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THEORY OF SELF-EXCITED MECHANICAL OSCILLATIONS OF HINGED ROTOR BLADES

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SUMMARY

Vibrations of rotary-wing aircraft may derive their energy from the rotation of the rotor rather than from the air forces. A theoretical analysis of these vibrations is described and methods for its application are explained herein.

The present paper also supersedes and extends the scope of the Advance Restricted Report entitled "Theory of Self-Excited Mechanical Oscillations of Hinged Rotor Blades," parts of which are in error. The theory has been extended to include the effects of unequal stiffness of the pylon for deflections in different directions and the effect of damping in the hinges and in the pylon. Both the derivation of the characteristic equation and the methods of application of the theory are given. In particular, the theory predicts the so-called "odd-frequency" self-excited speed range as well as the shaft-critical speed. Charts are presented from which the shaft-critical and the self-excited instabilities can be predicted for a great variety of cases. The influence of each physical parameter upon the instabilities has been obtained. The comprehensive treatment applies to a rotor that has any number of blades greater than two. Only a brief discussion and the formula for shaft-critical speed are given for the one- or two-blade rotor.

The use of complex variables in conjunction with Lagrange's equations has been found very convenient for the treatment of vibrations of rotating systems.

INTRODUCTION

A rotary-wing aircraft that has hinged blades will, under certain conditions, be subject to vibrations which derive their energy from the rotation of the rotor instead of from the air forces. The term "ground resonance" usually refers to vibrations of this type. Although such vibrations have apparently caused accidents in some rotary-wing aircraft and have impaired the flying qualities of others, very little attention has been given this problem in the literature. A theoretical analysis has therefore been undertaken, and the purpose of the present paper is to present the theory and to describe the application of the theory to rotary-wing aircraft.

General vibration theory and its application to allied problems as well as to the particular problem of rotor vibration are discussed in references 1 to 4. A good general background for the present problem is provided in the chapters on rotating machinery and on self-excited vibrations in reference 1. References 2 and 3 treat in more abstract fashion the topics of rotation and damping. A discussion of the variety of modes of vibration that exist in rotors and a number of frequency formulas obtained by considering separately each degree of freedom are given in reference 4. This discussion does not, however, lead to a prediction of self-excited modes of vibration.

Experience has shown that two types of mechanical vibration may occur in rotors. The vibration frequency of the pylon is equal to the rotational speed in one type, unequal in the other. The first type is sometimes called the even-frequency vibration or the one-to-one frequency, and the second type, the odd frequency. The one-to-one frequency vibration resembles the phenomenon occurring at a critical speed of the shaft of rotating machinery and will consequently be referred to in this paper as a "shaft-critical vibration." The odd-frequency vibration is properly called a self-excited vibration.

A derivation of the equations of motion for vibrations of a rotor for the case in which the pylon stiffness is equal in all directions of deflection is contained in reference 5. The equation for the shaft-critical speed is obtained and checked by tests of simple models. Reference 5, however, contains incorrect statements regarding the existence of self-excited vibrations. The error was due to a confusion in the use of conjugate complex quantities which has now been cleared up. The present paper therefore supersedes reference 5 and, in order to make the present paper independent of reference 5, the complete derivations are included herein without reference to the earlier report.

An alternative derivation of the characteristic equation for the whirling speeds of a three-blade rotor has been given by Wagner of the Kellett Autogiro Corporation. By considering only the case of a pylon having equal stiffness in all directions of deflection, Wagner has shortened the analysis by considering directly the equilibrium of forces and moments under conditions of steady circular whirling. Some examples of the dependence of whirling speed upon rotational speed are given, and the formula for the shaft-critical speed is obtained.

In the present report, the theory is extended to include the effects of damping in the hinges and in the hub and the effects of different stiffnesses of pylon deflection in different directions. The method of analysis, particularly the use of complex variables in the equations of motion, is explained in some detail and all the previous results are shown to be a special case of the more general problem here treated.

SYMBOLS

a	radial position of vertical hinge
A_{11}	elements of the characteristic determinant (see equation (31))
$\overline{A}_{11}$	
$\Delta A_{11} = \Delta\overline{A}_{11}$	
$A_{12} = \overline{A}_{12}$	
$A_{21} = \overline{A}_{21}$	
A_{22}	
$\overline{A}_{22}$	
b	distance from vertical hinge to center of mass of blade
B	damping force per unit velocity of pylon displacement $\left(B_f = \frac{B_x + B_y}{2}\right)$

ΔB	$= \dfrac{B_x - B_y}{2}$
B_I	coefficient defined in equation (35)
B_R	coefficient defined in equation (34)
$c, C_1, \ldots C_4$	arbitrary constants
C_I	coefficient defined in equation (35)
C_R	coefficient defined in equation (34)
D	time-derivative operator (d/dt)
F	dissipation function
I	moment of inertia of blade about hinge $\left[m_b b^2 \left(1 + \dfrac{r^2}{b^2}\right)\right]$
$I_1, \ldots I_5$	coefficients defined in equation (37)
j,k	indices and subscripts used with hinge coordinates (equation (14))
K	spring constant $\left(K_f = \dfrac{K_x + K_y}{2}\right)$
ΔK_f	$= \dfrac{K_x - K_y}{2}$
m	effective mass of pylon $\left(m_f = \dfrac{m_x + m_y}{2}\right)$
Δm	$= \dfrac{m_x - m_y}{2}$
M	total effective mass of blades and pylon $(m + nm_b)$
ΔM	mass added at hub for vibration test
n	total number of blades

r	radius of gyration of blade about its center of mass
R_1, ... R_5	coefficients defined in equation (37)
s	stiffness ratio (K_y/K_x)
t	time
T	kinetic energy
T_r	kinetic energy of rotation of blade about its center of mass
T_k	kinetic energy of translational motion of kth blade
T_s	kinetic energy of pylon
V	potential energy
x, y	displacements
x_o, y_o	values of x and y when $t = 0$
z	complex displacement $(x + iy)$
$\bar{z}$	complex conjugate of z $(x - iy)$
α	angle between blades $\left(\frac{2\pi}{n}\right)$
β_o, β_1, ... β_k	angular displacements of blades
β_{k_o}	value of β_k when $t = 0$
ζ_o, ζ_1, ... ζ_k	variables representing hinge deflections when equations are expressed in fixed coordinate system
θ_o, θ_1, ... θ_k	variables representing hinge deflections when equations are expressed in rotating coordinate system

$$\lambda_x == \frac{B_x}{M_x} \left(\frac{B_x}{M_x \omega_r} \text{ in applications}\right)$$

$$\lambda_y = \frac{B_y}{M_y} \left(\frac{B_y}{M_y \omega_r} \text{ in applications}\right)$$

$\lambda_\beta = \frac{B_\beta}{I} \left(\frac{B_\beta}{I\omega_r} \text{ in applications}\right)$

$\lambda_f = \frac{B_f}{M}$

$\lambda_a = \frac{B_a}{M}$

$\Delta\lambda_f = \frac{\Delta B_f}{M}$

$\Lambda_1 = \frac{a}{b\left(1 + \frac{r^2}{b^2}\right)}$

$\Lambda_2 = \frac{K_\beta}{I} \left(\frac{K_\beta}{I\omega_r^2} \text{ in applications}\right)$

$\Lambda_3 = \frac{\mu}{2\left(1 + \frac{r^2}{b^2}\right)}$

μ	mass ratio $\left(\frac{nm_b}{m + nm_b}\right)$
ν_1, ν_2	expressions defined in equation (3)
ω	angular velocity of rotor (the dimensionless ratio ω/ω_r is called ω in applications)
ω_a	angular whirling velocity measured in rotating coordinate system (used in nondimensional form in applications)
ω_f	angular whirling velocity measured in fixed coordinate system (used in nondimensional form in applications)
ω_r	reference frequency $\left(\sqrt{K_x/M_x}\right)$

Subscripts:

f	fixed coordinate system
a	rotating coordinate system

β	hinge deflection
x,y	component directions in fixed coordinate system
b	blade

APPROACH TO THE VIBRATION PROBLEM

Stability and Instability

If a vibrator were attached to a rotorcraft, several modes of vibration could be excited at any rotor speed. Only the modes that are likely to be excited during operation of the aircraft, however, are important.

In the present discussion, it is convenient to classify the modes of vibration according to the circumstances required for their excitation. The different types of vibration are identified analytically by the nature of the roots of the characteristic equation. A hinged rotor may encounter three types of vibration which, for convenience, are herein designated ordinary, self excited, and shaft critical. At zero or slow rotational speeds, an external force is required to excite vibration. The friction always present in such systems causes the vibration to be damped out when the force is removed. Modes of vibration requiring an external applied force to maintain them will be called ordinary vibrations. The mathematically idealized case of zero damping will also be considered an ordinary vibration when it is understood to be an approximation to a system actually damped. Self-excited modes of vibration are those with negative damping and are recognized analytically by the fact that a root of the characteristic equation is a complex number which has a negative imaginary part. A slight disturbance will tend to increase with time instead of damping out.

When a rotating system is not perfectly balanced, the centrifugal force of the unbalanced mass may excite vibrations that have peak amplitudes at certain rotational speeds. Vibration excited by unbalance and in resonance with the rotation will be called shaft-critical vibration. This type occurs at the rotational speed at which the spring stiffness of the pylon is neutralized by the centrifugal force. In the analysis, the shaft-critical vibration is recognized in rotating coordinates as a zero frequency and in fixed coordinates as a frequency equal to the rotational speed. The critical speeds of a rotating shaft are a common example of this class.

The purpose of a theory of rotor vibration is mainly to predict the occurrence of and, if possible, to show how to avoid self-excited and shaft-critical vibrations.

Choice of Degrees of Freedom

Of the large number of degrees of freedom of a hinged rotor, the important ones for the present problem have been found to be hinge deflection of the blades in the plane of rotation and horizontal deflections of the pylon. Other degrees of freedom, such as the flapping hinge motion of the blades, the bending or torsion of the blades, and the torsion of the drive shaft, are considered unimportant in the problem of self-excited oscillations. Some motions, such as landing-gear deflection, that produce lateral deflection at the top of the pylon may, however, be important.

Physical Parameters

The theoretical results given later provide a means of predicting the natural frequencies and, in particular, the critical speeds and unstable speed ranges in terms of certain physical parameters, such as mass, stiffness, and length. The successful application of the theory depends upon the determination of the proper values of these physical parameters for the aircraft or model being studied.

The important parameters to be determined are:

a	radial position of vertical hinge.
b	distance from vertical hinge to center of mass of blade.
m_b	mass of blade. Flexibility of the blade structure may have to be taken into account by the use of an effective value of m_b different from the actual blade mass. The effective blade mass can be taken as the value required to make the theory predict the correct pylon natural frequency when the rotor has a zero or very slow rotational speed.
I	moment of inertia of blade about hinge $\left[m_b b^2 \left(1+\frac{r^2}{b^2}\right)\right]$

K_β	spring constant of self-centering springs, which can be determined by a force test or from the hinge frequency with the hub rigidly supported.
m_x,m_y	effective mass of pylon for deflections in x- and y-directions.
K_x,K_y	effective stiffness of pylon.

The effective mass of the pylon is the value of a concentrated mass that would have the same kinetic energy expressed in terms of the deflections at the hub as the actual pylon and hub if it were placed at the rotor hub in the plane of rotation. The effective stiffness of the pylon is the stiffness of a spring that, if placed at the hub in the plane of rotation, would have the same potential energy in terms of deflections at the hub as the actual pylon. Equivalent definitions are that, if a simple spring and mass were imagined to be substituted at the hub in the plane of rotation for the pylon and hub, the natural frequency and the change of natural frequency with added mass would be the same as for the actual pylon.

