REPORT No. 172

DYNAMIC STABILITY AS AFFECTED BY THE LONGITUDINAL MOMENT OF INERTIA

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INTRODUCTION.

This report was submitted to the Subcommittee on Aerodynamics and by that committee recommended for publication as a technical report of the National Advisory Committee for Aeronautics.

In a recent technical note (No. 115, October, 1922) of the National Advisory Committee for Aeronautics, Norton and Carroll have reported experiments showing that a relatively large (15 per cent) increase in longitudinal moment of inertia made no noticeable difference in the stability of a standard S. E. 5A airplane. They point out that G. P. Thomson, Applied Aeronautics, page 208, stated that an increase in longitudinal moment of inertia would decrease the stability. Neither he nor they make any theoretical forecast of the amount of decrease. Although it is difficult, on account of the complications of the theory of stability of the airplane, to make any accurate forecast, it may be worth while to attempt a discussion of the matter theoretically with reference to finding a rough quantitative estimate.

GENERAL METHOD USED.

The notation used will be that of my Aeronautics (Wiley & Sons) particularly pages 135 ff. The effective quadratic factor of the stability biquadratic for the longitudinal motion which we are considering (the so-called phugoid) is

$$\lambda^2+\left(\frac{D}{C}-\frac{BE}{C^2}\right)\lambda+\frac{E}{C}=0$$

or

$$\lambda=-\frac{1}{2}\left(\frac{D}{C}-\frac{BE}{C^2}\right)\pm i\sqrt{\frac{E}{C}-\frac{1}{4}\left(\frac{DC-BE}{C^2}\right)^2}$$

the periodic time is

$$T=\frac{2\pi}{\sqrt{\frac{E}{C}-\frac{1}{4}\left(\frac{DC-BE}{C^2}\right)^2}}$$

and the time to damp one-half is

$$t=\frac{2\log_e 2}{\frac{D}{C}-\frac{BE}{C^2}}$$

The problem is to determine the effect on T and t due to a change in the longitudinal moment of inertia which is represented by the square k_B^2 of the radius of gyration. It is assumed that this change of k_B^2 is effected by a transfer of mass fore and aft in the airplane without altering the center of gravity, the total mass, the aerodynamic surfaces, or anything except k_B^2.

RESULTS OF THE TESTS.

For the experimental airplane $W=2,000$ pounds or slightly over 60 slugs and the moment of inertia is 1,860 slugs-feet, so that k_B^2 is about 30 in the standard form. The increase of k_B^2 is nearly 15 per cent, or about 4. The observed periodic times are 19.3 and 18.6 (mean value, 19) for the standard form; 18.8 and 20.3 (mean value, 19.5) for the modified distribution of mass. The increase of period is therefore about 2.5 per cent except for errors of observation. As a matter of fact in the two respective cases the observed periods differed among themselves by 0.7 and 1.5 seconds. With so few observations it is impossible safely to apply the theory of precision of measurements, but it is by no means certain that the error in the two means might not be as high as 0.5 second, which is the mean error. The conclusion from the experimental data is therefore that the increase of moment of inertia had no appreciable effect on stability. Further it may be inferred that unless many more observations were made or unless more precision in the individual measurements were attainable an increase of about 2.5 per cent in T would not be definitely noticeable. It is probable that the determination of t would be liable to quite as great an experimental error as T, if not greater.

INVESTIGATION OF THE PERIOD.

The experimental airplane was of a type of reasonably high longitudinal stability, and the damping time exceeded the period. Under these conditions it is known that the damping affects the period but slightly. Indeed, if $t=nT$, we have

$$T=\frac{2\pi}{\sqrt{\frac{E}{C}-\left(\frac{\log 2}{nT}\right)^2}} \text{ or } \frac{E}{C}=\left(\frac{2\pi}{T}\right)^2+\left(\frac{\log 2}{nT}\right)^2$$

Hence

$$\frac{E}{C}=\left(\frac{2\pi}{T}\right)\left[1+\frac{1}{80\,n^2}\right], \text{ nearly.}$$

Variation in the damping alone may be represented by variation in n, with $n=\infty$ for no damping. In the case of a fairly stable airplane, say $n=1$, the total effect of the damping on the period is only about 0.7 per cent and the change of n from 1 to 1.2, a change of 20 per cent, would change T by less than 0.2 per cent. If, then, it appears that the change in t is small, the effect of that change upon T will be very small, and it will be possible to treat the changes of T and t separately by the equations

$$T=2\pi\sqrt{\frac{C}{E}}, \qquad t=1.4\div\left(\frac{D}{C}-\frac{BE}{C^2}\right)$$

(Compare the discussion in art. 33, Aeronautics, pp. 68–70.)

