REPORT No. 728

ON THE USE OF RESIDUE THEORY FOR TREATING THE SUBSONIC FLOW OF A COMPRESSIBLE FLUID

By Carl Kaplan

SUMMARY

A new mathematical technique, due to Milne-Thomson, is used to obtain an improved form of the method of Poggi for calculating the effect of compressibility on the subsonic flow past an obstacle. By means of this new method, the difficult surface integrals of the original Poggi method can be replaced by line integrals. These line integrals are then solved by the use of residue theory. In this way an equation is obtained giving the second-order effect of compressibility on the velocity of the fluid. The method is, moreover, practicable for obtaining the higher-order effects of compressibility on the velocity field. As an illustration of the general result, the flow past an elliptic cylinder is discussed.

INTRODUCTION

There has been in recent years an increasing interest in the problems of flow in which the compressibility of the fluid is taken into account. The effect of compressibility on the subsonic flow past various simple shapes has been calculated by the method of Janzen and Rayleigh or by a method due to Poggi. Both of these methods, however, involve mathematical difficulties, which discourage their use for further study of compressible fluids. For example, the second-order effect of compressibility on the velocity field around an elliptic cylinder has been calculated approximately by the method of Janzen and Rayleigh but, owing to a certain limitation in the analysis, the result applies only to thick ellipses (reference 1). In order to eliminate this weakness in the theory, the calculation was repeated by the method of Poggi in reference 2, the result being expressed in a closed form. This method, however, also involved considerable difficulty owing to the necessity of evaluating many difficult integrals during the course of the analysis.

In the present paper a method, based on that of Poggi, is presented for dealing with problems involving the subsonic flow of a compressible fluid. Poggi's method consists in regarding a compressible fluid as an incompressible fluid with a continuous distribution of sinks and sources throughout the entire region external to the obstacle. In the determination of the second-order effect of compressibility on the velocity of the fluid, this concept of a compressible fluid leads to a series of double integrals extended over the entire region of flow. In the present paper, by means of a novel mathematical technique utilizing complex notation, the surface integrals are replaced by line integrals. By means of the well-known methods of the calculus of residues it is then shown that the second-order effect of compressibility on the velocity field around an arbitrary shape can be explicitly expressed in terms of residues. The method, moreover, is equally practicable for determining the higher-order effects of compressibility. As an example of the general result, the flow past an elliptic cylinder is discussed.

It is worthy of mention that a paper which also employs complex notation has appeared recently (reference 3). The purpose of that paper was to complete Hooker's treatment of the elliptic cylinder (reference 1). The method used is equivalent to the original Janzen-Rayleigh process except that the differential equation for the velocity potential is expressed in terms of the conjugate complex variables z and $\bar{z}$. The treatment in the paper is limited, however, to the elliptic cylinder, with no attempt made to obtain results applicable to arbitrary shapes.

THE METHOD OF POGGI

The equation of continuity for a compressible fluid moving irrotationally in two dimensions and with the adiabatic relationship between the pressure and the density can be written as follows:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{\frac{\partial\phi}{\partial x}\frac{\partial q^2}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial q^2}{\partial y}}{2c_0^2\left[1-\frac{\gamma-1}{2}\frac{1}{c_0^2}\left(q^2-U^2\right)\right]} \tag{1}$$

where

ϕ velocity potential
c_0 velocity of sound in undisturbed stream
U velocity of undisturbed stream
q magnitude of velocity of fluid
γ ratio of specific heats at constant pressure and constant volume

705899 O - 46 - 4

Equation (1) may also be written as

$$\left[1-\frac{\gamma-1}{2}M^2\left(\frac{q^2}{U^2}-1\right)\right]\Delta\phi$$
$$=\frac{1}{2}M^2\left[\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\left(\frac{q^2}{U^2}\right)+\frac{\partial\phi}{\partial y}\frac{\partial}{\partial y}\left(\frac{q^2}{U^2}\right)\right] \tag{2}$$

where the symbol Δ denotes the Laplacian operator

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

and where M is the Mach number U/c_0. It is next assumed that ϕ can be developed in a series of ascending powers of M^2. Thus

$$\phi=\phi_0+\phi_1M^2+\phi_2M^4+\ .\ .\ . \tag{3}$$

Then if a complex velocity w is defined as

$$w=-u+iv=\frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial y}$$

it follows that

$$w=w_0+w_1M^2+w_2M^4+\ .\ .\ . \tag{4}$$

where

$$w_0=\frac{\partial\phi_0}{\partial x}-i\frac{\partial\phi_0}{\partial y},\ w_1=\frac{\partial\phi_1}{\partial x}-i\frac{\partial\phi_1}{\partial y},\text{ and so on.}$$

The magnitude q of the velocity of the fluid is given by

$$q^2=w\overline{w}=(-u+iv)(-u-iv)=u^2+v^2$$

or from equation (4),

$$q^2=w_0\overline{w}_0+(w_0\overline{w}_1+\overline{w}_0w_1)M^2$$
$$+(w_0\overline{w}_2+\overline{w}_0w_2+w_1\overline{w}_1)M^4+\ .\ .\ . \tag{5}$$

If the expressions for ϕ and q^2 from equations (3) and (5) are substituted in equation (2) and the terms of the same powers of M on both sides of the equation are equated, it follows that

$$\Delta\phi_0=0 \tag{6a}$$

$$\Delta\phi_1=\frac{1}{2}\left[\frac{\partial\phi_0}{\partial x}\frac{\partial}{\partial x}\left(\frac{w_0\overline{w}_0}{U^2}\right)+\frac{\partial\phi_0}{\partial y}\frac{\partial}{\partial y}\left(\frac{w_0\overline{w}_0}{U^2}\right)\right] \tag{6b}$$

$$\Delta\phi_2=\frac{1}{2}(\gamma-1)\left(\frac{w_0\overline{w}_0}{U^2}-1\right)\Delta\phi_1$$
$$+\frac{1}{2}\left[\frac{\partial\phi_0}{\partial x}\frac{\partial}{\partial x}\left(\frac{w_0\overline{w}_1+\overline{w}_0w_1}{U^2}\right)+\frac{\partial\phi_0}{\partial y}\frac{\partial}{\partial y}\left(\frac{w_0\overline{w}_1+\overline{w}_0w_1}{U^2}\right)\right]$$
$$+\frac{1}{2}\left[\frac{\partial\phi_1}{\partial x}\frac{\partial}{\partial x}\left(\frac{w_0\overline{w}_0}{U^2}\right)+\frac{\partial\phi_1}{\partial y}\frac{\partial}{\partial y}\left(\frac{w_0\overline{w}_0}{U^2}\right)\right] \tag{6c}$$

and so on for the higher powers of M.