An experimental method of measuring the effective mass m_x and stiffness K_x of the pylon is to replace the rotor by an approximately equal, rigid, concentrated mass ΔM at the hub and to measure the natural frequency for two or more values of this added mass. The quantities are then related by the equation

$$\omega_f = \sqrt{\frac{K_x}{m_x + \Delta M}}$$

or

$$\frac{1}{\omega_f^2} = \frac{1}{K_x}(m_x + \Delta M)$$

If measured values of $1/\omega_f^2$ are plotted against added mass ΔM and a straight line is drawn through the points, the intercept and the slope of the line will determine the effective values of K_x and m_x.

For the parameters a and b, the actual geometric lengths should be used unless the flexibility of the hinge offset arm a is comparable in magnitude with the hinge spring stiffness. In this case, it is recommended that an effective value of a be guessed and that b be determined by subtraction from the correct geometric value of a + b.

The damping parameters may be defined by the form of a dissipation function F. The function

$$2F = B_x\dot{x}_f^2 + B_y\dot{y}_f^2 + \sum_{K=0}^{n-1} B_\beta\dot{\beta}_k^2$$

is equal to the rate of dissipation of energy by damping. The parameters B_x and B_y thus measure the damping force per unit velocity referred to linear displacements of the top of the pylon and B_β is the damping torque per unit angular velocity at a blade hinge. If the damping force per unit velocity is not a constant, effective values should be used that will represent the same dissipation of energy per cycle as actually occurs with a reasonable amplitude of vibration. The amplitude of free vibration in a single degree of freedom is given in terms of B_x, B_y, and B_β by

$$x_f = x_o\, e^{-\frac{B_x}{2M}t}$$

$$y_f = y_o\, e^{-\frac{B_y}{2M}t}$$

$$\beta_k = \beta_{k_o}\, e^{-\frac{B_\beta}{2I}t}$$

The damping parameters are probably the most difficult ones to measure accurately. In practice, it is advisable to make calculations for a given case, first on the basis of no damping and then with the use of the estimated values of the damping parameters.

MATHEMATICAL DEVELOPMENTS

Method of Analysis

The derivation of the characteristic equation that is used as the basis for predicting the unstable oscillations of a rotor is presented in this section of the report.

Readers interested solely in applications can omit this section and proceed immediately to the section entitled "Method of Applying Theory."

The method of analysis treats the equations of motion for small displacements from the equilibrium condition with steady rotation. A proper choice of coordinates leads to equations with constant coefficients. The solutions are exponential or trigonometric functions.

The mathematical manipulations involved in treating the motions of a mass in a plane of rotation are facilitated by the use of a complex variable. The typical disturbed motion obtained by solving the equations of motion is an elliptic whirling motion, which is represented in terms of a complex variable $z = x + iy$. At any instant, z represents the displacement of the pylon from its equilibrium position. An expression such as

$$z = ce^{i\omega_f t}$$

represents whirling of the pylon in a circle of radius c with angular velocity ω_f. The sign of ω_f determines the sense of the rotation. Two rotations in opposite sense with the same radius are equivalent to a vibratory motion in a straight line.

$$z = c\left(e^{i\omega_f t} + e^{-i\omega_f t}\right)$$

$$= 2c \cos \omega_f t$$

Two opposite rotations of different radii are equivalent to whirling in an ellipse. A complex value of ω_f represents whirling in a spiral, which may be either a damped or a self-excited motion depending upon the sign of the imaginary part. A self-excited motion exists when the imaginary part of ω_f is negative, and the magnitude of z increases with time.

The displacements may be referred to a fixed or to a rotating coordinate system. If z_f and z_a are the displacements with respect to a fixed and to a rotating reference system, respectively ,

$$z_f = z_a e^{i\omega t}$$

If

$$z_a = ce^{i\omega_a t}$$

then

$$z_f = ce^{i(\omega_a+\omega)t}$$

A whirling speed ω_a with respect to the rotating coordinates thus corresponds to a whirling speed $\omega_f = \omega_a + \omega$ with respect to the fixed coordinates. A shaft-critical vibration corresponds to $\omega_a = 0$ in the rotating coordinate system or to $\omega_f = \omega$ in the fixed coordinate system.

Example of Rotor with Locked Hinges

An example that involves a partial use of complex variables is given on page 253 of reference 2. The problem given there of a mass particle moving on the inner surface of a rotating spherical bowl is mathematically equivalent to the disturbed motion of a flywheel and shaft or of a rotor with locked hinges. The equations of motion obtained in real form in rotating coordinates

$$\left.\begin{aligned} \ddot{x}_a - 2\omega\dot{y}_a - \omega^2 x_a &= -\frac{B_a\dot{x}_a}{M} - \frac{K}{M}x_a \\ \ddot{y}_a + 2\omega\dot{x}_a - \omega^2 y_a &= -\frac{B_a\dot{y}_a}{M} - \frac{K}{M}y_a \end{aligned}\right\} \qquad (1)$$

are combined in the single equation

$$\ddot{z}_a + \left(2i\omega + \frac{B_a}{M}\right)\dot{z}_a + \left(\frac{K}{M} - \omega^2\right)z_a = 0 \qquad (2)$$

where

$$z_a = x_a + iy_a$$

is the complex position vector in the rotating coordinate system. The complete solution, if small damping is assumed, is

$$z_a e^{i\omega t} = C_1 e^{-\nu_1 t + i\omega_r t} + C_2 e^{-\nu_2 t - i\omega_r t} \qquad (3)$$

where

$$\nu_1 = \frac{1}{2}\frac{B_a}{M}\left(1 - \frac{\omega}{\omega_r}\right)$$

$$\nu_2 = \frac{1}{2}\frac{B_a}{M}\left(1 + \frac{\omega}{\omega_r}\right)$$

The path of the motion is represented by rotations of a complex vector in a plane.

The use of a complex variable has thus cut in half the number of equations to be handled and has yielded a solution from which the geometric path of the motion may easily be reconstructed. The advantage of the z-notation is not fully realized, however, unless it is used from the very beginning of the problem. The close similarity of this problem to the rotor-vibration problem makes it worth while to show the full application of the z-notation to the preceding example. The complex variable z_a at any instant determines the position of the mass particle relative to the rotating coordinate system. If the position in a fixed coordinate system is denoted by z_f,

$$z_f = z_a e^{i\omega t} \tag{4}$$

and z_a can be treated as a generalized coordinate in the Lagrangian equations of motion. The kinetic and potential energy expressions can be immediately written as

$$T = \frac{1}{2} M\dot{z}_f\dot{\bar{z}}_f$$

$$= \frac{1}{2} M(\dot{z}_a + i\omega z_a)\ (\dot{\bar{z}}_a - i\omega\bar{z}_a)$$

$$V = \frac{1}{2} Kz_a\bar{z}_a$$

A dissipation function for damping that depends upon motion relative to the rotating system can be written

$$F = \frac{1}{2} B_a\dot{z}_a\dot{\bar{z}}_a$$

The equations of motion are now obtained by considering z_a and $\bar{z}_a$ as generalized coordinates in the Lagrangian equations. Substitution in the equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\bar{z}}_a}\right) - \frac{\partial T}{\partial \bar{z}_a} + \frac{\partial F}{\partial \dot{\bar{z}}_a} + \frac{\partial V}{\partial \bar{z}_a} = 0$$

thus yields the equation previously given

$$\ddot{z}_a + \left(2i\omega + \frac{B_a}{M}\right)\dot{z}_a + \left(\frac{K}{M} - \omega^2\right) z_a = 0$$

The same method can be used to obtain the equations of motion in the fixed coordinate system. In this case,

$$\left.\begin{aligned} T &= \frac{1}{2} M \dot{z}_f \dot{\bar{z}}_f \\ V &= \frac{1}{2} K z_f \bar{z}_f \\ F &= \frac{1}{2} B_a \left(\dot{z}_f - i\omega z_f\right)\left(\dot{\bar{z}}_f + i\omega \bar{z}_f\right) \end{aligned}\right\} \qquad (5)$$

The equation of motion in terms of z_f becomes

$$\ddot{z}_f + \frac{B_a}{M}\left(\dot{z}_f - i\omega z_f\right) + \frac{K}{M} z_f = 0 \qquad (6)$$

and the solution for small values of damping is

$$z_f = C_1 e^{-\frac{B_a}{2M}\left(1 - \frac{\omega}{\sqrt{K/M}}\right)t + i\sqrt{K/M}\,t} + C_2 e^{-\frac{B_a}{2M}\left(1 + \frac{\omega}{\sqrt{K/M}}\right)t - i\sqrt{K/M}\,t} \qquad (7)$$

This solution shows that the motion consists of two circular vibrations in opposite directions and, moreover, that for $\omega > \sqrt{K/M}$ the first term represents unstable motion; that is, the vibration has negative damping.

This example illustrates a shaft-critical speed for $\omega = \sqrt{K/M}$ and a self-excited instability for $\omega > \sqrt{K/M}$.

A discussion of the physical picture of this instability due to damping is given on page 293 of reference 1.

The effect of damping in a nonrotating part of the system can be included in the analysis merely by adding to the previous dissipation function the term

$$\frac{1}{2} B_f \dot{z}_f \dot{\bar{z}}_f$$

The equation of motion then becomes

$$M\ddot{z}_f + B_f\dot{z}_f + B_a\left(\dot{z}_f - i\omega z_f\right) + K z_f = 0 \tag{8}$$

The solution for small values of damping becomes

$$z_f = C_1 e^{\left[-\frac{B_f}{2M} - \frac{B_a}{2M}\left(1 - \frac{\omega}{\sqrt{K/M}}\right) + i\sqrt{K/M}\right]t} + C_2 e^{\left[-\frac{B_f}{2M} - \frac{B_a}{2M}\left(1 + \frac{\omega}{\sqrt{K/M}}\right) - i\sqrt{K/M}\right]t} \tag{9}$$

The motion is now unstable above the speed

$$\omega = \sqrt{\frac{K}{M}}\left(1 + \frac{B_f}{B_a}\right) \tag{10}$$

Hinged Rotor

Inclusion of the effect of hinge motion in the plane of rotation increases the number of degrees of freedom and the number of equations of motion. For example, three hinged blades and two directions of pylon deflection give five degrees of freedom to be considered. If special linear combinations of the hinge deflections β_k are used as generalized coordinates, no more than four degrees of freedom need be considered simultaneously. The use of complex variables reduces these four equations to two equations.

Appropriate variables in the rotating system for a three-blade rotor are

$$\left.\begin{aligned}\theta_0 &= \frac{bi}{3}\left(\beta_0 + \beta_1 + \beta_2\right)\\ \theta_1 &= \frac{bi}{3}\left(\beta_0 + \beta_1 e^{\frac{2\pi i}{3}} + \beta_2 e^{\frac{4\pi i}{3}}\right)\\ \theta_2 &= \frac{bi}{3}\left(\beta_0 + \beta_1 e^{\frac{4\pi i}{3}} + \beta_2 e^{\frac{8\pi i}{3}}\right)\end{aligned}\right\} \qquad (11)$$

These variables and their complex conjugates satisfy the relations

$$\bar{\theta}_1 = -\theta_2 \qquad \bar{\theta}_2 = -\theta_1$$

and also

$$\theta_0\bar{\theta}_0 + \theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2 = -\theta_0^2 + 2\theta_1\bar{\theta}_1 = \frac{b^2}{3}\left(\beta_0^2 + \beta_1^2 + \beta_2^2\right)$$

The variables β_k, by virtue of their meaning, are referred to a rotating coordinate system. The special linear combinations of the β_k denoted by θ_k are also referred to a rotating coordinate system. The appropriate variables to represent the hinge deflections when fixed coordinates are used are defined by

$$\zeta_k = \theta_k e^{i\omega t} \qquad (12)$$

Geometrically, θ_1 or ζ_1 is the complex vector representing the displacement due to hinge deflection of the center of mass of all the blades, just as z represents the position of the shaft due to pylon deflection. It will be shown later that, in the equations of motion, θ_1 is coupled with z and θ_0 is an independent principal coordinate. Equations (11) when solved for β_0, β_1, and B_2 become

$$\begin{aligned}\theta_0 + \theta_1 + \theta_2 &= bi\beta_0\\ \theta_0 + \theta_1 e^{-i\alpha} + \theta_2 e^{i\alpha} &= bi\beta_1\\ \theta_0 + \theta_1 e^{i\alpha} + \theta_2 e^{-i\alpha} &= bi\beta_2\end{aligned}$$

Then, in a mode involving θ_1,

$$\theta_0 = 0$$

$$\theta_1 = \frac{b}{2} e^{i\omega_a t}$$

$$\theta_2 = - \overline{\theta}_1$$

$$\left.\begin{aligned} \beta_0 &= \sin \omega_a t \\ \beta_1 &= \sin (\omega_a t - \alpha) \\ \beta_2 &= \sin (\omega_\alpha t + \alpha) \end{aligned}\right\} \qquad (13)$$

Equations (13) show that, in the θ_1-mode, the blades are undergoing sinusoidal vibrations 120° out of phase with one another in a manner analogous to three-phase electrical currents.

