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COMPRESSIBLE POTENTIAL FLOW WITH CIRCULATION

ABOUT A CIRCULAR CYLINDER

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SUMMARY

The potential function for flow, with circulation, of a compressible fluid about a circular cylinder is obtained in series form including terms of the order of M^4 where M is the Mach number of the free stream. The resulting equations are used to obtain pressure coefficient as a function of Mach number at a point on the surface of the cylinder for different values of circulation. The coefficients derived are compared with the Glauert-Prandtl and Kármán-Tsien approximations which are functions of the pressure coefficients of an incompressible fluid. For the cases considered, the values of the pressure coefficients computed from the theory were found to lie somewhere between the two approximations, the first underestimating and the second overestimating the theoretical result.

INTRODUCTION

In the two-dimensional irrotational flow of a compressible fluid, where the expansion is assumed to be adiabatic, the velocity potential is known to satisfy a nonlinear partial differential equation of the second order. For subsonic velocities, at least three methods are known for the approximate solution of this equation. They are usually denoted as the method of small perturbations, the Rayleigh-Janzen method, and the hodograph method.

The method of small perturbations (references 1 and 2) assumes that velocity changes which are brought about by the airfoil in the uniform parallel air stream are small in comparison with the velocity of the undisturbed stream. Under this assumption it is possible to introduce new variables which reduce the differential equation

to a Laplace equation, and, as a consequence, the problem
becomes one concerning flow in an incompressible fluid,
provided the body is assumed distorted to correspond to
the change of variables. The assumed distortion consists
in expansion of the dimensions of the airfoil perpendic-
ular to the direction of the free stream in the ratio
$1/\sqrt{1-M^2}$, where M is the Mach number of the undisturbed
stream.

The Rayleigh-Janzen method (references 3 and 4) as-
sumes that the general expression for velocity potential
may be written as a series in rising powers of M and
with variable coefficients. These coefficients can be
shown to satisfy certain Poisson differential equations
and, if the equations are integrable, the solution be-
comes a matter of determining these coefficients. Suc-
cessive steps, however, become increasingly laborious and
the convergence of the series may be slow, even at rela-
tively small Mach numbers, if the shape of the body is
such that the speed of sound is approached locally. Solu-
tions, using this method of attack, have been carried out
by C. Kaplan (references 5 and 6), S. G. Hooker (reference
7), I. Imai (reference 8), K. Tamada and Y. Saito (refer-
ence 9), and L. Poggi (reference 10). Poggi introduced
certain refinements, and some of the preceding references
employ this process. It is tantamount to using the so-
called Neumann function in solving given Poisson equa-
tions and will be discussed later.

The hodograph method is ascribed by writers on that
subject to P. Molenbrock and A. Tschaplygin. Instead of
expressing the velocity potential as a function of coor-
dinates in the Cartesian or polar plane, the magnitude
of velocity V and its inclination θ to an assumed
axis are chosen as independent variables. The resulting
differential equation is linear and can be further simpli-
fied by replacing the pressure-volume relationship for
adiabatic expansion by the equation of a line tangent at
a point corresponding to the state of the fluid in the
ambient stream. This artifice was suggested by T. von
Kármán (references 2 and 11) and used most successfully
by H. T. Tsien (reference 12). K. Tamada (reference 13)
also has applied Tsien's more general results on elliptic
cylinders to compressible flow past a circular cylinder.

One noteworthy result of the hodograph method has
been the Kármán-Tsien expression for pressure coefficient

P in terms of Mach number M and $P_{M=0}$, the pressure coefficient for M=0. This expression may be written

$$P = P_{M=0} \cfrac{1}{\sqrt{1 - M^2} + \cfrac{M^2}{1+\sqrt{1 - M^2}} \cfrac{P_{M=0}}{2}} \tag{1}$$

It always gives, for negative pressure coefficients, a result greater in absolute value than the Glauert-Prandtl formula, which is based on the method of small perturbations,

$$P = P_{M=0} \frac{1}{\sqrt{1 - M^2}} \tag{2}$$

and is currently accepted as the more accurate of the two.

From equations (1) and (2) it is possible to compute the critical Mach number M_c, the value of M at which the local speed of sound is attained, in terms of $P_{M=0}$. The relations involving M_c and $P_{M=0}$, corresponding respectively to formulas (1) and (2), are

$$\frac{2}{\gamma M_c^2} \left\{ \left(\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1} M_c^2 \right)^{\frac{\gamma}{\gamma-1}} -1 \right\} = P_{M=0} \cfrac{1}{\sqrt{1-M_c^2} + \cfrac{M_c^2}{1+\sqrt{1-M_c^2}} \cfrac{P_{M=0}}{2}} \tag{3}$$

and

$$P_{M=0} = \frac{-2}{M_c} \sqrt{1 - M_c^2} \ (1 - M_c) \tag{4}$$

The difficulties inherent in the last two procedures are quite as distinctive as their respective approaches to the problem. As stated before, the Rayleigh-Janzen

method employs classical mathematics, the required terms
being solutions of Poisson equations with given boundary
conditions, but the work involved is arduous. In the
hodograph method the principal difficulty is to deter-
mine proper boundary conditions in the V, θ plane. In
available calculations the solution is given with a
slight distortion in the given boundary. It is possible
to correct this distortion, in some cases, so that the
final results are not too seriously affected. When the
flow around the body involves circulation, however, the
change in the boundary is more serious, for nonperiodic
terms appear and the boundary is no longer a closed curve.
At the present time, no way has been found to circumvent
this trouble. Added circulation does not involve any
essential variations in the Rayleigh-Janzen method, how-
ever, and in this report the velocity potential for such
compressible flow about a circular cylinder has been de-
rived. Since no theoretical study has been presented, as
far as is known, to determine the error in the Kármán-
Tsien pressure coefficient, the results obtained in this
report furnish a means of approaching this problem. The
results of such calculations, for various values of cir-
culation, are therefore included.

ANALYSIS

Consider a gas obeying the adiabatic law and flowing
irrotationally in two dimensions. Its equation of motion
may be written in polar coordinates in the form

$$\left[1 - \frac{\gamma-1}{2} M^2\left(\frac{V^2}{U^2} - 1\right)\right]\nabla^2\Phi = \frac{1}{2} \frac{M^2}{U^2}\left(\frac{\partial\Phi}{\partial r}\frac{\partial V^2}{\partial r} + \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta}\frac{\partial V^2}{\partial\theta}\right) \quad (5)$$

where

Φ velocity potential

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}$$

γ ratio of specific heats of gas

c_0 velocity of sound in undisturbed flow

U   velocity of free stream

$M = \dfrac{U}{c_0}$ Mach number of free stream

$V^2 = \left(\dfrac{\partial \Phi}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial \Phi}{\partial \theta}\right)^2$ local velocity squared

By introduction of the variables φ and v, so that

$$\varphi = \dfrac{\Phi}{U} \quad \text{and} \quad v = \dfrac{V}{U}$$

equation (5) may be written in the form

$$\left[1 - \dfrac{\gamma-1}{2} M^2 (v^2-1)\right] \nabla^2 \varphi = \dfrac{1}{2} M^2 \left(\dfrac{\partial \varphi}{\partial r} \dfrac{\partial v^2}{\partial r} + \dfrac{1}{r^2} \dfrac{\partial \varphi}{\partial \theta} \dfrac{\partial v^2}{\partial \theta}\right) \quad (6)$$

where

$$v^2 = \left(\dfrac{\partial \varphi}{\partial r}\right)^2 + \dfrac{1}{r^2} \left(\dfrac{\partial \varphi}{\partial \theta}\right)^2 \tag{7}$$

