REPORT 1324

COMPARISON OF SEVERAL METHODS FOR OBTAINING THE TIME RESPONSE OF LINEAR SYSTEMS TO EITHER A UNIT IMPULSE OR ARBITRARY INPUT FROM FREQUENCY-RESPONSE DATA [1]

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SUMMARY

Several methods of obtaining the time response of linear systems to either a unit impulse or an arbitrary input from frequency-response data are described and compared.

Comparisons indicate that all the methods give good accuracy when applied to a second-order system; the main difference is the required computing time. Several of the methods when applied to higher order systems require excessive computing time in order to obtain the same degree of accuracy. The methods generally classified as inverse Laplace transform methods were found to be most effective in determining the response to a unit impulse from frequency-response data of higher order systems.

Some discussion and examples are given of the use of the methods as flight-data-analysis techniques in predicting loads and motions of a flexible aircraft on the basis of simple calculations when the aircraft frequency response is known.

INTRODUCTION

The frequency-response type of analysis used on linear systems has found extensive application in the field of air-craft stability and in the determination of overall dynamic characteristics of an aircraft. In fact the current trend to perform analysis on flight test data in frequency-response form appears to be gaining favor especially in the case of flexible aircraft. Considerable emphasis has, therefore, been given to methods of determining the frequency response of a system from transient responses. A concise résumé and comparison of methods for obtaining the frequency response from transient responses are presented in reference 1.

This report briefly considers the next step in the process—the converting of the information contained in a frequency response to the time plane in the form of the response to a unit impulse. For certain purposes this conversion gives data in more useful form. The response of a linear system to a unit impulse may be used in conjunction with Duhamel's (superposition) integral to determine (1) the aircraft transient response to any type of input or (2) the input required to cause any required aircraft transient response.

It appears that methods of converting frequency-response data to transient data present a flight-data-analysis technique which permits the prediction of aircraft motions and loads for a flexible aircraft without knowledge of the equations of motion relating the input and output. These methods also bypass the need for computing transfer-function coefficients or stability derivatives in predicting these loads and motions. Such predictions are important in anticipating the motions and loads for more hazardous aircraft maneuvers. These methods are also useful in predicting time responses of complicated linear systems whose frequency response is known.

The purpose of this report is to collect and briefly compare a few of the methods now available for performing this operation. The methods are compared on the basis of accuracy, computing time required, and applicability of the method to higher order systems. Some extensions of these methods are also given. With such information available, engineers may then select the method which best fits their needs.

SYMBOLS

$F(t)$	function of time
$H(i\omega)$	frequency response of a system
$H(s)$	transfer function (in terms of Laplace variable s) relating input and output
$h(t)$	time response of a system to a unit impulse
$i=\sqrt{-1}$	
j	index of summation
K_1, K_2, K_3	transfer coefficients defined by equation (8)
K, L	limits on summation of P-transform
M	Mach number
m	index of summation of P-transform
N	total number of data points used in solution of equation (9)
n	limit of summation
P	polynomial transform operator
$\text{Re}\,[H(i\omega)]$	real part of frequency response defined by equation (21)
r	ordinate of pulse used to fit $\text{Re}\,[H(i\omega)]$
s	Laplace variable, $c+i\omega$
t	time, sec
x	output
$x(s)$	Laplace transform of $x(t)$
$x(t)$	response of a system to an arbitrary input
$\left\lvert\dfrac{x}{\delta}\right\rvert$	amplitude ratio of frequency response

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Δ	indicates increment
δ	input
$\delta(t)$	time history of input to a linear system
θ	pitching velocity, radians/sec
τ	dummy time variable of integration, sec
ϕ	phase angle, deg
$\phi_{x\delta}$	phase angle between output x and input δ of frequency response; negative phase angles indicate lag
ω	circular frequency, radians/sec
ω_d	damped natural frequency, radians/sec
ω_f	fundamental frequency, radians/sec
ω_n	undamped natural frequency, radians/sec

Subscripts:

e	elevator
i	index of summation indicating row
j	index of summation indicating column

A tilde indicates polynomial transform of function; for example, $\tilde{F}(x)$ denotes polynomial transform of $F(t)$.

DESCRIPTION AND DISCUSSION OF METHODS

In this section the methods for determining the response to a unit impulse and the methods for obtaining the response to arbitrary inputs are discussed.

The methods for obtaining the time response to a unit impulse from frequency-response data fall into several basic categories which may be generally classified as follows:

(1) Inverse Laplace transform methods
(2) Fourier method
(3) Other methods

These methods are briefly outlined to indicate the techniques involved. In order to describe the computations required and the accuracies obtained, each method is applied to the frequency-response data of figure 1 which defines a simple second-order system described by the transfer-function

$$\frac{x}{\delta}(s)=\frac{1}{s^2+6s+10} \tag{1}$$

It is usually intended that these methods be applied to higher order systems as will be shown later in the report. No attempt is made to repeat the development of the methods since this information may be obtained from the references.

INVERSE LAPLACE TRANSFORM METHODS

Floyd's method.—The method developed by George F. Floyd and described in detail in reference 2 is referred to as Floyd's method. Floyd shows that the inverse Laplace transform $h(t)$ of $H(s)$ given by the integral

$$h(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} H(s)e^{ts}\,ds \tag{2}$$

is for all positive values of time equivalent to

$$h(t)=\frac{2}{\pi}\int_0^\infty \mathrm{Re}\,[H(i\omega)]\cos\,t\omega\,d\omega \tag{3}$$

This operation is based on the assumptions that $H(s)$ may be written as the ratio of two rational polynomials in s with real and constant coefficients, that $\lim\limits_{s\to\infty} H(s)=0$, and that
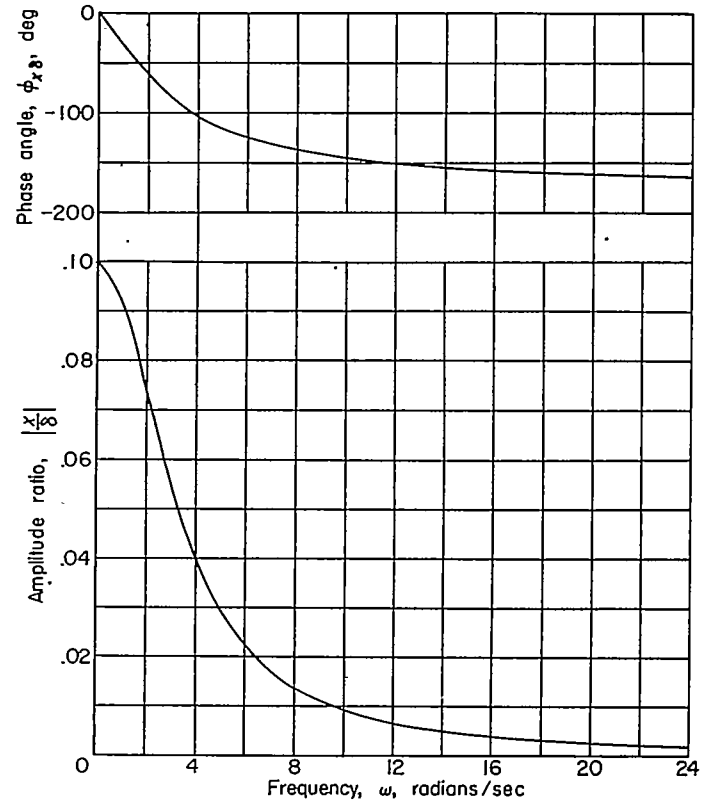

FIGURE 1.—Frequency response of second-order system.

$$\frac{x}{\delta}(s)=\frac{1}{s^2+6s+10}.$$

$H(s)$ has no poles in the right half of the s-plane or on the imaginary axis. The procedure for performing the integration required by equation (3) is to plot $\mathrm{Re}[H(i\omega)]$ against ω and then to approximate the exact shape with a series of straight-line segments. The straight-line approximation is written as a sum of trapezoidal functions and equation (3) is applied to each of the trapezoids; the resulting time functions due to each trapezoid are then added to obtain $h(t)$.

