'}^{t+\frac{\tau}{2}-\tau'} + \underline{v}_{\underline{k}'}(-T) \qquad (81)$$

If T is selected sufficiently large, the correlation between $\underline{v}_{-k'}(t)$ and $\underline{v}_{k'}(T)$ will disappear; it is therefore expedient to perform, after substitution into (80), the limiting process $T \longrightarrow \infty$. If, in addition, one takes the mean with respect to the directions - thus eliminating the term with $\underline{u}$ - one obtains from (80) and (81):

$$\sum_{k}^{k+\Delta k} \dot{\underline{v}}_{\underline{k}}\left(t + \frac{\tau}{2}\right)\underline{v}_{-\underline{k}}\left(t - \frac{\tau}{2}\right) = -\sum_{\underline{k}}^{k+\Delta k}\sum_{\underline{k}'}\sum_{\underline{k}''}\int_0^\infty d\tau'\left(\underline{v}_{\underline{k}''}^{t+\frac{\tau}{2}-\tau'}\ \underline{k}'\right)\left[\underline{v}_{\underline{k}'-\underline{k}''}^{t+\frac{\tau}{2}-\tau'}\ \underline{k} - \frac{(\underline{k}\ \underline{k}')\left(\underline{v}_{\underline{k}'-\underline{k}''}^{t+\frac{\tau}{2}-\tau'}\ \underline{k}'\right)}{k'^2}\right]\left(\underline{v}_{\underline{k}-\underline{k}'}^{t+\frac{\tau}{2}}, \underline{v}_{-\underline{k}}^{t-\frac{\tau}{2}}\right) \tag{82}$$

If one furthermore, as in section 4, takes into consideration the terms of the type (63) only, also replaces $\underline{k} - \underline{k}'$ by $\underline{k}'$, and integrates with respect to the cosine of the angle between $\underline{k}$ and this vector, there follows:

$$\sum_{k}^{k+\Delta k} \dot{\underline{v}}_{\underline{k}}\left(t + \frac{\tau}{2}\right)\underline{v}_{-\underline{k}}\left(t - \frac{\tau}{2}\right) = \sum_{k}^{k+\Delta k}\int dk' k'^2 \frac{V}{(2\pi)^3}\frac{\pi}{16}\int_0^\infty d\tau' \left|\overline{\underline{v}_{\underline{k}}^2}\right| \left|\overline{\underline{v}_{\underline{k}'}^2}\right| R_k\left(t - \frac{\tau'}{2}, \tau - \tau'\right) \times$$

$$R_{k'}\left(t + \frac{\tau - \tau'}{2}, \tau'\right) k^{-3}k'^{-3}(k^2 - k'^2)\left[2kk'\left(k^4 + k'^4 - \frac{2}{3}k^2k'^2\right) - (k^2 + k'^2)(k^2 - k'^2)\lg\frac{k + k'}{|k - k'|}\right] \tag{83}$$

This equation presents the possibility of expressing the differential quotients with respect to time $\frac{dR_k(t,\tau)}{dt}$ and $\frac{dR_k(t,\tau)}{d\tau}$ by the R_k themselves; when the R_k are known, one can, moreover, calculate the energy dissipation from (83), putting $\tau = 0$. For this purpose we shall assume that the entire turbulence phenomenon is either stationary or is damped very slowly so that the times during which the intensity $\left|\underline{v}_{\underline{k}}^2\right|$ noticeably varies are very long compared with the fluctuation period of $\underline{v}_{\underline{k}}$. The notation $\left|\underline{v}_{\underline{k}}^2\right|$ represents, therefore, the mean value over a time which is certainly much longer than the fluctuation period but is very much shorter than the damping time.

The equation for $\frac{dR_k(t,\tau)}{dt}$ gives a measure for the fluctuations of the quantity $R_k(t,\tau)$ as a function of t about its time mean:

$$R_k(\tau) = \overline{R_k(t,\tau)} \tag{84}$$

One may surmise that these fluctuations are small in the region of small τ which is determined by the small turbulence elements, and that they increase with growing τ; this question will be further investigated later on.