General formulas for any number of blades are

$$\left.\begin{aligned} \theta_j &= \frac{bi}{n} \sum_{k=0}^{n-1} \beta_k e^{ij\alpha k} \\ \alpha &= \frac{2\pi}{n} \\ \overline{\theta}_j &= - \theta_{n-j} \\ \theta_n &= \theta_o \\ \sum_{k=0}^{n-1} \theta_k \overline{\theta}_k &= \frac{b^2}{n} \sum_{k=0}^{n-1} \beta_k{}^2 \end{aligned}\right\} \qquad (14)$$

Derivation of Equations of Motion

The equations of motion and the characteristic equation of whirling speeds are herein derived for the general case of three or more equal blades on a pylon that may have different stiffness properties in different directions of deflection. The effects of damping in the blade hinges and in the pylon are included. The equations are first formulated in a nonrotating reference system. The required modifications are then given for the case of isotropic support stiffness. The corresponding equations referred to the rotating coordinates are then obtained.

Let the position of the center of mass of the kth blade be represented by the complex quantity z_k in the plane of rotation. (See fig. 1.) Let the bending deflection of the pylon be represented by z_f in a nonrotating coordinate system and let β_k be the hinge deflection of the kth blade. Then

$$z_k = z_f + \left(a + be^{i\beta_k}\right) e^{i(\alpha k+\omega t)} \qquad (15)$$

The complex velocity is

$$\dot z_k = \dot z_f + \left[bi\dot\beta_k e^{i\beta_k} + i\omega \left(a + be^{i\beta_k}\right)\right] e^{i(\alpha k+\omega t)} \qquad (16)$$

Because only small displacements are being considered, the exponential factors containing β_k can be expanded and only the terms that lead to quadratic terms for the kinetic-energy expression need be considered.

Some terms can be ignored either because they cancel after summation for all the blades or because the corresponding derivative expressions in the Lagrangian equations vanish. The substitution

$$e^{i\beta_k} = 1 + i\beta_k - \frac{\beta_k^2}{2}$$

leads to an expression for the kinetic energy of translational motion of the kth blade.

$$T_k = \frac{1}{2} m_b \dot z_k \dot{\bar z}_k \qquad (17)$$

where

$$\dot z_k \dot{\bar z}_k = \dot z_f \dot{\bar z}_f + \dot{\bar z}_f bi \left(\dot\beta_k + i\omega\beta_k\right) e^{i(\alpha k+\omega t)}$$

$$+ \dot z_f(-bi) \left(\dot\beta_k - i\omega\beta_k\right) e^{-i(\alpha k+\omega t)} + b^2\dot\beta_k^2 - \omega^2 ab\beta_k^2$$

The kinetic energy of rotation about the center of mass of the blade is

$$T_r = \frac{1}{2} m_b r^2 \dot{\beta}_k^2 \tag{18}$$

The effective mass of the pylon may be different in the x_f- and in the y_f-directions. Allowance for this possibility is made by writing the kinetic energy of the pylon as

$$\begin{aligned} T_s &= \frac{1}{2}\left(m_x \dot{x}_f^2 + m_y \dot{y}_f^2\right) \\ &= \frac{1}{2}\left[m \dot{z}_f \dot{\bar{z}}_f + \Delta m \left(\frac{\dot{z}_f^2 + \dot{\bar{z}}_f^2}{2}\right)\right] \end{aligned} \tag{19}$$

where

$$m = \frac{m_x + m_y}{2}$$

$$\Delta m = \frac{m_x - m_y}{2}$$

The total kinetic energy is the sum of the expressions for the separate kinetic energies.

The pylon spring constant may differ in the x_f- and in the y_f-directions and, consequently, the potential energy can be expressed as

$$V = \frac{1}{2}\left[K z_f \bar{z}_f + \Delta K \frac{z_f^2 + \bar{z}_f^2}{2} + \sum_{k=0}^{n-1} K_\beta \beta_k^2\right] \tag{20}$$

The effect of damping will be expressed with the aid of a dissipation function. If damping exists in the pylon, in the rotating shaft, and in the hinges, this function becomes

$$F = \frac{1}{2}\left[B \dot{z}_f \dot{\bar{z}}_f + \Delta B \frac{\dot{z}_f^2 + \dot{\bar{z}}_f^2}{2} + B_a \dot{z}_a \dot{\bar{z}}_a + \sum_{k=0}^{n-1} B_\beta \dot{\beta}_k^2\right] \tag{21}$$

The sum of the various energy expressions for all the blades, expressed in terms of the variables z_f and ζ_k in the nonrotating coordinates, becomes

$$\left.\begin{aligned}
T &= \frac{1}{2}\left[\Delta m\,\frac{\dot{z}_f^{\,2}+\dot{\bar{z}}_f^{\,2}}{2} + (m+nm_b)\,\dot{z}_f\dot{\bar{z}}_f + nm_b\left(\dot{\bar{z}}_f\dot{\zeta}_1 + \dot{z}_f\dot{\bar{\zeta}}_1\right)\right.\\
&\qquad\left. + \left(1+\frac{r^2}{b^2}\right)\sum\left(\dot{\zeta}_k - i\omega\zeta_k\right)\left(\dot{\bar{\zeta}}_k + i\omega\bar{\zeta}_k\right) - \omega^2\,\frac{a}{b}\sum\zeta_k\bar{\zeta}_k\right]\\
V &= \frac{1}{2}\left[\Delta K\,\frac{z_f^{\,2}+\bar{z}_f^{\,2}}{2} + K\,z_f\bar{z}_f + \frac{K_\beta}{b^2}\sum\zeta_k\bar{\zeta}_k\right]\\
F &= \frac{1}{2}\left[\Delta B\,\frac{\dot{z}_f^{\,2}+\dot{\bar{z}}_f^{\,2}}{2} + B\,\dot{z}_f\dot{\bar{z}}_f + B_a\left(\dot{z}_f - i\omega z_f\right)\left(\dot{\bar{z}}_f + i\omega\bar{z}_f\right)\right.\\
&\qquad\left. + \frac{B_\beta}{b^2}\sum_{k=0}^{n-1}\left(\dot{\zeta}_k - i\omega\zeta_k\right)\left(\dot{\bar{\zeta}}_k + i\omega\bar{\zeta}_k\right)\right]
\end{aligned}\right\}\quad(22)$$

The Lagrangian equations of motion are

$$\left.\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\bar{z}}_f}\right) - \frac{\partial T}{\partial\bar{z}_f} + \frac{\partial F}{\partial\dot{\bar{z}}_f} + \frac{\partial V}{\partial\bar{z}_f} &= 0\\
\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\bar{\zeta}}_1}\right) - \frac{\partial T}{\partial\bar{\zeta}_1} + \frac{\partial F}{\partial\dot{\bar{\zeta}}_1} + \frac{\partial V}{\partial\bar{\zeta}_1} &= 0
\end{aligned}\right\}\qquad(23)$$

and similar expressions for the other variables. The equations of motion in fixed coordinates then become

$$(m+nm_b)\,\ddot{z}_f + B\dot{z}_f + B_a(\dot{z}_f - i\omega z_f) + Kz_f + \Delta m\ddot{\bar{z}}_f + \Delta B\dot{\bar{z}}_f + \Delta K\bar{z}_f + nm_b\,\ddot{\zeta}_1 = 0$$

$$nm_b\ddot{z}_f + 2nm_b\left[\left(1+\frac{r^2}{b^2}\right)\left(\ddot{\zeta}_1 - 2i\omega\dot{\zeta}_1 - \omega^2\zeta_1\right) + \frac{B_\beta}{m_b b^2}\left(\dot{\zeta}_1 - i\omega\zeta_1\right) + \omega^2\frac{a}{b}\,\zeta_1 + \frac{K_\beta}{m_b b^2}\,\zeta_1\right] = 0$$

$$nm_b\left[\left(1+\frac{r^2}{b^2}\right)\left(\ddot{\zeta}_k - 2i\omega\dot{\zeta}_k - \omega^2\zeta_k\right) + \frac{B_\beta}{m_b b^2}\left(\dot{\zeta}_k - i\omega\zeta_k\right) + \omega^2\frac{a}{b}\,\zeta_k + \frac{K_\beta}{m_b b^2}\,\zeta_k\right] = 0 \qquad (24)$$

where ζ_k refers to the ζ-variables other than ζ_1. The complex conjugates of these equations are also obtained but give no additional information. Each complex equation is, of course, equivalent to two real equations. It is noticed that the first two equations contain only the variables z_f, $\overline{z}_f$, and ζ_1 and that the third equation represents n-2 equations, each containing one independent principal coordinate ζ_k. The physical meaning of this partial separation of variables is that a blade motion represented by ζ_1 involves a motion of the common center of mass of the blades and, thus, a coupling effect with the pylon. Blade motions in which the common center of mass does not move are represented by $\zeta_2, \ldots \zeta_n$. For three blades, the only such mode is the one corresponding to ζ_o. In this mode, all the blades move in phase; the motion is always damped and does not lead to instability.

The equations of motion of a one- or two-blade rotor are somewhat different from equations (24). The difference is connected with the circumstance that a rotor of three or more equal blades has no preferred direction in its plane; whereas, a one- or two-blade rotor has different dynamic properties in directions along and normal to the blades. Only a brief statement and the final equation for shaft-critical speed will be given for the one- or two-blade rotor.

The equations of motion involving z_f and ζ_1 can be written more compactly by use of the notation

$$D = \frac{d}{dt} \qquad D^2 = \frac{d^2}{dt^2}$$

and the substitutions

$$\frac{B_f}{M} = \lambda_f \qquad \frac{B_\beta}{m_b b^2 \left(1 + \frac{r^2}{b^2}\right)} = \lambda_\beta$$

$$\frac{B_a}{M} = \lambda_a \qquad \frac{a}{b \left(1 + \frac{r^2}{b^2}\right)} = \Lambda_1$$

$$\frac{\Delta B_f}{M} = \Delta\lambda_f \qquad \frac{K_\beta}{m_b b^2 \left(1 + \frac{r^2}{b^2}\right)} = \Lambda_2$$

$$\frac{nm_b}{m+nm_b} = \mu$$

Then

$$\left.\begin{aligned} &\left[D^2 + \lambda_f D + \lambda_a(D-i\omega) + \frac{K}{M}\right]z_f + \left(\frac{\Delta m}{M} D^2 + \Delta\lambda_f D + \frac{\Delta K}{M}\right)\bar{z}_f + \mu D^2 \zeta_1 = 0 \\ &\frac{1}{2\left(1 + \frac{r^2}{b^2}\right)} D^2 z_f + \left[(D-i\omega)^2 + \lambda_\beta(D-i\omega) + \omega^2\Lambda_1 + \Lambda_2\right]\zeta_1 = 0 \end{aligned}\right\} (25)$$

or, briefly,

$$\left.\begin{aligned} A_{11}(D)z_f + \Delta A_{11}\bar{z}_f + A_{12}(D)\zeta_1 &= 0 \\ A_{21}(D)z_f \qquad\qquad + A_{22}(D)\zeta_1 &= 0 \end{aligned}\right\} (26)$$