A simple illustration is shown in sketch A. The time
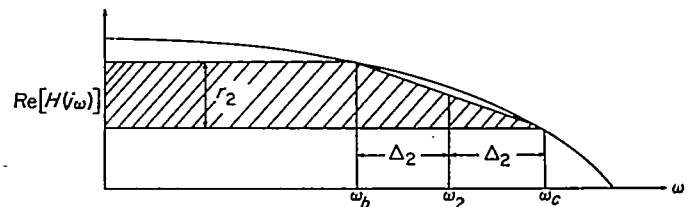
Sketch A.

function associated with a typical trapezoid (as shown by the cross-hatched section) may be expressed as

$$h_2(t)=\frac{2}{\pi}A_2\left(\frac{\sin\,\omega_2 t}{\omega_2 t}\right)\left(\frac{\sin\,\Delta_2 t}{\Delta_2 t}\right) \tag{4}$$

where

$$\omega_2=\frac{\omega_b+\omega_c}{2}$$

$$\Delta_2=\frac{\omega_c-\omega_b}{2}$$

$$A_2=r_2\omega_2$$

In general for n trapezoids the time response is given by

$$h(t)=\frac{2}{\pi}\sum_{j=1}^{n}A_j\left(\frac{\sin \omega_j t}{\omega_j t}\right)\left(\frac{\sin \Delta_j t}{\Delta_j t}\right) \qquad (5)$$

In figure 2 the real part Re $[H(i\omega)]$ of the simple second-order system computed from figure 1 is shown plotted against ω. The function $h(t)$ is then easily evaluated since tables of $\frac{\sin x}{x}$ are given in reference 2. In the upper part of figure 2, two straight-line fits to the Re $[H(i\omega)]$ are shown, one a five-line fit and the other an eleven-line fit. For clarity, however, only the points and not the connecting lines of the fit are shown. In the lower part of figure 2 the resulting responses to a unit impulse computed by Floyd's method are compared with the exact response to a unit impulse. The accuracy of Floyd's method depends on the number of lines used to fit the Re $[H(i\omega)]$ and the location of the cutoff frequency. The cutoff frequency is defined as the maximum frequency at which the Re $[H(i\omega)]$ was fitted. It should be noted that the limits of the definite integral of equation (3) are 0 to ∞; however, in the practical case the Re $[H(i\omega)]$ is cut off at some finite frequency. This error is reflected at the low values of time, especially at $t=0$. For a given number of lines, the accuracy also depends on the judicious fit of the lines.

Numerical-integration method.—In order to use automatic computing machines to perform the inverse Laplace transform method, the necessary operations indicated by equation (3) are performed in the following manner by using a numerical-integration method. For a given value of time t, the
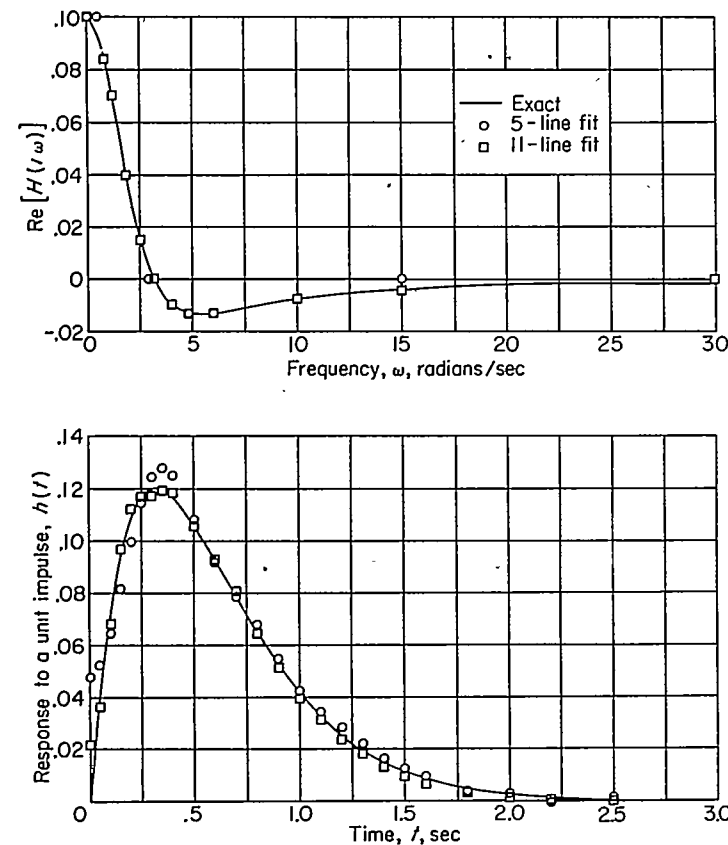
product curve Re $[H(i\omega)]$ cos $t\omega$ is evaluated over the ω range and integrated by numerical-integration techniques which give one point on the time history of the response to a unit impulse. By repeating the above computation for all the desired values of time, the time response can be obtained.

This method was applied to the frequency-response data of figure 1. The accuracy of the method is shown in figure 3. In the upper part of figure 3 the error due to the interval $\Delta\omega$ chosen for the computation is shown by the comparison between the circle and square symbols. The accuracy of the computation for the case of $\Delta\omega=1.0$, as indicated by the circle symbols, was not satisfactory beyond $t=2.0$ seconds. This result is to be expected since the interval $\Delta\omega=1.0$ was too large to permit numerical-integration methods to perform adequately the integration required by equation (3).

Rectangular-pulse method.—A method for determining the time response to a unit impulse from frequency-response data has been given in reference 3 and has been recently extended in reference 4. It is referred to herein as the rectangular-pulse method. The method involves the use of tables for time-plane values equivalent to unit rectangular
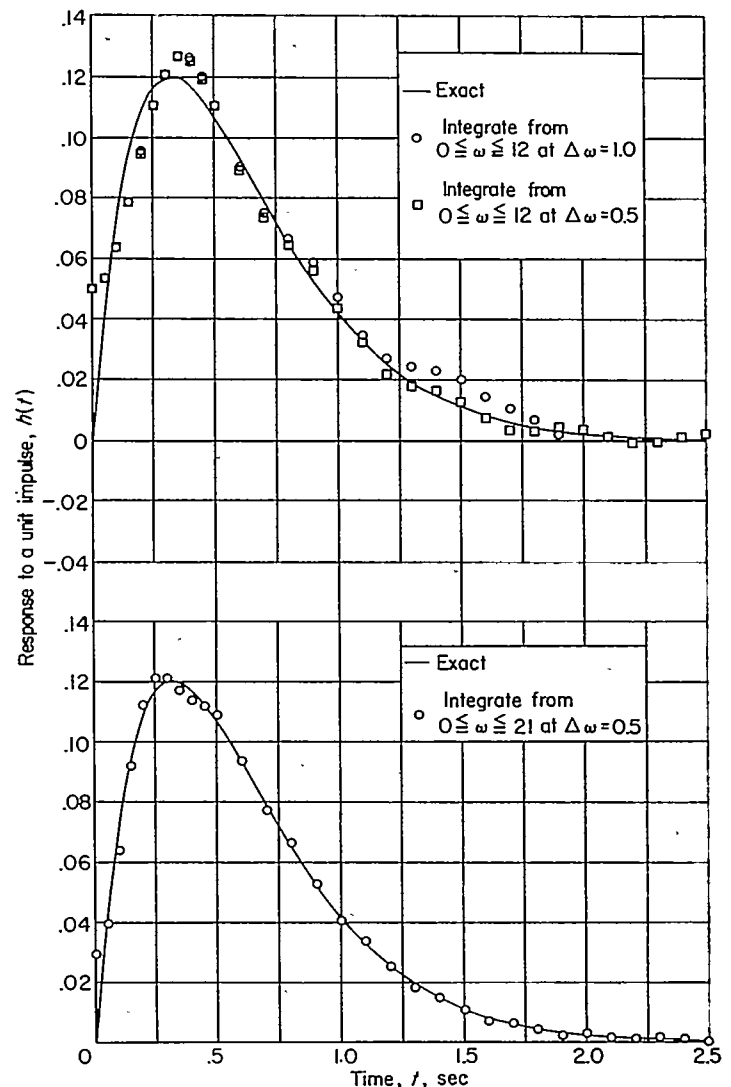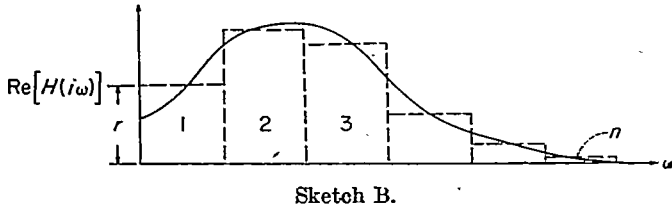

FIGURE 3.—Time response to a unit impulse obtained from the numerical-integration method compared to the exact values for the second-order system.

FIGURE 2.—Time response to a unit impulse obtained from Floyd's method compared to the exact values for the second-order system.

pulses of the Re $[H(i\omega)]$. The method requires fitting the Re $[H(i\omega)]$ with a series of rectangular pulses or a staircase function so that the area under the curve is equal to the area of the pulse in each case as is shown in sketch B. This

Sketch B.

fitting although made visually should be made carefully. If the ordinate of pulse 1 is designated as r_1, of pulse 2 as r_2, and of pulse n as r_n, then the time response to a unit impulse corresponding to this Re $[H(i\omega)]$ is given by

$$h(t)=\sum_{j=1}^{n}r_j h_j(t) \tag{6}$$

The time functions $h_1(t)$, $h_2(t)$, $h_n(t)$ associated with each of the rectangular pulses shown in sketch B are tabulated in reference 3. These time functions are the inverse Fourier transforms of unit rectangular pulses of the Re $[H(i\omega)]$.