Following the method of Rayleigh and Janzen, assume that φ may be developed in a series of ascending powers of M^2, so that

$$\varphi = \varphi_0 + M^2\varphi_1 + M^4\varphi_2 + \ldots \tag{8}$$

After substitution of equation (8) in equation (7), elementary calculations show that

$$v^2 = v_0^2 + v_1^2 M^2 + v_2^2 M^4 + \ldots \tag{9}$$

where

$$v_0^2 = \left(\dfrac{\partial \varphi_0}{\partial r}\right)^2 + \dfrac{1}{r^2} \left(\dfrac{\partial \varphi_0}{\partial \theta}\right)^2 \tag{10a}$$

$$v_1^2 = 2\left\{\frac{\partial\varphi_0}{\partial r}\frac{\partial\varphi_1}{\partial r} + \frac{1}{r^2}\frac{\partial\varphi_0}{\partial\theta}\frac{\partial\varphi_1}{\partial\theta}\right\} \qquad (10b)$$

$$v_2^2 = 2\left\{\frac{\partial\varphi_0}{\partial r}\frac{\partial\varphi_2}{\partial r} + \frac{1}{r^2}\frac{\partial\varphi_0}{\partial\theta}\frac{\partial\varphi_2}{\partial\theta}\right\} + \left\{\left(\frac{\partial\varphi_1}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\varphi_1}{\partial\theta}\right)^2\right\} \qquad (10c)$$

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In a similar manner, equations (8) and (9) may be substituted in equation (6) and on equating coefficients of the same powers of M, the following relations for φ_0, φ_1, φ_2, . . . result:

$$\nabla^2\varphi_0 = 0 \qquad (11a)$$

$$\nabla^2\varphi_1 = \frac{1}{2}\left(\frac{\partial\varphi_0}{\partial r}\frac{\partial v_0^2}{\partial r} + \frac{1}{r^2}\frac{\partial\varphi_0}{\partial\theta}\frac{\partial v_0^2}{\partial\theta}\right) \qquad (11b)$$

$$\nabla^2\varphi_2 = \frac{1}{2}(\gamma - 1)(v_0^2 - 1)\nabla^2\varphi_1$$

$$+ \frac{1}{2}\left(\frac{\partial\varphi_0}{\partial r}\frac{\partial v_1^2}{\partial r} + \frac{1}{r^2}\frac{\partial\varphi_0}{\partial\theta}\frac{\partial v_1^2}{\partial\theta}\right)$$

$$+ \frac{1}{2}\left(\frac{\partial\varphi_1}{\partial r}\frac{\partial v_0^2}{\partial r} + \frac{1}{r^2}\frac{\partial\varphi_1}{\partial\theta}\frac{\partial v_1^2}{\partial\theta}\right) \qquad (11c)$$

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If equations (11a), (11b), and (11c) can be solved successively for $\varphi_0, \varphi_1, \varphi_2$, . . . the values may be substituted in equation (8) to get the potential function for the flow of a compressible fluid. A step-by-step procedure is therefore established whereby any desired degree of approximation to φ may be obtained, provided the

value of M is within the region of convergence of the resulting series. Equation (11a) is the differential equation satisfied by the potential function in the case of incompressibility. Once this potential function is known, it is used to evaluate the right-hand member of the second equation, the solution of which furnishes the second term in the development of φ. The method of obtaining further terms follows the same general procedure.

Consider now the case of a right circular cylinder of infinite length in a compressible fluid, the axis of the cylinder being at right angles to the direction of steady flow. In determining the velocity distribution about the cylinder, the problem may be treated two-dimensionally with a circle as the boundary curve and the equations established in the Rayleigh-Janzen method may be applied directly, in the following manner. The radius of the circle is arbitrarily assumed equal to 1, and a polar coordinate system is chosen with origin at the center of the circle and polar axis extending down stream. The flow about the circle is assumed to be that resulting from the combination of uniform stream velocity and circulation about the cylinder. Under these conditions, the classical expression for φ_0 is well known. It may be written

$$\varphi_0 = \left(r + \frac{1}{r}\right) \cos \theta - \frac{\Gamma}{2\pi U} \theta$$

where Γ is the circulation around the circle, measured positive in a clockwise direction. For ease of computation it is convenient to set

$$\frac{\Gamma}{\pi U} = K$$

and, as a consequence,

$$\varphi_0 = \left(r + \frac{1}{r}\right) \cos \theta - \frac{K}{2} \theta \qquad (12)$$

The boundary conditions, in general, are

$$\frac{\partial \varphi}{\partial r} = 0 \quad \text{for} \quad r = 1 \qquad (13a)$$

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and

$$\frac{\partial \varphi}{\partial r} = \cos \theta \quad \text{for} \quad r = \infty \tag{13b}$$

From equations (12) and (10a)

$$v_o^2 = \left(1 + \frac{1}{r^4}\right) - \frac{2}{r^2} \cos 2\theta + K\left(\frac{1}{r} + \frac{1}{r^3}\right) \sin \theta + \frac{K^2}{4r^2} \tag{14}$$

This result, together with equation (11b), gives

$$\nabla^3 \varphi_1 = \left(\frac{-4}{r^5} + \frac{2}{r^7}\right) \cos \theta + \frac{2}{r^3} \cos 3\theta$$

$$+ K \sin 2\theta \left(\frac{-1}{2r^2} - \frac{2}{2r^4} + \frac{1}{2r^6}\right) - \frac{K^2}{2r^3} \cos \theta \tag{15}$$

The more elementary methods of integration lead to certain difficulties when an attempt is made to solve for φ_1 in equation (15). These difficulties result from nonperiodic terms in the particular integral and resultant trouble in determining such constants of integration that the necessary periodicity, in terms of θ, is maintained in the final expression for the potential function. This difficulty may be obviated, however, by established methods. (See appendix.) It follows that the solution of

$$\nabla^2 \Omega = \frac{\sin m\theta}{r^s} \tag{16}$$

satisfying the boundary conditions

$$\left(\frac{\partial \Omega}{\partial r}\right)_{r=1} = 0 \qquad \left(\frac{\partial \Omega}{\partial r}\right)_{r=\infty} = 0$$

is

$$\Omega = \frac{\sin m\ \theta}{m(m-s+2)\ (m+s-2)} \left\{ \frac{(s-2)}{r^m} - \frac{m}{r^{s-2}} \right\} \qquad (\text{when } m+2 \neq s) \qquad (17a)$$

and

$$\Omega = \frac{-\sin m\ \theta}{mr^m} \left\{ \frac{1}{2m} + \frac{1}{2}\ \log r \right\} \qquad (\text{when } m+2=s) \qquad (17b)$$

The veracity of these solutions, together with analogous ones existing when sin m θ is replaced by cos m θ, may be checked easily by substitution in equation (16).