Of the four coefficients B, C, D, E which here enter, the formulas (Aeronautics, p. 135) show that B and C alone depend on k_B^2, whereas D and E are independent of k_B^2. The percentage changes of T are therefore (neglecting the effect of changes in damping) one-half the percentage changes in C.

Now

$$C=k_B^2\,(X_u\,Z_w-X_w\,Z_u)-M_q\,(-Z_w-X_u)-(Z_q+U)\,M_w-M_u\,X_q.$$

The large terms here are $Z_w\,M_q-UM_w$ so that for approximate calculation k_B^2 is entirely neglected. Ordinarily $Z_w\,M_q-UM_w$ is in the hundreds, whereas $X_u\,Z_w-X_w\,Z_u$ is of the magnitude 1. A change of 4 units in k_B^2 would therefore ordinarily represent a change of under 1 per cent in C or under $\frac{1}{2}$ per cent in T. It would seem a well-founded conclusion to infer that changes in k_B^2 of the order of 15 per cent would, except as they affected T through the damping, be for a stable airplane only of the order of magnitude of one-tenth the precision of the experiment. So far as I have the data at hand, the coefficient of k_B^2 seems to fall off at decreasing speeds as fast as C, and the conclusion would seem to be very widely valid that *no practicable changes in k_B^2 are likely in ordinary types of airplanes to make observable changes in the period T.*

INVESTIGATION OF THE DAMPING.

If the attention next be turned to the damping time t, it may be assumed safely, in view of the above considerations upon the changes in C due to changes in k_B^2, that in the expression for

$$\frac{1.4}{t}=\frac{D}{C}-\frac{BE}{C^2}=\frac{1}{C}\left(D-\frac{4\pi^2 B}{T^2}\right)$$

changes in C (or T) will be inappreciable: for although this expression is a difference, the magnitude of the terms is decidedly different and the variation in C would in any event tend to counterbalance in the two terms, as may be clearer from the second form than from the first. The chief variation in t would therefore be

$$\frac{1.4}{t^2}\,\delta t=\frac{4\pi^2}{T^2}\,\frac{\delta B}{C} \qquad \text{or } \delta t=\frac{28n^2}{C}\,\delta B, \text{ nearly.}$$

Now $B=-M_q+k_B^2(-Z_w-X_u)$ and $\delta B=(-Z_w-X_u)\,\delta k_B^2$. The value of $-Z_w-X_u$ is, let us say, around 5 and the change in B is around 20. Hence the change in t is of the order of magnitude of 1 second, or 5 per cent. This is a much larger change in t than in T, but its influence upon T is negligible. We do notice, however, that the damping time might easily be noticeably increased, though the increase of the period be imperceptible, provided the same degree of precision attached to the measurement of t as to T. At any rate unless it is decidedly harder to determine t accurately, to look for the effect of diminished stability in the value of t would be more promising.

If the aerodynamic constants of the airplane are approximately known and the value of $n=t/T$ is also approximately known the forecast of δt is given by the equation

$$28\delta t=n^2\,\frac{-Z_w\,\delta k_B^2}{Z_w M_q-UM_w}=\frac{28n^2\,\delta k_B^2}{-M_q+UM_w/Z_w}.$$

In this simple equation the ratio M_w/Z_w is likely to be of the order of magnitude of from 1 to 1/3, so that the denominator is, say, about 200. Airplanes differ so much that only the roughest estimates can be expected to hold in general, but the order of magnitude of δt can be readily estimated for a particular case to determine whether the experimental determination of δt is worth attempting. And with reference to the particular data of Norton and Carroll (loc. cit.), the fact that there is no perceptible variation in t (except that due to the change in T, since it is n which is tabulated and does not change) would indicate to me that the change of 0.5 second in T on the average is illusory (as the authors seem to infer) in that it must be within the experimental error; there should theoretically be a decidedly larger percentage increase in n than in T.

ACTUAL COMPUTATION OF THE CHANGES.