The method was applied to the frequency-response data of figure 1. The fitting of the staircase function to the Re $[H(i\omega)]$ is illustrated in the upper part of figure 4 and the accuracy of the method is demonstrated in the lower part of figure 4. Again the error in the computed response to the unit impulse at $t=0$ is due to cutting off the Re $[H(i\omega)]$ at $\omega=20$ radians/sec and not integrating to ∞ as indicated by equation (3).

FOURIER METHOD

Fourier response to unit impulse.—In reference 5, the response to a unit step input is derived by the Fourier method. Differentiating this response gives an expression by which the response to a unit impulse may be approximated. For a linear system characterized by its frequency response with amplitude ratio $\left|\dfrac{x}{\delta}\right|$ and phase angle $\phi_{x\delta}$, this expression is

$$h(t)=\frac{2\omega_f}{\pi}\sum_{n=1}^{\infty}\left|\frac{x}{\delta}\right|_{(2n-1)\,\omega_f}\cos\left[(2n-1)\omega_f t+(\phi_{x\delta})_{(2n-1)\,\omega_f}\right] \tag{7}$$

In using equation (7) the choice of the fundamental frequency ω_f determines the accuracy and length of the computations. Since the accuracy is affected by ω_f it has been found by experience that a value $\omega_f=\dfrac{\omega_n}{5}$ can be used as a good first estimate, where ω_n is the undamped natural frequency of the system. Instead of ω_n, the value of frequency at which the amplitude ratio peaks for a lightly damped system may be used. This value may be determined from the frequency response of the system.

In table I the numerical computation of $h(t)$ for the system defined by the frequency response of figure 1 is shown. In this computation, 14 terms are carried in the expression for
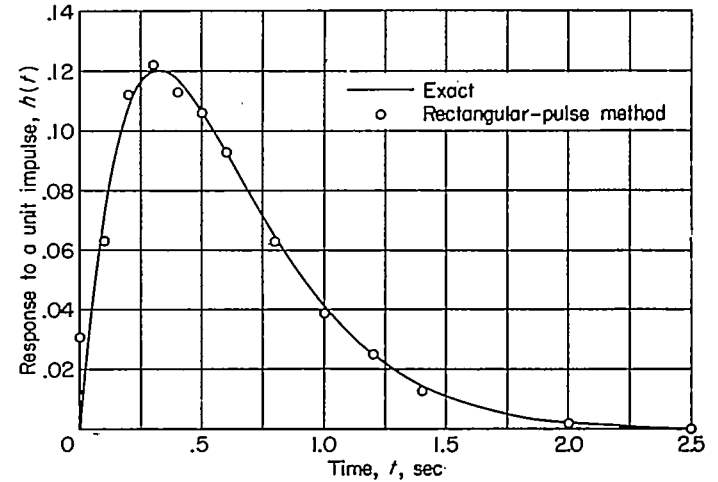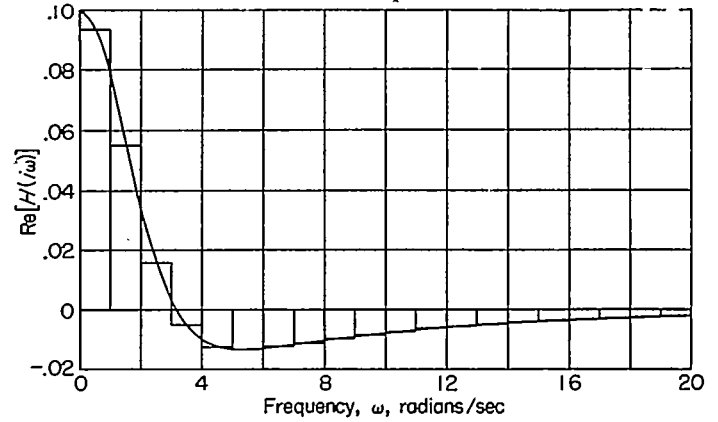

FIGURE 4.—Time response to a unit impulse obtained from the rectangular-pulse method compared to the exact values for the second-order system.

$h(t)$, and more accuracy may be obtained, of course, by carrying more terms. The accuracy of the method is shown in figure 5 by the comparison between the computed and exact response to the unit impulse.

OTHER METHODS

Schumacher's method.—The method of reference 6 permits the computation of transfer-function coefficients by assuming the shape of the transfer function relating the input and output and then curve fitting this relation to the aircraft data in frequency-response form. Once the transfer-function coefficients are known, the system is completely specified since the response to a unit impulse input or any other arbitrary input may then be computed by the normal methods available for solving differential equations. To illustrate the method a sample computation is performed by using the system defined in figure 1. The differential equation relating input δ and output x for the frequency response shown in figure 1 is

$$D^2x+K_1Dx+K_2x=K_3\delta \tag{8}$$

By applying the vector least-squares method of reference 6,

TABLE I

NUMERICAL CALCULATION ILLUSTRATING COMPUTATION OF THE FUNCTION FOR THE RESPONSE TO A UNIT IMPULSE BY THE FOURIER METHOD

$$\left[\omega_f = \frac{1}{5}\omega_n = 0.64\right]$$

n	$2n-1$	$(2n-1)\omega_f$	$\left|\frac{x}{\delta}\right|$	$\phi_{x\delta}$, deg
1	1	0.64	0.0970	−22
2	3	1.92	.0760	−61
3	5	3.20	.0520	−91
4	7	4.48	.0350	−111
5	9	5.76	.0240	−124
6	11	7.04	.0170	−133
7	13	8.32	.0130	−140
8	15	9.60	.0100	−145
9	17	10.88	.0079	−149
10	19	12.16	.0064	−152
11	21	13.44	.0053	−155
12	23	14.72	.0044	−157
13	25	16.00	.0038	−159
14	27	17.28	.0033	−160

$$h(t) = \frac{1.28}{\pi}[0.097 \cos (0.64t-22) + 0.076 \cos (1.92t-61) +$$

$$0.052 \cos (3.2t-91) + 0.035 \cos (4.48t-111) +$$
$$0.024 \cos (5.76t-124) + 0.017 \cos (7.04t-133) +$$
$$0.013 \cos (8.32t-140) + 0.010 \cos (9.6t-145) +$$
$$0.0079 \cos (10.88t-149) + 0.0064 \cos (12.16t-152) +$$
$$0.0053 \cos (13.44t-155) + 0.0044 \cos (14.72t-157) +$$
$$0.0038 \cos (16t-159) + 0.0033 \cos (17.28t-160)]$$

the following set of simultaneous equations is obtained:

$$\begin{bmatrix} N & +\sum_{j=\omega_1}^{\omega_n}(\omega B_x)_j & -\sum_{j=\omega_1}^{\omega_n}(A_x)_j \\ +\sum_{j=\omega_1}^{\omega_n}(\omega B_x)_j & \sum_{j=\omega_1}^{\omega_n}\left(\omega\left|\frac{x}{\delta}\right|\right)_j^2 & 0 \\ -\sum_{j=\omega_1}^{\omega_n}(A_x)_j & 0 & \sum_{j=\omega_1}^{\omega_n}\left(\left|\frac{x}{\delta}\right|^2\right)_j \end{bmatrix}\begin{Bmatrix} K_3 \\ K_1 \\ K_2 \end{Bmatrix} = \begin{Bmatrix} -\sum_{j=\omega_1}^{\omega_n}(\omega^2 A_x)_j \\ 0 \\ \sum_{j=\omega_1}^{\omega_n}\left(\omega\left|\frac{x}{\delta}\right|\right)_j^2 \end{Bmatrix}$$

$$(9)$$

where

$$A_x = \left|\frac{x}{\delta}\right| \cos \phi_{x\delta}$$

and

$$B_x = \left|\frac{x}{\delta}\right| \sin \phi_{x\delta}$$

These equations are then solved simultaneously for the transfer coefficients K_3, K_1, and K_2. From the form of equation (8), it is seen that the response to a unit impulse input is given by the equation

$$h(t) = \frac{K_3}{\omega_d} e^{-at} \sin \omega_d t \qquad (10)$$

where

$$\omega_d = \sqrt{K_2 - \frac{K_1^2}{4}}$$

and

$$a = \frac{K_1}{2}$$

A sample computation is shown in table II. Table II demonstrates the computational steps involved in solving for the transfer coefficients K_1, K_2, and K_3 for the system
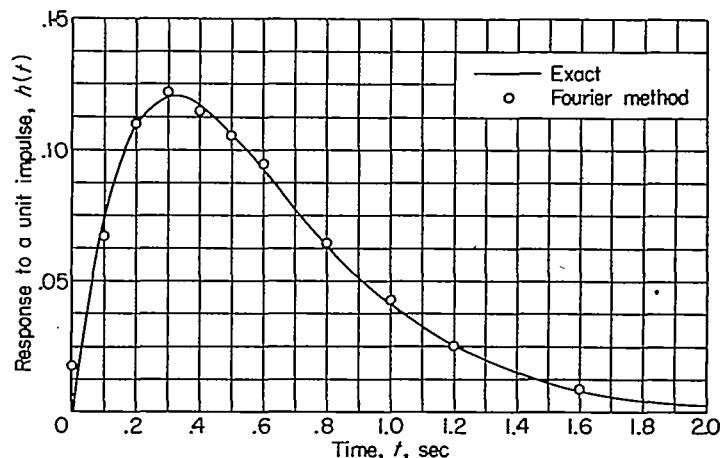

FIGURE 5.—Time response to a unit impulse obtained from the Fourier method compared to the exact values for the second-order system.