Since equation (15) is a linear differential equation, its solution is determined by considering each term of the right-hand member and summing the individual integrals obtained by means of equations (17a) and (17b). The final result is

$$\varphi_1 = \cos\theta \left(\frac{13}{12r} - \frac{1}{2r^3} + \frac{1}{12r^5} \right) + \cos 3\theta \left(\frac{-1}{4r} + \frac{1}{12r^3} \right)$$

$$+ K \sin 2\theta \left(\frac{1}{8} + \frac{1}{6r^2} + \frac{1}{24r^4} + \frac{\log r}{2r^2} \right)$$

$$+ K^2 \cos\theta \left(\frac{1}{4r} + \frac{\log 4}{4r} \right) \qquad (18)$$

In the evaluation of φ_2 the calculation follows the same pattern of development. From equation (10b), together with equations (12) and (18)

$$v_1^2 = \left(\frac{+19}{6r^4} - \frac{7}{3r^6} + \frac{1}{2r^8} \right) + \cos 2\theta \left(\frac{-8}{3r^2} + \frac{1}{r^4} - \frac{1}{r^6} + \frac{1}{3r^8} \right)$$

$$+ \cos 4\theta \left(\frac{1}{r^2} \right) + K \sin\theta \left(\frac{+1}{4r} + \frac{11}{6r^3} - \frac{5}{12r^5} + \frac{1}{3r^7} + \frac{2\log r}{r^5} \right)$$

$$+ \, \dot{K} \sin 3 \, \theta \left(- \frac{1}{4r} - \frac{7}{6r^3} - \frac{1}{2r^5} + \frac{1}{12r^7} - \frac{2 \log r}{r^3} \right)$$

$$+ \, K^2 \left(\frac{1}{4r^2} + \frac{1}{4r^4} + \frac{\log r}{2r^4} \right)$$

$$+ \, K^2 \cos 2 \, \theta \left(\frac{-1}{2r^2} - \frac{7}{12r^4} - \frac{1}{12r^6} - \frac{\log r}{2r^2} - \frac{\log r}{r^4} \right)$$

$$+ \, K^3 \sin \theta \left(+ \frac{1}{4r^3} + \frac{\log r}{4r^3} \right) \tag{19}$$

This result, together with equations (12), (14), and (18), substituted in equation (11c), gives

$$\nabla^2 \varphi_2 = (\gamma - 1) \left\{ \cos \theta \left(\frac{1}{r^5} + \frac{2}{r^7} - \frac{3}{r^9} + \frac{1}{r^{11}} \right) \right.$$

$$+ \, \cos 3 \, \theta \left(\frac{3}{r^7} - \frac{1}{r^9} \right) + \cos 5 \, \theta \left(\frac{-1}{r^5} \right)$$

$$+ \, K \sin 2 \, \theta \left(\frac{-1}{2r^4} - \frac{7}{4r^6} - \frac{3}{2r^8} + \frac{3}{4r^{10}} \right)$$

$$+ \, K \sin 4 \, \theta \left(\frac{3}{4r^4} + \frac{3}{2r^6} - \frac{1}{4r^8} \right)$$

$$+ \, K^2 \cos \theta \left(\frac{-1}{8r^3} - \frac{3}{8r^5} - \frac{9}{8r^7} + \frac{3}{8r^9} \right)$$

$$+ \, K^2 \cos 3 \, \theta \left(\frac{1}{8r^3} + \frac{9}{8r^5} + \frac{3}{8r^7} - \frac{1}{8r^9} \right)$$

$$+ K^3 \sin 2\theta \left(\frac{-3}{16r^4} - \frac{3}{8r^6} + \frac{1}{16r^8} \right) + K^4 \cos \theta \left(\frac{-1}{16r^5} \right) \Bigg]$$

$$+ \Bigg[\cos \theta \left(\frac{-32}{3r^5} + \frac{39}{2r^7} - \frac{15}{r^9} + \frac{11}{3r^{11}} \right) + \cos 3\theta \left(\frac{19}{6r^3} + \frac{3}{r^7} - \frac{5}{3r^9} + \frac{1}{2r^{11}} \right)$$

$$+ \cos 5\theta \left(\frac{-3}{2r^3} - \frac{1}{r^5} \right)$$

$$+ K \sin 2\theta \left(\frac{-1}{4r^2} - \frac{11}{3r^4} + \frac{25}{12r^6} - \frac{14}{3r^8} + \frac{3}{2r^{10}} - \frac{6 \log r}{r^6} + \frac{4 \log r}{r^8} \right)$$

$$+ K \sin 4\theta \left(\frac{1}{4r^2} + \frac{11}{4r^4} + \frac{2}{r^6} - \frac{5}{12r^8} + \frac{1}{12r^{10}} + \frac{3 \log r}{r^4} \right)$$

$$+ K^2 \cos \theta \left(\frac{-3}{8r^3} - \frac{2}{3r^5} - \frac{13}{24r^7} - \frac{\log r}{r^5} + \frac{\log r}{2r^7} \right)$$

$$+ K^2 \cos 3\theta \left(\frac{5}{8r^3} + \frac{2}{r^5} + \frac{3}{4r^7} - \frac{1}{4r^9} + \frac{\log r}{2r^3} + \frac{3 \log r}{r^5} - \frac{\log r}{r^7} \right)$$

$$+ K^3 \sin 2\theta \left(\frac{-1}{2r^4} - \frac{1}{3r^6} - \frac{\log r}{2r^4} \right) + K^4 \cos \theta \left(\frac{-1}{16r^5} \right) \Bigg\} \quad (20)$$

To integrate, formulas (17a) and (17b) are again
resorted to. The method of integration given in the
appendix also provides integrals corresponding to the
new type of terms appearing in the right-hand member of
equation (20). Thus, the solution of

$$\nabla^2\Omega = \frac{\log r \sin m\theta}{r^s} \tag{21}$$

satisfying boundary conditions $\left(\dfrac{\partial\Omega}{\partial r}\right)_{r=1} = 0$, $\left(\dfrac{\partial\Omega}{\partial r}\right)_{r=\infty} = 0$, is

$$\Omega = -\frac{\sin m\theta}{2m\ r^m}\left\{\frac{1}{(m+s-2)^2} + \frac{1}{(m-s+2)^2}\right\} - \frac{\sin m\theta}{2mr^{s-2}}\left\{\frac{2\ m\ \log r}{(m+s-2)(m-s+2)}\right.$$

$$\left. + \frac{1}{(m+s-2)^2} - \frac{1}{(m-s+2)^2}\right\} \tag{22a}$$

when $m \neq s-2$. When $m = s-2$,

$$\Omega = \frac{-\sin m\theta}{2m\ r^m}\left(\frac{1}{2}\log^2 r + \frac{1}{2m}\log r + \frac{1}{2m^2}\right) \tag{22b}$$

Proceeding directly with the integration results in the following expression

$$\varphi_a = (\gamma-1)\left\{\cos\theta\left(\frac{17}{60r} - \frac{1}{8r^3} + \frac{1}{12r^5} - \frac{1}{16r^7} + \frac{1}{80r^9}\right)\right.$$

$$+ \cos 3\theta\left(\frac{-61}{240r^3} + \frac{3}{16r^5} - \frac{1}{40r^7}\right)$$

$$+ \cos 5\theta\left(\frac{+1}{16r^3} - \frac{3}{80r^5}\right) + K\sin 2\theta\left(\frac{427}{960r^2} - \frac{7}{48r^4}\right.$$

$$\left. - \frac{3}{64r^6} + \frac{1}{80r^8} + \frac{\log r}{8r^2}\right)$$

$$+ K \sin 4\theta \left(\frac{-1}{16r^2} + \frac{1}{320r^4} - \frac{1}{80r^6} - \frac{3 \log r}{16r^4} \right)$$

$$+ K^2 \cos \theta \left(\frac{49}{128r} - \frac{3}{64r^3} - \frac{3}{64r^5} + \frac{1}{128r^7} + \frac{\log r}{16r} \right)$$

$$+ K^2 \cos 3\theta \left(\frac{-1}{64r} - \frac{57}{640r^3} + \frac{3}{128r^5} - \frac{1}{320r^7} - \frac{3 \log r}{16r^3} \right)$$

$$+ K^3 \sin 2\theta \left(\frac{41}{512r^2} - \frac{1}{32r^4} + \frac{1}{512r^6} + \frac{3 \log r}{64r^2} \right)$$

$$\left. + K^4 \cos \theta \left(\frac{+3}{128r} - \frac{1}{128r^3} \right) \right\}$$

$$+ \left\{ \cos \theta \left(\frac{+137}{80r} - \frac{4}{3r^3} + \frac{13}{16r^5} - \frac{5}{16r^7} + \frac{11}{240r^9} \right) \right.$$

