REPORT 958

LAMINAR MIXING OF A COMPRESSIBLE FLUID

By Dean R. Chapman

SUMMARY

A theoretical investigation of the velocity profiles for laminar mixing of a high-velocity stream with a region of fluid at rest has been made assuming that the Prandtl number is unity. A method which involves only quadratures is presented for calculating the velocity profile in the mixing layer for an arbitrary value of the free-stream Mach number.

Detailed velocity profiles have been calculated for free-stream Mach numbers of 0, 1, 2, 3, and 5. For each Mach number, velocity profiles are presented for both a linear and a 0.76-power variation of viscosity with absolute temperature. The calculations for a linear variation are much simpler than those for a 0.76-power variation. It is shown that by selecting the constant of proportionality in the linear approximation such that it gives the correct value for the viscosity in the high-temperature part of the mixing layer, the resulting velocity profiles are in excellent agreement with those calculated by a 0.76-power variation.

INTRODUCTION

The velocity profile for turbulent mixing at constant pressure of an incompressible stream with a dead-air region has been calculated by several investigators, principally Tollmien (reference 1). These calculations agree well with the available experimental data, although the conventional assumptions regarding the mixing length of a turbulent flow have since been shown by experiments to be incorrect (reference 2). The many difficulties encountered in making precise turbulent-mixing calculations are, of course, a consequence of the extremely complicated mechanism govering all turbulent flows. In contradistinction to the case of turbulent mixing, the mechanism involved in laminar mixing is relatively simple, and the mathematical relation between stresses and velocity gradients for laminar flow is well known. The velocity profiles for laminar mixing, however, apparently have not as yet been calculated even for the case of incompressible flow. It is the purpose of the present paper to calculate the velocity profiles for laminar mixing (starting with zero boundary-layer thickness) of an air stream of arbitrary temperature with a dead-air region also of arbitrary temperature. In cases where a laminar boundary layer of appreciable thickness exists at the point where mixing begins, the results given herein are not directly applicable in the initial part of the mixing region. For such cases, it is necessary to make some supplementary approximation in order to apply the results.

Since the practical applications of laminar-mixing phenomenon usually involve the flow of a gas, the present analysis includes the effects of compressibility. Examples of typical flows wherein laminar mixing occurs can be found in the flow of small-scale jet pumps, in the flow behind the intersection of shock waves of unequal strength, and in the flow immediately behind the base of a body which has a laminar boundary layer.[1]

SYMBOLS AND NOTATIONS

C	constant of proportionality between viscosity and temperature
c_p	specific heat at constant pressure
k	coefficient of heat conduction
L	characteristic length
M	Mach number
p	static pressure
Pr	Prandtl number $\left(\frac{c_p \mu}{k}\right)$
S	Sutherlands constant, approximately 216° F for air
T	absolute temperature
U_∞	free-stream velocity
u, v	velocity components in x, y directions, respectively
x, y	Cartesian coordinates
δ	thickness of mixing layer, taken between points where the velocity is 0.01 and 0.99 of the free-stream velocity
ω	exponent of viscosity variation with temperature
ρ	mass density
μ	coefficient of viscosity
ν	kinematic coefficient of viscosity
ψ	stream function
ζ	dimensionless independent variable

SUBSCRIPTS AND SUPERSCRIPTS

*	dimensionless variables as defined in equation (14)
∞	free-stream conditions
0	stagnation conditions of the free stream
d	conditions in the dead-air region

BASIC EQUATIONS AND ASSUMPTIONS

BASIC EQUATIONS

A schematic illustration of the flow under consideration is shown in figure 1. In order to make the laminar-mixing process amenable to calculation, the usual assumptions are made that the layer affected by viscosity is thin and has zero

[1] The present analysis was undertaken as part of an investigation of this latter problem, and originally appeared as Appendix B of a thesis "Base Pressure at Supersonic Velocities," submitted to the California Institute of Technology, June 1948. The results of some supplementary computations not given in the thesis have been added for sake of completeness in the present report.

pressure gradient. Under these conditions the formal procedure for estimating the order of magnitude of the various terms in the complete Navier-Stokes equations for viscous compressible flow can be carried through in precisely the same manner as is done in the classical (Prandtl) treatment of laminar boundary-layer flows. In so doing, the dynamic equation for the x direction reduces to the familiar boundary-layer momentum equation

$$\rho u \frac{\partial u}{\partial x}+\rho v \frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{1}$$

while the dynamic equation for the y direction reduces to zero on both sides. In passing from the Navier-Stokes equation to (1) it is to be noted that the usual boundary-layer assumption $\delta/x<<1$ is violated in the immediate vicinity of point 0 (fig. 1) just as in the case of boundary-layer flow over a plate.