It is to be observed that equation (6a) is Laplace's equation for the flow of an incompressible fluid. Thus, if the known solution for the incompressible fluid is used as the first approximation, the higher approximations for the compressible fluid may be obtained successively from equations (6b), (6c), etc. Poggi's method consists essentially in considering the compressible fluid to be an incompressible fluid with a continuous distribution of sinks and sources in the entire region external to the solid boundary. According to Poggi, then, the right-hand sides of equations (6) represent successive terms in an infinite series giving this sink-source distribution. The first approximation to the strength of the sink-source distribution in the plane of flow is therefore given by

$$-\frac{1}{4\pi U^2}\left[\frac{\partial\phi_0}{\partial x}\frac{\partial}{\partial x}(w_0\overline{w}_0)+\frac{\partial\phi_0}{\partial y}\frac{\partial}{\partial y}(w_0\overline{w}_0)\right]dx\,dy$$

If new independent variables $z=x+iy$ and $\overline{z}=x-iy$ are introduced, then symbolically,

$$\left.\begin{aligned}\frac{\partial}{\partial x}&=\frac{\partial}{\partial z}+\frac{\partial}{\partial\overline{z}}\\ \frac{\partial}{\partial y}&=i\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial\overline{z}}\right)\end{aligned}\right\} \tag{7}$$

and the expression for the strength of the sink-source distribution becomes

$$-\frac{1}{2\pi U^2}\left[\frac{\partial\phi_0}{\partial z}\frac{\partial}{\partial\overline{z}}(w_0\overline{w}_0)+\frac{\partial\phi_0}{\partial\overline{z}}\frac{\partial}{\partial z}(w_0\overline{w}_0)\right]dx\,dy \tag{8}$$

This expression can be further simplified by means of the following considerations: From the definition of the complex velocity,

$$w=\frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial y}$$

it follows, by use of equations (7), that

$$w=2\frac{\partial\phi}{\partial z}\text{ and }\overline{w}=2\frac{\partial\phi}{\partial\overline{z}} \tag{9}$$

In particular, w_0, being the complex velocity of an incompressible fluid, is a function of z only. Expression (8) can therefore be written as

$$-\frac{1}{4\pi U^2}\left(w_0^2\frac{d\overline{w}_0}{d\overline{z}}+\overline{w}_0^2\frac{dw_0}{dz}\right)dx\,dy \tag{10}$$

Let the plane z of the obstacle be represented conformally on the plane Z of the corresponding circle. Since the strengths of the sink-source distribution of corresponding elements of the two planes are equal, it follows that the expression for the strength of the sink-source distribution of an element of the plane Z is given by

$$-\frac{1}{4\pi U^2}\left[W_0^2\frac{dZ}{dz}\frac{d}{d\overline{Z}}\left(\overline{W}_0\frac{d\overline{Z}}{d\overline{z}}\right)+\overline{W}_0^2\frac{d\overline{Z}}{d\overline{z}}\frac{d}{dZ}\left(W_0\frac{dZ}{dz}\right)\right]dX\,dY \tag{11}$$

where $dx\,dy$ has been replaced by $\frac{dz}{dZ}\frac{d\overline{z}}{d\overline{Z}}dX\,dY$ and w_0 by $W_0\frac{dZ}{dz}$, where W_0 is the complex velocity of an incompressible fluid past the circle in the plane Z.

If the radius of the circle into which the profile of the obstacle is mapped is assumed to be R_1, the complex velocity induced at any point Z_P external to the circle by a source of unit strength is given by

$$\frac{1}{Z-Z_P}+\frac{1}{\frac{R_1^2}{\overline{Z}}-Z_P}+\frac{1}{Z_P}=\frac{1}{Z-Z_P}+\frac{R_1^2/Z_P}{R_1^2-Z_P\overline{Z}}$$

where there are unit sources at the point $Q(Z)$ and the inverse point $R\left(\frac{R_1^2}{\overline{Z}}\right)$ and a unit sink at the center of the circle (fig. 1). It follows, therefore, that the complex velocity induced at the point Z_P by the sink-source distribution given by expression (11) is

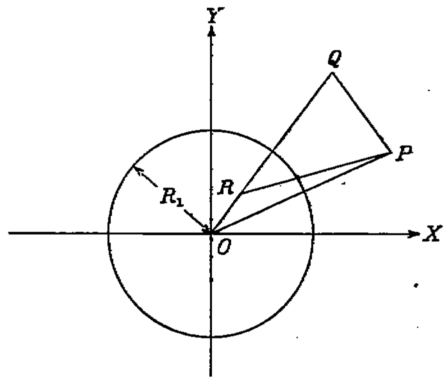

FIGURE 1.—Image of a simple source with regard to a circle.

$$(W_1)_P=-\frac{1}{4\pi U^2}\int\int\frac{d}{d\overline{Z}}\left(\frac{1}{Z-Z_P}W_0^2\overline{W}_0\frac{dZ}{dz}\frac{d\overline{Z}}{d\overline{z}}\right)dX\,dY$$
$$-\frac{1}{4\pi U^2}\int\int\frac{d}{dZ}\left(\frac{R_1^2/Z_P}{R_1^2-Z_P\overline{Z}}\overline{W}_0^2W_0\frac{dZ}{dz}\frac{d\overline{Z}}{d\overline{z}}\right)dX\,dY$$
$$-\frac{1}{4\pi U^2}\int\int\frac{1}{Z-Z_P}\frac{d}{dZ}\left(\overline{W}_0^2W_0\frac{dZ}{dz}\frac{d\overline{Z}}{d\overline{z}}\right)dX\,dY$$
$$-\frac{1}{4\pi U^2}\int\int\frac{R_1^2/Z_P}{R_1^2-Z_P\overline{Z}}\frac{d}{d\overline{Z}}\left(W_0^2\overline{W}_0\frac{dZ}{dz}\frac{d\overline{Z}}{d\overline{z}}\right)dX\,dY \qquad (12)$$

where the integrations are performed over the entire region external to the circle.