The Characteristic equation

The general form of solution of equations (26) is an elliptic whirling motion that can be represented by

$$\left.\begin{aligned} z_f &= C_1 e^{i\omega_f t} + C_2 e^{-i\bar{\omega}_f t} \\ \bar{z}_f &= \bar{C}_1 e^{-i\bar{\omega}_f t} + \bar{C}_2 e^{i\omega_f t} \\ \zeta_1 &= C_3 e^{i\omega_f t} + C_4 e^{i\bar{\omega}_f t} \end{aligned}\right\} (27)$$

Special cases of this motion include whirling in a circle - $C_2 = C_4 = 0$, and linear vibration. $C_1 = C_2$, $C_3 = C_4$. Substitution of equations (27) in equation (26) gives

$$\left.\begin{aligned} &\left[A_{11}(i\omega_f)\ C_1 + \Delta A_{11}(i\omega_f)\ \bar{C}_2 + A_{12}(i\omega_f)\ C_3\right] e^{i\omega_f t} \\ &\quad + \left[A_{11}(-i\bar{\omega}_f)\ C_2 + \Delta A_{11}(-i\bar{\omega})\bar{C}_1 + A_{12}(-i\bar{\omega}_f)C_4\right] e^{-i\bar{\omega}_f t} = 0 \\ &\left[A_{12}(i\omega_f)\ C_1 + A_{22}(i\omega_f)\ C_3\right] e^{i\omega_f t} \\ &\quad + \left[A_{12}(-i\bar{\omega}_f)\ C_2 + A_{22}(-i\bar{\omega}_f)\ C_4\right] e^{i\bar{\omega}_f t} = 0 \end{aligned}\right\} (28)$$

In order for equations (27) to be a solution of equations (26), equations (28) must be satisfied for each value of t. The coefficient of each time factor $e^{i\omega_f t}$ or $e^{-i\bar{\omega}_f t}$ must therefore separately vanish. Because each bracketed expression represents a complex quantity that vanishes, its complex conjugate also must vanish. The condition for a solution can therefore be expressed by the vanishing of the first bracketed terms and the complex conjugates of the second bracketed terms. Hence,

$$\left.\begin{array}{llll} A_{11}(i\omega_f)C_1 & +\Delta A_{11}(i\omega_f)\bar{C}_2+A_{12}(i\omega_f)C_3 & & =0 \\ A_{12}(i\omega_f)C_1 & \qquad\qquad +A_{22}(i\omega_f)C_3 & & =0 \\ \Delta\bar{A}_{11}(i\omega_f)C_1 & +\bar{A}_{11}(i\omega_f)\bar{C}_2 & +\bar{A}_{12}(i\omega_f)\bar{C}_4 & =0 \\ & \bar{A}_{12}(i\omega_f)\bar{C}_2 & +\bar{A}_{22}(i\omega_f)\bar{C}_4 & =0 \end{array}\right\} \quad (29)$$

where $\bar{A}_{11}(i\omega_f)$ is the complex conjugate of $A_{11}(-i\bar{\omega}_f)$ and is obtained from $A_{11}(i\omega_f)$ by changing $i\omega$ to $-i\omega$ without changing $i\omega_f$. The characteristic equation giving the rotational speeds is the determinant of the coefficients of C_1, $\bar{C}_2$, C_3, and $\bar{C}_4$ equated to zero. With the second and third columns interchanged for symmetry, the determinant becomes

$$\begin{vmatrix} A_{11} & A_{12} & \Delta A_{11} & 0 \\ A_{12} & A_{22} & 0 & 0 \\ \Delta\bar{A}_{11} & 0 & \bar{A}_{11} & \bar{A}_{12} \\ 0 & 0 & \bar{A}_{12} & \bar{A}_{22} \end{vmatrix} = 0 \qquad (30)$$

The expanded form of this determinant is

$$(A_{11}A_{22}-A_{12}A_{21})\,(\bar{A}_{11}\bar{A}_{22}-\bar{A}_{12}\bar{A}_{21})\,-\Delta A_{11}\Delta\bar{A}_{11}A_{22}\bar{A}_{22}=0 \quad (31)$$

where

$$A_{11} = -\omega_f^2 + i\omega_f\lambda_f + i\lambda_a(\omega_f-\omega) + \frac{K}{M}$$

$$\overline{A}_{11} = -\omega_f^2 + i\omega_f\lambda_f + i\lambda_a(\omega_f+\omega) + \frac{K}{M}$$

$$\Delta A_{11} = \Delta\overline{A}_{11} = -\frac{\Delta m}{M}\omega_f^2 + i\omega_f\Delta\lambda_f + \frac{\Delta K}{M}$$

$$A_{12}A_{21} = \overline{A}_{12}\overline{A}_{21} = \frac{\mu}{2\left(1 + \frac{r^2}{b^2}\right)}\omega_f^4 = \Lambda_3\,\omega_f^4$$

$$A_{22} = -(\omega_f-\omega)^2 + i\lambda_\beta(\omega_f-\omega) + \omega^2\Lambda_1 + \Lambda_2$$

$$\overline{A}_{22} = -(\omega_f+\omega)^2 + i\lambda_\beta(\omega_f+\omega) + \omega^2\Lambda_1 + \Lambda_2$$

The roots ω_f of this equation are the characteristic whirling speeds of the rotor.

For the case of isotropic supports,

$$\Delta A_{11} = 0$$

and the equations of motion are satisfied by equations (27) with $C_2 = C_4 = 0$.

The characteristic equation is then simply

$$A_{11}A_{22} - A_{12}A_{21} = 0 \tag{32}$$

In a rotating coordinate system, the complex coordinates are z_a and θ_1, where

$$z_f = z_a e^{i\omega t}$$

$$\zeta_1 = \theta_1 e^{i\omega t}$$

Then

$$Dz_f = (Dz_a + i\omega z_a)\,e^{i\omega t}$$

$$D\zeta_1 = (D\theta_1 + i\omega\theta_1)\,e^{i\omega t}$$

If the whirling speed in rotating coordinates is represented by ω_a,

$$z_a = C_1 e^{i\omega_a t}$$

$$\theta_1 = C_2 e^{i\omega_a t}$$

The characteristic equation is then obtained by substituting $\omega_a + \omega$ for ω_f.

$$A_{11}(\omega_a+\omega)\ A_{22}(\omega_a+\omega) - A_{12}(\omega_a+\omega)\ A_{21}(\omega_a+\omega) = 0 \qquad (33)$$

The characteristic equation can thus be stated in terms of a whirling speed in either the fixed or the rotating coordinate system.

METHOD OF APPLYING THEORY

Application Neglecting Damping

In plotting curves for use in applications of the theory, it is convenient to consider one of the pylon bending frequencies $\omega_r = \sqrt{K_x/M_x}$ as a reference frequency and to refer all other frequencies as well as the rotational speed ω to the reference frequency as unit. The number of independent parameters is thus reduced by 1. All quantities in equations (31) to (33) are then expressed nondimensionally.

The natural whirling speeds and the three types of vibration - ordinary, self-excited, and shaft-critical - can now be predicted from a study of the roots of equation (31) in which ω_f is considered a function of ω for fixed values of the other parameters.

The case of no damping will be considered first. Because equation (31) with damping terms omitted is of the fourth degree in ω_f^2 and of only the second degree in ω^2, it may be solved conveniently by first choosing values of ω_f and then solving the equation for ω^2. Similar indirect methods can be used with equations (32) and (33). Special methods to be used when damping is included will be discussed later.

The meaning of equations (31) to (33) will be illustrated by examples. The real part of ω_f will be plotted against ω for selected values of the parameters Λ_1, Λ_2, Λ_3, and s. The simplest case is that in which the mass of the blades is so small that any force on the pylon due to blade motions is negligible. The pylon motions are then independent of the blade motions. This case is obtained by putting $\Lambda_3 = 0$. The characteristic equation (31), (32), or (33) then factors into expressions yielding straight lines and hyperbolas.

An example of a rotor with particular values of the parameters is plotted as long-dash lines in figure 2. The horizontal straight lines correspond to pylon bending and the slanting hyperbolas correspond to hinge deflection. Each curve represents the trend of one of the real roots ω_f. As Λ_3 increases slightly from zero, the greatest changes in the curves occur in the vicinity of the intersections of the straight lines with the hyperbolas. Here each branch breaks away from the intersection and rejoins the other branch. At a gap, such as C in figure 2, the number of real roots of the frequency equation is reduced by 2. The missing roots are complex conjugate numbers; and one of them must have a negative imaginary part, which implies a self-excited vibration.

Consider the interpretation of figure 2 as ω is gradually increased from zero. At zero rotational speed, the values of ω_f are the natural frequencies that could be excited as ordinary vibration by applied vibrating force. Positive and negative values occur in pairs of equal magnitude and correspond to linear vibration modes represented in complex notation as

$$z_f = c\left(e^{i\omega_f t} + e^{-i\omega_f t}\right)$$

As ω increases from zero, the positive and negative values of ω_f no longer are equal in magnitude. The normal modes are therefore whirling motions with angular velocities equal to the plotted values of ω_f.

The shaft-critical speed is the rotational speed at which $\omega_f = \omega$ and hence is given by the point A where a 45° line through the origin intersects the ω_f-curve. This speed corresponds to the peak for vibrations excited by unbalance in the rotating system. As ω increases above the shaft-critical speed, the modes of whirling are stable until, for the case of no damping, the value of ω_f becomes complex at the value of ω at which a vertical line is tangent to the plotted curve. This point B is the beginning of the self-excited range. At the point D, the motion again becomes stable. The real part of ω_f is plotted in the region C as a short-dash line. The complex roots in the region C have been calculated and plotted in figure 3.

The point E, at which $\omega_f = 0$, is of some interest. At this speed, a vibration of the blades could be excited by a steady force $(\omega_f = 0, \omega_a = -\omega)$ such as the force of gravity if the plane of the rotor is not horizontal.

Because the most important information to be obtained from the frequency equation is the critical value of ω for the shaft-critical and self-excited vibrations, a set of charts that gives this information for a large variety of values of the physical parameters has been prepared. These charts are given in figures 4 to 6, which correspond to values of stiffness ratio $K_y/K_x = s$ of 1, ∞, and 0, respectively. The use of the charts is illustrated by a numerical example. Suppose the values of the parameters for a certain rotor are $\Lambda_1 = 0.07$, $\Lambda_2 = 0.22$, $\Lambda_3 = 0.1$, $s = 1$, and $\omega_r = 155$ cycles per minute. A straight line, such as AB in figure 4, is first drawn to represent the function $\omega^2\Lambda_1 + \Lambda_2$. This line intersects contours $\Lambda_3 = 0.1$ at $\omega^2 = 0.77$ for the shaft-critical point and $\omega^2 = 1.6$ and 4.85 for the beginning and for the end of the self-excited range, respectively. With a reference frequency of 155 cycles per minute, these values correspond to actual rotational speeds of 136, 196, and 342 rpm.

All possible values of Λ_1, Λ_2, and Λ_3 are thus covered by suitably changing the straight line AB. The general effect of the stiffness ratio s is not large; any case can therefore be estimated with a fair degree of accuracy by use of figures 4 to 6.

Possibility of Avoiding Occurrence of Vibration

Figures 4 to 6 can also be used for the inverse problem of finding the values of the parameters that are required to obtain given values of critical rotational speed. These figures show that to eliminate entirely the self-excited instability requires that Λ_1 be equal to or greater than 1. The shaft-critical instability can be entirely eliminated only with a value of Λ_1 in a small range near 4 and with $s = \infty$ or $s = 0$. These values of Λ_1 differ radically from present designs in which a typical value is 0.07.

The satisfactory requirement of keeping the instabilities outside the operating range of rotational speed is found by first picking a reasonable value of the pylon frequency $\sqrt{K_x/M}$ to fix the scale unit for ω and by then observing the combinations of Λ_1 and Λ_2 that can be used to avoid the critical Λ_3-contours.