By employing the same considerations on order of magnitudes as were used for the complete Navier-Stokes equations, the complete differential equation representing the balance of energy in viscous compressible flow reduces to

$$\rho u \frac{\partial (c_p T)}{\partial x}+\rho v \frac{\partial (c_p T)}{\partial y}=\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{2}$$

which is, of course, the usual energy equation for laminar boundary-layer flow. In addition to equations (1) and (2), the equation expressing conservation of mass is needed:

$$\frac{\partial (\rho u)}{\partial x}+\frac{\partial (\rho v)}{\partial y}=0 \tag{3}$$

For a given gas the variation of μ and c_p with temperature is known; hence, the foregoing system of three partial differential equations is completed by the addition of the equation of state for a region of constant pressure

$$\frac{T}{T_\infty}=\frac{\rho_\infty}{\rho} \tag{4}$$

ASSUMPTIONS

In order to solve the above system of equations, the following assumptions are made:

1. c_p=constant
2. $Pr=c_p\mu/k=1$
3. $\frac{\mu}{\mu_\infty}=C\left(\frac{T}{T_\infty}\right)^\omega$, where C is a constant depending on T and T_∞

The second of these assumptions if often made in calculating boundary-layer flows when only the velocity profile is desired and not the thermal characteristics. The difference between the boundary-layer velocity distributions for $Pr=1$ and $Pr=0.73$ is small, as is clearly shown by the numerical results of Emmons and Brainerd (reference 3). Since the mixing-layer and boundary-layer flows differ only in the boundary conditions and not in the differential equations, the effect of assumption 2 may be expected to be similar in the two types of viscous flow. At moderate supersonic Mach numbers, the use of $Pr=1$ for air does not introduce more than 1- or 2-percent error in the boundary-layer velocity profile; and hence, for all practical purposes, the mixing-layer velocity distribution calculations for $Pr=1$ should be sufficiently accurate for air.

Assumption 3 needs some explanation since the introduction of a constant C differing from unity in the approximate relation between viscosity and temperature apparently has not been used in previous work. Usually C is taken as unity, and in such cases the approximation $\mu/\mu_\infty=(T/T_\infty)^\omega$ gives reliable results for a fixed ω, provided the free-stream temperature is restricted to a certain range. By introducing the factor C, the approximating equation can be made to give the same value as a more exact equation at any desired

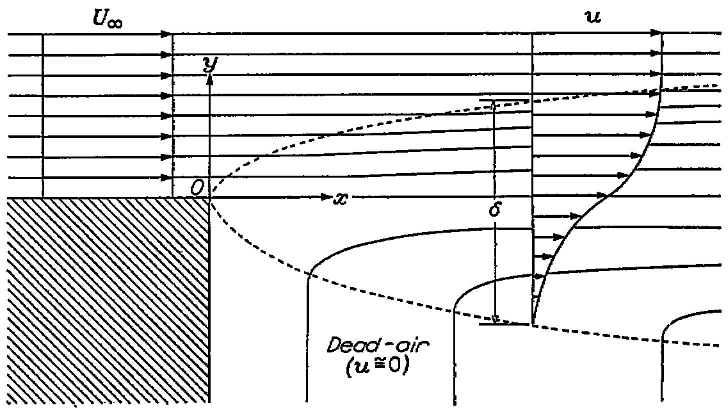

FIGURE 1.—Schematic drawing of the flow.

temperature in the mixing layer regardless of T_∞ or ω. Assuming that Sutherland's equation

$$\frac{\mu}{\mu_\infty}=\left(\frac{T}{T_\infty}\right)^{3/2}\frac{T_\infty+S}{T+S} \tag{5}$$

represents the true variation of viscosity with temperature, then the approximate equation can be made exact at any given temperature T by means of the relation

$$C\left(\frac{T}{T_\infty}\right)^\omega=\left(\frac{T}{T_\infty}\right)^{3/2}\frac{T_\infty+S}{T+S} \tag{6}$$

In particular, if the approximating equation is linear in temperature ($\omega=1$, thereby greatly simplifying the boundary-layer equations) and the viscosity is matched at the temperature T_d, then the above equation gives

$$C=\sqrt{\frac{T_d}{T_\infty}}\frac{T_\infty+S}{T_d+S} \tag{7}$$

as the value of the constant C. By selecting C in this manner, rather than taking it as unity, a linear variation of viscosity with temperature then becomes an accurate approximation in the inner part of the viscous layer, rather than in the outer part where the viscous stresses are less important.