Before carrying out the further calculations one has to determine how the partial volume V and its velocity $\underline{u}$ are to be chosen. One could first try to put $\underline{u} = 0$ and to identify V with the total volume. However, one would obtain an erroneous picture of the actual conditions: The decrease with time of the correlation function $\overline{R_k(t,\tau)} = R_k(\tau)$ as a function of τ is determined in this coordinate system by the largest turbulence elements, and is therefore very rapid. One can show that the correlation function in this coordinate system is given with sufficient approximation by

$$R_k(\tau) = 1 - \frac{kv_0\tau}{\sqrt{3}} e^{-\frac{k^2v_0^2\tau^2}{12}} \int_0^{\frac{kv_0\tau}{\sqrt{12}}} dx e^{x^2}$$

The calculations which have led to this expression need not be discussed in more detail since the expression is not further used later. The physical interpretation of the expression is given by the following consideration: The function $R_k(\tau)$ in it decreases after a time of the order $\frac{2\pi}{kv_0}$; that is the time during which, for instance, precisely an eddy of the wave length $\frac{2\pi}{k}$, due to the high velocity in the largest turbulence elements, passes by the point of observation. The fact that the correlation function decreases after that time signifies therefore simply that the velocity in the largest turbulence elements is of the order of magnitude v_0, but, statistically, fluctuates about values of this order. This phenomenon is not connected with the disintegration of the eddy of the wave length $\frac{2\pi}{k}$. If the $F(k) \sim k^{-5/3}$-law is valid, it is rather to be expected, according to the similitude considerations of v. Weizsaecker, that the disintegration of the eddy takes place only after time intervals of the order $2\pi v_0^{-1}k^{-2/3}k_0^{-1/3}$. On the other hand, however, the energy dissipation is connected with the disintegration of the eddies, not with the motion on a large scale. If one wants to describe the disintegration of the eddies in equations, one must move the coordinate system at the same time. One then must make the linear dimensions of the partial volume V somewhat, but not very much larger than $\frac{2\pi}{k}$ and move it simultaneously in proportion to the mean velocity within it. We shall assume experimentally that one can give for every k a volume V corresponding to it in such a manner that Vk^3 becomes independent of k and that for the thus selected, simultaneously moved volume elements, the correlation function $R(t,\tau)$ is a universal function of the variables $v_0k^{2/3}k_0^{1/3}t$ and $v_0k^{2/3}k_0^{1/3}\tau$ as is to be expected according to v. Weizsaecker's similitude consideration. We shall show that the relations for $R_k(t,\tau)$ arising from (83) actually can be satisfied by this assumption if the F_k are distributed according to the $k^{-5/3}$-law.

If one puts $R_k(t,\tau) = g(\zeta,\eta)$ wherein

$$\zeta = \frac{v_0}{6}k^{2/3}k_0^{1/3}t \qquad \eta = \frac{v_0}{6}k^{2/3}k_0^{1/3}\tau \qquad \frac{k'}{k} = y \tag{85}$$

there follows from (83)

$$\left.\begin{aligned} \frac{dg(\zeta,\eta)}{d\eta} &= \frac{3}{16}\int_0^\infty dyf(y)\int_0^\infty d\eta'\left[g\left(\zeta - \frac{\eta'}{2},\eta' - \eta\right)g\left(\left(\zeta + \frac{\eta - \eta'}{2}\right)y^{2/3},\eta'y^{2/3}\right) - \right. \\ &\qquad \left. g\left(\zeta - \frac{\eta'}{2},\eta + \eta'\right)g\left(\left(\zeta - \frac{\eta + \eta'}{2}\right)y^{2/3},\eta'y^{2/3}\right)\right] \\ \frac{dg(\zeta,\eta)}{d\zeta} &= \frac{3}{8}\int_0^\infty dyf(y)\int_0^\infty d\eta'\left[g\left(\zeta - \frac{\eta'}{2},\eta' - \eta\right)g\left(\left(\zeta + \frac{\eta - \eta'}{2}\right)y^{2/3},\eta'y^{2/3}\right) - \right. \\ &\qquad \left. g\left(\zeta - \frac{\eta'}{2},\eta + \eta'\right)g\left(\left(\zeta - \frac{\eta + \eta'}{2}\right)y^{2/3},\eta'y^{2/3}\right)\right] \end{aligned}\right\} \tag{86}$$