$$+ \cos 3\theta \left(\frac{-19}{48r} - \frac{5}{48r^3} + \frac{3}{16r^5} - \frac{1}{24r^7} + \frac{1}{144r^9} \right)$$

$$+ \cos 5\theta \left(\frac{1}{16r} + \frac{1}{16r^3} - \frac{1}{20r^5} \right)$$

$$+ K \sin 2\theta \left(\frac{1}{16} + \frac{2267}{2880r^2} - \frac{23}{144r^4} - \frac{19}{192r^6} + \frac{1}{40r^8} \right.$$

$$\left. + \frac{11 \log r}{12r^2} - \frac{\log r}{2r^4} + \frac{\log r}{8r^6} \right)$$

$$+ K \sin 4\theta \left(\frac{-1}{64} - \frac{7}{48r^2} - \frac{7}{288r^4} - \frac{1}{48r^6} + \frac{1}{576r^8} - \frac{\log r}{4r^2} - \frac{\log r}{4r^4} \right)$$

$$+ K^2 \cos \theta \left(\frac{197}{288r} - \frac{17}{96r^3} - \frac{1}{72r^5} + \frac{3 \log r}{16r} - \frac{\log r}{8r^3} + \frac{\log r}{48r^5} \right)$$

$$+ K^2 \cos 3\theta \left(\frac{-1}{16r} - \frac{911}{5760r^3} + \frac{1}{128r^5} - \frac{1}{160r^7} - \frac{\log r}{16r} - \frac{5 \log r}{12r^3} \right.$$

$$\left. - \frac{\log r}{16r^5} - \frac{\log^2 r}{4r^3} \right)$$

$$+ K^3 \sin 2\theta \left(\frac{77}{576r^2} - \frac{1}{36r^4} + \frac{5 \log r}{32r^2} + \frac{\log^2 r}{16r^2} \right)$$

$$+ K^4 \cos \theta \left(\frac{3}{128r} - \frac{1}{128r^3} \right) \tag{23}$$

Applications of Theory

With the expressions for φ_0, φ_1, and φ_2, the two-term approximation for velocity potential is

$$\Phi = U \left(\varphi_0 + \varphi_1 M^2 + \varphi_2 M^4 \right) \tag{24}$$

and from this function the values of velocity at any point in the plane may be computed. Of particular interest is the evaluation of

$$- \frac{1}{r} \frac{\partial \Phi}{\partial \theta}$$

for this gives velocity normal to the radius vector of the point in question and thus, when $r = 1$, is equal to the velocity at the surface of the cylinder.

Neglecting all terms containing powers of $1/r$ greater than the first, Glauert (reference 1) has given this well-known result

$$\left(-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) = U\left(\sin\theta + \frac{K}{2r}\frac{\sqrt{1-M^2}}{1-M^2\sin^2\theta}\right)$$

and under the same restrictions equation (24) gives

$$\left(-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) = U\left(\sin\theta + \frac{K}{2r} - \frac{KM^2\cos2\theta}{4r} - \frac{KM^4\cos2\theta}{8r} + \frac{KM^4\cos4\theta}{16r}\right)$$

These results are identical to the order of M^4.

Velocity at the surface of the cylinder is

$$V(1,\theta) = U\left\{2\sin\theta + \frac{K}{2} + M^2\left[\frac{2\sin\theta}{3} - \frac{\sin3\theta}{2}\right.\right.$$

$$\left. - \frac{2K\cos2\theta}{3} + \frac{K^2\sin\theta}{4}\right] + M^4\left[(\gamma-1)\left(\frac{23}{120}\sin\theta\right.\right.$$

$$- \frac{11}{40}\sin3\theta + \frac{1}{8}\sin5\theta - \frac{127}{240}K\cos2\theta$$

$$+ \frac{23}{80}K\cos4\theta + \frac{19}{64}K^2\sin\theta - \frac{81}{320}K^2\sin3\theta$$

$$\left.- \frac{13}{128}K^3\cos2\theta + \frac{1}{64}K^4\sin\theta\right) + \left(\frac{37}{40}\sin\theta\right.$$

$$- \frac{25}{24} \sin 3 \theta + \frac{3}{8} \sin 5 \theta - \frac{887}{720} K \cos 2 \theta$$

$$+ \frac{59}{72} K \cos 4 \theta + \frac{71}{144} K^2 \sin \theta - \frac{631}{960} K^2 \sin 3 \theta$$

$$\left. \left. \left. - \frac{61}{288} K^3 \cos 2 \theta + \frac{1}{64} K^4 \sin \theta \right) \right] \right\} \qquad (25)$$

Results derived from this equation will be presented in terms of pressure coefficient, which is defined as

$$\frac{p - p_o}{1/2 \rho_o U^2} \qquad (26)$$

where the zero subscripts refer to free-stream conditions. By means of Bernoulli's equation it follows that

$$P_{M=o} = 1 - \left(\frac{V}{U} \right)^2 \qquad (27)$$

where $P_{M=o}$ denotes the pressure coefficient for incompressible fluids. If P denotes pressure coefficient for a compressible fluid obeying the adiabatic law, then

$$P = \frac{2}{\gamma M^2} \left\{ \left[1 + \frac{\gamma-1}{2} M^2 \left(1 - \frac{V^2}{U^2} \right) \right]^{\frac{\gamma}{\gamma-1}} - 1 \right\} \qquad (28)$$

where M is the Mach number of the free stream and γ is the ratio of specific heats (1.40 for air).

As an approximation for P the Glauert-Prandtl result (reference 2) is given by equation (2) and the Kármán-Tsien result (reference 2) is given by equation (1).

The velocity at the topmost part of the cylinder may be found by setting $\theta = 90°$ in equation (25) and the resultant expression is a function of K and M. In figure 1, pressure coefficient at this point is plotted against M for $K = 0$. As a test for rapidity of convergence the expressions for velocity, using only M^2 (one-term approximation) as well as M^2 and M^4 (two-term approximation), are used. It is to be noted that the curves diverge greatly near the critical Mach number, but that, for smaller values of M, the curves derived from equation (25) are together and definitely lie between the results derived from the Glauert-Prandtl and Karman-Tsien relations. Figures 2 and 3 show the same equations applied for $K = 1/4$ and $1/2$, respectively. It thus appears from these calculations that the true value of P lies somewhere between the approximations applied. On the other hand, experimental data, as determined from airfoils, have shown a much better agreement with the Karman-Tsien equation than have the theoretical results obtained here for the cylinder.

In figures 4 and 5 the same point on the cylinder is under consideration, but circulation is made negative by setting K equal to $-1/4$ and $-1/2$ in the two cases. As the pressure coefficient gets smaller in absolute value, the theoretical data agree more nearly with equation (1). For all the calculations, the one-term and two-term approximations diverge widely as the Mach number increases. This is to be expected for, as has been pointed out by Messrs. G. I. Taylor and C. F. Sharman in reference 14, the convergence of the series fails when M reaches its critical value. For near-critical velocities, several more terms would be required to furnish an accurate evaluation of the true potential-flow pressure coefficient.

Figure 6 shows the value of pressure coefficient at all points on the surface of the cylinder. These results were derived from equation (25) with K set equal to $1/4$ and at a Mach number of $1/5$. Crosses on the graph are at positions obtained from equation (1), and the circles were determined by equation (2). The disagreement at the extreme pressure coefficients is again in evidence.