TABLE II

NUMERICAL CALCULATIONS ILLUSTRATING SCHUMACHER'S METHOD

ω, radians/sec	$\left|\frac{x}{\delta}\right|$	$\phi_{x\delta}$, deg	$A_x = \left|\frac{x}{\delta}\right| \cos \phi_{x\delta}$	$B_x = \left|\frac{x}{\delta}\right| \sin \phi_{x\delta}$
0	0.1000	0	0.10000	0
1	.0924	−33.7	.07687	−.05127
2	.0745	−63.4	.03336	−.06661
3	.0555	−86.8	.00310	−.05541
4	.0404	−104.0	−.00977	−.03920
5	.0298	−116.6	−.01334	−.02665
6	.0225	−125.8	−.01316	−.01825
7	.0174	−132.9	−.01184	−.01275
8	.0139	−138.4	−.01039	−.00923

$$\begin{bmatrix} 9.0 & -0.91336 & -0.15483 \\ -0.91336 & 0.15220 & 0 \\ -0.15483 & 0 & 0.03069 \end{bmatrix}\begin{Bmatrix} K_3 \\ K_1 \\ K_2 \end{Bmatrix} = \begin{Bmatrix} 1.97049 \\ 0 \\ 0.15220 \end{Bmatrix}$$

$$K_3 = 1.0002$$

$$K_1 = 6.0023$$

$$K_2 = 10.0053$$

$$\omega_d = 0.992$$

$$a = 3.0011$$

$$h(t) = 1.0010e^{-3.0011t} \sin 0.999t$$

defined in figure 1. These transfer coefficients are then substituted into equation (10), and a plot of this function is compared with the exact response of the system to a unit impulse input in figure 6.

This method gave additional information when applied to the second-order system since the transfer-function coefficients were determined in the process of the computation. A reasonable amount of computing time was required, and good accuracy was obtained. The method, however, requires previous knowledge of the form of the transform function relating the input and output. Use of the curve-fitting method on the higher order system did not prove too effective in determining all the parameters of the flexible system. If the form of the transfer function of the short period is assumed and fitted to that portion of the aircraft frequency response associated with the short period then the method is very effective in determining the short-period transfer-function coefficients.

The P-transform method.—The P-transform method, as described in appendix A, differs from the other methods presented in this report in that the time response to a unit impulse can be determined directly from a known response to a known arbitrary input. The computation is carried out entirely in the time domain, bypassing the frequency plane entirely, and does not require knowledge of the transfer function relating the input and output. The method also represents a simple procedure for using the response to a unit impulse and determining the response to a given arbitrary input.

An example of the method is shown in figure 7 in which the assumed input $\delta(t)$ and output $x(t)$ are given from which the response to the unit impulse $h(t)$ is computed. A comparison of the computed response to a unit impulse and the exact response to a unit impulse is shown in the lower part of figure 7.

The z-transform method.—The z-transform method as described in appendix A, when applied to the continuous linear systems, is very similar to the P-transform method. In fact, for a given time interval, the z-transform gives the same results as the P-transform for these systems. The computation obtained by using the z-transform is so similar
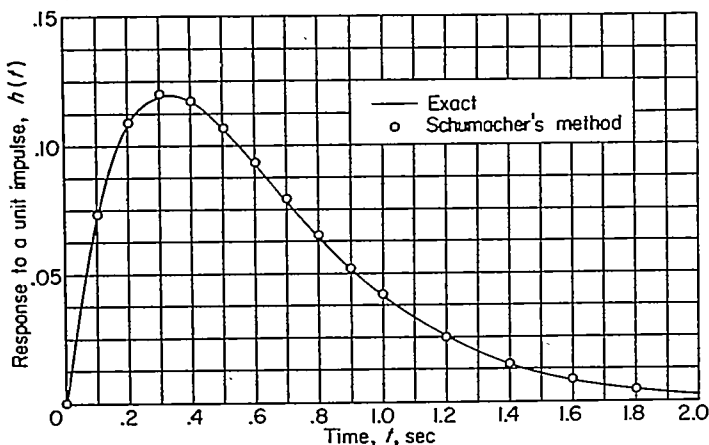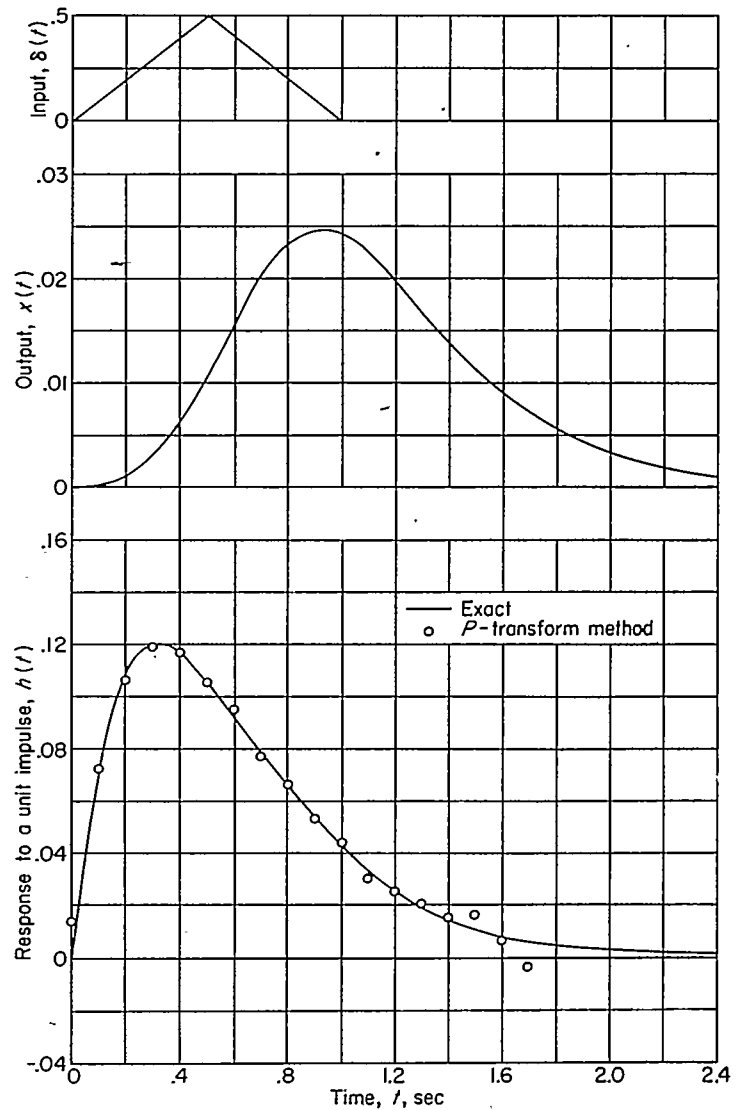

FIGURE 7.—Time response to a unit impulse obtained from the P-transform method compared to the exact values for the second-order system.

to that obtained by using the P-transform that it is not repeated herein.

RESPONSE TO ARBITRARY INPUTS

The procedure for obtaining the response to arbitrary inputs when the response to the unit impulse is known is simply a matter of applying Duhamel's integral to the response to the unit impulse and time history of the arbitrary input. A numerical method of performing this operation is shown in appendix A of reference 3.

The response to a unit step input may be obtained by numerically integrating the response to a unit impulse by using the integrating matrix given in reference 3. In like manner, the response to a ramp input may be obtained by numerically integrating the response to a step function. The response to a triangular input is obtained by superposition of the responses to various ramp inputs translated along the time scale.

The response to an arbitrary input may be obtained from the response to a unit impulse by the P-transform method by use of equation (A4) of the appendix.