where

$$f(y) = \frac{1 - y^2}{y^{14/3}}\left[2y\left(1 - \frac{2}{3}y^2 + y^4\right) - (1 - y^2)(1 - y^4)\lg\frac{1 + y}{|1 - y|}\right] \tag{87}$$

These equations actually do no longer contain the constants k_0, v_0. The reason is that the integral with respect to y converges for small as well as for large y; $f(y)$ disappears sufficiently for $y = 0$ as well as for $y \longrightarrow \infty$. Therefore one may take the integral over k', instead of from k_0, simply from 0, without considerable error; moreover, the convergence of the integral for large values of k shows that the molecular friction actually is of no importance in this problem; the behavior of the spectrum in the region of the smallest turbulence elements is unimportant for the correlation functions $R(t,\tau)$ and the energy dissipation for medium k-values.

Before attempting the numerical solution of (86), we shall use the equation (83) for calculating the energy dissipation in the approximation here aspired to. For this purpose we put $\tau = 0$ and integrate the equation (83) with respect to k between two arbitrary limits K_1 and K_2:

$$\frac{d}{dt}\int_{K_1}^{K_2} 4\pi k^2 dk\left|\frac{\overline{v_k^2}}{2}\right| = \frac{\pi^2 V}{4(2\pi)^3}\int_{K_1}^{K_2}\frac{dk}{k}\int_0^\infty \frac{dk'}{k'}\left|\overline{v_k^2}\right|\left|\overline{v_{k'}^2}\right|\int_0^\infty d\tau' \overline{R_k\left(t - \frac{\tau'}{2},\tau'\right)R_{k'}\left(t - \frac{\tau'}{2},\tau'\right)} \times$$

$$\left(k^2 - k'^2\right)\left[2kk'\left(k^4 + k'^4 - \frac{2}{3}k^2k'^2\right) - \left(k^2 + k'^2\right)\left(k^2 - k'^2\right)^2 \lg\frac{|k - k'|}{k + k'}\right] \tag{88}$$

The integrand on the right side is an antisymmetrical function in k and k'. If one calculates the variation with time of the total energy, that is, if one puts $K_1 = 0$, $K_2 = \infty$, there results therefore zero, as far as the integral on the right converges at all. That is, the total energy is constant in time; this is a necessary requirement since the molecular friction has not been taken into consideration. However, if one considers the variation with time of the energy which is contained in the part of the spectrum lying between K_1 and K_2, the integral on the right side may be transformed in the following manner (we shall call the antisymmetrical integrand simply J):

$$\frac{d}{dt}\int_{K_1}^{K_2} 4\pi k^2 dk\left|\frac{\overline{v_k^2}}{2}\right| = \int_{K_1}^{K_2} dk\int_0^\infty dk' J = \int_{K_1}^{K_2} dk\int_0^{K_1} dk' J - \int_{K_1}^{K_2} dk\int_{K_2}^\infty dk'(-J) \tag{89}$$