As noted before, $(W_1)_P$ is the complex velocity induced at any point Z_P external to the circular boundary by a sink-source distribution originating in the plane z of the obstacle. The actual velocity $(w_1)_P$ of the fluid at the corresponding point z_P in the plane z of the obstacle is given by

$$w_1=W_1\frac{dZ}{dz} \qquad (13)$$

the subscript P having been dropped. It will be observed that w_1 is expressed as a function of the conjugate complex variables Z and $\overline{Z}$ of the plane of the circle.

THE METHOD OF RESIDUES

Equation (12) in its present form appears to be unmanageable. It can be transformed, however, into a form suitable for further treatment by means of the following theorem:

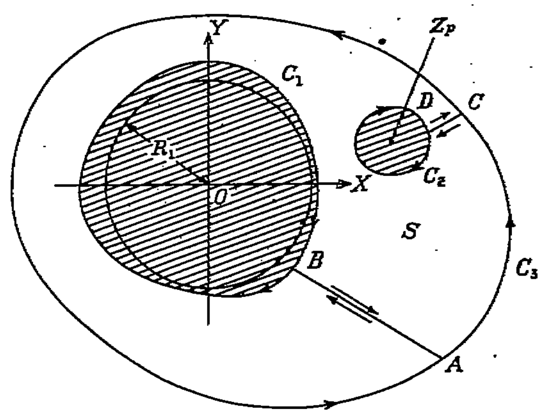

FIGURE 2.—Illustration of Stokes' theorem for the plane.

Consider a contour in the plane Z of the circle consisting of a closed curve C_1 enclosing the circle of radius R_1, a small closed curve C_2 enclosing the point Z_P, a very large closed curve C_3 enclosing both C_1 and C_2, and the lines AB and CD connecting, respectively, the curves C_3 and C_1, and C_3 and C_2 (fig. 2). The contour is described in such a manner as to leave the area S, regarded as enclosed by it, on the left. Now, suppose $F(X,Y)$ and $G(X,Y)$ to be vector components along the X and Y axes, respectively. Then Stokes' theorem for the plane states that

$$\int\int_S\left(\frac{\partial G}{\partial X}-\frac{\partial F}{\partial Y}\right)dX\;dY=\int_C(FdX+GdY)$$
$$=\int_{C_3}(FdX+GdY)-\int_{C_2}(FdX+GdY)-\int_{C_1}(FdX+GdY) \qquad (14)$$

where the line integrals on the right-hand side are taken in the counterclockwise sense.

If $g(Z,\overline{Z})$ is a function of the independent variables $Z,\overline{Z}$ continuous and differentiable in the region S enclosed by the contour C, then since

$$2\frac{\partial g}{\partial Z}=\frac{\partial g}{\partial X}-i\frac{\partial g}{\partial Y}$$

it follows that

$$2\int\int_S\frac{\partial g}{\partial Z}dX\;dY=\int\int_S\left(\frac{\partial g}{\partial X}-i\frac{\partial g}{\partial Y}\right)dX\;dY$$

By Stokes' theorem, equation (14), it is seen that

$$\int\int_S\frac{\partial g}{\partial Z}dX\;dY=\frac{1}{2}\int_C(gdY+igdX)=\frac{1}{2}i\int_C gd\overline{Z}$$

or

$$\int\int_S\frac{\partial g}{\partial Z}dX\;dY=\frac{1}{2}i\int_{C_3}gd\overline{Z}-\frac{1}{2}i\int_{C_2}gd\overline{Z}-\frac{1}{2}i\int_{C_1}gd\overline{Z} \qquad (15)$$

Similarly

$$\int\int_S\frac{\partial g}{\partial \overline{Z}}dX\;dY=-\frac{1}{2}i\int_{C_3}gdZ+\frac{1}{2}i\int_{C_2}gdZ+\frac{1}{2}i\int_{C_1}gdZ \qquad (16)$$

These equations were first given by L. M. Milne-Thomson (reference 4).

In the present problem where the area S is the entire region external to the circle of radius R_1, the curve C_1 is the circle itself, the curve C_2 is a vanishingly small circle with the point Z_P as center and the curve C_3 is an infinitely large circle with its center at the origin. By means of equations (15) and (16) it is possible to replace the surface integrals of equation (12) by line integrals around the circles C_1, C_2, and C_3. For the purpose of evaluating the line integrals it will be observed that it is possible to make the integrands analytic. The solution of any given problem therefore reduces to the evaluation of residues, for they form the only contributions to the integral of a function that is analytic at all points except singularities. In order to transform the surface integrals into line integrals, it is first necessary to express the integrands as derivatives with regard to either Z or $\overline{Z}$. It is to be noted that the first two integrands of equation (12) are already in this form. The last two integrands can be brought into the required form in the following way: It is observed that

$$\left.\begin{aligned}\overline{W}_0^2\frac{d\overline{Z}}{d\overline{z}}&=\frac{d}{d\overline{Z}}\left(\int\overline{W}_0^2\frac{d\overline{Z}}{d\overline{z}}d\overline{Z}\right)\\ W_0^2\frac{dZ}{dz}&=\frac{d}{dZ}\left(\int W_0^2\frac{dZ}{dz}dZ\right)\end{aligned}\right\}\tag{17}$$

The third and the fourth integrands of equation (12) become, respectively,

$$\left.\begin{aligned}&\frac{d}{d\overline{Z}}\left[\frac{1}{Z-Z_P}\frac{d}{dZ}\left(W_0\frac{dZ}{dz}\right)\int\overline{W}_0^2\frac{d\overline{Z}}{d\overline{z}}d\overline{Z}\right]\\ &\frac{d}{dZ}\left[\frac{R_1^2/Z_P}{R_1^2-Z_P\overline{Z}}\frac{d}{d\overline{Z}}\left(\overline{W}_0\frac{d\overline{Z}}{d\overline{z}}\right)\int W_0^2\frac{dZ}{dz}dZ\right]\end{aligned}\right\}\tag{18}$$