Effect of Damping

The effect of damping has been included in equation (31) through the parameters λ_x, λ_y, and λ_β. A method of computation similar to that used in flutter theory appears preferable to attempting to solve the equation directly for ω_f. The beginning and the end of an unstable range can be found by the following method: At a limit point between a stable and an unstable speed range, the value of ω_f is real. Equation (31) is first separated into real and imaginary parts with ω_f considered real. Each part is considered a functional relation between ω_f and ω and is plotted for a given set of values of the parameters. The intersections of the real and the imaginary equations give the rotor speeds and frequencies corresponding to the beginning and the end of the unstable ranges. In the computations, it is preferable to choose values of ω_f and to solve the equations for the corresponding values of ω.

The explicit form for computation in the simplest case of isotropic supports, with damping in the pylon and in the hinges but not in the rotating shaft $(\lambda_a = 0)$, is obtained from equation (32) rearranged as follows:

Real equation

$$\omega^2 - 2B_R\omega + C_R = 0 \tag{34}$$

$$\omega = B_R \pm \sqrt{B_R{}^2 - C_R}$$

where

$$B_R = \frac{\omega_f}{1 - \Lambda_1}\left[1 + \frac{\lambda_f\lambda_\beta}{2\left(-\omega_f{}^2 + \frac{K}{M}\right)}\right]$$

and

$$C_R = -\frac{\omega_f{}^2}{1 - \Lambda_1}\left(-1 + \frac{\Lambda_2}{\omega_f{}^2} - \frac{\Lambda_3\omega_f{}^2 + \lambda_f\lambda_\beta}{-\omega_f{}^2 + \frac{K}{M}}\right)$$

Imaginary equation

$$\omega^2 - 2B_I\omega + C_I = 0 \tag{35}$$

where

$$B_I = \frac{1}{1 - \Lambda_1}\left[\omega_f - \frac{\lambda_\beta}{2\lambda_f\omega_f}\left(-\omega_f{}^2 + \frac{K}{M}\right)\right]$$

and

$$C_I = -\frac{1}{1 - \Lambda_1}\left[\frac{\lambda_\beta}{\lambda_f}\left(-\omega_f{}^2 + \frac{K}{M}\right) + \left(-\omega_f{}^2 + \Lambda_2\right)\right]$$

The most general case obtained from equation (31) is written:

Real equation

Coefficient of	ω^6	$\lambda_a(1-\Lambda_1)^2$	
	ω^4	$R_1(1-\Lambda_1)^2 + \lambda_a^2\ R_3$	
	ω^2	$R_1R_3 - I_1I_3 + \lambda_a\ R_2 - R_5$	(36)
	1	$R_1R_2 - I_1I_2 - R_4 + \Lambda_3^2\,\omega_f{}^2$	

Imaginary equation

Coefficient of ω^4 $\quad I_1(1-\Lambda_1)^2 + \lambda_a^2 I_3$

ω^2 $\quad R_1 I_3 + R_3 I_1 + \lambda_a^2 I_2 - I_5$ $\quad$ (37)

1 $\quad R_1 I_2 + R_2 I_1 - I_4$

where

$$R_1 = \frac{M_x M_y}{M^2}\left(-\omega_f^2 + \frac{K_x}{M_x}\right)\left(-\omega_f^2 + \frac{K_y}{M_y}\right) - \omega_f^2\left(\lambda_x \frac{M_x}{M} + \lambda_a\right)\left(\lambda_y \frac{M_y}{M} + \lambda_a\right)$$

$$I_1 = \frac{M_x}{M}\left(-\omega_f^2 + \frac{K_x}{M_x}\right)\omega_f\left(\lambda_y \frac{M_y}{M} + \lambda_a\right) + \frac{M_y}{M}\left(-\omega_f^2 + \frac{K_y}{M_y}\right)\omega_f\left(\lambda_x \frac{M_x}{M} + \lambda_a\right)$$

$$R_2 = \left(-\omega_f^2 + \Lambda_2\right)^2 - \omega_f^2 \lambda_\beta^2$$

$$I_2 = 2\left(-\omega_f^2 + \Lambda_2\right)\omega_f\lambda_\beta$$

$$R_3 = -2\left(1 - \Lambda_1\right)\left(-\omega_f^2 + \Lambda_2\right) - 4\omega_f^2 + \lambda_\beta^2$$

$$I_3 = -2\left(1 - \Lambda_1\right)\omega_f\lambda_\beta + 4\omega_f\lambda_\beta$$

$$R_4 = 2\omega_f^4\Lambda_3\left[\left(-\omega_f^2 + \frac{K}{M}\right)\left(-\omega_f^2 + \Lambda_2\right) - \omega_f^2\left(\lambda_f + \lambda_a\right)\lambda_\beta\right]$$

$$I_4 = 2\omega_f^4\Lambda_3\left[\left(-\omega_f^2 + \frac{K}{M}\right)\omega_f\lambda_\beta + \omega_f\left(\lambda_f + \lambda_a\right)\left(-\omega_f^2 + \Lambda_2\right)\right]$$

$$R_5 = 2\omega_f^4\Lambda_3\left[-\left(1 - \Lambda_1\right)\left(-\omega_f^2 + \frac{K}{M}\right) - \lambda_a\lambda_\beta\right]$$

$$I_5 = 2\omega_f^4\Lambda_3\left[-\left(1 - \Lambda_1\right)\omega_f\left(\lambda_f + \lambda_a\right) - 2\lambda_a\omega_f\right]$$

Examples of calculated cases with damping are shown in figures 7 to 9. The presence of small amounts of damping in both the pylon and the hinge degrees of freedom does not greatly change the predictions that would be made from the equations with no damping. The plot of the real equation is practically the same as the plot obtained when damping is neglected. The intersections of the curves of the imaginary and the real equations with any reasonable value of λ_f/λ_β are near the points that would be considered the limits of the unstable range if damping were neglected. Increasing the amount of damping decreases the gap between the limits of stability until the unstable range is finally eliminated. An approximate solution for the amount of damping required to eliminate the self-excited instability is obtained by requiring that the damping be at least large enough to make the curve of the real equation pass through the point where $\omega_f = 1$ and ω is the value given by the equation

$$1 = \omega - \sqrt{\omega^2 \Lambda_1 + \Lambda_2}$$

The values required in the case of $s = \infty$ have been computed and plotted in figure 10. The elimination of self-excited vibration by damping thus looks promising and merits further study with reference to specific application.

LIMITATIONS AND FURTHER DEVELOPMENTS OF THE THEORY

Polar Symmetry

An important idea in the rotor vibration theory is the concept of polar symmetry. This concept implies the absence of a preferred direction in the plane of the rotor. A rotor of three or more equal blades has polar symmetry. A rotor of two blades or one with unequal centering springs does not have polar symmetry. A pylon for which $K_x = K_y$, $B_x = B_y$, and $m_x = m_y$ has polar symmetry. The possibility of solving the rotor vibration problem in terms of exponential or trigonometric functions depends upon the existence of polar symmetry in the rotating parts or in the nonrotating parts or in both. The general case of no polar symmetry would lead to Mathieu functions or something similar.

Two Blades

A brief comparison between the two-blade and the general case is presented herein. Polar symmetry of the pylon is assumed. The shaft-critical speed is obtained by substituting $\omega_a = 0$ in the characteristic equation as expressed in a rotating coordinate system. For one or two blades, the equation obtained is

$$\left[\left(-\omega^2 + \frac{K}{M}\right)\left(\omega^2 \frac{a}{b} + \frac{K_\beta}{m_b b^2}\right) - \mu\ \omega^4\right]\left(-\omega^2 + \frac{K}{M}\right) = 0 \qquad (38)$$

The first bracketed factor gives the beginning of a self-excited range and the second factor gives the end of the range.

Equation (38) can be compared with the equation for the shaft-critical speed of three or more equal blades and for polar symmetry

$$\left(-\omega^2 + \frac{K}{M}\right)\left(\omega^2 \frac{a}{b} + \frac{K_\beta}{m_b b^2}\right) - \frac{\mu}{2}\ \omega^4 = 0 \qquad (39)$$

A useful chart based on equation (39) is given in figure 11; some experimental results of tests of a simple model in figure 12. These tests demonstrate the essential difference between the two-blade and the general case.

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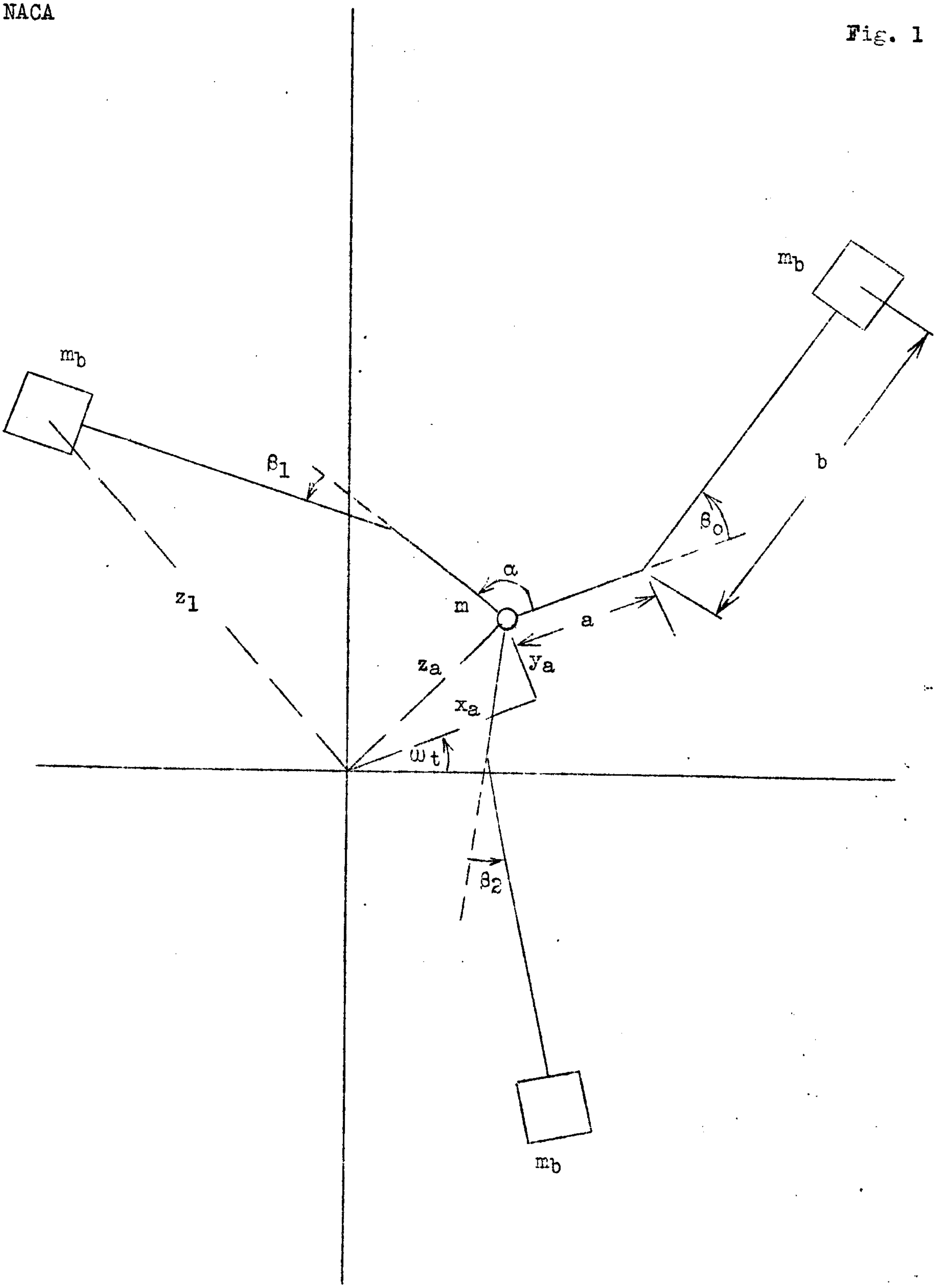

Figure 1.- Simplified mechanical system representing rotor.