An actual calculation of the changes in T and t for the airplane in question can not be made unless all the necessary aerodynamical coefficients are available, and I have not succeeded in finding these coefficients nor material from which they may be calculated. However, if we take the case of the *JN–2* flying at 51.8 M. P. H., from my Aeronautics, page 141, we have a speed not far different from that at which Norton and Carroll operated their S. E. 5A, the ratio of t/T is $n=1.05$, which is very close to their ratio and the actual values of t and T, are not far removed from theirs. The equivalences are sufficiently good for illustrative purposes. We have the following data:

$V=51.8$ M. P. H.	$X_u=-121.$	$X_w=+.113.$	$Z_u=.849.$
$Z_w=-2.26.$	$M_w=+2.45.$	$M_q=-113.$	$k_B^2=34.$
$B_1=194.$	$C_1=467.$	$D_1=64.3.$	$E_1=67.$
$T=16.7.$	$t=17.7.$	M_u, X_q, Z_q all neglected.	

It is not at all likely that many of these data are as precise as the figures indicated; but it is not the absolute values that are under discussion—it is rather the order of magnitude of the changes which are introduced by a variation of k_B^2, other things being kept constant. We have for these changes

$$\delta C = \delta k_B^2 \ (X_u Z_w - X_w Z_u) = 0.37 \ \delta k_B^2,$$
$$\delta B = \delta k_B^2 \ (-Z_w - X_u) = 2.13 \ \delta k_B^2.$$

Note that in this case the change in B is nearly 6 times as great as that in C; whereas the relative change is nearly 15 times as great, because C is so much larger than B. The calculations give the following table:

$\delta k_B^2 =$	−4.0	0	+ 4.0	+ 8.0
% change =	−11.8	0	+11.8	+23.5
$\delta B =$	−8.5	0	+ 8.5	17.0
$B =$	185.5	194	202.5	211.0
$\delta C =$	−1.5	0	+ 1.5	+ 3.0
$C =$	465.5	467	468.5	. 470.0
$t =$	17.2	17.7	18.4	19.2
$\delta t =$	−0.5	0	+ 0.7	+ 1.5
% change =	−2.8	0	+ 4.0	+ 8.5
calc. δt* =	−0.6	0	+ 0.6	+ 1.1
$T =$	16.66	16.68	16.70	16.72
$\delta T =$	−0.02	0	+ 0.02	0.04
% change =	−0.12	0	+ 0.12	+ 0.24

*Calculation by formula $\delta t = 28n^2 \delta B/C$.

CONCLUSION.

This table shows, as was indicated on theoretical grounds, that the change in T is insignificant relative to that in t. The difference in this particular case is more pronounced than could be inferred from the general argument. That argument led to the prediction of a change of less than 0.5 per cent in T for an increase of 15 per cent in k_B^2 and of the consequent impossibility of detecting the change experimentally; the calculated change in T is only 0.12 per cent. On the other hand the table shows clearly, as was demonstrated in the text, that the change of t might be of the order of magnitude of 5 per cent and that the change in n would be practically wholly due to this cause. These results differ from the experimental figures of Norton and Carroll in such a way as to indicate that all their results were identical within the experimental error.

NOTE ON THE SHORT OSCILLATIONS.

In simple harmonic motion, slightly damped $(Wk^2\theta''/g) + R\theta' + F\theta = 0$, the period T is proportional to $(k^2/F)^{1/2}$ and the damping time t to k^2/R. Hence a small percentage increase in k^2 produces an equal percentage increase in t but only half that percentage increase in T and a like amount in the ratio $n = t/T$. The airplane shows the same qualitative phenomenon of a greater sensitivity to k^2 in t than in T, but the quantitative relation is very different; the percentage change in t is only 1/4 to 1/2 that in k^2 whereas the percentage change in T is reduced to a negligible amount. This sort of difference is not surprising in view of the complicated coupled system found in the airplane. It might be interesting to observe that in the short period heavily damped oscillation, which we have ignored, the relative changes are much nearer those found in the simple uncoupled harmonic case.

INVESTIGATION OF THE LATERAL STABILITY.

It might be interesting to see what effect the change in the position of matter should have on lateral stability; for the increase of about 4 units in k_B^2 should produce the same numerical change in k^2 which enters into all the coefficients, except the last, of the biquadratic regulating the lateral stability. There are three types of motion: "Roll," which is so strongly damped that its discussion is uninteresting; "Spiral," which is represented by a single small positive or negative root of the biquadratic; "Dutch roll," which is an oscillatory damped motion. (See Aeronautics, pp. 147–148.) For the spiral motion $\lambda = -E/D$, where E is independent of k_C^2 and D is a large number, measured in the thousands in the notation of my book. The expression for D contains k_C^2 in the product of $gk_C^2L_v$, and a change of 4 units in k_C^2 would make a change of 120 L_v. The numerical value of L_v is of the order of magnitude of 1, but decidedly variable, so that 120 L_v might be anything from 1 to 5 per cent of D. The damping time would therefore increase with k_C^2 by a small amount, say of about the same order of magnitude relatively as in the longitudinal case. However, it would probably be more difficult to measure experimentally on account of the very slow and one-sided (nonoscillatory) subsidence of the motion.