By means of equations (15) and (16) it follows that the line-integral form of equation (12) is

$$\begin{aligned}(W_1)_P=&-\frac{1}{8\pi iU^2}\int\frac{1}{Z-Z_P}F(Z,\overline{Z})dZ\\ &-\frac{R_1^2/Z_P^2}{8\pi iU^2}\int\frac{1}{\overline{Z}-R_1^2/Z_P}\overline{F}(\overline{Z},Z)d\overline{Z}\end{aligned}\tag{19}$$

where

$$\left.\begin{aligned}F(Z,\overline{Z})&=W_0^2\overline{W}_0\frac{dZ}{dz}\frac{d\overline{Z}}{d\overline{z}}+\frac{d}{dZ}\left(W_0\frac{dZ}{dz}\right)\int\overline{W}_0^2\frac{d\overline{Z}}{d\overline{z}}d\overline{Z}\\ \overline{F}(\overline{Z},Z)&=\overline{W}_0^2W_0\frac{dZ}{dz}\frac{d\overline{Z}}{d\overline{z}}+\frac{d}{d\overline{Z}}\left(\overline{W}_0\frac{d\overline{Z}}{d\overline{z}}\right)\int W_0^2\frac{dZ}{dz}dZ\end{aligned}\right\}\tag{20}$$

The integrals of equation (19) are taken successively around the circle C_1 of radius R_1, the circle C_2 of vanishingly small radius R_2, and the circle C_3 of infinitely large radius R_3. Each of these integrals is taken in the counterclockwise sense, and therefore due regard must be given to the sign in accordance with equation (15); that is, the sign of the integrals around C_1 and C_2 is negative and that around C_3 is positive.

The result of the integrations around the circular contours C_1, C_2, and C_3 can be explicitly expressed by means of Cauchy's theorem on residues. This theorem states that, if $f(Z)$ be analytic on a contour C and throughout its interior except at a number of poles inside the contour, then

$$\int_C f(Z)dZ=2\pi iM\tag{21}$$

where M denotes the sum of the residues of $f(Z)$ at those poles which lie within the contour C. Similarly,

$$\int_C g(\overline{Z})d\overline{Z}=-2\pi iN\tag{22}$$

where N denotes the sum of the residues of $g(\overline{Z})$ at those poles that lie within the contour C.

Consider, for the moment, only the first integral of equation (19). In general, $F(Z,\overline{Z})$ is a function of both Z and $\overline{Z}$. On the contour of integration C, however, $\overline{Z}=R^2/Z$ and therefore $F(Z,\overline{Z})=F(Z,R^2/Z)$. Thus, an analytic function of Z has been created and the theorem of residues given by equation (21) can be applied. Similar considerations hold for the second integral of equation (19) with $Z=R^2/\overline{Z}$, but in this case the theorem of residues takes the form given by equation (22). It is to be further noted that $\overline{F}(\overline{Z},Z)$ is the conjugate complex of $F(Z,\overline{Z})$. This fact simplifies matters to a great extent in that the result of the second integration can be written down immediately from the result of the first integration.

Consider the first line integral of equation (19):

In the integral around the circle C_1 of radius R_1 with center at the origin, the poles of $F(Z,\overline{Z})$ associated with the terms involving only the variable Z lie within the contour, while the poles associated with the terms involving only the variable $\overline{Z}$ lie outside the contour because $\overline{Z}=R_1^2/Z$. The poles lying outside the contour together with the simple pole at $Z=Z_P$ do not contribute to the integral around C_1. In addition, since $\overline{Z}=R_1^2/Z$, there will correspond to each zero of $\overline{Z}$ a pole of Z at the origin. Then, according to equation (21), the result of the integral around C_1 is given by

$$\int_{C_1}\frac{1}{Z-Z_P}F(Z,\overline{Z})dZ=2\pi iS(Z_P)\tag{23}$$

where $S(Z_P)$ denotes the sum of the residues of $\frac{1}{Z-Z_P}F\left(Z,\frac{R_1^2}{Z}\right)$ at the poles within the contour C_1.

In the integral around the circle C_2 of vanishingly small radius R_2 with center at $Z=Z_P$, the only pole within the contour is the simple pole at $Z=Z_P$. Since, in the limit as the radius $R_2\rightarrow0$, $Z\rightarrow Z_P$ and $\overline{Z}\rightarrow\overline{Z}_P$, it follows from equation (21) that

$$\int_{C_2}\frac{1}{Z-Z_P}F(Z,\overline{Z})dZ=2\pi iF(Z_P,\overline{Z}_P)\tag{24}$$

In the calculation of the integral around the circle C_3 of very large radius R_3, the first step is to replace $\overline{Z}$ by R_3^2/Z and thus to render $F(Z,\overline{Z})$ analytic. Furthermore, since the radius R_3 is ultimately made to approach infinity, it is expedient to expand the integrand

$$\frac{1}{Z-Z_P}F(Z,R_3^2/Z)$$

in the neighborhood of infinity. The residue is then simply the limit of the coefficient of $1/Z$ as $R_3\rightarrow\infty$. It will be observed that this coefficient is the constant term in the expansion of $F(Z,R_3^2/Z)$. This calculation can be performed for an arbitrary profile in the z plane in the following way:

The conformal transformation that converts the profile in the z plane into a circle with center at the origin of the Z plane and leaves the region at infinity unaltered is of the type

$$z=Z+a_0+\frac{a_1}{Z}+\frac{a_2}{Z^2}+\ .\ .\ . \tag{25}$$

where the coefficients a_0, a_1, a_2, . . . are, in general, complex numbers.

Now suppose that the undisturbed flow of velocity U is inclined at an angle α to the negative direction of the real axis and that the circulation Γ is arbitrary. In terms of the complex coordinate Z, with origin at the center of the circle of radius R_1, the potential function of the incompressible flow past the circle is

$$f_0(Z)=U\left(Ze^{i\alpha}+\frac{R_1^2e^{-i\alpha}}{Z}\right)+\frac{i\Gamma}{2\pi}\log Z \tag{26}$$

For this flow, the complex velocity is given by

$$W_0=\frac{df_0}{dZ}=U\left(e^{i\alpha}-\frac{R_1^2e^{-i\alpha}}{Z^2}\right)+\frac{i\Gamma}{2\pi}\frac{1}{Z} \tag{27}$$

By means of equations (25) and (27) it can be shown very easily that in the limit $R_3\rightarrow\infty$ the constant term in the expansion of $F(Z,R_3^2/Z)$ is simply $U^3e^{i\alpha}$. It follows therefore from the residue theorem, equation (21), that

$$\int_{C_3}\frac{1}{Z-Z_P}F(Z,\overline{Z})dZ=2\pi iU^3e^{i\alpha} \tag{28}$$

Consider the second line integral of equation (19): As noted before, the second integral of equation (19) can be obtained immediately from the first. Thus, corresponding to equation (23), the integral around the circle C_1 is given by

$$\int_{C_1}\frac{1}{\overline{Z}-\frac{R_1^2}{Z_P}}\overline{F}(\overline{Z},Z)d\overline{Z}=-2\pi i\overline{S}\left(\frac{R_1^2}{Z_P}\right) \tag{29}$$

where $\overline{S}(R_1^2/Z_P)$ is the conjugate complex of $S(Z_P)$ except that Z_P is replaced by R_1^2/Z_P and not by $\overline{Z}_P$, and where, for convenience, the internal pole at $\overline{Z}=R_1^2/Z_P$ has been excluded in this calculation.

