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THEORY OF REVERSIBLE AND NONREVERSIBLE CRACKS IN SOLIDS

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THEORY OF GRIFFITH

The existence of incipient cracks on the surface or inside of solid bodies is, as is known, the reason for the sharp decrease in their actual strength in unilateral tension as compared with the theoretical strength value derived from the electron theory of the forces of a molecular chain. According to the theory of Griffith (ref. 1), in the presence of a crack the tensile force applied to the body, which usually is represented as a cylindrical rod, is distributed near the edge of the crack. When this 'superstress' attains a value corresponding to the theoretical strength of the body the crack starts to deepen suddenly as though 'slitting' the body; this constitutes the mechanism of its rupture.

The theory of Griffith is usually presented in a somewhat different form, associated with the importance not of the forces but, of the potential energy of the body under consideration, more accurately, with the change in this energy which is produced by the cracks of given length (or depth). In the case of the two-dimensional problem this change, according to Griffith and other authors, may be expressed by a two-termed formula

$$U = -\frac{F^2 l^2}{2E} + 2\sigma l \qquad (1)$$

where l is the length of the crack, F is the mean value of the tensile stress (in the absence of a crack or at a large distance from the latter), E is Young's modulus, and σ is the surface stress. The first term U is the decrease in energy due to the formation of cracks of length l (in the separating of its edges). The second term represents the corresponding increase in the surface energy on the assumption that the edges of the crack are separated by a distance where the forces of the molecular chain may be neglected. The dependence of U on l expressed by formula (1) is graphically represented in figure 1; the mean stress F playing the part of a parameter. The maximum of the curve $U(l)$ corresponds

*"Teoriya obratimykh i neobratimykh treshchin v tverdykh telakh," Zhurnal Tekhnicheskoi Fiziki, Vol. 22, No. 11, Nov. 1952, pp. 1857-1866.

to the critical value of this parameter $F_k = F(l_k)$, at which the further lengthening of the crack becomes favorable, that is, to the stress failure which can be measured under the usual test conditions. The experimental value $l = l_k$, corresponding to the given value $F = F_k$, characterizes the critical length of the crack at which its further lengthening becomes favorable. The relation between F_k and l_k is expressed by the well known formula of Griffith (derived from the condition $\partial U/\partial l = 0$)

$$F_k = \sqrt{\frac{2\sigma E}{l_k}} \tag{2}$$

As F_k increases, l_k decreases as seen in figure 1; the curves F_1 and F_2 correspond to two different values of F for the condition $F_2 > F_1$. In the theory of Griffith the crack is a through hole of elliptical shape with its major semiaxis equal to $l/2$ and its minor semiaxis proportional to the mean value of the applied stress F. The magnitude of the true stress F_0 at the edges of the crack (that is, at the ends of the major axis) is inversely proportional to the radius of curvature δ of the ellipse at the corresponding vertices and is expressed in terms of F_k by the formula

$$F_0 = F_k \sqrt{\frac{l_k}{\delta}} \tag{3}$$

For F_0 to have a finite value, the radius of curvature δ must likewise be finite. If it is assumed, in conformity with the molecular concept of the structure of solids, that the minimum value of the radius of curvature of the crack should be equal in order of magnitude to the distance between the neighboring atoms or molecules ($\delta = 10^{-8}$ cm) and if, further, the theoretical strength is identified with the value of Young's modulus E which is in agreement with the electron theory of the forces of a molecular chain and if, finally, account is taken that the surface stress is equal in order of magnitude to the product of the modulus E by δ, the value of F_0 obtained from formula (3) is practically identical with that which is determined by formula (2).

CRITICISM OF GRIFFITH'S THEORY

Griffith's theory is unsatisfactory both in principle and in regard to its agreement with experimental facts. Leaving these facts aside for the moment, the defects of Griffith's theory with respect to fundamental principle will be considered first in order to construct a theory of cracks which is free from defects.