APPENDIX

For the integration of the differential equation

$$\frac{\partial^2 \varphi}{\partial r} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} = f(r, \theta) \qquad (29)$$

with boundary conditions

$$\left(\frac{\partial \varphi}{\partial r}\right)_{r=1} = 0 \qquad \text{and} \qquad \left(\frac{\partial \varphi}{\partial r}\right)_{r=\infty} = 0 \qquad (30)$$

it is assumed, as in Poggi's method, that the unit circle, with center at the origin is surrounded externally by a continuous distribution of sources such that the source strength of an element $RdRd\omega$ is

$$f(R, \omega)RdRd\omega \qquad (31)$$

Equation (29) may then be interpreted as the equation for incompressible flow in such a region.

The velocity potential of an incompressible fluid at point (r, θ) due to a unit source at (R, ω) may be calculated by the method of images. If this potential is denoted by $\bar{\varphi}$, then

$$\bar{\varphi} = \frac{1}{2\pi} \left\{ \frac{1}{2} \log \left[r^2 + R^2 - 2Rr \cos(\theta - \omega) \right] \right.$$

$$\left. + \frac{1}{2} \log \left[\frac{1}{r^2} + R^2 - \frac{2R}{r} \cos(\theta - \omega) \right] + \log \frac{1}{R} \right\} \qquad (32)$$

The required potential φ, satisfying equation (29), is therefore

$$\varphi(r,\theta) = \frac{1}{4\pi} \int_s \int \left\{ \log\left[r^2 + R^2 - 2Rr \cos (\theta - \omega) \right] \right.$$

$$+ \log\left[\frac{1}{r^2} + R^2 - 2\frac{R}{r} \cos(\theta - \omega) \right]$$

$$\left. + 2 \log \frac{1}{R} \right\} f(R,\omega) \, R dR d\omega \tag{33}$$

where the integration extends over the region of the plane lying external to the unit circle.

In the equations under consideration in this report the function $f(R,\omega)$ is restricted to one of the forms

$$\frac{\sin m\omega}{R^s} \qquad\qquad \frac{\cos m\omega}{R^s}$$

$$\log R \, \frac{\sin m\omega}{R^s} \qquad\qquad \log R \, \frac{\cos m\omega}{R^s}$$

$$m, \; s \gtreqless 1$$

As an example of the integration process, take the first case listed. Then, set $r' = 1/r$, which results in

$$\varphi(r,\theta) = \frac{1}{4\pi} \int_1^\infty \int_0^{2\pi} \log\left(r^2 + R^2 - 2Rr \cos(\theta - \omega) \right) \frac{\sin m\omega}{R^s} \, R dR d\omega$$

$$+ \frac{1}{4\pi} \int_1^\infty \int_0^{2\pi} \log\left(r'^2 + R^2 - 2Rr' \cos(\theta - \omega) \right) \frac{\sin m\omega}{R^s} R dR d\omega$$

$$- \frac{1}{2\pi} \int_1^\infty \int_0^{2\pi} \log R \, \frac{\sin m\omega}{R^s} \, R \, dR \, d\omega \tag{34}$$

$$= I_1 + I_2 + I_3$$

Integrating I_3 first with respect to ω shows immediately that its value is zero.

For purposes soon evident I_2 is written in the form

$$I_2 = \frac{1}{4\pi} \int_1^\infty \int_0^{2\pi} \left\{ \log R^2 \right.$$

$$\left. + \log \left[1 + \left(\frac{r'}{R} \right)^2 \cos (\theta - \omega) \right] \right\} \frac{\sin m\omega}{R^s} \, R \, dR \, d\omega \tag{35}$$

Since the $\log R^2$ term vanishes, after integration with respect to ω, the expression for I_2 may be simplified further by the substitution

$$\log \left[1 + \left(\frac{r'}{R} \right)^2 - 2\frac{r'}{R} \cos (\theta - \omega) \right]$$

$$= -2 \sum_{n=1}^\infty \frac{1}{n} \left(\frac{r'}{R} \right)^n \cos n (\theta - \omega) \tag{36}$$

since $r' < R$, and

$$I_2 = - \frac{1}{2\pi} \int_1^\infty \int_0^{2\pi} \sum_{n=1}^\infty \frac{1}{n} \left(\frac{1}{rR} \right)^n \frac{1}{R^{s-1}} \sin m\omega \, (\cos n\, \theta \cos n\omega$$

$$+ \sin n\theta \sin n\omega) \, dRd\omega = - \frac{1}{2\pi} \int_1^\infty \frac{1}{m} \frac{1}{r^m R^{m+s-1}} \pi \sin m\theta \, dR$$

$$= - \frac{1}{2} \frac{1}{mr^m} \frac{\sin m\theta}{m+s-2} \tag{37}$$

To integrate I_1, the region exterior to the unit circle is broken into two parts: The first part is a circular ring external to the unit circle and extends to the fixed point r; the second part is the remaining portion of the plane and extends infinitely. Then

$$I_1 = \frac{1}{4\pi} \int_1^r \int_0^{2\pi} \left\{ \log r^2 + \log \left[1 + \left(\frac{R}{r}\right)^2 \right. \right.$$

$$\left. \left. - 2\frac{R}{r} \cos (\theta - \omega) \right] \right\} \frac{\sin m\omega}{R^s} R dR d\omega$$

$$+ \frac{1}{4\pi} \int_r^\infty \int_0^{2\pi} \left\{ \log R^2 + \log \left[1 + \left(\frac{r}{R}\right)^2 \right. \right.$$

$$\left. \left. - 2\left(\frac{r}{R}\right) \cos (\theta - \omega) \frac{\sin m\omega}{R^s} \right] \right\} R dR d\omega = J_1 + J_2 \tag{38}$$

By use of the same series expansion as was previously used,

$$J_1 = - \frac{1}{2\pi} \int_1^r \frac{\pi}{m} \frac{R^{m-s+1}}{r^m} \sin m\theta \, dR$$

$$= - \frac{\sin m\theta}{2mr^m} \frac{1}{m-s+2} (r^{m-s+2} - 1) \quad \text{when } m-s+2 \neq 0$$

$$= - \frac{\sin m\theta}{2mr^m} \cdot \log r \quad \text{when } m-s+2 = 0 \tag{39}$$

In the same manner

$$J_2 = - \frac{1}{2\pi} \int_r^\infty \frac{\pi}{m} \frac{r^m}{R^{m+s-1}} \sin m\,\theta\, dR$$

$$= - \frac{r^m}{2m} \frac{\sin m\,\theta}{(m-s-2)r^{m+s-2}} \tag{40}$$

From equations (37), (38), (39), and (40), the solution of equation (29), for the case in which

$$f(r,\theta) = \frac{\sin m\theta}{r^s}$$

is consequently

$$\varphi = \frac{\sin m\,\theta}{m(m-s+2)(m+s-2)} \left\{ \frac{(s-2)}{r^m} - \frac{m}{r^{s-2}} \right\} \quad (\text{when } m-2 \neq s)$$

$$\varphi = \frac{-\sin m\theta}{mr^m} \left\{ \frac{1}{2m} + \frac{1}{2} \log r \right\} \quad (\text{when } m+2=s) \tag{41}$$

For the other cases the integration process follows exactly the same procedure.