The integral around the circle C_2 yields nothing because there are no poles of $\overline{Z}$ within this contour. However, analogous to the residue at the simple pole at $Z=Z_P$ within the small circle C_2 given by equation (24), there is a residue at the simple pole at $\overline{Z}=R_1^2/Z_P$ within the circle C_1 (excluded from equation (29)) given by

$$\int_{C_1}\frac{1}{\overline{Z}-\frac{R_1^2}{Z_P}}\overline{F}(\overline{Z},Z)d\overline{Z}=-2\pi i\overline{F}\left(\frac{R_1^2}{Z_P},Z_P\right) \tag{30}$$

where, in the evaluation of the integral, $\overline{Z}$ has been replaced by R_1^2/Z_P and, on account of the substitution $Z=R_1^2/\overline{Z}$, Z has been replaced by Z_P.

Finally, for the integral around the circle C_3 of infinitely large radius R_3, since $\overline{F}(\overline{Z},Z)$ is the conjugate complex of $F(Z,\overline{Z})$, it follows, according to equation (28), that the residue of

$$\frac{1}{\overline{Z}-\frac{R_1^2}{Z_P}}\overline{F}(\overline{Z},Z)$$

is $U^3e^{-i\alpha}$ and therefore

$$\int_{C_3}\frac{1}{\overline{Z}-\frac{R_1^2}{Z_P}}\overline{F}(\overline{Z},Z)d\overline{Z}=-2\pi iU^3e^{-i\alpha} \tag{31}$$

In the summing up of these results, it should be remembered that a negative sign is attached to the integrals around C_1 and C_2 and a positive sign to the integral around C_3. It follows that

$$W_1=\frac{1}{4U^2}\left[S(Z)-\frac{R_1^2}{Z^2}\overline{S}\left(\frac{R_1^2}{Z}\right)-\frac{R_1^2}{Z^2}\overline{F}\left(\frac{R_1^2}{Z},Z\right)\right]+\frac{1}{4U^2}F(Z,\overline{Z})-\frac{U}{4}\left(e^{i\alpha}-\frac{R_1^2e^{-i\alpha}}{Z^2}\right) \tag{32}$$

where the subscript P has been dropped and where

$F(Z,\overline{Z})$ given by first of equations (20)

$\overline{F}\left(\frac{R_1^2}{Z},Z\right)$ obtained from expression for $\overline{F}(\overline{Z},Z)$ by replacing $\overline{Z}$ by R_1^2/Z

$S(Z)$ given by equation (23)

$\overline{S}\left(\frac{R_1^2}{Z}\right)$ obtained from expression for $\overline{S}(\overline{Z})$ by replacing $\overline{Z}$ by R_1^2/Z

The last term of equation (32) is equal to $-\frac{1}{4}$ the complex velocity of the incompressible flow without circulation past a circular cylinder of radius R_1.

In equation (32), the terms in the brackets and the last term (representing, respectively, the integrals around the circles C_1 and C_3) are analytic functions of Z and are therefore solutions of Laplace's equation. On the other hand, the second term in equation (32) (representing the integral around the circle C_2) is a non-analytic function of $Z,\overline{Z}$ and is a particular solution of Poisson's equation. Furthermore, the complementary solution and the particular solution are such that, taken together, they satisfy the appropriate boundary conditions of the problem; that is, the velocity of the fluid normal to the boundary C_1 is zero and the velocity of the fluid at infinity is zero.

Finally, according to equations (4) and (13), it follows that the second approximation to the actual velocity of a compressible fluid at any point z external to the obstacle is given by

$$w=(W_0+W_1M^2)\frac{dZ}{dz} \tag{33}$$

where W_1 is given by equation (32) and where W_0, the complex velocity of an incompressible fluid past the circular cylinder of radius R_1, is given by equation (27).

In conclusion, it can be stated that, when the set of surface integrals given by equation (12) are replaced by the solution in the form of equation (32), the labor involved in the calculation of the second-order effect of compressibility has been reduced to a minimum, for it is necessary only to evaluate residues. The same process can be used, moreover, to derive equations analogous to equation (32) for the higher-order effects of compressibility on the flow past arbitrary shapes.

The general result of this paper given by equation (32) will now be applied, mainly for the purpose of illustration, to the case of an elliptic cylinder.

APPLICATION TO THE ELLIPTIC CYLINDER

It is known that the region external to a circle with center at the origin of the Z plane is mapped on the region external to an ellipse with foci at $z=-c$ and $z=c$ by the transformation

$$z=Z+\frac{c^2}{4Z} \tag{34}$$

Thus, to a circle of radius $c/2$ with center at the origin of the Z plane, there corresponds a line segment extending from $z=-c$ to $z=c$ in the z plane; and, to a concentric circle of radius $R_1(>c/2)$ in the Z plane, there corresponds an ellipse in the z plane with foci at $z=-c$ and $z=c$ and with semiaxes given by

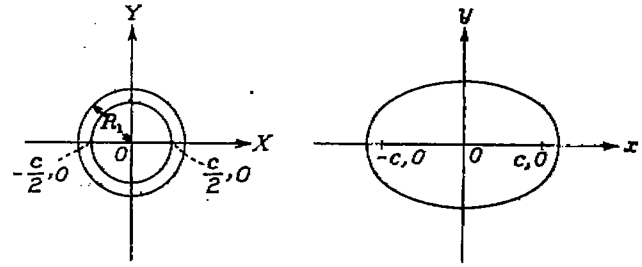

FIGURE 3.—Conformal mapping of an ellipse into a circle.

$$a=R_1+\frac{c^2}{4R_1},\ b=R_1-\frac{c^2}{4R_1} \qquad \text{(See fig. 3.)}$$

By means of equation (34) the streamlines of the flow past the circle can be transformed into those of the corresponding flow past the ellipse. The irrotational flow of an incompressible fluid past a circle of radius R_1 is given by the complex potential

$$f_0(Z)=U\left(Ze^{i\alpha}+\frac{R_1^2e^{-i\alpha}}{Z}\right) \tag{35}$$

where the undisturbed flow is of velocity U inclined at an angle α to the negative direction of the real axis and where the circulation is zero.