It is first necessary to note the following qualification. If for a given value $F = F_k$ the length of the crack would be found greater than the corresponding value l_k (formula (2) or (3)) then, according to Griffith's theory this crack should have increased in a catastrophic manner. This conclusion which expresses the essential character of Griffith's theory is generally considered entirely natural and undisputable. However, an entirely different result is obtained if the initial length of the crack is assumed less than l_k. In this case, according to the very meaning of the theory of Griffith, the crack should have spontaneously closed, that is, gradually decreased in length and would have finally disappeared entirely. As far as is known to the author, such a conclusion from the theory of Griffith has never been drawn by any one at any time. Griffith himself assumed, and in this, all his followers agreed with him, that for a stress F less than the critical value F_k, which corresponds to the given length of the crack $l = l_k$, the latter maintains its length for an unlimited time; in other words, for $l < l_k$, the length l instead of decreasing catastrophically (similarly to its catastrophic increase for the case $l > l_k$) remains constant. This assumption is a necessary condition for the applicability of Griffith's theory to real solid bodies but is at the same time contradictory to it (generally without explicit awareness of it). Early in the concept of the theory it was established that the initial incipient crack has, even in the absence of tensile forces, a finite width considerably greater than the interatomic distance δ, representating, as it were, a cut in the body not connected with the action of the tensile forces and therefore, incapable of spontaneously closing when the tensile forces are removed. With increase in the latter up to the value F_k, the initial crack maintaining a constant length can only widen, and only when F exceeds this value would it begin to lengthen.

This contrasting of the initial structure of the crack ('crack with hollow') with the structure which it develops on further lengthening is however, entirely arbitrary and is inconsistant with experimental facts. In fact, from this supposition the result follows that if on passing through the critical value F_k and not allowing the crack the possibility of starting to lengthen during a certain time interval Δt, and when the applied tensile force is suddenly removed, the crack will spontaneously close to its initial length. A test of this kind has not up to this time been conducted. There exists, however, every reason to believe that actually the initial length of the crack plays no essential part, and that in the absence of tensile forces the crack should be entirely closed.

This defect in the principle of the theory of Griffith may be formally reduced to the statement that the curve $U(l)$ for $F = \text{constant}$ has only a single extreme point which corresponds to the maximum of the energy, that is, to an unstable state of the body in the presence of a given tensile force.

This qualification deprives Griffith's theory of any physical meaning. To obtain a physical meaning of the theory, it is necessary to establish a relation between U and l such that, together with the maximum, which corresponds to the unconditional existing limiting value $l = l''$, there should be a minimum for a somewhat smaller value $l = l'$ which corresponds to the stable state of the body. Such a relation may be graphically represented by the curve shown in figure 2, which is reminiscent of one of the isothermals in the theory of Van der Waals (turned upside down).

Curves of precisely this type are, in fact, obtained when the theory is constructed on the basis of a true representation of the geometry of the cracks and not on the entirely arbitrary and actually incorrect representation by Griffith, which is that elliptical holes with finite curvature at the ends of the major axis are formed. The present theory related to the representation of stable elastic cracks arising from the action of tensile forces (external or internal) is in agreement with the investigations of P. A. Rebinder on the effect of adsorbing substances or of a gaseous medium in which the solid under tension is located on its different mechanical properties. This theory corresponds also to the tests of Obreimoff on the splitting of sheets of mica along the cleavage planes (ref. 2).

CRACKS FORMED IN AXIAL SPLITTING OF CRYSTALLINE PLATES

The cracks formed in a solid should not have rounded edges, as is implicitly assumed in the theory of Griffith, but sharp edges corresponding to the gradual asymptotic approach of both surfaces separating the crack to the normal distance δ, which in a crystalline body may be defined as the constant of the crystal lattice in the corresponding direction, and which in general is identical in order of magnitude to the normal distance between the neighboring atoms or molecules. From a purely geometrical point of view these normal distances between neighboring layers of particles could be treated as cracks, plane in the case of crystalline bodies, and more or less curved in the case of bodies with vitreous structure, and the entire solid could be considered as pierced in all possible (in particular crystallographic) directions by a system of holes of atomic width. However, from a dynamic point of view these intervals between neighboring layers of particles (atoms, molecules) become actual cracks only when the distance between them is increased relative to the normal distance, that is, becomes equal to $\delta + y$; if the widening y (the difference between the actual distance and its normal value) is considerably greater than δ, even though at a certain limited part of the body near the surface or far from it rather than at the corresponding parts, the edges of the cracks may be considered completely disconnected. Such a slit-shaped crack is represented in figure 3(a) for the case of a crack on the surface of the body and in figure 3(b) for the case of an internal crack. The abscissa here denotes the distance measured along the crack and the ordinate the corresponding widening y or, more accurately, half of it (thus the value $y = 0$ characterizes the normal distance between the neighboring layers of particles).[1]