SOLUTION TO BASIC EQUATIONS

As was first pointed out by Prandtl in reference 4, and later used to advantage by Busemann and Crocco (references 5 and 6, respectively), the consequence of the assumption $Pr=1$ when applied to boundary-layer flow is that the temperature becomes a function only of the velocity. Hence

$$c_pT=f(u) \tag{8}$$

By substituting this relation into equation (2) and using equation (1) in conjunction with the assumption $Pr=1$, it follows that the energy equation is automatically satisfied if the function $f(u)$ satisfies the ordinary differential equation

$$\frac{d^2f}{du^2}+1=0 \tag{9}$$

Integrating this equation, using the boundary conditions,

$$\begin{aligned} T&=T_\infty \text{ for } u=U_\infty \\ T&=T_d \text{ for } u=0 \end{aligned} \tag{10}$$

gives

$$f(u)=c_pT=c_pT_d-\frac{u^2}{2}+\frac{u}{U_\infty}\left[c_p(T_\infty-T_d)+\frac{U_\infty^2}{2}\right] \tag{11}$$

as the relationship between velocity and temperature. Since the temperature determines the density, equation (11) also provides a means for calculating the density as a function of the velocity.

Following the method first given for incompressible flow by von Mises (reference 7) and later used for compressible flow by von Kármán and Tsien (reference 8), a transformation is made to a new set of independent variables (x, ψ), where ψ is the stream function. By using ψ as one of the independent variables, the continuity equation (3) is identically satisfied, and the velocity components are given by

$$u=\frac{\rho_\infty}{\rho}\frac{\partial\psi}{\partial y},\ v=-\frac{\rho_\infty}{\rho}\frac{\partial\psi}{\partial x} \tag{12}$$

Since the requirements of conservation of energy are fulfilled by equation (11), and conservation of mass by equation (12), the only equation now remaining to be satisfied is the momentum equation (1). If a transformation were made to a completely new set of independent variables (s, ψ), the transformation formula would be

$$\frac{\partial}{\partial y}=\frac{\partial\psi}{\partial y}\frac{\partial}{\partial\psi}+\frac{\partial s}{\partial y}\frac{\partial}{\partial s}=\frac{\rho u}{\rho_\infty}\frac{\partial}{\partial\psi}+\frac{\partial s}{\partial y}\frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial x}=\frac{\partial\psi}{\partial x}\frac{\partial}{\partial\psi}+\frac{\partial s}{\partial x}\frac{\partial}{\partial s}=-\frac{\rho v}{\rho_\infty}\frac{\partial}{\partial\psi}+\frac{\partial s}{\partial x}\frac{\partial}{\partial s}$$

Setting $s\equiv x$,

$$\frac{\partial s}{\partial y}=0 \text{ and } \frac{\partial s}{\partial x}=1$$

so that the transformation formulas are [2]

$$\frac{\partial}{\partial y}=\frac{\rho u}{\rho_\infty}\frac{\partial}{\partial\psi}$$

$$\left(\frac{\partial}{\partial x}\right)_y=-\frac{\rho v}{\rho_\infty}\frac{\partial}{\partial\psi}+\left(\frac{\partial}{\partial x}\right)_\psi$$

[2] The variables held constant in a differentiation process are explicitly indicated in those cases where ambiguity could result if the subscript notation were not used.

It follows that

$$\rho u\left(\frac{\partial u}{\partial x}\right)_y+\rho v\left(\frac{\partial u}{\partial y}\right)_x=\rho u\left(\frac{\partial u}{\partial x}\right)_\psi$$

and

$$u\left(\frac{\partial u}{\partial y}\right)_x=u\frac{\rho u}{\rho_\infty}\left(\frac{\partial u}{\partial\psi}\right)_x$$

Hence the momentum equation (1) in the (x,ψ) system becomes

$$\rho_\infty\frac{\partial u}{\partial x}=\frac{\partial}{\partial\psi}\left(u\mu\frac{\rho}{\rho_\infty}\frac{\partial u}{\partial\psi}\right) \tag{13}$$

This can be put in dimensionless form by introducing the variables

$$\begin{aligned} u^*&=\frac{u}{U_\infty} & T^*&=\frac{T}{T_\infty} \\ x^*&=\frac{x}{L} & \mu^*&=\frac{\mu}{\mu_\infty}=CT^{*\omega} \\ \rho^*&=\frac{\rho}{\rho_\infty} & \psi^*&=\frac{\psi}{\sqrt{\nu_\infty U_\infty LC}} \end{aligned} \tag{14}$$

Except for the parameter C appearing in the definition of ψ^* and μ^*, these variables are the same as those used by Kármán and Tsien (reference 8). Remembering that T_0^*, the free-stream total-temperature ratio, is given by

$$T_0^*=\frac{T_0}{T_\infty}=1+\frac{\gamma-1}{2}M_\infty^2$$

then the relation (equation (11)) between temperature and velocity can be written as

$$T^*=T_d^*-\frac{\gamma-1}{2}M_\infty^2u^{*2}+(T_0^*-T_d^*)u^* \tag{15}$$

The momentum equation (13) becomes, using $\rho^*T^*=1$,

$$\frac{\partial u^*}{\partial x^*}=\frac{\partial}{\partial\psi^*}\left(u^*T^{*\omega-1}\frac{\partial u^*}{\partial\psi^*}\right) \tag{16}$$