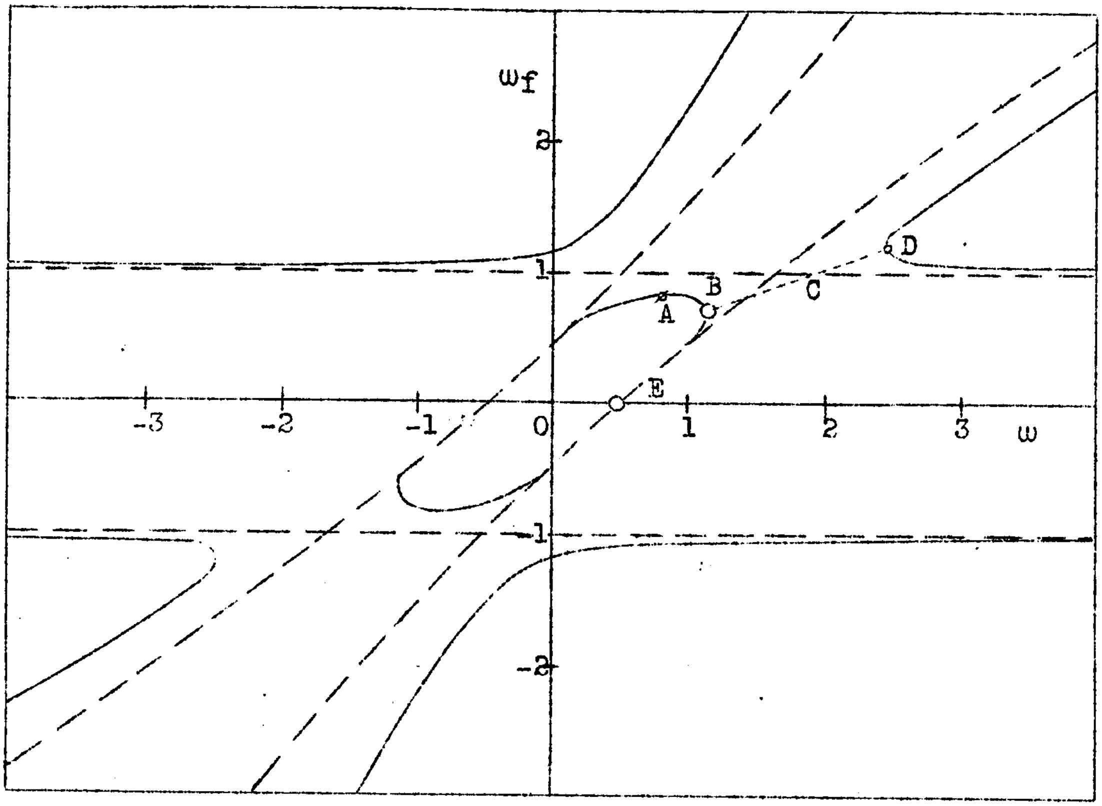

Figure 2.- The effect of coupling between pylon and hinge motions.

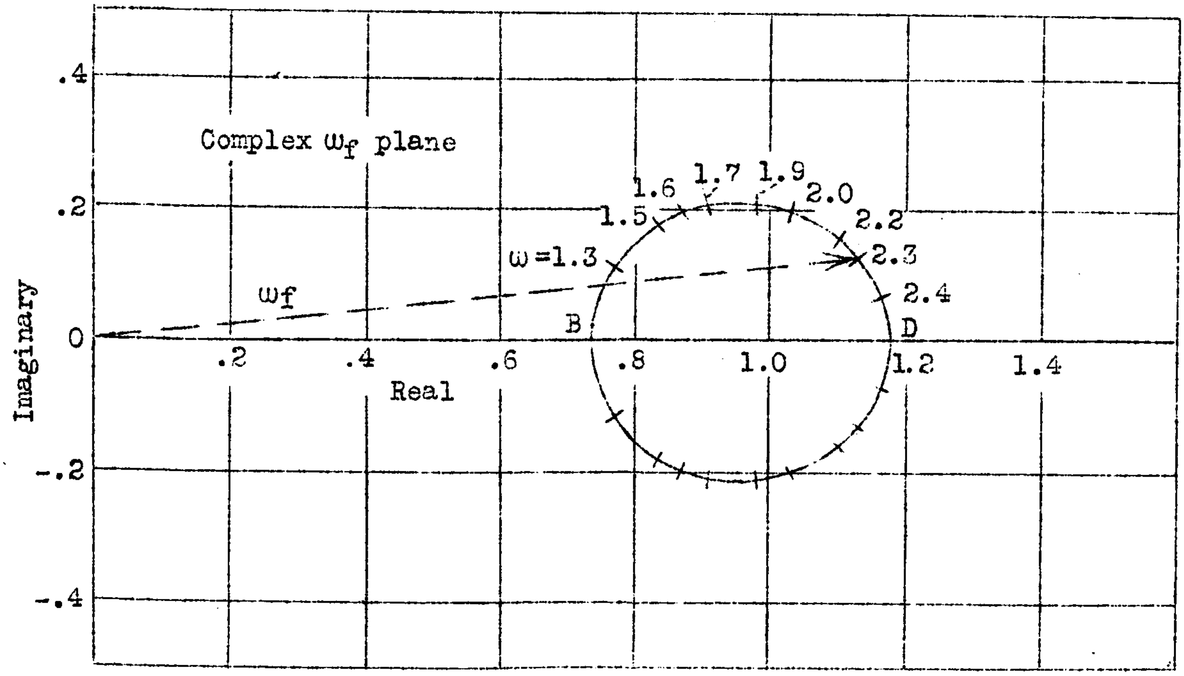

Figure 3.- The complex frequency in the unstable range for $\Lambda_1 = .07$; $\Lambda_2 = .22$; $\Lambda_3 = .1$; $s = 1$.

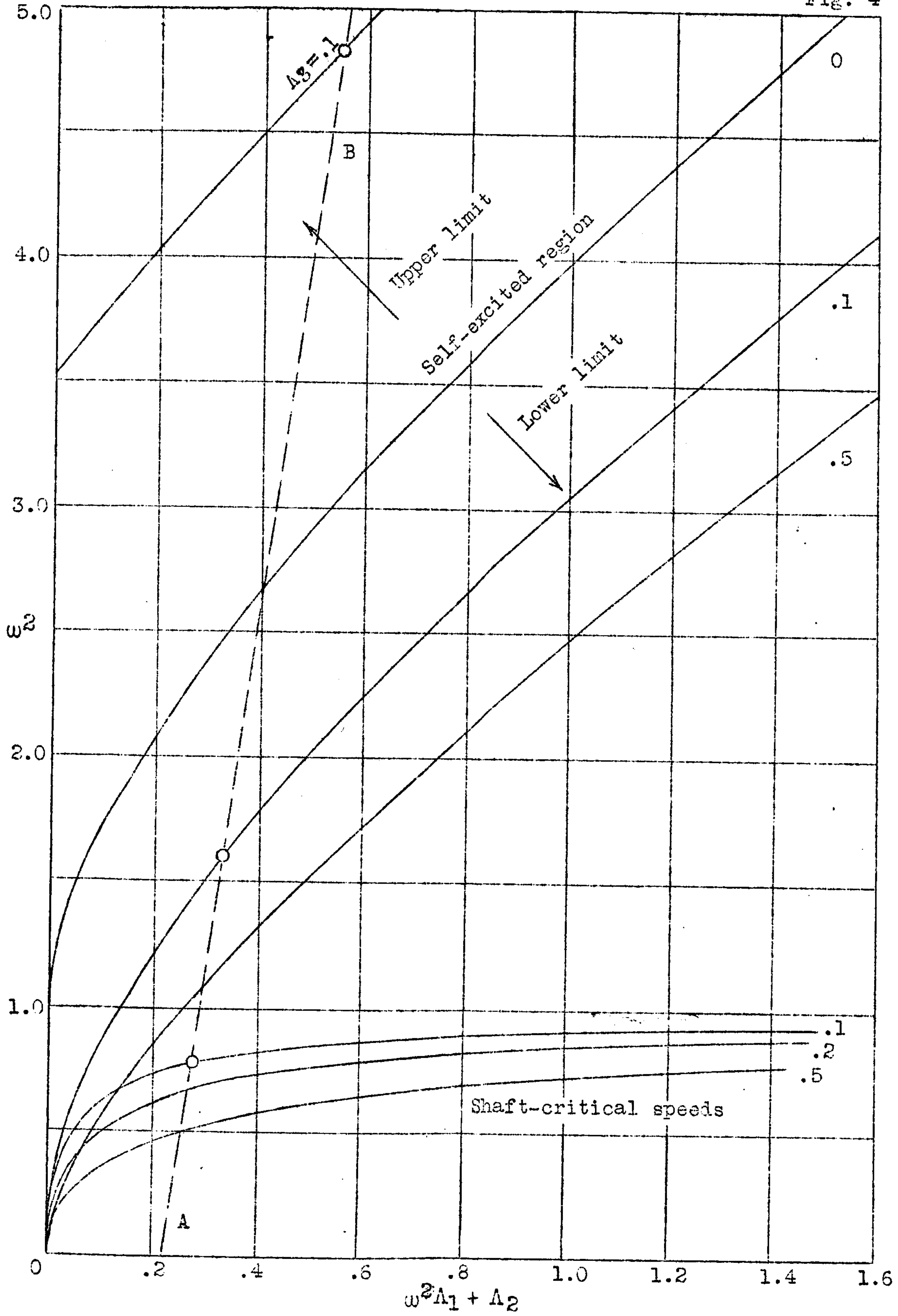

Figure 4.- Stability chart for s = 1.

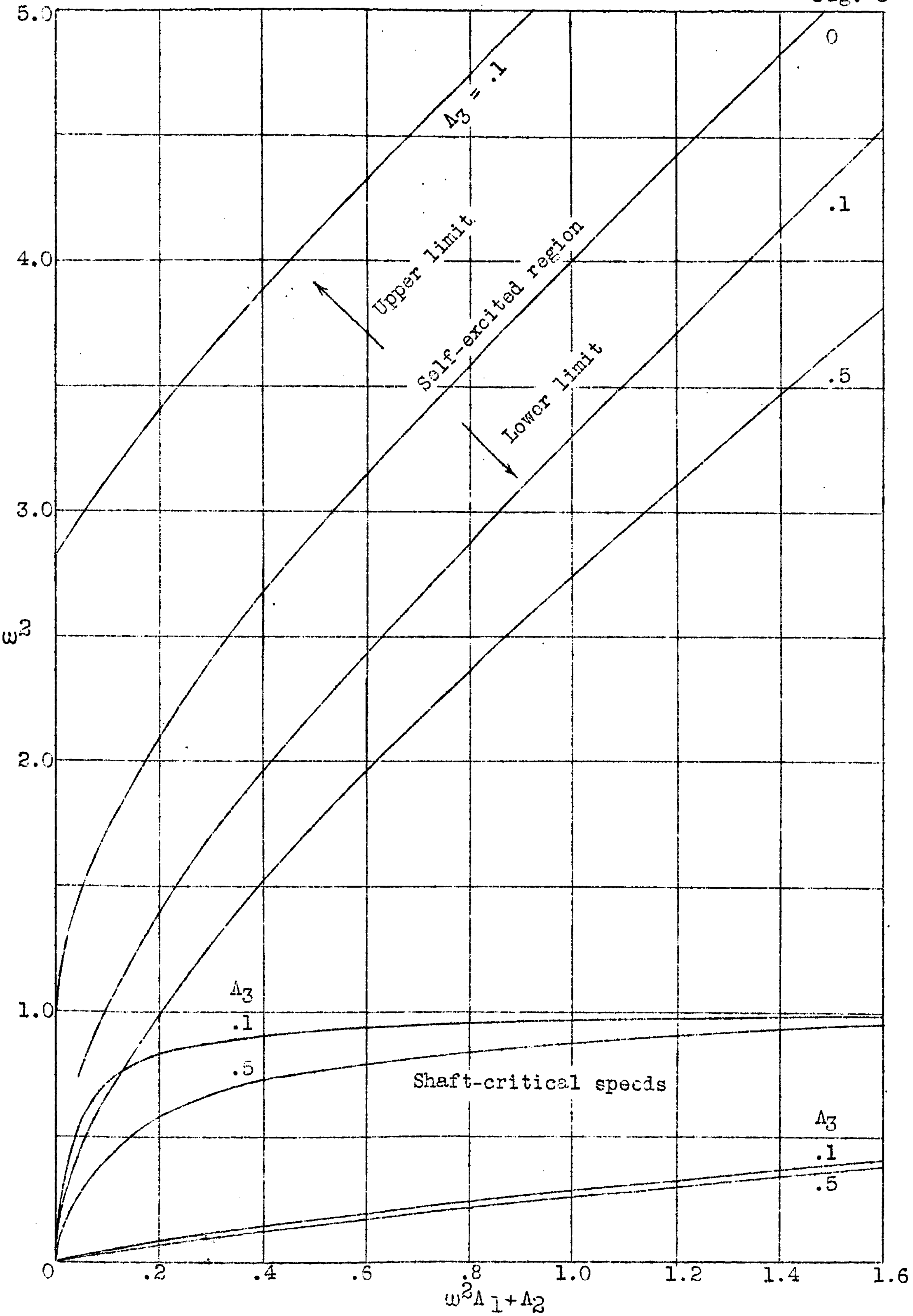

Figure 5.- Stability chart for s = ∞.