In the first of the two integrals on the right J is always positive; in the second (-J) is always positive. From this notation there follows that the first integral may be interpreted as the energy which, per unit time, flows from smaller wave numbers ($k' < K_1$) into the region between K_1 and K_2; the second integral as the energy which flows toward larger wave numbers ($k' > K_2$). If one puts, in particular, $K_1 = 0$ and $K_2 \gg k_0$, the second integral represents the entire energy dissipation; for the normal spectrum $\left[F(k) \sim k^{-5/3}\right]$ it must prove to be independent of K_2. Thus one obtains from (83), (86), and (89) for the energy dissipation the expression

$$S = \frac{\rho}{24} v_0^3 k_0 \int_{K_2}^{\infty} \frac{dk''}{k''} \int_0^{\frac{K_2}{k''}} dy f(y) \int_0^{\eta} d\eta \overline{g\left(\zeta - \frac{\eta}{2}, \eta\right) g\left(\left(\zeta - \frac{\eta}{2}\right) y^{2/3}, \eta y^{2/3}\right)}$$

$$= \frac{\rho}{24} v_0^3 h_0 \int_0^1 dy(-\lg y) f(y) \int_0^{\infty} d\eta \overline{g\left(\zeta - \frac{\eta}{2}, \eta\right) g\left(\left(\zeta - \frac{\eta}{2}\right) y^{2/3}, \eta y^{2/3}\right)} \qquad (90)$$

This expression is actually independent of K_2 as it must be. Since the entire energy dissipation according to equation (44) is also given by $\rho\kappa\frac{\sqrt{3}}{8} v_0^3 k_0$, there follows

$$\kappa = \frac{1}{3\sqrt{3}} \int_0^1 dy(-\lg y) f(y) \int_0^{\infty} d\eta \overline{g\left(\zeta - \frac{\eta}{2}, \eta\right) g\left(\left(\zeta - \frac{\eta}{2}\right) y^{2/3}, \eta y^{2/3}\right)} \qquad (91)$$

From this equation κ can be calculated numerically if the function $g(\zeta,\eta)$ is known.

We now turn to the treatment of the equation system (86). This system represents a considerable simplification compared to the initial equation (77) in so far as it does not any more contain any dimensional quantities, and is already derived from the equilibrium spectrum $k^{-5/3}$. On the other hand, (86) also still contains statements regarding the fluctuations of the $g(\zeta,\eta)$ as a function of the ζ and is, for that reason, doubtlessly too complicated to permit rigorous solutions. One could attempt to completely neglect the fluctuations in a first approximation, and to calculate with the mean values. Unfortunately, however, it turns out that the contribution of the fluctuations in certain regions is large. One recognizes this from the second equation (86).

For, if one puts

$$g(\zeta,\eta) = g(\eta) + \Delta g(\zeta,\eta) \tag{92}$$

where $g(\eta)$ signifies the mean value over ζ of $g(\zeta,\eta)$:

$$g(\eta) = \overline{g(\zeta,\eta)} \tag{93}$$

there follows from the time mean of the second equation (86):

$$\frac{3}{8}\int_0^\infty dyf(y)\int_0^\infty d\eta'\left[g(\eta' - \eta) + g(\eta' + \eta)\right]g\left(\eta' y^{2/3}\right)$$

$$= -\frac{3}{8}\int_0^\infty dyf(y)\int_0^\infty d\eta'\left[\overline{\Delta g\left(\zeta - \frac{\eta'}{2},\eta' - \eta\right)\Delta g\left(\left(\zeta + \frac{\eta - \eta'}{2}\right)y^{2/3},\eta' y^{2/3}\right) +}\right.$$

$$\left.\overline{\Delta g\left(\zeta - \frac{\eta'}{2},\eta' + \eta\right)\Delta g\left(\left(\zeta - \frac{\eta + \eta'}{2}\right)y^{2/3},\eta' y^{2/3}\right)}\right] \tag{94}$$

This relationship shows that the fluctuations $\Delta g(\zeta,\eta)$ cannot always be small. It is true that the left side of (94) vanishes for $\eta = 0$; this follows from the relation

$$f\left(\frac{1}{y}\right) = -y^{4/3}f(y) \tag{95}$$

which will have to be discussed later, and signifies that the spectrum $k^{-5/3}$ is actually in equilibrium; however, for larger $|\eta|$ the left side assumes appreciable values. Therefore, it is doubtful whether one will obtain a sufficient approximation if, in the transition from (82) to (83), one takes only the mean values of the type (63) into consideration. However, I did not succeed in improving here the approximation or in obtaining more than a very crude estimation of (86).