This is the basic equation which must be solved. The boundary conditions of the problem are such that no boundary layer exists at the point where mixing first begins. Under these conditions the velocity profiles will be similar at all points downstream of the origin, and hence the velocity u^* will be a function only of some dimensionless variable ζ. This dimensionless variable must involve both ψ^* and x^*, and must be zero at the origin of coordinates since the mixing-layer thickness is also zero at the origin. Therefore, let

$$\zeta=\psi^{*a}x^{*b}$$

where a and b are pure numbers which must be determined by the condition that both sides of equation (16) for u^* are functions only of the single variable ζ. Setting

$$g(\zeta)\equiv u^*T^{*\omega-1} \tag{17}$$

then the right side of equation (16) can be written as

$$\frac{\partial}{\partial\psi^*}\left(g\frac{du^*}{d\zeta}\frac{\partial\zeta}{\partial\psi^*}\right)=\frac{\partial}{\partial\psi^*}\left(g\frac{du^*}{d\zeta}a\psi^{*a-1}x^{*b}\right)=$$
$$x^{*b}\frac{\partial}{\partial\psi^*}\left(g\frac{du^*}{d\zeta}a\psi^{*a-1}\right)$$

from which it is obvious that in order for the right side of equation (16) to be a function only of ζ, it is necessary that $a=1$. With $u^*=u^*(\zeta)$ and $a=1$, the partial differential equation (16) reduces to the ordinary differential equation

$$bx^{*-2b-1}\zeta\frac{du^*}{d\zeta}=\frac{d}{d\zeta}\left(g\frac{du^*}{d\zeta}\right)$$

Consequently, in order for the entire equation to be a function only of ζ, it is also necessary that $b=-\frac{1}{2}$, hence

$$\zeta=\frac{\psi^*}{\sqrt{x^*}}=\frac{\psi}{\sqrt{U_\infty\nu_\infty xC}} \tag{18}$$

and the ordinary differential equation for the velocity distribution now reduces to

$$-\frac{\zeta}{2}\frac{du^*}{d\zeta}=\frac{d}{d\zeta}\left(g\frac{du^*}{d\zeta}\right) \tag{19}$$

Equation (19) is the same differential equation that was obtained for boundary-layer flow in reference 8. It is a nonlinear differential equation since g depends upon the velocity u^*. This equation, however, can easily be converted into an integral equation which can be solved by the method of successive approximations. The conversion is made by temporarily assuming that g is a known function of ζ (instead of u^*) and formally applying the standard methods for solving first-order linear differential equations. The result is

$$u^*=C_1\int_0^\zeta\frac{F}{g}d\zeta+C_2 \tag{20}$$

where

$$F=e^{-\int_0^\zeta\frac{\zeta}{2g}d\zeta} \tag{21}$$

The boundary conditions are

$$u^*=1 \text{ at } \zeta=\infty \tag{22a}$$

$$u^*=0 \text{ at } \zeta=-\infty \tag{22b}$$

Letting u_0^* be the value of u^* at $\zeta=0$, equation (20) can be written as

$$u^*=C_1\int_0^\zeta\frac{F}{g}d\zeta+u_0^* \tag{23}$$

The constant C_1 must satisfy two requirements

$$\frac{1-u_0^*}{\int_0^\infty\frac{F}{g}d\zeta}=C_1=\frac{u_0^*}{\int_{-\infty}^0\frac{F}{g}d\zeta} \tag{24}$$

Equation (23) is an integral equation for u^*, since both F and g are functions of u^*. By simply estimating a reasonable solution for u^* as a function of ζ, a first approximation ${}_1u^*$ to the true solution can be calculated from

$${}_1u^*=C_1\int_0^\zeta\frac{{}_0F}{{}_0g}d\zeta+{}_0u_0^*$$

The zero-order approximations ${}_0F$ and ${}_0g$ can be calculated directly from ${}_0u^*$ by using equations (15), (17), and (21). If this process is repeated untill a given approximation is the same as the preceding one (to the degree of accuracy desired), and equation (24) is simultaneously satisfied, then the solution to the problem is obtained. The iteration process turns out to be rapdily convergent, requiring two or three iterations to obtain the function $u^*(\zeta)$ accurate to within 1 percent, and about four or five iterations to obtain $u^*(\zeta)$ accurate to within a few tenths of 1 percent.

In order to change the function $u^*(\zeta)$ back to the physical coordinates (x, y) a simple quadrature is necessary. By definition of the stream function,

$$\left(\frac{\partial\psi}{\partial x}\right)_\zeta dx+\left(\frac{\partial\psi}{\partial\zeta}\right)_x d\zeta=d\psi=\left(\frac{\partial\psi}{\partial y}\right)_x dy+\left(\frac{\partial\psi}{\partial x}\right)_y dx$$

or

$$\frac{1}{2}\zeta\sqrt{\frac{U_\infty\nu_\infty C}{x}}dx+\sqrt{U_\infty\nu_\infty xC}\,d\zeta=\frac{\rho u}{\rho_\infty}dy-\frac{\rho v}{\rho_\infty}dx$$

Hence, with x held constant, integration gives

$$y\sqrt{\frac{U_\infty}{\nu_\infty xC}}=\int_0^\zeta\frac{T^*}{u^*}d\zeta \tag{25}$$

from which u/U_∞ as a function of $y\sqrt{\frac{U_\infty}{\nu_\infty x}}$ can be determined.