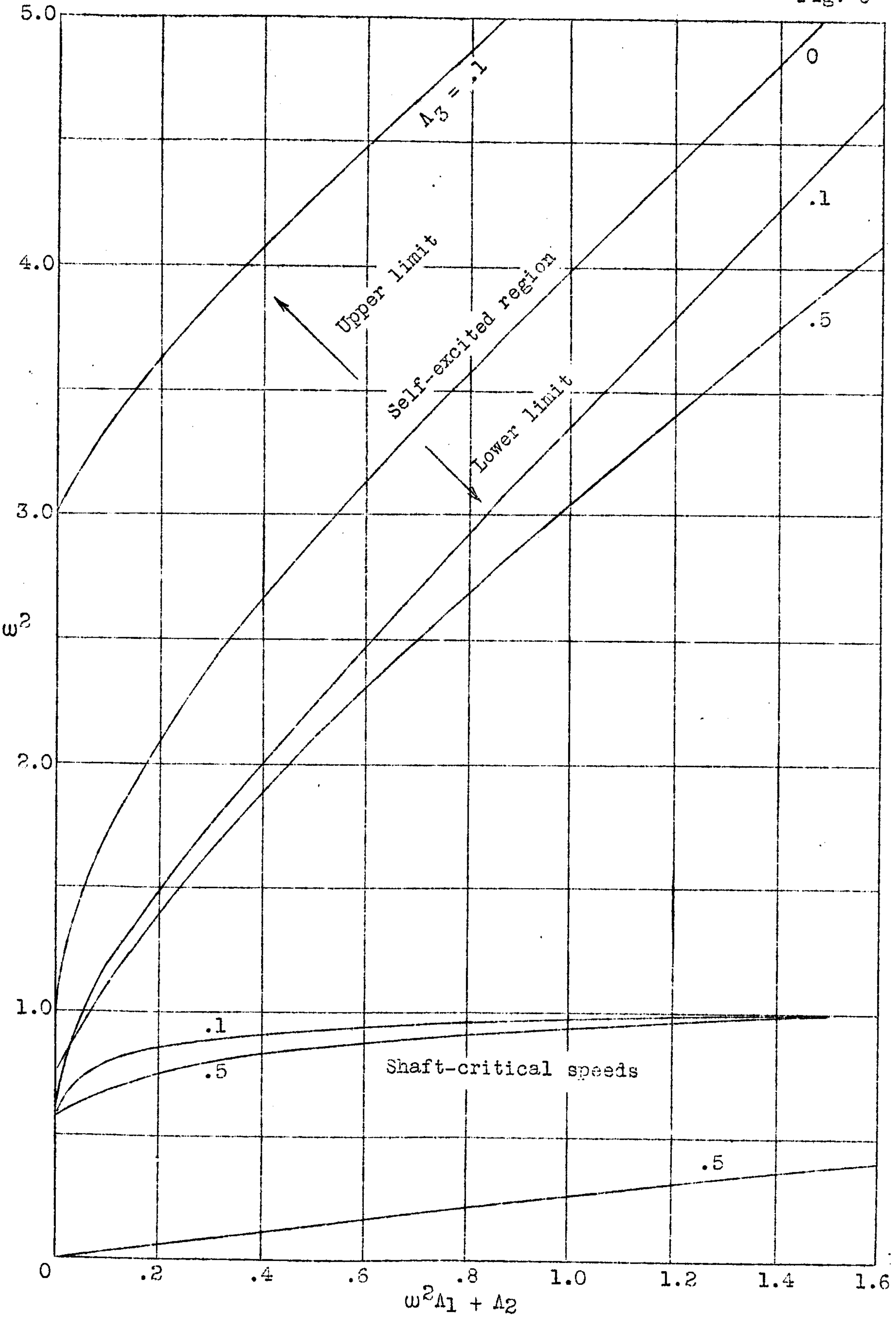

Figure 6.- Stability chart for s = 0.

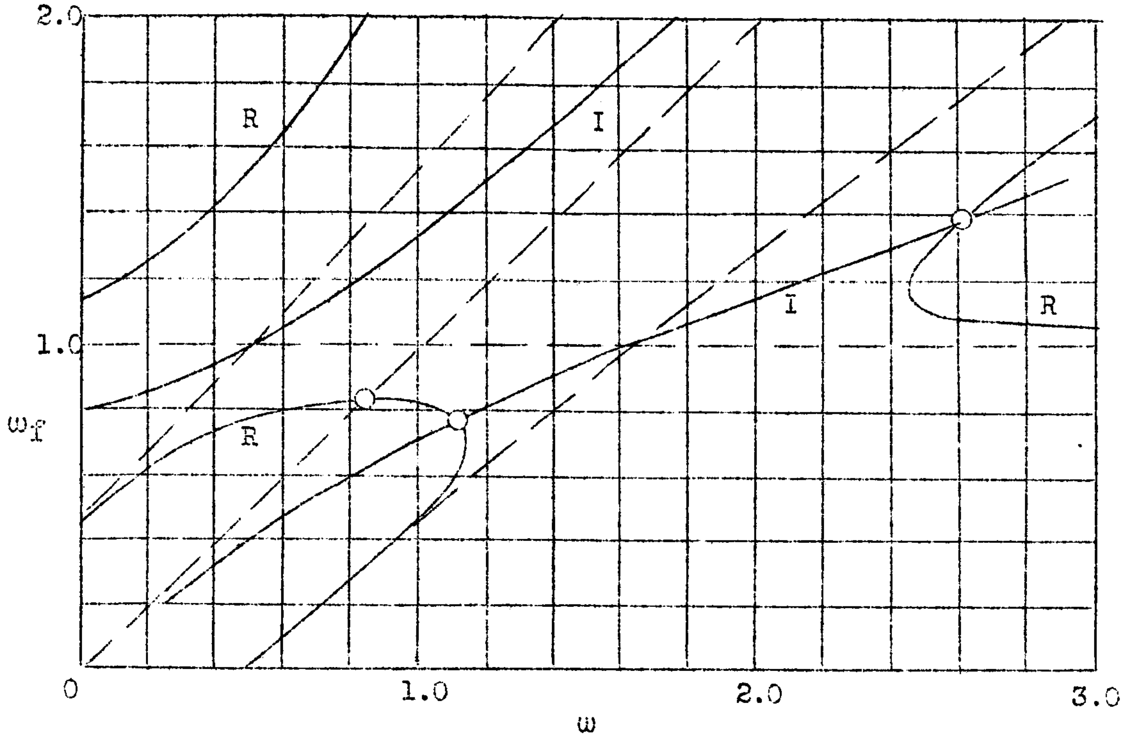

Figure 7.- Plot of real and imaginary equations for a typical case. s = 1; Λ_1 = .07; Λ_2 = .22; Λ_3 = .198.

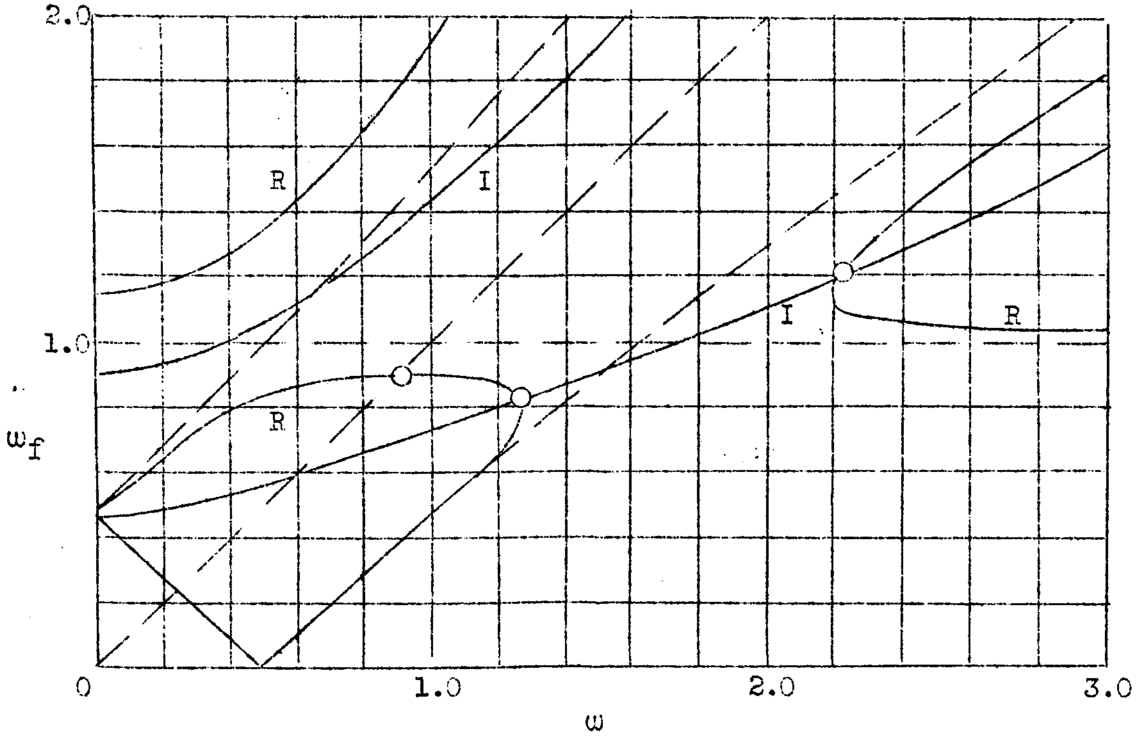

Figure 8.- Plot of real and imaginary equations for case of s = ∞; Λ_1 = .07; Λ_2 = .22; Λ_3 = .198.