Although under the observed conditions, the length of the crack strictly speaking is infinite, practically it may be assumed that the crack, in the actual sense of the word, ends where the distance between its edges becomes equal, say, to twice the value of the normal distance δ, that is, where $y = \delta$.

A less arbitrary and physically more justified definition of the effective length of the crack may be obtained on the basis of the dependence of the mutual interaction between the oppositely lying parts of the walls of the crack, and on the distance between them y (or, more accurately, $\delta + y$). This force, referred to unit area, is denoted by $f(y)$. For sufficiently small (in comparison with δ) values of y, the force should, evidently, be proportional to y, reducing to an attraction in the case $y > 0$, and to a repulsion in the case $y < 0$. This dependence may be expressed by the formula:

$$f = \frac{Ey}{\delta} \tag{4}$$

where $f > 0$ corresponds to an attraction, and $f < 0$ to a repulsion. Since for $y >> \delta$, f is equal to 0, it follows that for a certain value $y = y^*$ the force f attains its maximum value f^*, which may be identified with the theoretical strength of the material in rupture. Since y^* is near in value to δ, f^* should be near in value to E, as is clear from formula (4) which may be approximately extended to this case but becomes entirely unsuitable for $y > y^*$.

The dependence of f on y is graphically shown in figure 4. It may be analytically represented by the very simple formula

$$f = Aye^{-\alpha y} \tag{5}$$

where $A = E/\delta$ and the coefficient α is equal to the reciprocal of the distance y^*. For simplicity in this report α will be identified with $1/\delta$.

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[1]The difference between our proposed description of the crack and the description that follows from the theory of Griffith does not require explanation. We may remark only that the 'rounding' of the edges of the Griffith cracks with a radius of curvature δ would correspond, from our point of view, to the cracks for which the widening y, instead of asymptotically approaching zero, becomes zero in a more or less sharp manner. The assumption made by equating the minimum value of the radius of curvature to δ is entirely without foundation.

Formula (5) may be replaced by a two-term expression of the type

$$f = \frac{a}{(\delta + y)^{\mu}} - \frac{b}{(\delta + y)^{\nu}} \quad (\nu - \mu > 0) \tag{5a}$$

in which the first term characterizes the forces of attraction and the second (generally omitted in considering the surface energy), the forces of repulsion.

Evidently, integrating f with respect to y between the limits 0 and ∞, results in the work of the total division of the body into two parts, that is, twice the value of the surface energy (or of the surface stress in the case of absolute zero temperature). Therefore,

$$2\sigma_0 = \int_0^{\infty} f dy$$

If, as the upper limit in this integral an infinite value of y is taken as a measure of the surface energy for the cracks (slits) not having entirely separated edges, this expression applies

$$2\sigma(y_1) = \int_0^{y_1} f dy$$

As in the case of formula (5)

$$2\sigma(y_1) = \frac{A}{\alpha^2}\left[1 - (y_1\alpha + 1)e^{-\alpha y_1}\right] \tag{6}$$

For $y_1 \rightarrow \infty$, this expression reduces to $2\sigma_0 = A/\alpha^2$; hence, in the relation $A = E/\delta$ (that is, $A/\alpha = E$), there is obtained the formula

$$\sigma_0 = \frac{1}{2} E\delta \tag{6a}$$

which practically agrees with the relation between the surface stress and the rupture strength that was identified earlier with E.

More accurately, this strength is expressed by the formula

$$f^* = \frac{A}{\alpha} e = Ee = 2.72E$$

The accuracy is, however, illusory because formula (5) possesses a roughly approximate character the same as the identification of $y^*(= 1/\alpha)$ with δ.