FIGURE 6.—Time response to a unit impulse obtained from Schumacher's method compared to the exact values for the second-order system.

For directly determining the time response to an arbitrary input from frequency-response data, the Fourier method is perhaps the best known. The general method is indicated in references 7 and 8. In general, an input which can be expressed as a Fourier series can be represented as

$$\delta(t)=A_0+\sum_{n=1}^{\infty} C_{n\omega_f}(\sin n\omega_f t+\phi_{n\omega_f}) \qquad (11)$$

The response to the input given by equation (11) of a linear system whose amplitude ratio is $\left|\dfrac{x}{\delta}\right|$ and phase angle is $\phi_{x\delta}$ can be written as

$$x(t)=B_0+\sum_{n=1}^{\infty} C_{n\omega_f}\left|\frac{x}{\delta}\right|_{n\omega_f} \sin[n\omega_f t+\phi_{n\omega_f}+(\phi_{x\delta})_n] \qquad (12)$$

For a few specific shapes of inputs the values of B_0, $C_{n\omega_f}$, and $\phi_{n\omega_f}$ of equation (12) are known. Two wave forms frequently used are the square wave and triangular wave and examples of these are given.

Response to a square wave.—As indicated in reference 5 the Fourier series for a unit square-wave input $\delta(t)$ may be written as

$$\delta(t)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)\omega_f t \qquad (13)$$

and the response to this unit step input of a linear system of which the amplitude ratio is $\left|\dfrac{x}{\delta}\right|$ and the phase angle is $\phi_{x\delta}$ is given by

$$x(t)=\frac{1}{2}\left|\frac{x}{\delta}\right|_{\omega=0}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left|\frac{x}{\delta}\right|_{(2n-1)\omega_f}}{2n-1}\sin[(2n-1)\omega_f t+(\phi_{x\delta})_{(2n-1)\omega_f}] \qquad (14)$$

The accuracy of the computation depends on the choice of the optimum fundamental square-wave frequency ω_f and, as before, a suitable value is usually $\omega_f=\dfrac{\omega_n}{5}$ where ω_n is the lowest undamped natural frequency of the system.

Response to a triangular-wave input.—The specific form of the input used in determining the time response to a triangular-wave input is shown in sketch C where T_f is the

Sketch C.
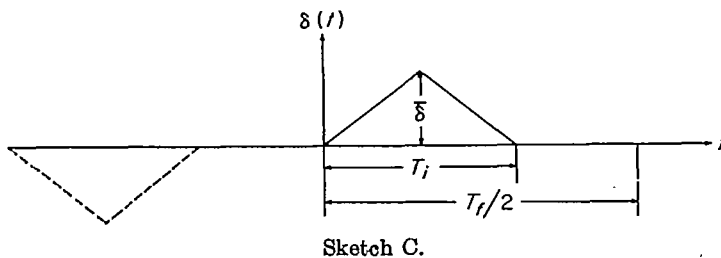
total period of the input, T_i is the length of the base in seconds, and $\bar\delta$ is the maximum value of the input.

Without giving the analytical development, representation of the Fourier series for this triangular-wave input is given by

$$\delta(t)=\sum_{n=1}^{\infty}C_{2n-1}\sin[(2n-1)\omega_f t+\phi_{2n-1}] \qquad (15)$$

where

$$\omega_f=\frac{2\pi}{T_f}\text{ radians/sec} \qquad (16)$$

$$C_{2n-1}=\frac{4\bar\delta}{\pi^2(2n-1)^2\left(\frac{T_i}{T_f}\right)}\left[1-\cos(2n-1)\pi\left(\frac{T_i}{T_f}\right)\right] \qquad (17)$$

and

$$\phi_{2n-1}=180\left[\frac{1}{2}-(2n-1)\left(\frac{T_i}{T_f}\right)\right] \qquad (18)$$

The response to this triangular-wave input of a linear system of which the amplitude is $\left|\dfrac{x}{\delta}\right|$ and the phase angle is $\phi_{x\delta}$ is then

$$x(t)=\sum_{n=1}^{\infty}C_{2n-1}\left|\frac{x}{\delta}\right|_{(2n-1)\omega_f}\sin[(2n-1)\omega_f t+\phi_{(2n-1)\omega_f}+(\phi_{x\delta})_{(2n-1)\omega_f}] \qquad (19)$$

Here again a suitable first approximation is $\omega_f=\dfrac{1}{5}\omega_n$.

For a higher order system such as a linear system with several structural modes (as indicated by peaks in the amplitude-ratio curve), the choice of a fundamental frequency ω_f becomes difficult inasmuch as odd multiples of ω_f must give the natural frequency ω_n and the frequency of each of the higher structural modes.

If an input is nonperiodic and cannot be expanded into a Fourier series directly, the Fourier transform of the input is used. The tables of reference 3 may be used to determine numerically the Fourier transform of an arbitrary input. The procedure for computing the response of a linear system to such an arbitrary input is briefly described as follows. The Fourier transform of the output is formed by multiplying the Fourier transform of the arbitrary input by the frequency response of the system, which is the ratio of the Fourier transforms of the output and input. The inverse Fourier transform of this product is then taken which is the time response of the system to the arbitrary input. The tables of reference 3 may also be used to perform the operation of the inverse Fourier transform.

COMPARISON AND USE OF METHODS
COMPARISON OF METHODS

A comparison of the methods as applied to the second-order system, defined by the frequency response shown in figure 1, may be made by noting the differences between the exact response to a unit impulse and the response computed by each of the methods shown in figures 2 to 7. From these figures there appears to be little difference between the accuracy of any one method over the others, and in each case greater accuracy may be obtained at the cost of more computation. For the comparisons shown, however, the computing time required by the rectangular-pulse method was significantly less than the time required by the other methods. The Fourier response to a unit impulse required the most time. A listing of the methods in the order of the computing time required for the comparisons shown in figures 2 to 7 is given as follows:

Rectangular pulse (refs. 3 and 4)
Schumacher's (ref. 6)
Floyd's (ref. 2)
Numerical integration
Fourier reponse to a unit impulse
(All computations were performed on a desk-type computer.)

A more severe test of the methods occurs when they are applied to linear systems of higher order. In order to demonstrate an application of this type, a further comparison of the methods was made by applying them to the system defined by the transfer function

$$\frac{x}{\delta}(s) = \frac{1}{s^2+6s+10} \frac{100}{s^2+0.4s+100} \frac{225}{s^2+0.2s+225} \qquad (20)$$

The frequency response of this system is shown in figure 8. The real part of the frequency response shown in figure 9 was computed from figure 8 by using the relation

$$\mathrm{Re}[H(i\omega)] = \left|\frac{x}{\delta}\right| \cos \phi_{x\delta} \qquad (21)$$

In figure 10 a comparison of the response to a unit impulse computed by four of the methods is shown. For Floyd's method a 27-line fit to the $\mathrm{Re}[H(i\omega)]$ shown in figure 9 was used. For the rectangular-pulse method, the staircase fit to the $\mathrm{Re}[H(i\omega)]$ was made with the interval $\Delta\omega=1$

FIGURE 9.—Real part of the frequency response of the higher order system.
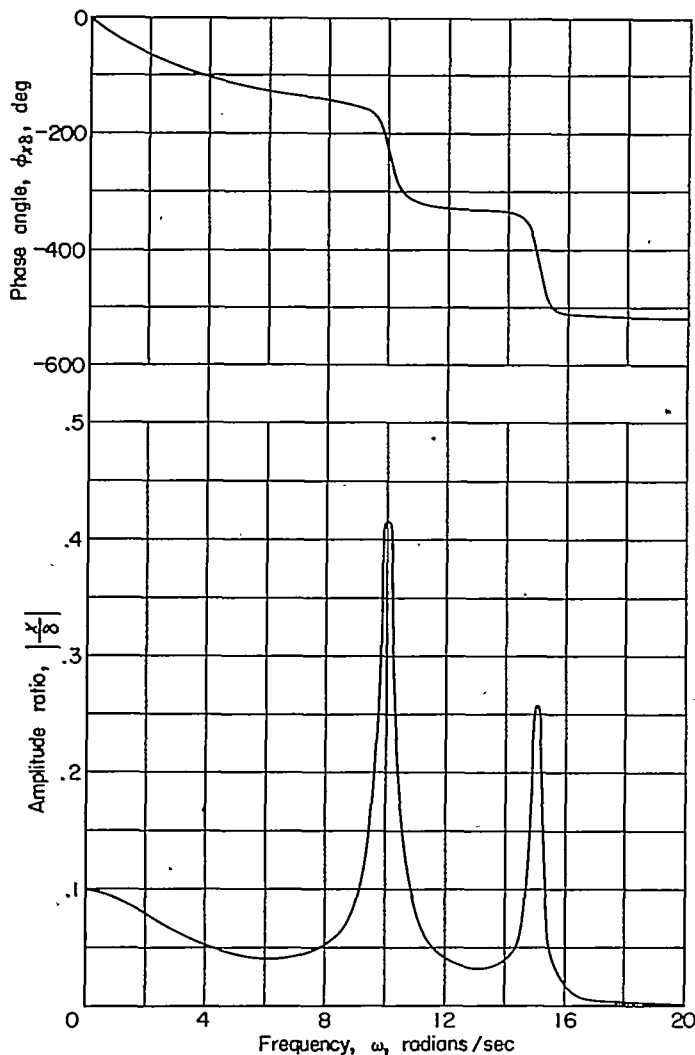
FIGURE 8.—Frequency response of higher order system.