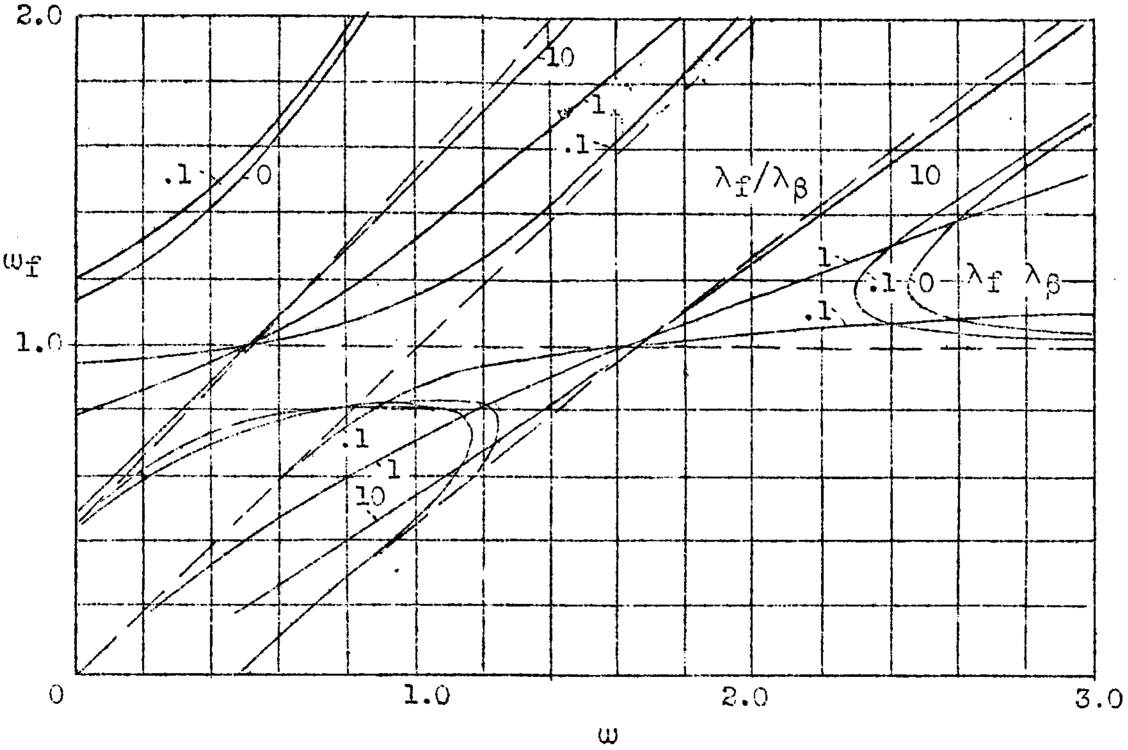

Figure 9.- Effect of damping for case of s = 1; Λ_1 = .07; Λ_2 = .22; Λ_3 = .198.

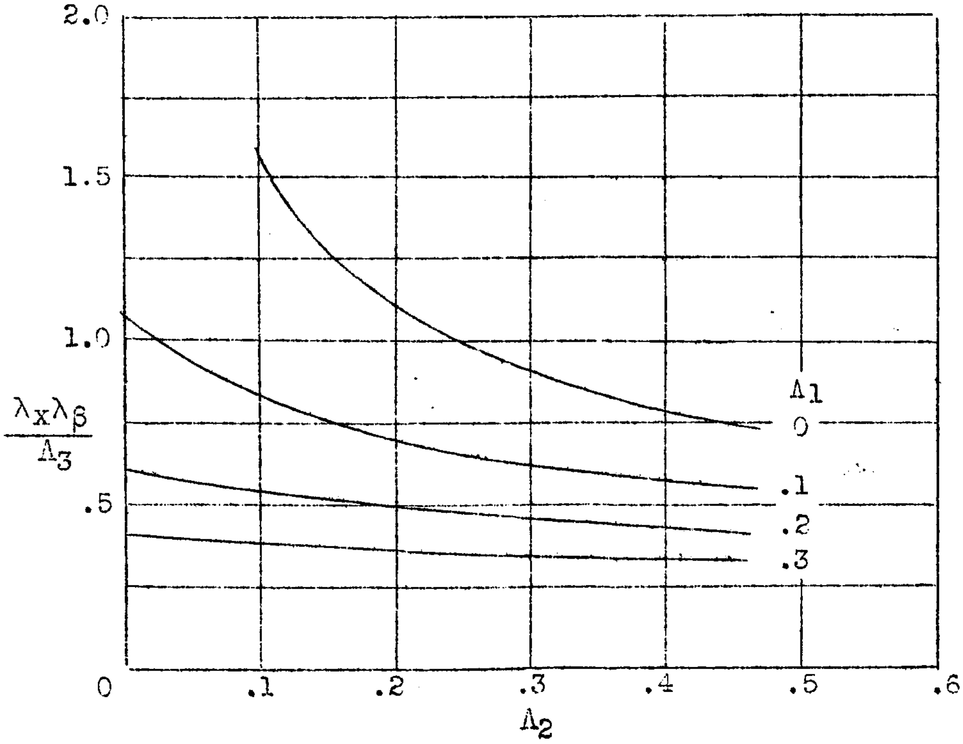

Figure 10.- Damping required to eliminate self-excited oscillation for s = ∞.

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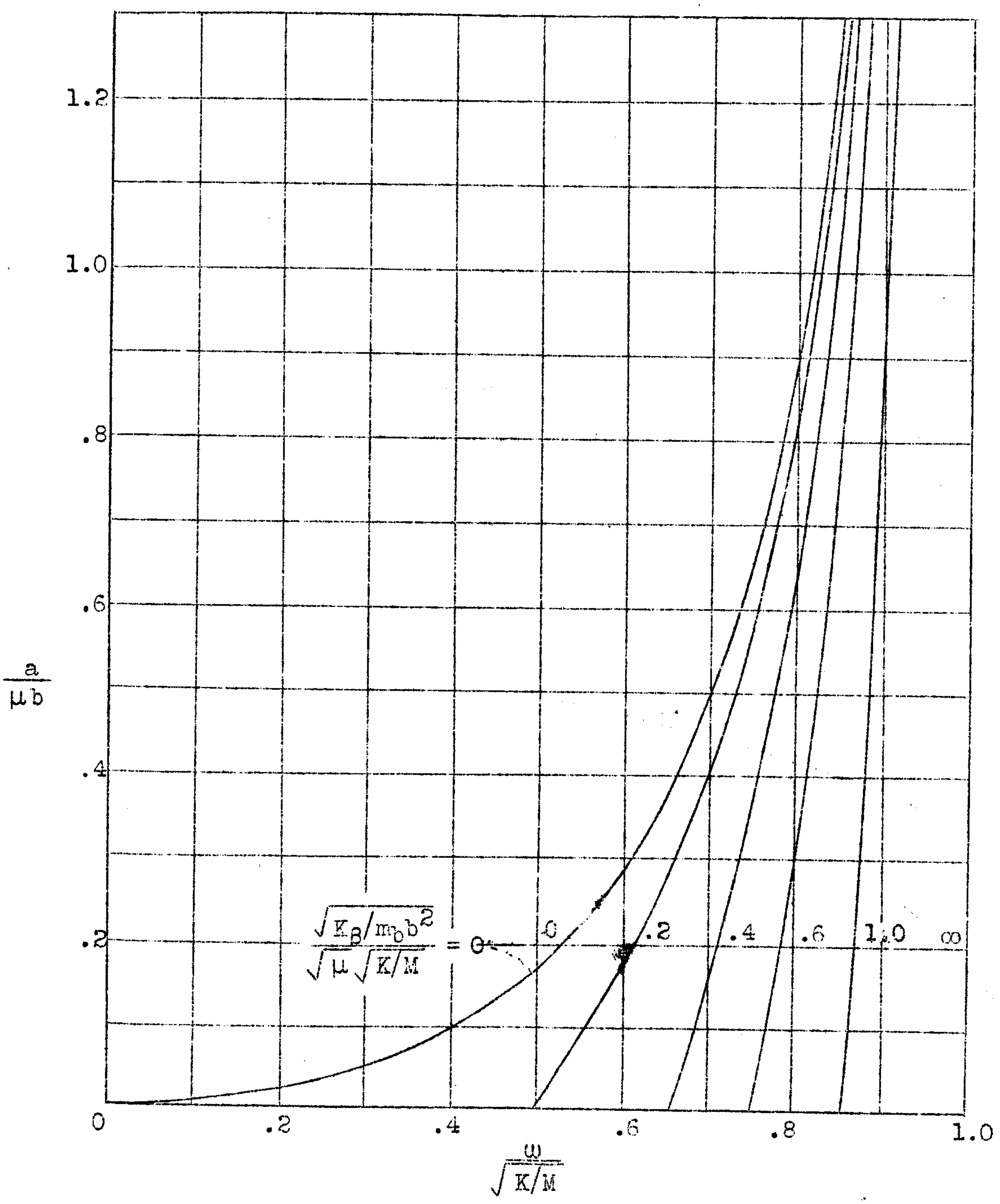

Figure 11.- Shaft-critical speeds, s = 1.

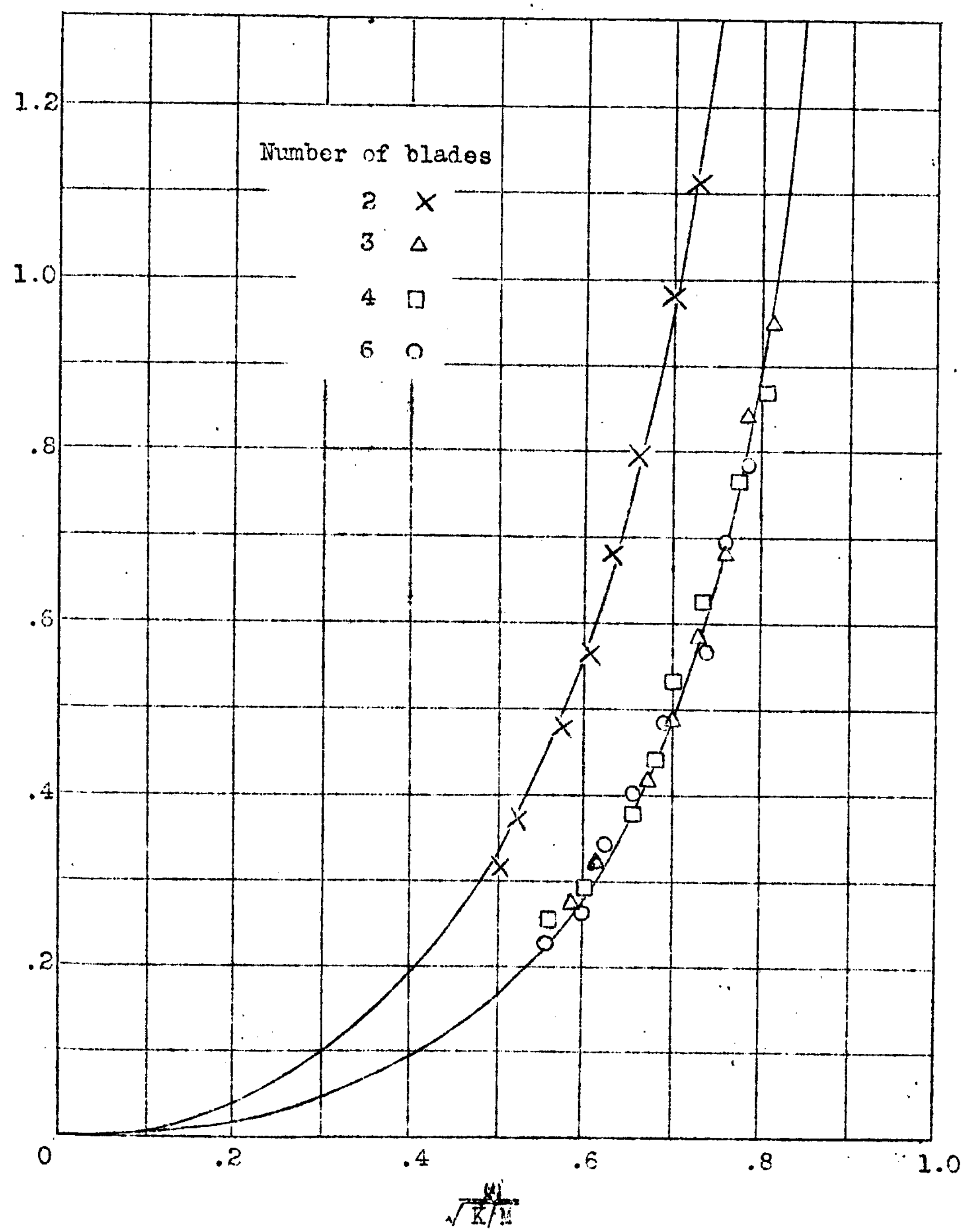

Figure 12.- Experimental critical speeds on small models.