When the slit-shaped crack represented in figure 3(a) or (b) is taken into consideration, it is assumed that the surface energy of its principal part (corresponding to $y > \delta$) is equal in unit width to $2\sigma_0 l$, where l is the effective length; while the surface energy of the deeper part of the crack may be represented with a sufficient degree of accuracy by the integral of the expression

$$2 \int_0^y f dy = 2 \frac{E}{\delta} \int_0^y y dy = \frac{E}{\delta} y^2$$

taken along x from $x = l$ to $x = \infty$:

$$U = \frac{E}{\delta} \int_l^\infty y^2 \, dx \tag{7}$$

That the surface energy of a narrow crack should be less than that of a wide crack was noted by Rebinder in his work on the strength of solids under the action of substances adsorbed on their surfaces. Rebinder, however, restricted himself to only a qualitative determination of the function $\sigma(y)$.

DEPENDENCE OF LENGTH OF CRACK ON THE MAGNITUDE OF TENSILE FORCES FOR A SPLIT PLATE

Imagine a horizontal plate of infinite length split under the effect of two oppositely directed forces $\pm F$ applied perpendicular to its plane and to its edge from above and from below, respectively (fig. 5). The width of the plate (in the direction perpendicular to the plane) is taken to be unity. The thickness of the plate is denoted by $2b$ which is assumed to be small in comparison with the effective length l of the forming crack (lengthened in the form of a gradually narrowing slit).

To determine l as a function of F requires that each of the two similar halves into which the plate splits be considered as a beam bending under the action of a force concentrated on the free end and the cohesive forces on the other half of the plate, which are distributed along the surface directed toward this half, and attaining a maximum value at the required distance l. In the work of Obreimoff (ref. 2), account was taken only of the wide part of the crack or its 'mouth', while its slit-shaped part, extending from $x = l$ to $x = \infty$, which is the usual concept of a crack of finite length, was not taken into account at all. These surface forces take the place of the forces which in the usual theory of the bending of a beam are reduced to the reaction of the wall to which one of its ends is clamped.

Under these conditions the shape of the plate, or more accurately, of one of its halves (the lower for example) is determined by the differential equation

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{8}$$

where y denotes the displacement of the corresponding point of the half-plate under consideration, because the lower half-plate is taken to be positive downwards, E is Young's modulus, $I = b^3/3$ is the moment of inertia of a plane section, and M the moment of the forces applied to the half-plate with respect to the point $x = 0$. This moment is expressed by the following formula

$$M = Fx - \int_0^x f\xi \, d\xi \tag{8a}$$

where f is the surface force referred to unit length. Differentiating this expression with respect to x gives

$$\frac{dM}{dx} = F - fx \tag{8b}$$

Equation (8) is thus, equivalent to the following third-order equation

$$\frac{d^3y}{dx^3} = \frac{1}{EI} (F - fx) \tag{9}$$

where f is a function of y determined by formula (5) or (5a).

For the solution to the problem of the dependence of the effective length of the crack on the external force F, an exact solution of equation (9) is not necessary. The problem may be solved by an approximate method taking into account that for large values of x the derivative d^3y/dx^3 should approach zero. Since it is then possible to put $f = Ay = \frac{E}{\delta} y = \frac{F}{x}$, the dependence of y on x in the slit-shaped part should be expressible by the asymptotic formula

$$y = \frac{F}{E} \frac{\delta}{x} \tag{10}$$

obtained from equation (9) under the condition $d^3y/dx^3 = 0$. In fact, from equation (10) it follows that $d^3y/dx^3 = - \frac{6F\delta}{E} \frac{1}{x^4}$, which leads to the following corrected expression for y

$$y = \frac{F\delta}{Ex} \left(1 + \frac{3}{2} \frac{\delta b^3}{x^4}\right) \tag{10a}$$

For the purpose of this report the correction term will be neglected.