It is to be noted that no graphical or numerical differentiations are needed at any point in the above iteration process, only quadratures are required.

As is evident from equation (16), the assumption $\mu^*=CT^*$, that is, $\omega=1$, makes the momentum equation (16) independent of temperature, and hence density. Consequently, with $\omega=1$, the solution to equation (16) in (x,ψ) coordinates is independent of Mach number. For zero Mach number, $T^*=1$, $C=1$, and

$$\left(y\sqrt{\frac{U_\infty}{\nu_\infty x}}\right)_{M=0}=\int_0^\zeta\frac{d\zeta}{u^*}$$

Using this relation the solution in physical (x,y) coordinates for a linear variation of viscosity with temperature is obtained from the solution for zero Mach number by substituting equation (15) into equation (25). This gives

$$y\sqrt{\frac{U_\infty}{\nu_\infty xC}}=T_d^*\left(y\sqrt{\frac{U_\infty}{\nu_\infty x}}\right)_{M=0}-\frac{\gamma-1}{2}M_\infty^2\int_0^\zeta u^*d\zeta+(T_0^*-T_d^*)\zeta \tag{26}$$

where the integral in the second term on the right side is carried out for $u^*(\zeta)$ corresponding to $M_\infty=0$.

RESULTS

Numerical calculations of the velocity distribution have been made for the following cases:

1. $\mu^*=T^{*0.76}$; $M_\infty=0$, 1, 2, 3, and 5

2. $\mu^*=CT^*$; $M_\infty=0$, 1, 2, 3, and 5

The various solutions for case 2 are obtained directly (equation (26)) from the solution for $M_\infty=0$ without carrying out the laborious iteration process that is necessary to obtain solutions for case 1. All numerical results in physical coordinates ($y\sqrt{U_\infty/\nu_\infty x}$ as independent variable) have been calculated for the case $T_d=T_0$. If the dead-air temperature is radically different from the free-stream stagnation temperature, the proper velocity-distribution curves can be obtained by carrying out the integration indicated in equation (25), since the function $u^*(\zeta)$ in (x,ψ) coordinates is independent of the thermal boundary conditions of the problem.

Curves of $u^*(\zeta)$ are shown in figure 2 for various Mach numbers. The corresponding curves in the physical plane are shown in figure 3 for the case $\mu^*=T^{*0.76}$, and in figure 4 for the case $\mu^*=CT^*$. In the latter two figures the familiar Blasius curve for the incompressible laminar boundary-layer flow is shown for purposes of comparison. The constant C that is used in figure 4 is determined by matching the viscosity coefficient at the temperature $T_d^*=T_0^*$ according to equation (7).

The particular curves shown in figure 4 apply for $T_\infty=400°$ R. Curves for any other temperature level T_∞ differ only in the constant factor C.

CONCLUDING REMARKS

A comparison is shown in figure 5 which illustrates the good agreement between velocity distributions calculated for the two approximations, $\mu^*=T^{*0.76}$ and $\mu^*=CT^*$. At a Mach number of 2 or less, the curves for a linear variation of viscosity with temperature virtually coincide with the curves for a 0.76-power variation. For general use the linear approximation is recommended since it gives results which are practically as accurate as the former, yet does not require a laborious iteration solution to be worked out for each Mach number.

In general the laminar-mixing layer is several times thicker than the laminar boundary layer, as is illustrated in figures 3 and 4 where, for purposes of camparison, the Blasius profile is also shown. The rate of growth of mixing-layer thickness with increasing Mach number is somewhat larger than the corresponding rate of growth for a laminar boundary layer. The curves in figure 3 indicate a value of roughly

$$\frac{\delta_M}{\delta_{M=0}}=1+0.11M_\infty^2$$

for rate of growth of the mixing layer; whereas the corresponding value for a laminar boundary layer (see reference 9, for example) is approximately

$$\frac{\delta_M}{\delta_{M=0}}=1+0.09M_\infty^2$$

This difference is to be expected since a larger percentage of low-density air exists in a mixing layer than in a boundary layer.