One may perhaps assume for such an estimation that, for large values of η, the first term of the summation on the right side of (94) is much larger than the second. For in the first one, the integral is taken over the region $\eta' \sim \eta$ which probably contributes a great deal whereas in the second, for large η, the factors Δg have already strongly decayed in the entire integration range. One may therefore attempt the assumption, at least for large η, that the second term of the summation on the right side of (94) may be neglected. In this approximation the time average of the first equation (86) then becomes, with use of (94):

$$\frac{dg(\eta)}{d\eta} = -\frac{3}{8}\int_0^{\infty}(fy\ dy)\int_0^{\infty} d\eta' g(\eta + \eta')g\left(\eta' y^{2/3}\right) \qquad (\text{for}\quad \eta \gg 0) \quad (96)$$

One may utilize this equation, for instance, in such a manner that one assumes a plausible form for $g(\eta)$, leaving the scale in the η-direction undetermined at first, and determining it subsequently so that the equation (96) is valid as exactly as possible for large η. In this way one will describe the steepness of the decrease of $g(\eta)$ for large η with some correctness, and precisely this steepness is decisive for the value of κ.

In the practical execution of the calculation it is expedient to introduce, in the place of y and $f(y)$, new variables

$$s = y^{2/3}; \quad \varphi(s)ds = f(y)dy \quad (97)$$

Then there applies as one recognizes from (87) (compare also (95)):

$$\varphi\left(\frac{1}{s}\right) = -s\varphi(s) \quad (98)$$

This equation is based on the fact that the energy dissipation from the wave number $\frac{1}{\alpha}k$ to the wave number k coincides with the one from k to αk, except for a factor qualified by the similitude transformation. Furthermore, there then applies in good approximation

$$\varphi(s) = \frac{32}{5}s(1 - s^3)\left(1 - \frac{2}{7}s^3\right) \qquad \left(\text{for } 0 \leq s \leq 1\right) \tag{99}$$

and for larger s one can reduce $\varphi(s)$ by means of (98) to the range $0 \leq s \leq 1$.

Figure 4 gives a plausible curve for $g(\eta)$, and in addition the right side of (96) as $\int_0^\infty d\eta \ldots$ and $g'(\eta)$. The scale is selected in such a manner that the two last curves coincide for large η. Considerable differences then exist for small η but there the equation (96) also can no longer be correct. If one substitutes the function $g(\eta)$ thus obtained into (91) and neglects the fluctuations, there results for κ:

$$\kappa = \frac{1}{2\sqrt{3}} \int_0^1 ds(-\lg s)\varphi(s) \int_0^\infty d\eta g(\eta)g(\eta s) = 0.98 \tag{100}$$

This crude estimation therefore gives the correct order of magnitude for κ, but the exact value may well differ from 0.98 by as much as a factor 2. The calculations of this section thus have not led to an exact calculation of the constant κ but they did provide a qualitative mathematical representation of the processes on which the energy dissipation is based. Perhaps it will be possible to arrive at a rather exact experimental determination of κ by means of a comprehensive discussion of the various experiments of Simmons, Dryden (cited before), Prandtl[15], and others regarding the spectrum and the damping of turbulence.

Translated by Mary L. Mahler
National Advisory Committee
for Aeronautics

---

[15]Prandtl, L.: Proc. VI. Intern. Congr. f. Appl. Mech. Cambridge, Mass., 1938, p. 340.