$$\frac{x}{\delta}(s) = \frac{1}{s^2+6s+10} \frac{100}{s^2+0.4s+100} \frac{225}{s^2+0.2s+225}.$$

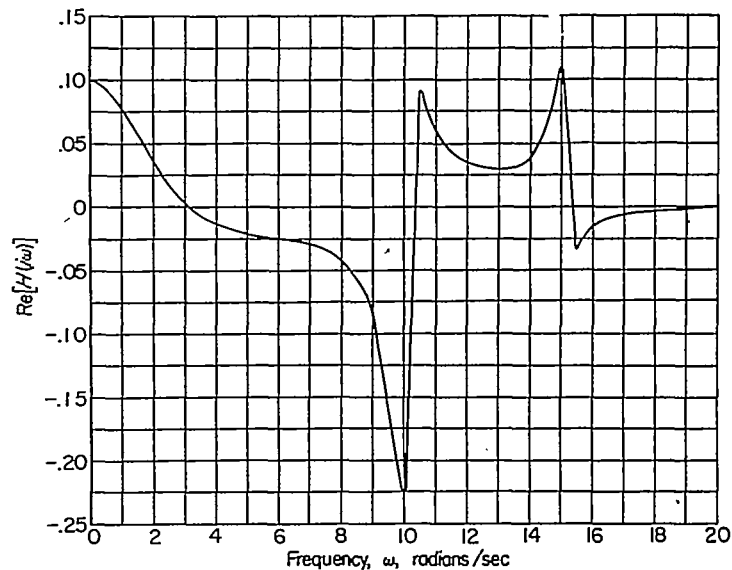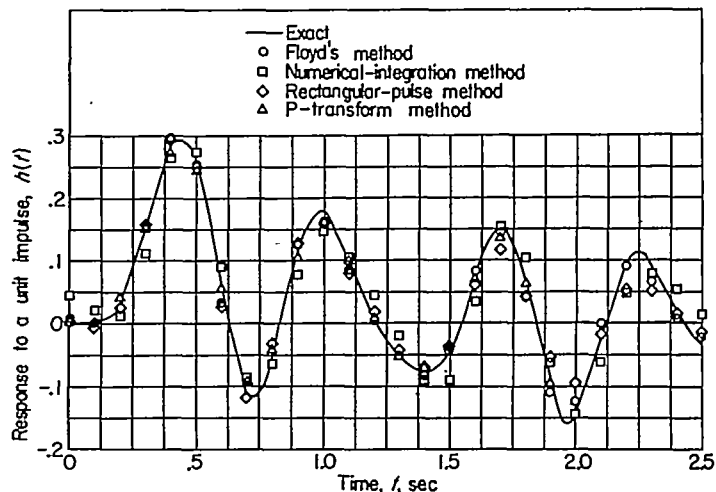FIGURE 10.—Time response to a unit impulse obtained by four methods compared to the exact values for the higher order system.

radian/sec, and for the numerical-integration method, an interval $\Delta\omega=0.5$ radian/sec was used. The response of the linear system defined by equation (20) to a triangular input shown in figure 7 of reference 3 was used to compute the points shown in figure 10 for the P-transform method. A time interval of $\Delta t=0.1$ second was used. The Fourier response to a unit-impulse method was found to be impractical in this case because of the large amount of computation required to obtain any accuracy. The accuracies of each of the four methods illustrated appear to be equivalent.

USE OF THE RESPONSE TO A UNIT IMPULSE AS A FLIGHT-DATA-ANALYSIS TECHNIQUE

As a further comparison of the methods applied to a higher order system, a typical longitudinal maneuver ($M=0.82$) for a flexible swept-wing airplane has been analyzed. In this case the output was the pitching-velocity response θ at the center of gravity of the airplane and the input was

the elevator angle δ_e. The analysis of this maneuver serves to demonstrate the use of the method of obtaining the time response of linear systems to a unit impulse from frequency-response data as a flight-data-analysis technique. The Fourier integrals of the output θ and input δ_e were evaluated by using automatic computing equipment and the methods of integration of product curves were used. The frequency response obtained by dividing the Fourier integral of the output by the Fourier integral of the input is shown in figure 11. The $\mathrm{Re}\left[\dfrac{\theta}{\delta_e}(i\omega)\right]$ computed from the frequency response of figure 11 is shown in figure 12. In figure 13 a comparison of the response to a unit impulse computed by three of the methods is shown. For Floyd's method a 37-line fit to the $\mathrm{Re}\left[\dfrac{\theta}{\delta_e}(i\omega)\right]$ shown in figure 12 was used and $h_\theta(t)$ was computed at enough values of time to define adequately its shape. For the rectangular-pulse method, the staircase fit to the $\mathrm{Re}\left[\dfrac{\theta}{\delta_e}(i\omega)\right]$ was made with the interval $\Delta\omega=1$ radian/sec and, for the numerical-integration method, an interval of $\Delta\omega=0.5$ radian/sec was used. In the case of the numerical method, points are shown in figure 13 only at time intervals of 0.1 second in order to compare accuracies. Several attempts were made to compute the response to
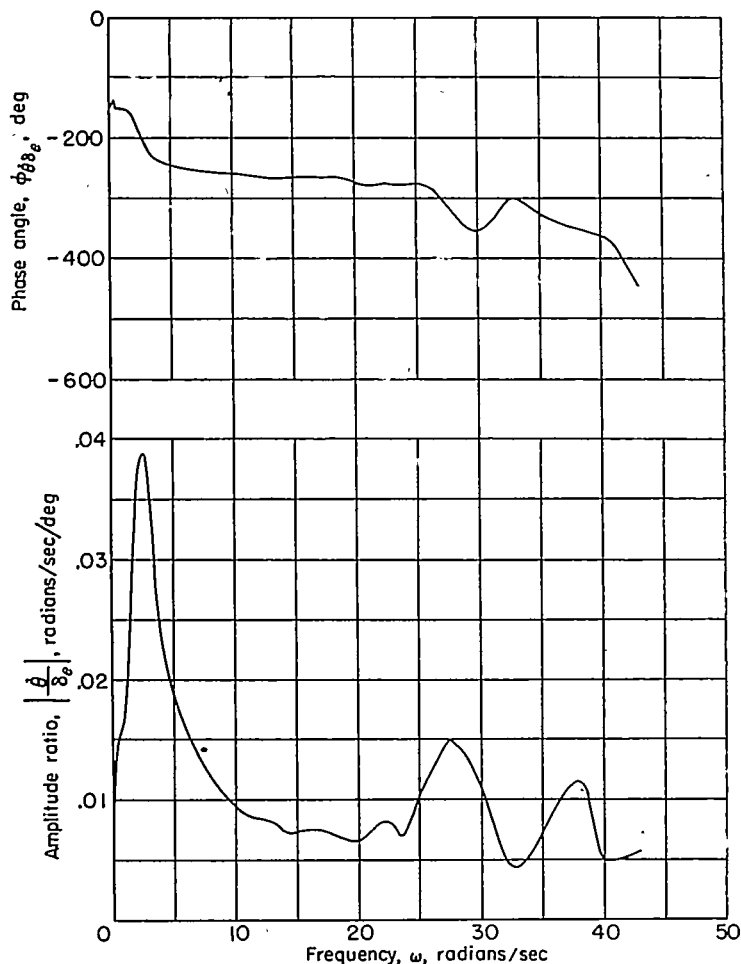

FIGURE 11.—A typical frequency response obtained from flight data of a flexible airplane at $M=0.82$.
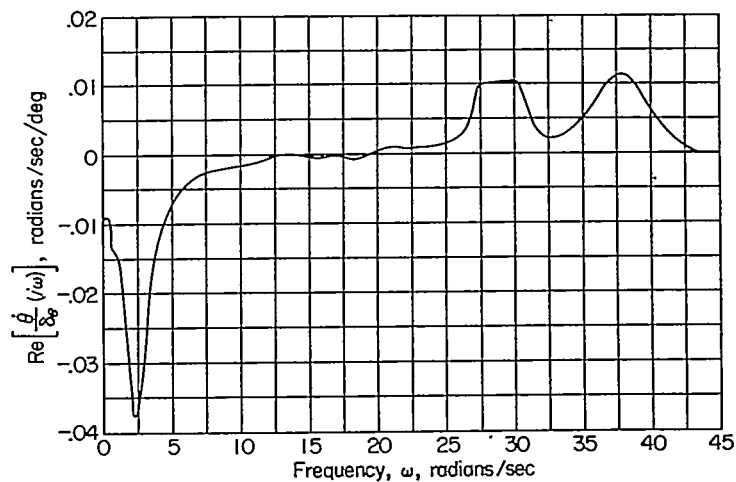