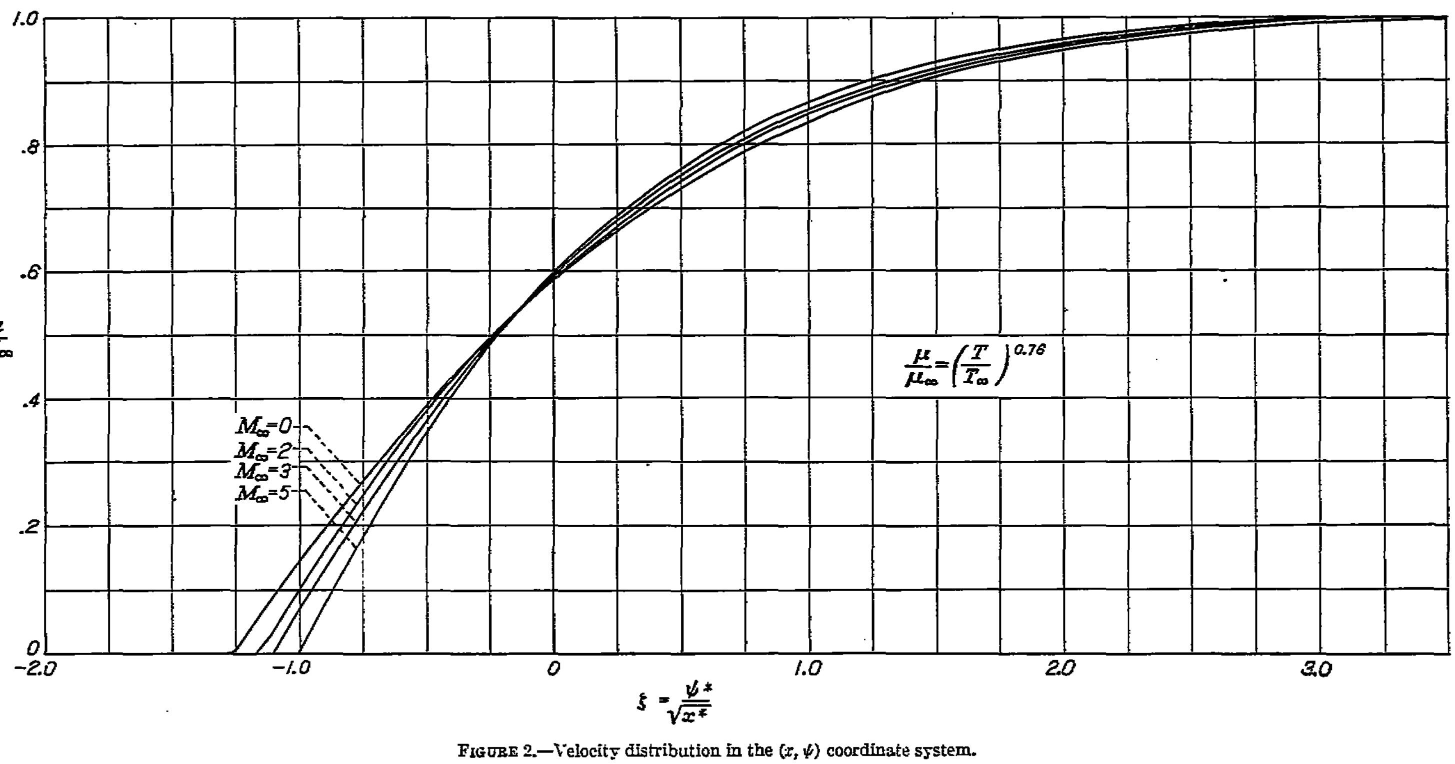

FIGURE 2.—Velocity distribution in the (x, ψ) coordinate system.