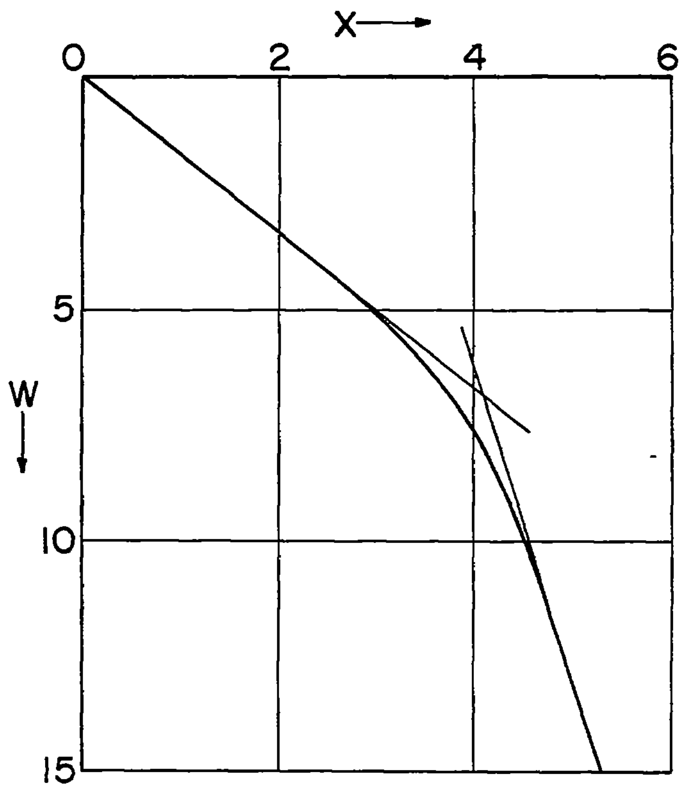

Figure 1.- Representation of the function w(x).