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Fig. 1
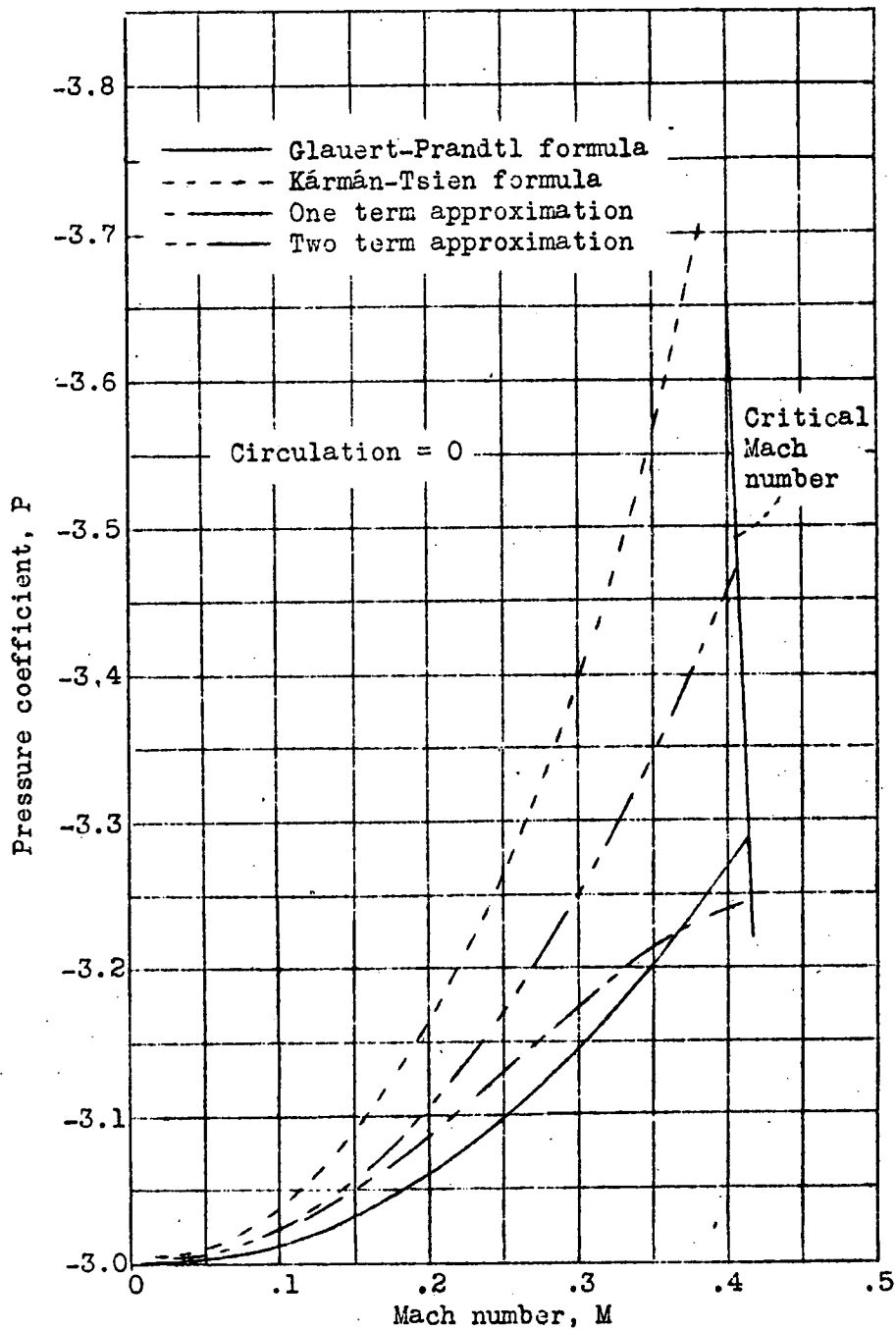
Figure 1.- Variation of minimum pressure coefficient with
Mach number when circulation is 0.
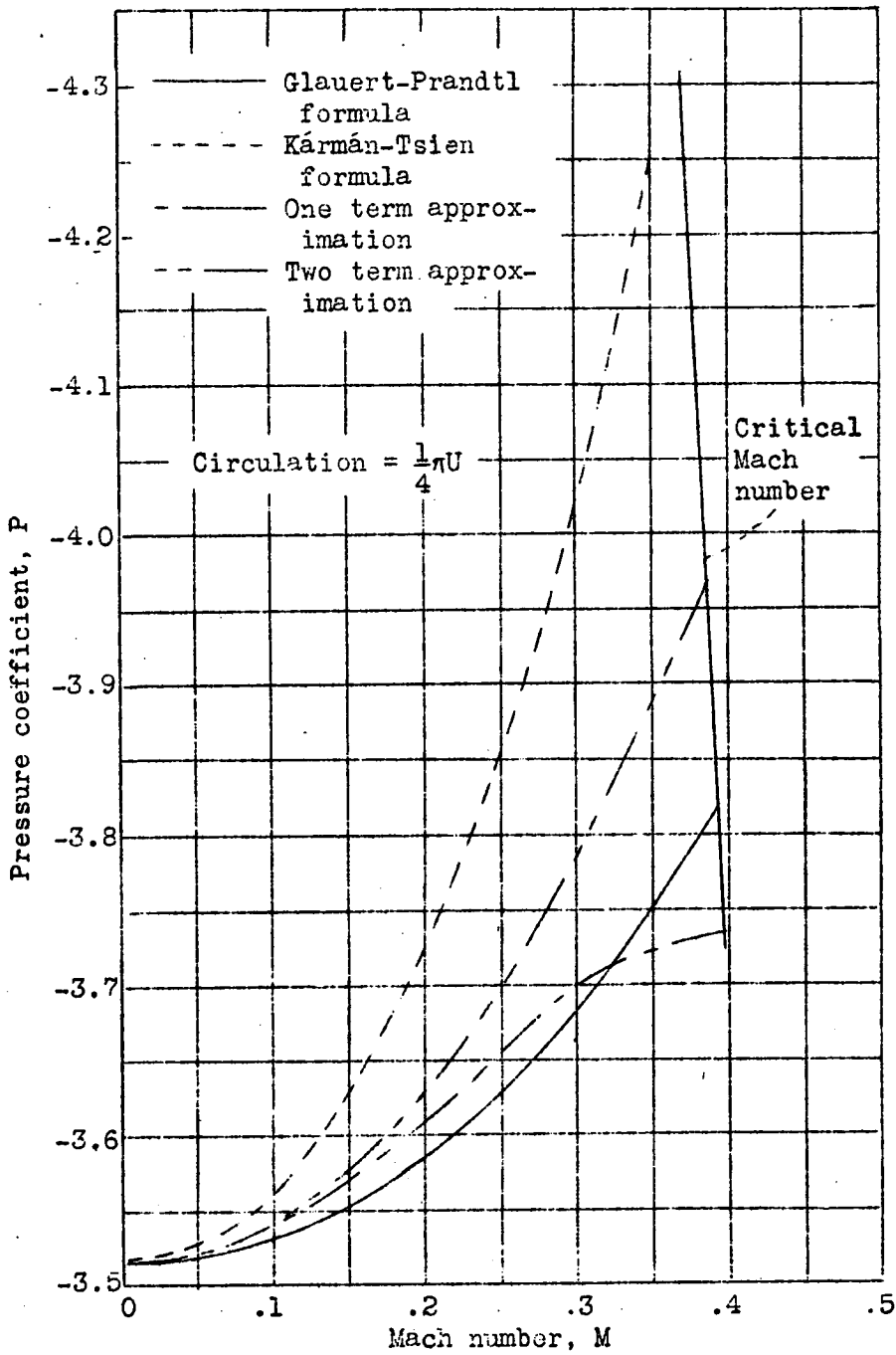
Figure 2.- Variation of minimum pressure coefficient
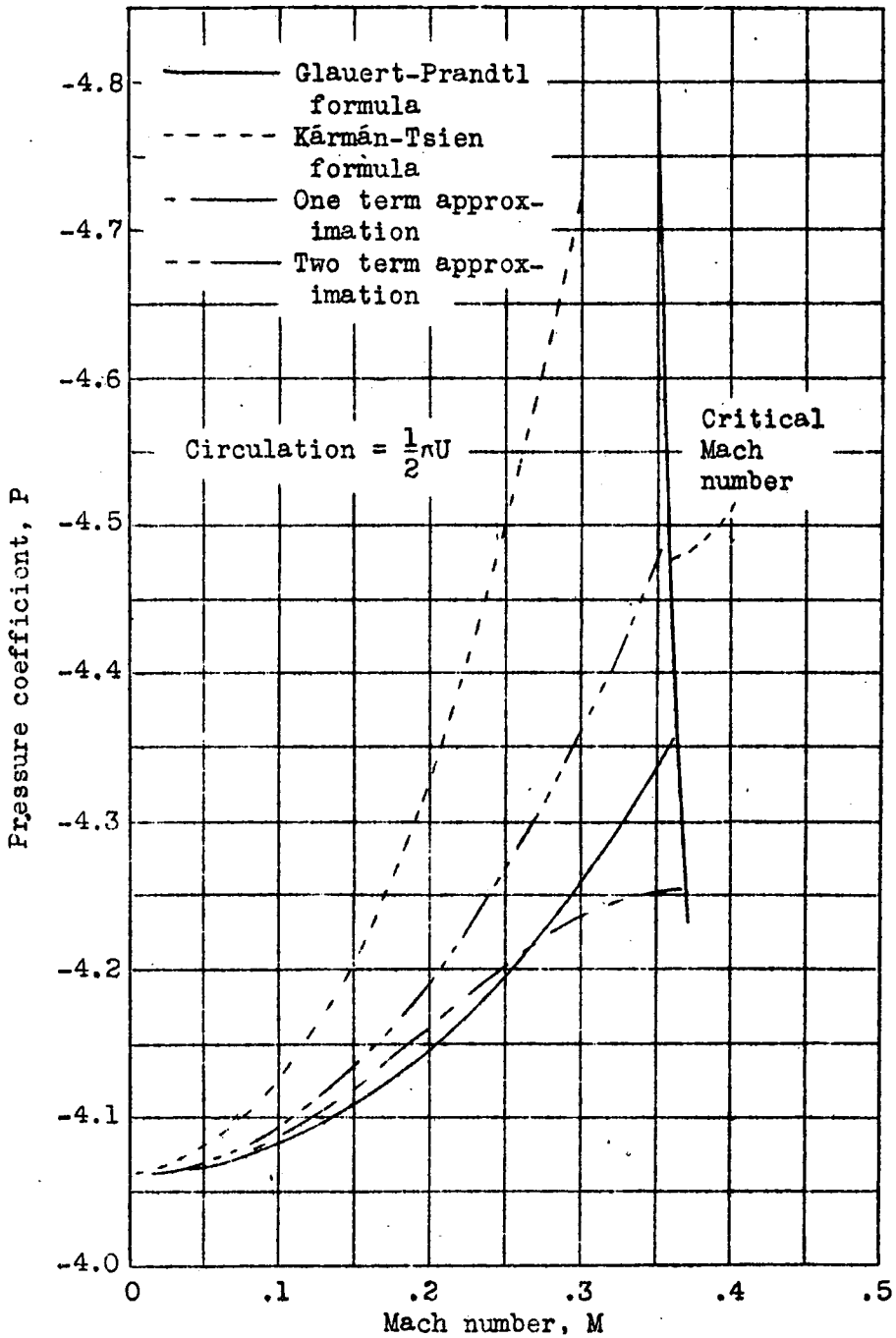with Mach number when circulation is $\frac{1}{4}\pi U$.