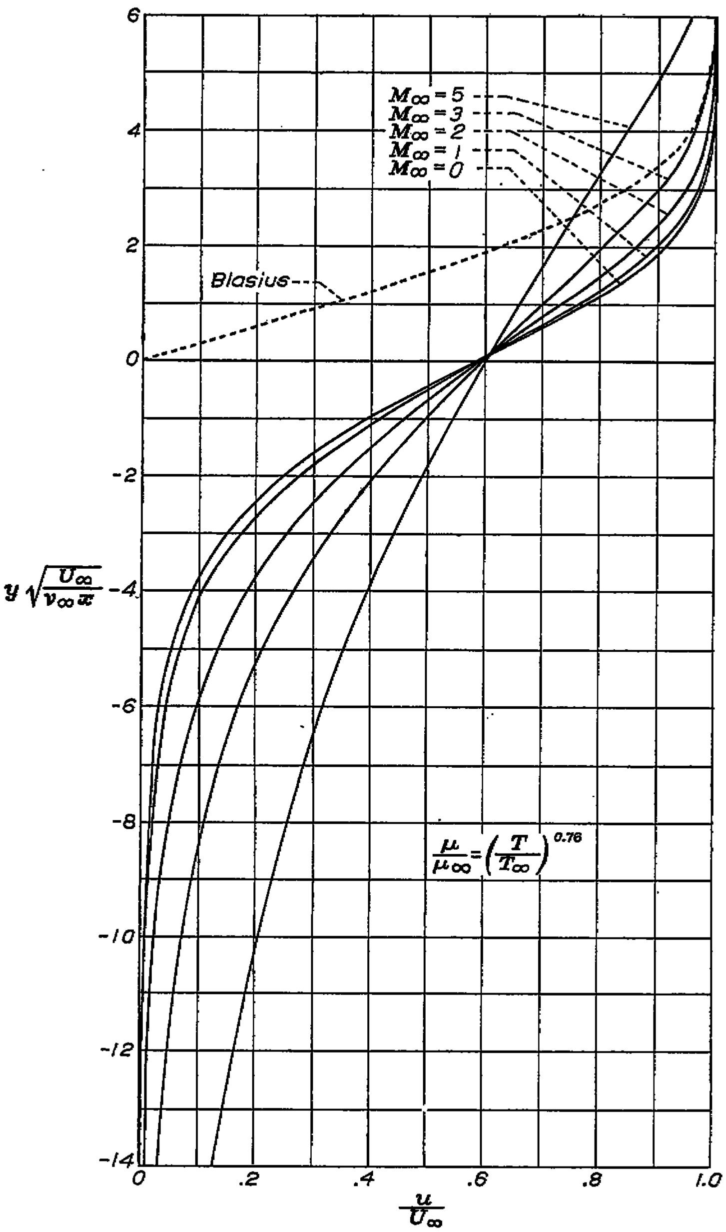

FIGURE 3.—Velocity distribution for 0.76-power variation of viscosity with temperature.

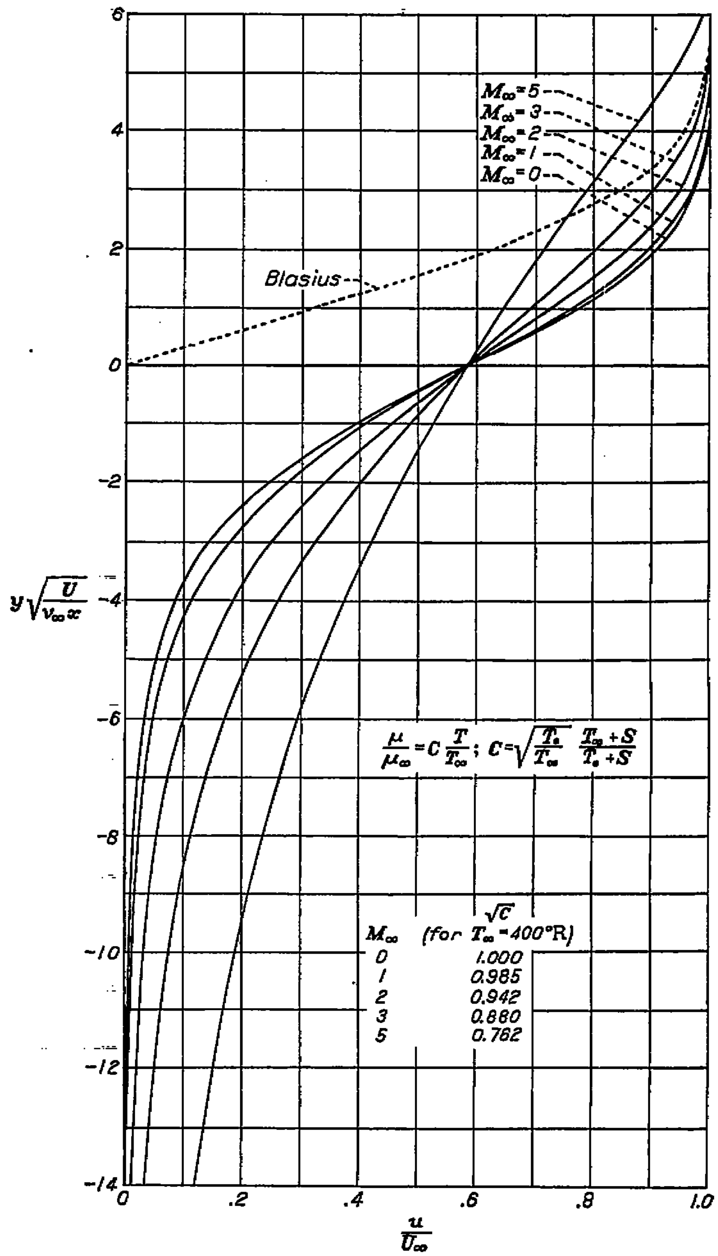

FIGURE 4.—Velocity distribution for linear variation of viscosity with temperature.