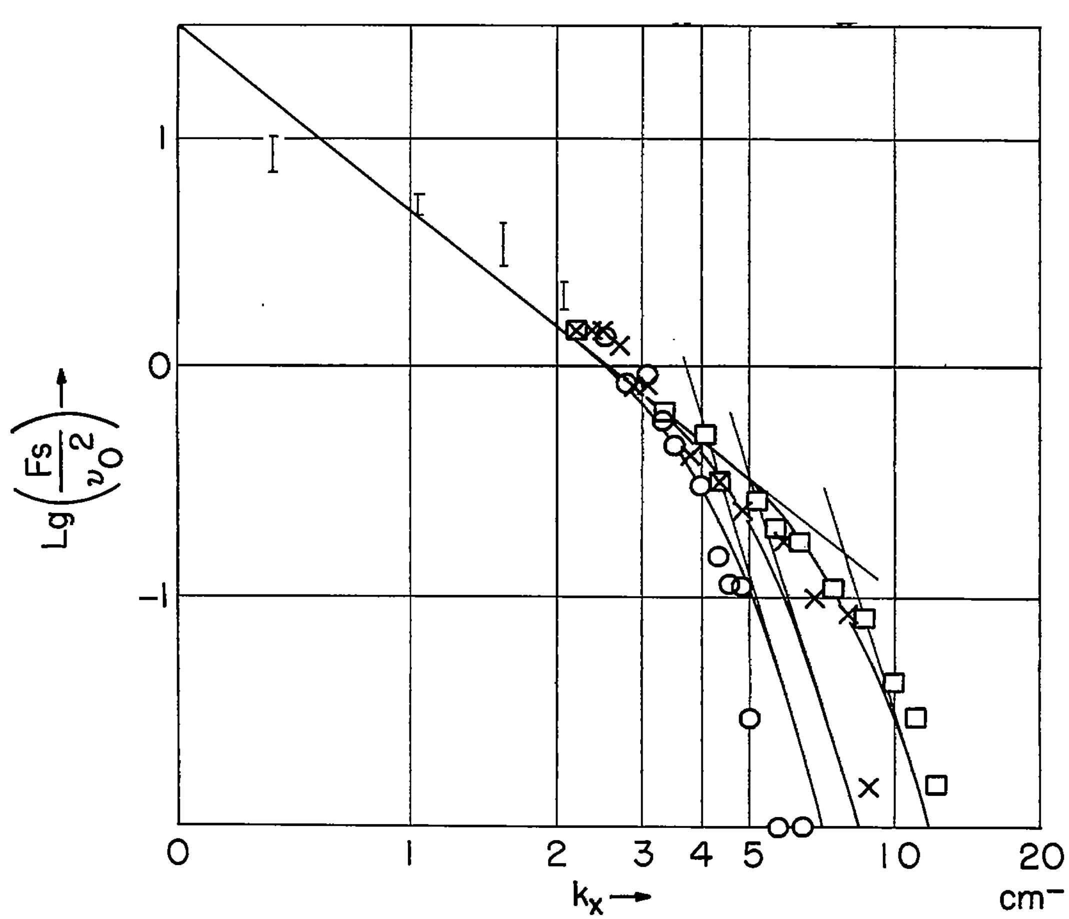

Figure 2.- The turbulent energy distribution as a function of the wave number.

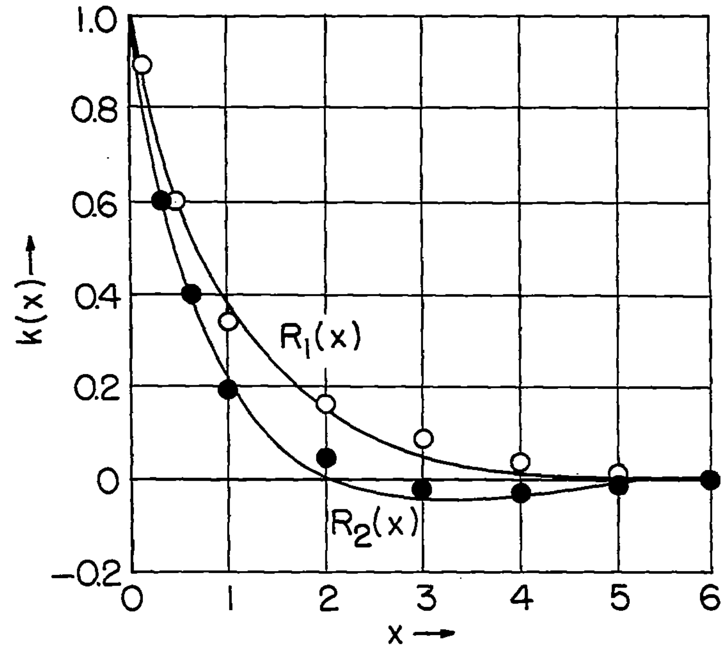

Figure 3.- The correlation functions.

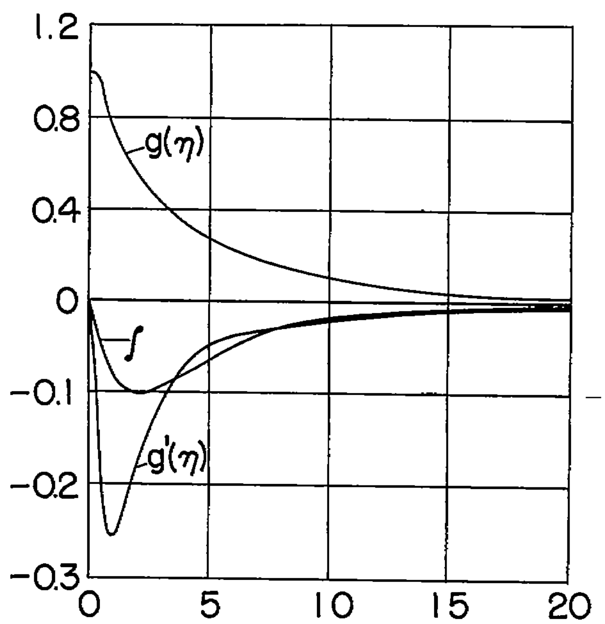

Figure 4.- The correlation function $g(\eta)$.