Figure 3.- Variation of minimum pressure coefficient
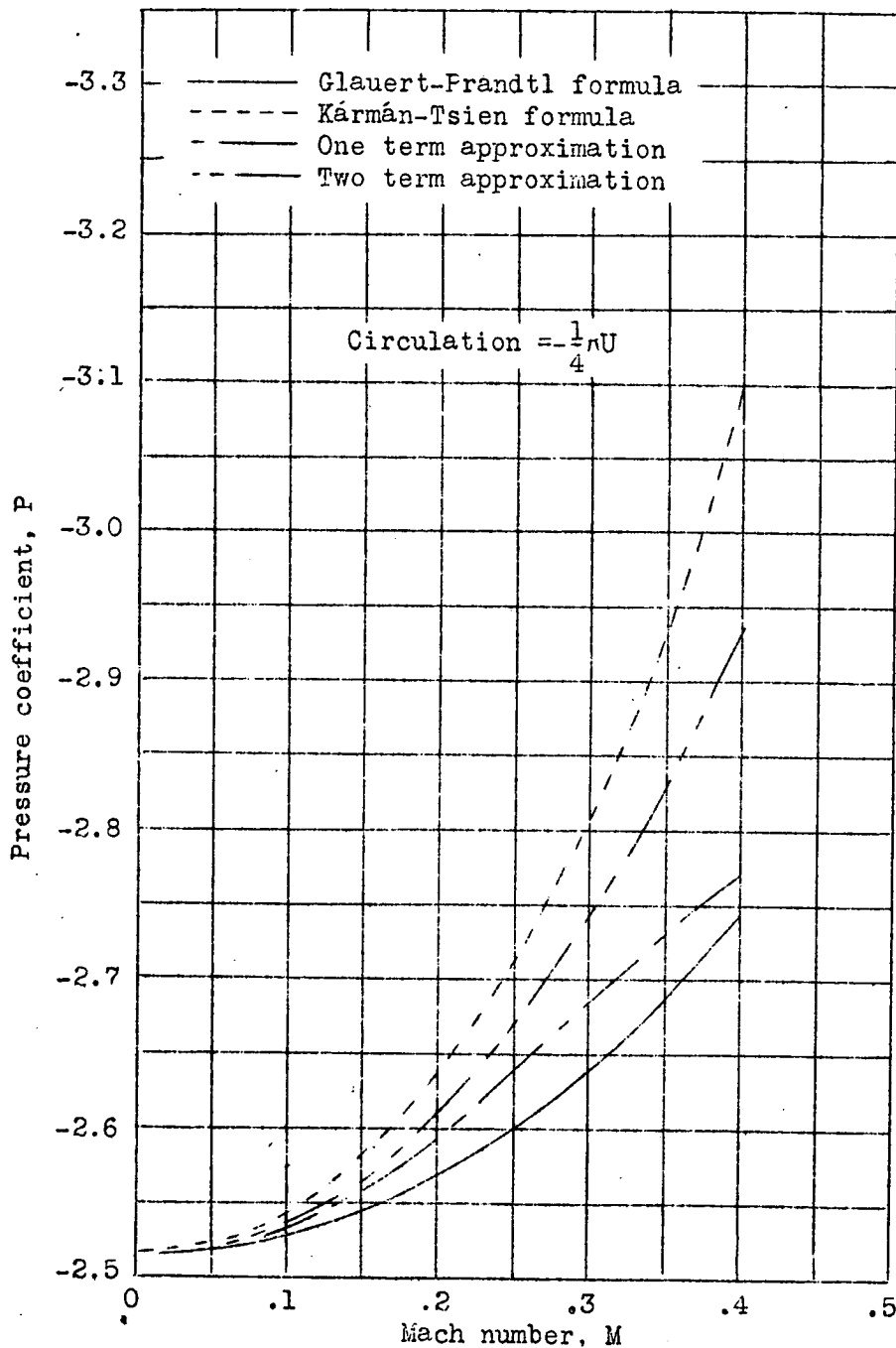with Mach number when circulation is $\frac{1}{2}\pi U$.