Before the various terms of equation (32) for the case of the elliptic cylinder are computed, the process will be simplified by the introduction of the nondimensional quantities $\frac{R_1}{c/2}$, $\frac{Z}{c/2}$, and $\frac{z}{c/2}$, with the letters R_1, Z, and z being retained to denote the corresponding nondimensional quantities. It follows then from equation (34) that

$$\left.\begin{aligned}\frac{dZ}{dz}&=\frac{1}{1-\frac{1}{Z^2}}\\ \frac{d\overline{Z}}{d\overline{z}}&=\frac{1}{1-\frac{1}{\overline{Z}^2}}\end{aligned}\right\} \tag{36}$$

and from equation (35) that

$$\left.\begin{aligned}W_0&=U\left(e^{i\alpha}-\frac{R_1^2e^{-i\alpha}}{Z^2}\right)\\ \overline{W}_0&=U\left(e^{-i\alpha}-\frac{R_1^2e^{i\alpha}}{\overline{Z}^2}\right)\end{aligned}\right\} \tag{37}$$

By means of equations (36) and (37) it follows easily, after the elementary integration indicated in the first of equations (20) is performed, that

$$F(Z,\bar{Z})=U^3\left(e^{i\alpha}-\frac{R_1^2e^{-i\alpha}}{Z^2}\right)^2\left(e^{-i\alpha}-\frac{R_1^2e^{i\alpha}}{\bar{Z}^2}\right)\frac{1}{\left(1-\frac{1}{Z^2}\right)\left(1-\frac{1}{\bar{Z}^2}\right)}$$

$$-2U^3(e^{i\alpha}-R_1^2e^{-i\alpha})\frac{1}{Z^3\left(1-\frac{1}{Z^2}\right)^2}\left[e^{-2i\alpha}\bar{Z}+\frac{R_1^4e^{2i\alpha}}{\bar{Z}}+\frac{1}{2}(e^{-i\alpha}-R_1^2e^{i\alpha})^2\log\frac{\bar{Z}-1}{\bar{Z}+1}\right] \tag{38}$$

In order to evaluate $S(Z_P)$, given by equation (23), the following expression is needed:

$$F\left(Z,\frac{R_1^2}{Z}\right)=\frac{U^3R_1^2(Z^2e^{i\alpha}-R_1^2e^{-i\alpha})^3}{Z^2(Z+1)(Z-1)(Z+R_1^2)(Z-R_1^2)}$$

$$-2U^3(e^{i\alpha}-R_1^2e^{-i\alpha})\left[R_1^2e^{-2i\alpha}+R_1^2e^{2i\alpha}Z^2+\frac{1}{2}(e^{-i\alpha}-R_1^2e^{i\alpha})^2Z\log\frac{R_1^2-Z}{R_1^2+Z}\right]\frac{1}{(Z+1)^2(Z-1)^2} \tag{39}$$

In the first term of this expression, there is a double pole at the origin, simple poles at $Z=-1$ and $Z=1$ inside the circle C_1 of radius $R_1(>1)$, and simple poles at $Z=-R_1^2$ and $Z=R_1^2$ outside the circle C_1. The poles outside the contour C_1 do not contribute to the residue $S(Z_P)$. In the second term, there are double poles at $Z=-1$ and $Z=1$ inside the circle C_1.

The necessary technique for evaluating the residue at a multiple pole is as follows:

Suppose the function $f(Z)$ has a pole of order n at $Z=a$ so that $(Z-a)^nf(Z)$ is analytic within and on the contour enclosing the point a. Then the residue at the multiple pole is

$$\frac{1}{(n-1)}\frac{d^{n-1}}{dZ^{n-1}}\left[(Z-a)^nf(Z)\right]_{Z=a} \tag{40}$$

By the use of this rule, it follows readily from equations (23) and (39) that

$$S(Z)=U^3\left\{\frac{R_1^2}{R_1^4-1}\frac{(e^{i\alpha}-R_1^2e^{-i\alpha})^3}{Z^2-1}+\frac{R_1^4e^{-3i\alpha}}{Z^2}+\frac{2R_1^2(e^{i\alpha}-R_1^2e^{-i\alpha})}{(Z^2-1)^2}(Z^2e^{2i\alpha}+e^{-2i\alpha})\right.$$

$$\left.-\frac{1}{(Z^2-1)^2}(e^{i\alpha}-R_1^2e^{-i\alpha})(e^{-i\alpha}-R_1^2e^{i\alpha})^2\left[Z^2\left(\frac{R_1^2}{R_1^4-1}+\frac{1}{2}\log\frac{R_1^2+1}{R_1^2-1}\right)-\left(\frac{R_1^2}{R_1^4-1}-\frac{1}{2}\log\frac{R_1^2+1}{R_1^2-1}\right)\right]\right\} \tag{41}$$

where the subscript P has been dropped.

The expression for $\bar{S}(R_1^2/Z)$ follows from the expression for $S(Z)$ if i is replaced by $-i$ and Z by R_1^2/Z. Thus

$$\bar{S}\left(\frac{R_1^2}{Z}\right)=U^3\left\{\frac{R_1^2}{R_1^4-1}\frac{Z^2(e^{-i\alpha}-R_1^2e^{i\alpha})^3}{R_1^4-Z^2}+e^{3i\alpha}Z^2+\frac{2R_1^2Z^2(e^{-i\alpha}-R_1^2e^{i\alpha})}{(R_1^4-Z^2)^2}(R_1^4e^{-2i\alpha}+Z^2e^{2i\alpha})\right.$$

$$\left.-\frac{Z^2}{(R_1^4-Z^2)^2}(e^{-i\alpha}-R_1^2e^{i\alpha})(e^{i\alpha}-R_1^2e^{-i\alpha})^2\left[R_1^4\left(\frac{R_1^2}{R_1^4-1}+\frac{1}{2}\log\frac{R_1^2+1}{R_1^2-1}\right)-Z^2\left(\frac{R_1^2}{R_1^4-1}-\frac{1}{2}\log\frac{R_1^2+1}{R_1^2-1}\right)\right]\right\} \tag{42}$$