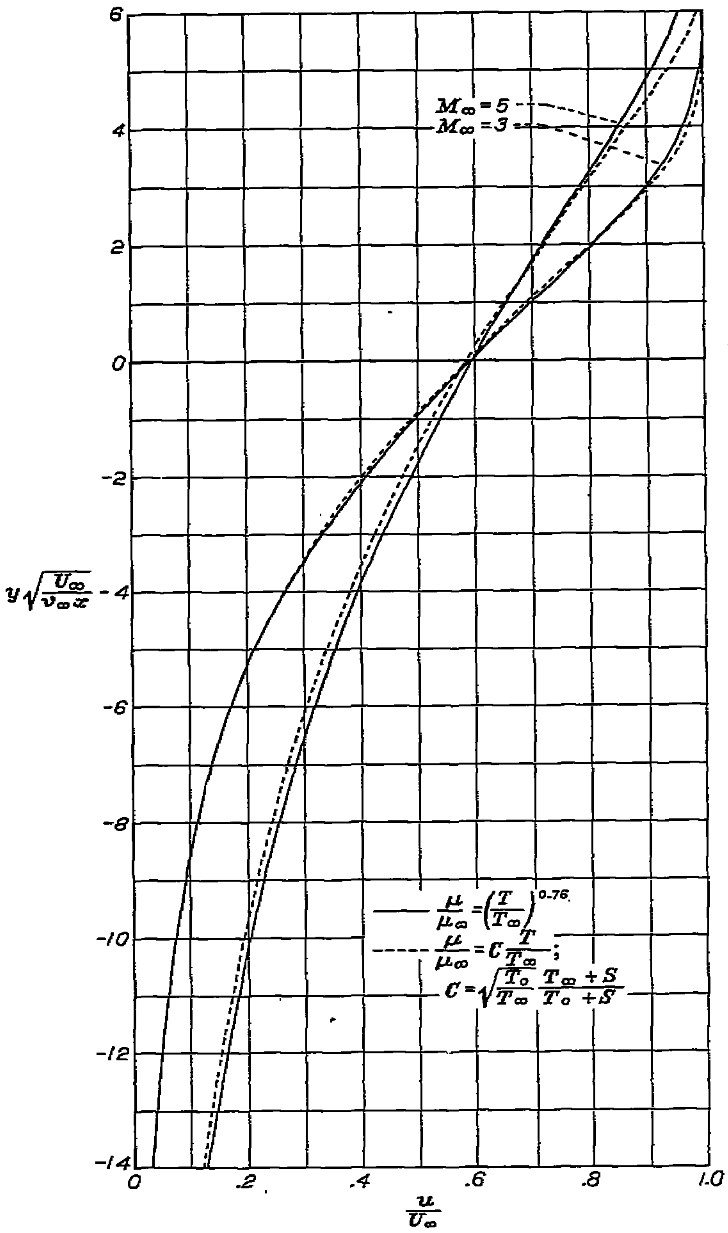

Figure 5.—Comparison of velocity profile for a linear and a 0.76-power variation of viscosity with temperature.

The foregoing statements, which indicate an increase in mixing-layer thickness with increasing Mach number, are based on the assumption that the Reynolds number ($U_\infty x/\nu_\infty$) is held constant while the Mach number is varied. In most experimental apparatus the Reynolds number changes considerably with a variation in Mach number. Consequently, depending upon the particular experimental method employed, the observed rate of mixing in the x direction may be either increased or decreased if the Mach number is increased.

Ames Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Moffett Field, Calif., *Jan. 5, 1949.*

REFERENCES

1. Tollmien, Walter: Calculation of Turbulent Expansion Processes. NACA TM 1085, 1945.
2. Liepmann, Hans Wolfgang, and Laufer, John: Investigations of Free Turbulent Mixing. NACA TN 1257, 1947.
3. Emmons, H. W., and Brainerd, J. G.: Temperature Effects in a Laminar Compressible-Fluid Boundary Layer Along a Flat Plate. Jour. App. Mach., vol. 8, no. 3, Sept. 1941, pp. A-105–A-110.
4. Prandtl, Ludwig: Eine Beziehung Zwischen Wärme austausch und Strömungswiderstand der Flussigkeiten. Physikalische Zeitschrift, vol. 11, 1910, p. 1072.
5. Busemann, Adolph: Gas Dynamics, Theory and Concepts. Chapter I, Paragraph 9 in Handbuch der Experimentalphysik, vol. 4, Part I. Leipzig, Akademische Verlagsgesellschaft, 1931.
6. Crocco, Luigi: Transmission of Heat From a Flat Plate to a Fluid Flowing at a High Velocity. NACA TM 690, 1932.
7. von Mises, Richard: Bemerknung zur hydrodynamik, Zeitschrift für angewandte Mathematik und Mechanik, vol. 7, 1927.
8. von Kármán, Th., and Tsien, H. S.: Boundary Layer in Compressible Fluids. Jour. Aero. Sci., vol. 5, no. 6, April 1938, pp. 227–232.
9. Lees, L.: The Stability of the Laminar Boundary Layer in a Compressible Fluid. NACA Rep. 876, 1947.