FIGURE 12.—Real part of the typical frequency response obtained from flight data of a flexible airplane at $M=0.82$.
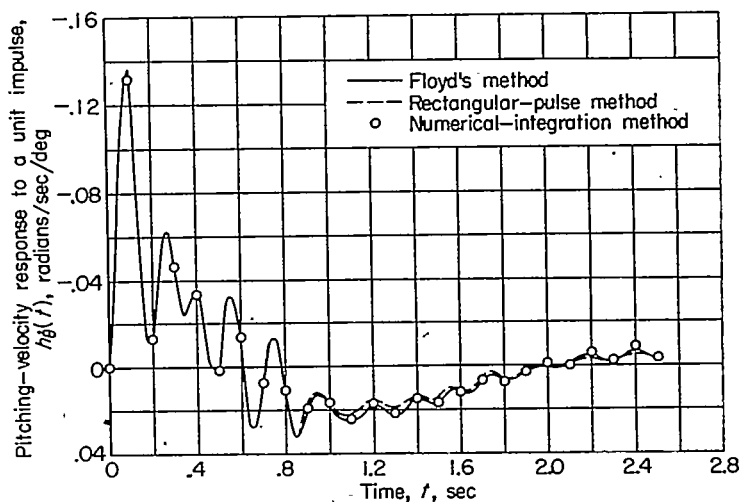

FIGURE 13.—Time response to a unit impulse obtained by three methods for the typical frequency response obtained from flight data of a flexible airplane at $M=0.82$.

the unit impulse by the P-transform method using the original elevator input $\delta_e(t)$ and output $\theta(t)$ shown in figure 14. The computations were made at time intervals ranging from 0.02 second $\leq \Delta t \leq 0.1$ second but did not yield satisfactory accuracy. It appears that this inaccuracy is primarily due to the sensitivity of the method to small errors in the first few terms of the response or input.

Time histories of the original elevator input δ_e and pitching-velocity response θ are shown in figure 14. The elevator input δ_e has also been used as a forcing function with the response to a unit impulse shown in figure 13 to compute a time history of θ. The Duhamel method outlined in appendix A of reference 3 with a time interval of 0.05 second was used for these calculations. A comparison of this computed θ response with the original θ response from a flight test is shown in the lower part of figure 14. This comparison shows the amount of error involved in the total computation procedure (transferring the data in time-history form to frequency-response form, then to the response to a unit impulse, and finally to the response to an arbitrary input by means of Duhamel's integral).
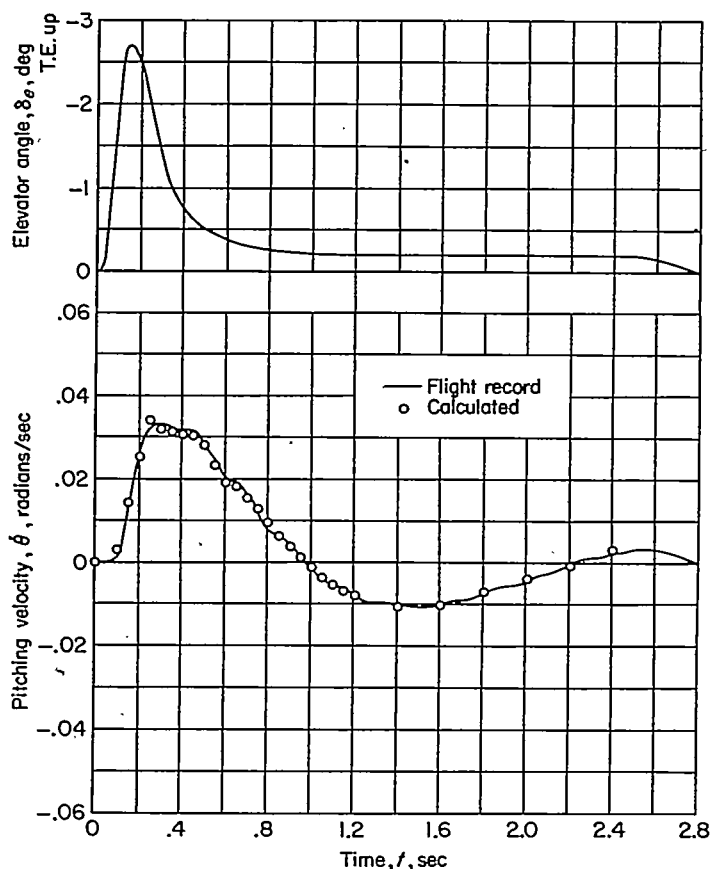
FIGURE 14.—Comparison with the actual flight record of the calculated pitching-velocity response using the same input-output data as was used to obtain the typical frequency response of the flexible airplane at M=0.82.

FIGURE 15.—Comparison of the predicted pitching-velocity response with the actual flight record obtained from the flexible airplane at M=0.80.

Another flight-test maneuver at M=0.80 was selected with the other conditions approximately the same as the previous data in order to see how well the response to a unit impulse computed from one maneuver could be used to predict the time response to an arbitrary input from a different maneuver. The time histories of this maneuver are shown in figure 15. Also shown in the lower part of figure 15 is the $\dot{\theta}(t)$ response calculated by application of Duhamel's integral to the response to a unit impulse given in figure 13, and the elevator motion for the maneuver is shown in the upper part of figure 15. A comparison of the computed and measured $\dot{\theta}$ response in the lower part of figure 15 gives some indication of how well the motions of an aircraft can be predicted by a detailed analysis of a single maneuver.

CONCLUDING REMARKS

Several methods were compared for obtaining the time response of linear systems to either a unit impulse or arbitrary input from frequency-response data. The methods were compared on the basis of accuracy, computing time required, and applicability to higher order linear systems.

The application of each of the methods to a simple second-order system indicated little difference between the accuracy of one method over the others, and, in general, it would be expected that greater accuracy could be obtained for each
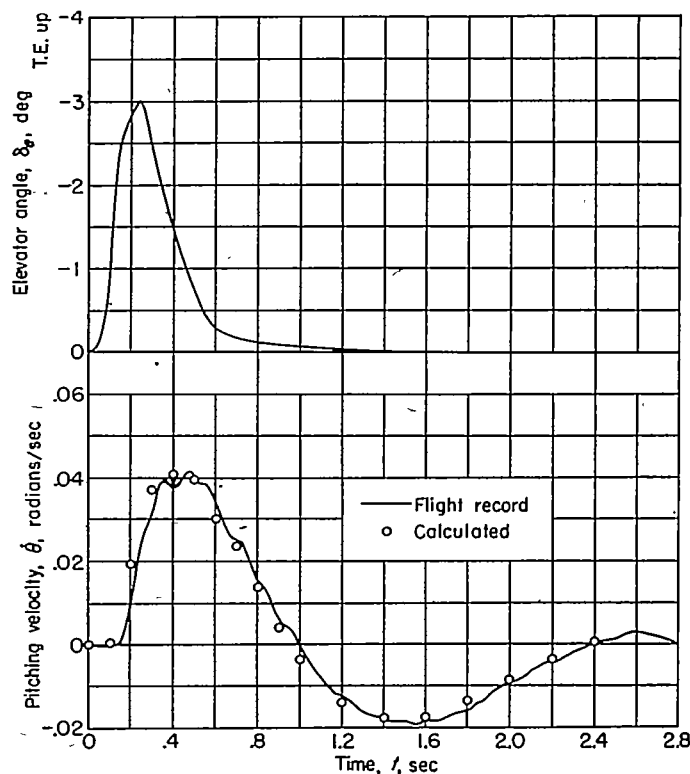
of the methods at the cost of more computing time.

For higher order systems the three methods generally classified as inverse Laplace transform methods were most effective. They gave good results for a moderate amount of computation. All of these methods are based on the evaluation of the form of the inverse Laplace transform equation for positive values of time.

The methods generally classified as Fourier methods gave good accuracy when applied to the second-order system. The accuracy of these methods was found to depend on the choice of a fundamental frequency ω_f. For simple systems a suitable value was found to be $\omega_f=\frac{\omega_n}{5}$ where ω_n is the lowest natural frequency. For higher order systems the choice of ω_f becomes more difficult. The Fourier response to a unit-impulse method was found to be impractical in the case of higher order systems because of the large amount of computation required to obtain accuracy.