The expression for $\bar{F}(R_1^2/Z,Z)$ is obtained from equation (38) by first forming the conjugate complex expression $\bar{F}(\bar{Z},Z)$ and then replacing $\bar{Z}$ by R_1^2/Z. Thus,

$$\bar{F}\left(\frac{R_1^2}{Z},Z\right)=U^3\left[\frac{(e^{i\alpha}Z^2-R_1^2e^{-i\alpha})^3}{(Z^2-1)(R_1^4-Z^2)}+\frac{2R_1^2Z^2(R_1^2e^{i\alpha}-e^{-i\alpha})(e^{2i\alpha}Z^2+R_1^4e^{-2i\alpha})}{(R_1^4-Z^2)^2}\right.$$

$$\left.+\frac{R_1^2Z^3(e^{i\alpha}-R_1^2e^{-i\alpha})^2(R_1^2e^{i\alpha}-e^{-i\alpha})}{(R_1^4-Z^2)^2}\log\frac{Z-1}{Z+1}\right] \tag{43}$$

The expression for $F(Z,\bar{Z})$ is given directly by equation (38) and can be rewritten in the following form:

$$F(Z,\bar{Z})=U^3\left\{-\left(e^{i\alpha}-\frac{R_1^2e^{-i\alpha}}{Z^2}\right)^2\left(\frac{R_1^2e^{i\alpha}}{\bar{Z}^2}-e^{-i\alpha}\right)\frac{Z^2\bar{Z}^2}{(Z^2-1)(\bar{Z}^2-1)}\right.$$
$$\left.-2(e^{i\alpha}-R_1^2e^{-i\alpha})\frac{Z}{(Z^2-1)^2}\left[e^{-2i\alpha}\bar{Z}+\frac{R_1^4e^{2i\alpha}}{\bar{Z}}+\frac{1}{2}(R_1^2e^{i\alpha}-e^{-i\alpha})^2\log\frac{\bar{Z}-1}{\bar{Z}+1}\right]\right\} \qquad (44)$$

If the foregoing expressions for $S(Z)$, $\bar{S}(R_1^2/Z)$, $\bar{F}\left(\frac{R_1^2}{Z},Z\right)$, and $F(Z,\bar{Z})$ are inserted into equation (32), it follows that

$$\frac{W_1}{U}=-\frac{1}{4}\left(e^{i\alpha}-\frac{R_1^2e^{-i\alpha}}{Z^2}\right)-\frac{1}{4}R_1^2\left(e^{3i\alpha}-\frac{R_1^2e^{-3i\alpha}}{Z^2}\right)-\frac{1}{4}R_1^2\frac{(e^{2i\alpha}-e^{-2i\alpha})(e^{i\alpha}-R_1^2e^{-i\alpha})}{Z^2-1}+\frac{1}{2}R_1^2\frac{(e^{i\alpha}-R_1^2e^{-i\alpha})(Z^2e^{2i\alpha}+e^{-2i\alpha})}{(Z^2-1)^2}$$
$$-\frac{1}{4}R_1^4\frac{(e^{2i\alpha}-e^{-2i\alpha})(R_1^2e^{i\alpha}-e^{-i\alpha})}{Z^2-R_1^4}+\frac{1}{4}R_1^2\frac{(e^{i\alpha}Z^2-R_1^2e^{-i\alpha})^2}{Z^2(Z^2-1)(Z^2-R_1^4)}$$
$$-\frac{1}{8}(R_1^2e^{i\alpha}-e^{-i\alpha})(e^{i\alpha}-R_1^2e^{-i\alpha})\left[\frac{Z^2+1}{(Z^2-1)^2}(R_1^2e^{i\alpha}-e^{-i\alpha})+R_1^2\frac{Z^2+R_1^4}{(Z^2-R_1^4)^2}(e^{i\alpha}-R_1^2e^{-i\alpha})\right]\log\frac{R_1^2+1}{R_1^2-1}$$
$$+\frac{1}{4}R_1^4\frac{Z(e^{i\alpha}-R_1^2e^{-i\alpha})^2(R_1^2e^{i\alpha}-e^{-i\alpha})}{(Z^2-R_1^4)^2}\log\frac{Z+1}{Z-1}-\frac{1}{4}\left\{\left(e^{i\alpha}-\frac{R_1^2e^{-i\alpha}}{Z^2}\right)^2\left(\frac{R_1^2e^{i\alpha}}{\bar{Z}^2}-e^{-i\alpha}\right)\frac{Z^2\bar{Z}^2}{(Z^2-1)(\bar{Z}^2-1)}\right.$$
$$\left.+2(e^{i\alpha}-R_1^2e^{-i\alpha})\frac{Z}{(Z^2-1)^2}\left[e^{-2i\alpha}\bar{Z}+\frac{R_1^4e^{2i\alpha}}{\bar{Z}}+\frac{1}{2}(R_1^2e^{i\alpha}-e^{-i\alpha})^2\log\frac{\bar{Z}-1}{Z+1}\right]\right\} \qquad (45)$$

VELOCITY AT THE SURFACE OF THE ELLIPTIC CYLINDER

The velocity at the circular boundary C_1 is obtained from equation (45) by putting $Z=R_1e^{i\theta}$ and $\bar{Z}=R_1e^{-i\theta}$. Since at the boundary the normal velocity of the fluid is zero, the velocity there must be purely tangential. Then, by means of the relation

$$Q_{1t}=-iW_1e^{i\theta}$$

where Q_{1t} is the tangential velocity at the boundary, it follows from equation (45) that