Figure 4.- Variation of pressure coefficient with Mach number at topmost point of cylinder when circulation is $-\frac{1}{4}\pi U$.
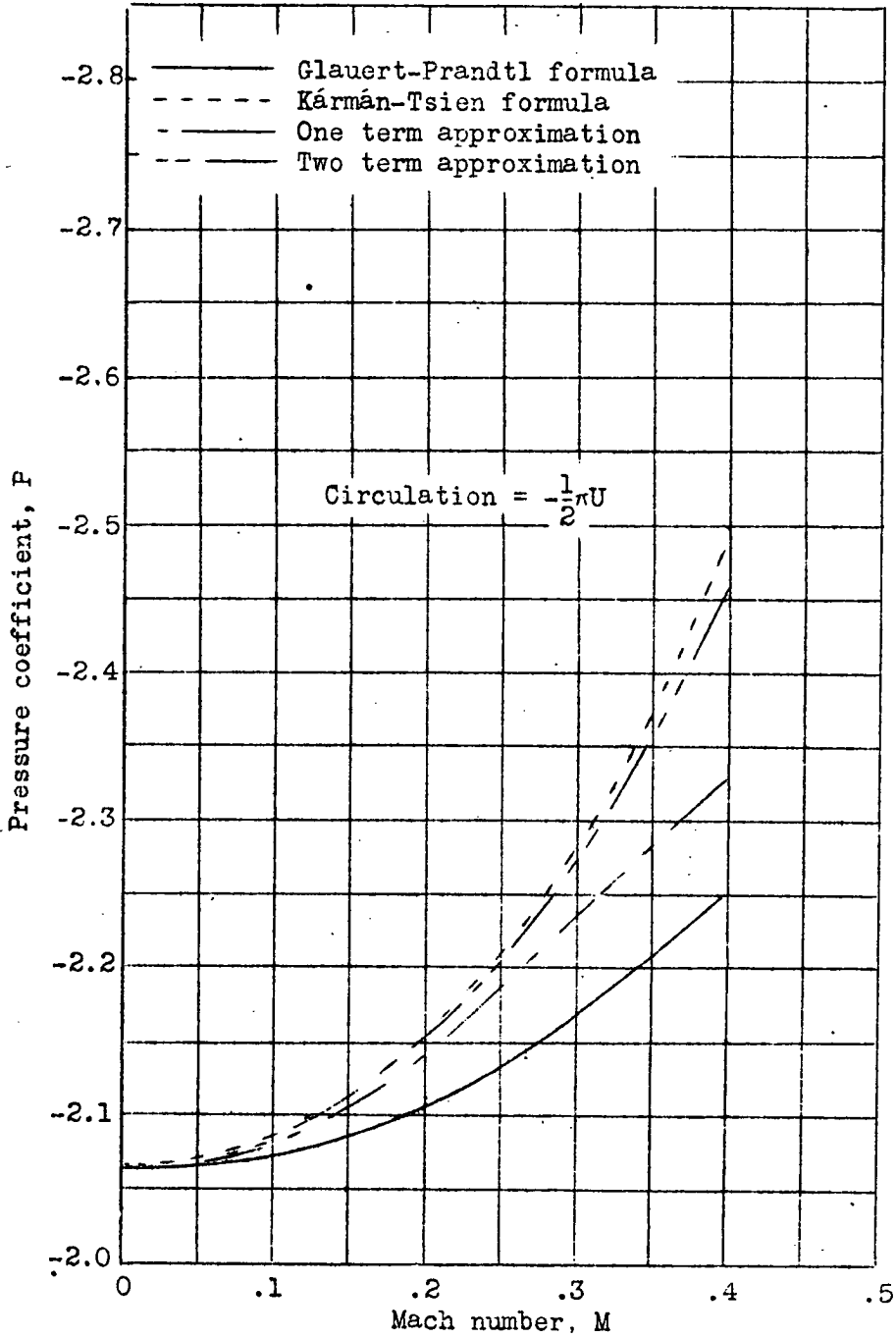
Fig. 5

Figure 5.- Variation of pressure coefficient with Mach
number at topmost point of cylinder when
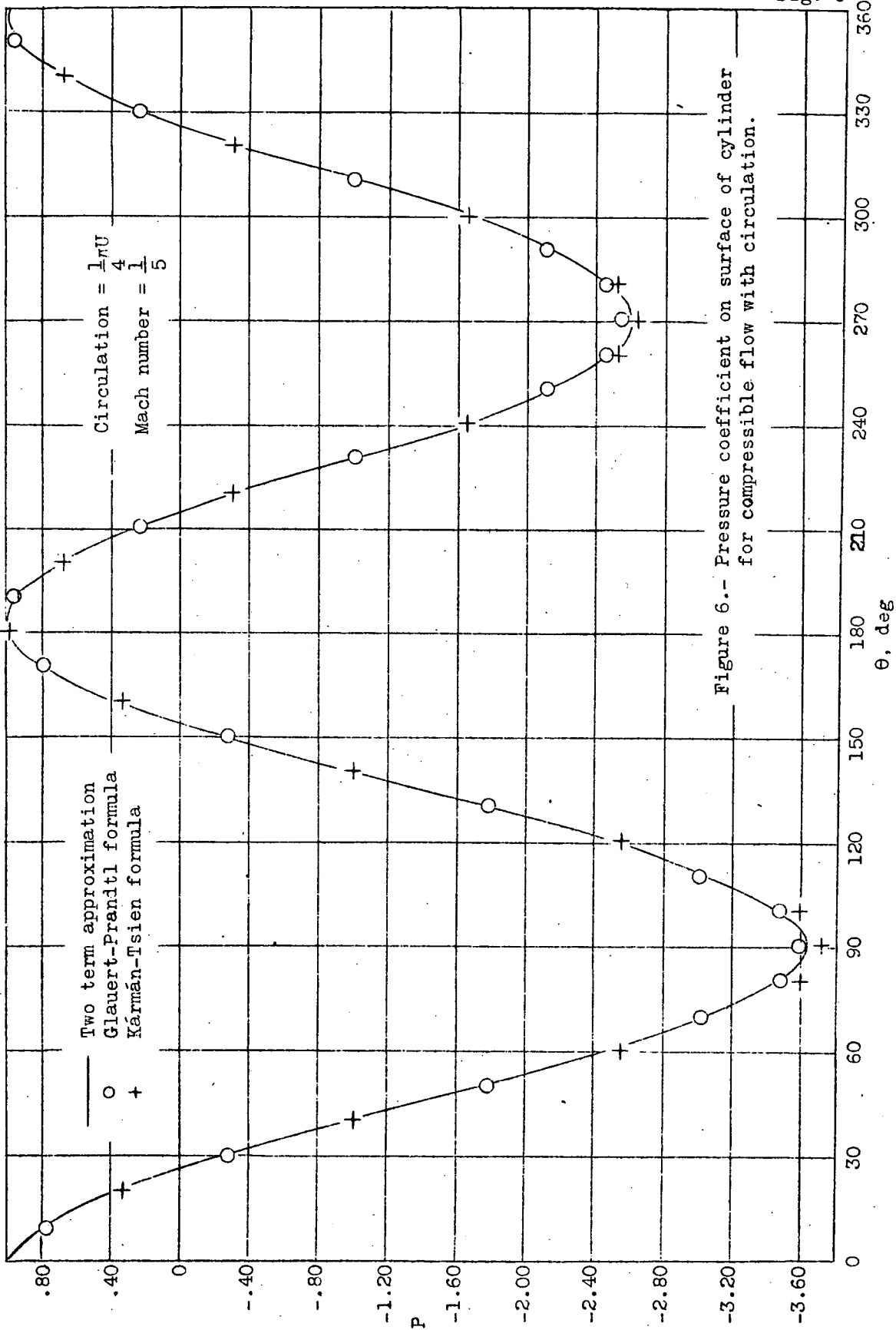circulation is $-\frac{1}{2}\pi U$.

Fig. 6

Figure 6.- Pressure coefficient on surface of cylinder for compressible flow with circulation.