With respect to the "Dutch roll" the approximate quadratic is

$$\lambda^2 + \left(\frac{C}{B} - \frac{E}{D}\right)\lambda + \frac{BD}{B^2 - AC} = 0$$

and the coefficients B, C, D are tolerably complicated.

$$B = -Y_v\, k_A^2\, k_C^2 - L_p\, k_C^2 - N_r\, k_A^2.$$

Here the first two terms are by far the largest and vary directly as k_C^2 so that the percentage increase in B is about the same as in k_C^2. The change in C is much less, and in the same direction which would indicate for C/B a percentage decrease somewhat less than for k_C^2 itself. The increase in D would tend numerically to decrease E/D, which for a fairly stable airplane is considerably less than C/B. Whether the change in E/D conspires with that in C/B depends therefore on whether the airplane is stable or unstable spirally. The net tendency of the increase in k_C^2 would surely be to a decreased stability in Dutch roll if the stability be measured by the time required to damp to half amplitude. It would also seem tolerably clear that in some airplanes of fairly common type the change in the time of damping might be in the neighborhood of 5 to 10 per cent, i. e., in the neighborhood of the percentage change in k_C^2. If, then, the experimental determination of this damping time were of about the same difficulty as the determination of the damping time for the phugoid (estimated relatively to the time and not absolutely), there is a possibility that the effect of the changed distribution of mass fore and aft could be seen fully as easily in the Dutch roll as in the phugoid.

In many cases AC is so small relative to B^2 and E/D relative to C/B that the quadratic may be written

$$\lambda^2 + \frac{C}{B}\lambda + \frac{D}{B} = 0 \text{ or } \lambda = -\frac{C}{2B} \pm i\sqrt{\frac{D}{B} - \frac{1}{4}\left(\frac{C}{B}\right)^2}.$$

In many cases, too, the damping is so great that its effect upon the periodic time can not be neglected as in the case of longitudinal motion. It has been seen that in a general way the percentage change of B is about the same as that in k_C^2 whereas the percentage changes in D and C are in general much less, and all in the same direction. The periodic time may be written

$$T = \frac{2\pi}{\sqrt{\frac{D}{B}}\sqrt{1 - \frac{1}{4}\frac{C}{D}\frac{C}{B}}}.$$

If the change in C/D be ignored and the percentage changes in D/B and C/B be taken as of about the same magnitude and somewhat less than that of B, it is seen that the changes in the factors in the denominator tend to offset each other. It is therefore unlikely that, in an airplane fairly stable in the Dutch roll, the change in k_C^2 should make a percentage change in T as great as one-half that in k_C^2, and under certain circumstances, it might be much less.

With respect to the numerical illustration of the effect of a change in k_C^2 upon stability the conditions are less satisfactory than for the longitudinal motion. I called attention in my article on "The Variation of Yawing Moment Due to Rolling," Report No. 26, from the Fourth Annual Report of the National Advisory Committee for Aeronautics, to the possibility that previous calculations of the coefficient N_p might be incorrect in sign and in numerical magnitude and further that on account of lag in the adjustment of stream lines to a moving airplane or model the values should be checked experimentally. Now it so happens that the coefficient D_2 contains in addition to the term $gk_C^2 L_v$, which affects the damping in spiral motion, the terms $(Y_v L_r + UL_v) N_p$ which in magnitude far exceed any changes in $gk_C^2 L_v$ so that a reversal of sign N_p would be of far greater significance than any practicable change in $gk_C^2 L_v$. A similar remark holds for the coefficient C_2. It is therefore not alone on account of the greater complexity of the changes of damping and period in the case of lateral motion as compared with longitudinal motion that I have found it difficult in the general discussion to be as definite in statement for the lateral case as for the longitudinal, but also because of a lesser confidence in the accuracy of the numerical values for the fundamental coefficients. For this reason it would also appear that until better data are available, the general considerations offered above are as satisfactory as an apparently more accurate display of tabulated calculations, and with less liability to misinterpretation. From a careful consideration of the experimental difficulties I should judge that even though the percentage changes in the periods of damping of the lateral motion be considerably greater than for the longitudinal there would be not so good an opportunity to detect them, let alone interpret them if detected.