$$\left(\frac{Q_{1t}}{U}\right)_{circle}=-\frac{1}{2}\sin(\theta+\alpha)-\frac{1}{2}R_1^2\sin(\theta+3\alpha)+R_1^2\frac{(R_1^4+1)\cos(\theta+\alpha)-2R_1^2\cos(\theta-\alpha)}{R_1^4-2R_1^2\cos 2\theta+1}\sin 2\alpha+\frac{4R_1^4\sin^3(\theta+\alpha)}{R_1^4-2R_1^2\cos 2\theta+1}$$
$$-\frac{R_1^4-2R_1^2\cos 2\alpha+1}{4(R_1^4-2R_1^2\cos 2\theta+1)^2}[(R_1^4-1)^2\sin(\theta-\alpha)+2R_1^2(R_1^4+1)\sin 2\cos(\theta-\alpha)-4R_1^4\sin 2\theta\cos(\theta+\alpha)]\log\frac{R_1^2+1}{R_1^2-1}$$
$$+\frac{1}{4}R_1(R_1^2+1)\frac{R_1^4-2R_1^2\cos 2\alpha+1}{(R_1^4-2R_1^2\cos 2\theta+1)^2}[(1-R_1^2+R_1^4)\sin(2\theta-\alpha)+2R_1^2\sin\alpha-R_1^2\sin(2\theta+\alpha)]\log\frac{R_1^2+2R_1\cos\theta+1}{R_1^2-2R_1\cos\theta+1}$$
$$+\frac{1}{2}R_1(R_1^2-1)\frac{R_1^4-2R_1^2\cos 2\alpha+1}{(R_1^4-2R_1^2\cos 2\theta+1)^2}[-(1+R_1^2+R_1^4)\cos(2\theta-\alpha)+2R_1^2\cos\alpha+R_1^2\cos(2\theta+\alpha)]\tan^{-1}\frac{2R_1\sin\theta}{R_1^2-1} \qquad (46)$$

As a special case of this equation, suppose $\alpha=0$; that is, the undisturbed stream is parallel to the major axis of the ellipse. Then

$$\left(\frac{Q_{1t}}{U}\right)_{circle}=\frac{1}{2}(R_1^2-1)\sin\theta-R_1^2(R_1^2-1)^2\frac{\sin\theta}{R_1^4-2R_1^2\cos 2\theta+1}-\frac{(R_1^2-1)^3}{4(R_1^4-2R_1^2\cos 2\theta+1)^2}\left\{(R_1^2-1)[(1+3R_1^2+R_1^4)\sin\theta\right.$$
$$\left.+R_1^2\sin 3\theta]\log\frac{R_1^2+1}{R_1^2-1}-R_1(R_1^4-1)\sin 2\theta\log\frac{R_1^2+2R_1\cos\theta+1}{R_1^2-2R_1\cos\theta+1}+2R_1[(R_1^4+1)\cos 2\theta-2R_1^2]\tan^{-1}\frac{2R_1\sin\theta}{R_1^2-1}\right\}$$

This expression for the velocity at the circular boundary agrees with the result obtained in reference 2 (with $1/R_1$ replaced by σ).

Another special case of equation (46) occurs when $\alpha=\pi/2$; that is, the undisturbed stream is parallel to the minor axis of the ellipse. Then,

$$\left(\frac{Q_{1t}}{U}\right)_{circle}=-\frac{1}{2}(R_1^2+1)\cos\theta+R_1^2(R_1^2+1)^2\frac{\cos\theta}{R_1^4-2R_1^2\cos 2\theta+1}+\frac{(R_1^2+1)^3}{4(R_1^4-2R_1^2\cos 2\theta+1)^2}\left\{(R_1^2+1)[(1-3R_1^2+R_1^4)\cos\theta\right.$$
$$\left.+R_1^2\cos 3\theta]\log\frac{R_1^2+1}{R_1^2-1}-R_1[(R_1^4+1)\cos 2\theta-2R_1^2]\log\frac{R_1^2+2R_1\cos\theta+1}{R_1^2-2R_1\cos\theta+1}-R_1(R_1^4-1)\sin 2\theta\tan^{-1}\frac{2R_1\sin\theta}{R_1^2-1}\right\}$$

This expression for the velocity at the circular boundary agrees with that obtained in reference 3.

According to equation (37) the velocity of an incompressible fluid at the surface of the circular cylinder of radius R_1 is

$$\left(\frac{Q_{0t}}{U}\right)_{circle}=2\sin(\theta+\alpha)$$

The velocity at the circular boundary, including the second-order effect of compressibility, is therefore given by

$$\left(\frac{Q_t}{U}\right)_{circle}=2\sin(\theta+\alpha)+M^2\left(\frac{Q_{1t}}{U}\right)_{circle} \tag{47}$$

where $\left(\frac{Q_{1t}}{U}\right)_{circle}$ is given by equation (46) and where M is the Mach number U/c_0.

Now, according to equations (36), it follows that, for the elliptic cylinder,

$$\frac{dz}{dZ}\frac{d\bar{z}}{d\bar{Z}}=1-\left(\frac{1}{Z^2}+\frac{1}{\bar{Z}^2}\right)+\frac{1}{Z^2\bar{Z}^2}$$

On the boundary itself, $Z=R_1e^{i\theta}$ and $\bar{Z}=R_1e^{-i\theta}$. Then

$$\left(\frac{dz}{dZ}\frac{d\bar{z}}{d\bar{Z}}\right)_{circle}=\frac{1}{R_1^4}(R_1^4-2R_1^2\cos 2\theta+1)$$

It follows, therefore, from equation (33) that, at the surface of the elliptic cylinder

$$\left(\frac{q_t}{U}\right)_{ellipse}=\frac{R_1^2}{(R_1^4-2R_1^2\cos 2\theta+1)^{\frac{1}{2}}}\left(\frac{Q_t}{U}\right)_{circle} \tag{48}$$

where $\left(\frac{Q_t}{U}\right)_{circle}$ is given by equation (47).

As a final observation it is remarked that the result for the example of the elliptic cylinder was obtained by the use of the fundamental equation (32) of this paper with far less effort than were the results obtained by the methods of references 1, 2, and 3.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., *September 8, 1941.*

REFERENCES

1. Hooker, S. G.: The Two-Dimensional Flow of Compressible Fluids at Sub-Sonic Speeds Past Elliptic Cylinders. R. & M. No. 1684, British A. R. C., 1936.
2. Kaplan, Carl: Two-Dimensional Subsonic Compressible Flow Past Elliptic Cylinders. Rep. No. 624, NACA, 1938.
3. Imai, Isao, and Aihara, Takasi: On the Subsonic Flow of a Compressible Fluid Past an Elliptic Cylinder. Rep. No. 194 (vol. XV, 8), Aero. Res. Inst., Tokyo Imperial Univ., Aug. 1940.
4. Milne-Thomson, L. M.: Theoretical Hydrodynamics. MacMillan and Co., Ltd., 1938, p. 238.