Schumacher's method gave additional information when applied to the second-order system since the transfer-function coefficients were determined in the process of the computation. This method required a reasonable amount of computing time and gave good accuracy. The method, however, requires previous knowledge of the form of the transfer function relating the input and output. Use of the method on higher order systems did not prove too effective in determining all the parameters of the flexible system. A valuable use of the method was found in fitting the known short-period transfer function to only the short-period portion of the flexible-system frequency response to determine

the short-period transfer coefficients.

The P-transform method is different from the other methods presented in that the time response to a unit impulse can be determined directly from a known response to a known arbitrary input. The computation is carried out entirely in the time domain and bypasses the frequency plane entirely. The method also represents a simple procedure for using the response to a unit impulse and determining the response to a given arbitrary input. The method, however, when applied to the flight data of this report did not yield satisfactory accuracy. It appears this is primarily due to the sensitivity of the method to errors in the first few terms of the response or input.

For all the examples computed in this report the rectan-gular-pulse method of NACA Technical Note 3598 required less computing time, for the same degree of accuracy, than any of the other methods used to obtain the time response to a unit impulse from frequency-response data.

When an aircraft frequency response is known, it appears that these methods may be used as a flight-data-analysis technique which permits prediction of aircraft motions and loads without knowledge of the equations of motion relating the inputs and outputs for a flexible aircraft.

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    LANGLEY FIELD, VA., *March 16, 1956.*

APPENDIX A

P-TRANSFORM AND z-TRANSFORM METHODS

THE P-TRANSFORM METHOD

Recently a new general operational calculus has been introduced, the elements of which were given by Tustin in reference 9, and the mathematics was formalized by Bubb in reference 10. In this calculus, which appears well adapted to the numerical analysis and synthesis of linear systems, a polynomial transform or P-transform of a function $F(t)$ is defined by the equation

$$P[F(t)] = \tilde{F}(x)$$
$$= \sum_{m=-K}^{L} F(m \, \Delta t) x^m$$
$$= \sum_{m=0}^{\infty} F(m \, \Delta t) x^m \tag{A1}$$

where $F(m \, \Delta t)$ is the ordinate of $F(t)$ at an integral multiple of a time interval Δt. It may be noted that equation (A1) may be obtained from the Laplace transform by setting $t = m \, \Delta t$ and $x = e^{-s \, \Delta t}$, replacing the integral sign by a summation sign, and summing over the limits $m=0$ to $m=\infty$. The inverse P-transform is given by

$$F(t) = P^{-1}[\tilde{F}(x)] \tag{A2}$$

The superposition (convolution or Faltung) integral, which is also known as Duhamel's integral, relates the input $\delta(t)$, output $x(t)$, and response to a unit impulse $h(t)$ (also called a memory function) of a linear system and is given by

$$x(t) = \int_{-\infty}^{t} \delta(\tau) h(t-\tau) d\tau \tag{A3}$$

Bubb (ref. 10) shows that the P-transform of equation (A3) is

$$\tilde{x}(x) = \Delta t \tilde{\delta}(x) \tilde{h}(x) \tag{A4}$$

In the synthesis problem the input $\delta(t)$ and the output $x(t)$ are given and the response to the unit impulse $h(t)$ is to be calculated. This calculation is performed by forming the P-transforms of $x(x)$ and $\delta(x)$ and dividing by ordinary polynomial division, $\tilde{x}(x)$ by $\tilde{\delta}(x)$, to get the P-transform of the response to the unit impulse

$$\tilde{h}(x) = \frac{1}{\Delta t} \frac{\tilde{x}(x)}{\tilde{\delta}(x)}$$
$$= \sum_{m=-K}^{L} h_m x^m$$
$$= \sum_{0}^{\infty} h_m x^m \tag{A5}$$

For practical cases, however, the summation is made over a finite range of t values. The inverse P-transform to $\tilde{h}(x)$ then gives $h(t)$, the time response to the unit impulse. An example of this operation is shown in table III; the known input $\delta(t)$ and output $x(t)$ are shown in the upper part of figure 7, and the computed response to the unit impulse is compared with the analytic solution in the lower part of figure 7.

This operational calculus is also well adapted to the solution of the analysis problem in which the system response to a unit impulse $h(t)$ and the input function $\delta(t)$ are known and the calculation of the system output $x(t)$ is desired. The P-transforms of $h(t)$ and $\delta(t)$ are formed and multiplied together by ordinary polynomial multiplication as indicated in equation (A4). Since this operation is just the inverse of the operation shown in table III, an illustrative example is not shown. This method has been found to be a simple and rapid means of applying the Duhamel process.

The value of this operational calculus lies in the fact that all computations remain in the time domain and no translation to the frequency plane is required. Also, only simple direct arithmetical procedures are required for solving practical problems.

THE z-TRANSFORM METHOD

The z-transformation developed originally for the analysis and synthesis of sampled-data systems is also applicable to numerical solutions of continuous linear systems, as shown in references 11 and 12. The z-transform is defined by the

TABLE III

NUMERICAL CALCULATION ILLUSTRATING THE P-TRANSFORM METHOD

$$\left[\bar{h}(x)=\frac{1}{\Delta t}\frac{\bar{x}(x)}{\bar{\delta}(x)}\right]$$

t, sec	Time histories		P-transforms			Inverse P-transform
	Input, $\delta(t)$	Output, $x(t)$	Input, $\bar{\delta}(x)$	Output, $\bar{x}(x)$	Response to a unit impulse, $\bar{h}(x)$	Response to a unit impulse, $h(t)$
0	0	0	0	0	0. 014	0. 014
.1	.1	.00014	$.1x$	$.00014x$	$.072x$	.072
.2	.2	.00100	$.2x^2$	$.00100x^2$	$.106x^2$	.106
.3	.3	.00292	$.3x^3$	$.00292x^3$	$.119x^3$	.119
.4	.4	.00603	$.4x^4$	$.00603x^4$	$.117x^4$	.117
.5	.5	.01031	$.5x^5$	$.01031x^5$	$.106x^5$	.106
.6	.4	.01537	$.4x^6$	$.01537x^6$	$.095x^6$	.095
.7	.3	.01994	$.3x^7$	$.01994x^7$	$.077x^7$	.077
.8	.2	.02316	$.2x^8$	$.02316x^8$	$.066x^8$	.066
.9	.1	.02466	$.1x^9$	$.02466x^9$	$.053x^9$	.053
1.0	0	.02435	0	$.02435x^{10}$	$.044x^{10}$	.044
1.1	0	.02250	0	$.02250x^{11}$	$.030x^{11}$	.030
1.2	0	.01977	0	$.01977x^{12}$	$.025x^{12}$	.025
1.3	0	.01681	0	$.01681x^{13}$	$.021x^{13}$	.021
1.4	0	.01393	0	$.01393x^{14}$	$.015x^{14}$	.015
1.5	0	.01131	0	$.01131x^{15}$	$.016x^{15}$	.016
1.6	0	.00903	0	$.00903x^{16}$	$.006x^{16}$	.006
1.7	0	.00716	0	$.00716x^{17}$	$-.004x^{17}$	$-.004$
1.8	0	.00552	0	$.00552x^{18}$	$.012x^{18}$	.012
⋮	⋮	⋮	⋮	⋮	⋮	⋮

$$\bar{h}(x)=\frac{1}{\Delta t}\frac{\bar{x}(x)}{\bar{\delta}(x)}$$

$$=\frac{1}{0.1}\left(\frac{0+0.00014x+0.00100x^2+0.00292x^3+0.00603x^4+0.01031x^5+\ldots+0.00552x^{18}\ldots}{0+0.1x+0.2x^2+0.3x^3+0.4x^4+\ldots+0.1x^9}\right)$$

$$\bar{h}(x)=0.014+0.072x+0.106x^2+0.119x^3+0.117x^4+\ldots+0.012x^{18}\ldots$$

equation

$$z[F(t)]=F^*(z)$$

$$=\sum_{n=0}^{\infty}f(n\Delta t)z^{-n} \qquad (A6)$$

Comparison of equation (A6) and (A1) indicates the similarity of the two transforms. The z-transformation may be obtained from the Laplace transformation by setting $t=n\Delta t$ and $z=e^{s\Delta t}$, replacing the integral sign by a summation sign, and summing over the limits $n=0$ to $n=\infty$. The numerical computation applying the z-transform to the continuous systems used in this report is so similar to that in which the P-transform is used that it is not repeated.

It should be noted that in reference 13 Salzer defines the symbol z by

$$z=e^{-s\Delta t}$$

This definition makes the z-transform, when applied to continuous linear systems the same as the P-transform.

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