It is understood by the effective length l of the crack that the value of x, for which the value of y is equal to δ, corresponds (approximately) to the maximum value of the force. Thus the following linear relation between l and F can be obtained

$$l = \frac{F}{E} \tag{11}$$

ENERGY OF CRACK IN SPLITTING OF A PLATE AND STRENGTH OF THE LATTER

The preceding results may also be obtained from the condition of the maximum total energy of the system under consideration (half-plate). This energy U breaks up into three parts: (a) the elastic energy U_1 of the external part treated as a beam of length l with load F on the free end; (b) surface energy of the open part of the crack $U_2 = 2\sigma_0 l$; and (c) the energy of the incipient slit-shaped part corresponding to $x > l$. For part (c), formula (7) may be used with formula (10), which gives

$$U_3 = \frac{\delta F^2}{E}\int_l^{\infty}\frac{dx}{x^2} = \frac{\delta F^2}{El} \tag{12}$$

Part (a) is obtained by the expression

$$U_1 = -\frac{1}{2}Fy_0$$

where y_0 is the maximum deflection, that is, the value of y for $x = 0$. This may be expressed by the approximate formula

$$y_0 = \frac{l^3F}{3EI}$$

which is obtained by the integration of equation (8) substituting in it $M = Fx$, that is, not taking into account explicity the internal forces but schematically using the conditions $y = 0$ and $dy/dx = 0$ for $x = l$.

Thus, the total energy of the plate, or more accurately, the half-plate of the splitting force F over the effective depth l may be approximately represented by the following formula

$$U = -\frac{F^2l^3}{6EI} + 2\sigma_0 l + \frac{F^2\delta}{El} \tag{13}$$

which of course is applicable to the case of very small values of e. (These values will not, in this report be taken into account.)

The dependence of U on l expressed by formula (13) is graphically shown by the curve of the same type as in figure 3 with the difference that when l approaches 0, U approaches ∞ (in reality it approaches a finite value). In figure 6 is shown a family of curves $U(l)$ corresponding to different values of the splitting force F (isodynes) in the order of increase of the latter. For a value that is not too large the curve has two extremities: a minimum for $l = l'$ and a maximum for $l = l'' > l'$. By increasing F these points approach each other and finally for a certain value $F = F_c$ (the critical point analogous to the critical temperature in the theory of Van der Waals) they coalesce.

For $F << F_c$, the minimum of the function U practically coincides with the minimum of the sum

$$U_2 + U_3 = 2\sigma_0 l + \frac{F^2\delta}{El}$$

and corresponds to the following value

$$l' = F\sqrt{\frac{\delta}{2E\sigma_0}} \tag{14}$$

Since, according to equation (6a) $\sigma_0 = E\delta/2$ this expression may be written in the form

$$l' = \frac{F}{E} \tag{14a}$$

which agrees with the previously derived expression (11) for the effective length of the crack due to the action of the force F. From this it is clear that the length of crack, which was considered earlier, represents a stable value, corresponding to the minimum of the potential energy U and characterized by the stable equilibrium of the split plate.

The maximum value of U corresponds to the unstable equilibrium of the plate at which the length of the crack, having attained a certain value $l'' > l'$ under the action of a very small disturbance, either returns to the value $l = l'$, or suddenly increases. For the condition $F = F_c$, this length practically reduces to that which corresponds to the maximum of the function $U_1 + U_2 = -\frac{F^2 l^3}{6EI} + 2\sigma_0 l$ and is expressed by the formula

$$l'' = \frac{2}{F}\sqrt{\sigma_0 EI} = \frac{E}{F}\sqrt{2\sigma I} \tag{15}$$

Comparing equations (14a) and (15) we may write the relation between l and l' in the form

$$l'l'' = \sqrt{\frac{1}{2}\,\delta b^3} \tag{15a}$$

Evidently the value F_c corresponding to the agreement of l' and l'' simply represents the strength limit of the plate relative to the action considered, that is, the minimum force introduced by the sudden formation of the lengthening crack.[2] In order of magnitude, this force may be determined by identifying expression (14a) with (15). Still more simply it may be computed with the aid of formula

$$l_c = \sqrt[4]{\frac{1}{2}\,\delta b^3} \tag{16}$$

and

$$F_c = El_c \tag{16a}$$

When $\delta = 10^{-8}$ centimeter and $b = 1$ centimeter, $l_c \approx 10^{-2}$ centimeter and $F_c = 10^{+10}$ dynes (for $E = 10^{+12}$ dynes/cm^2) are obtained.

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[2] In contrast to the theory of Griffith this force is not connected with the existence of an already existing crack of a certain finite length.

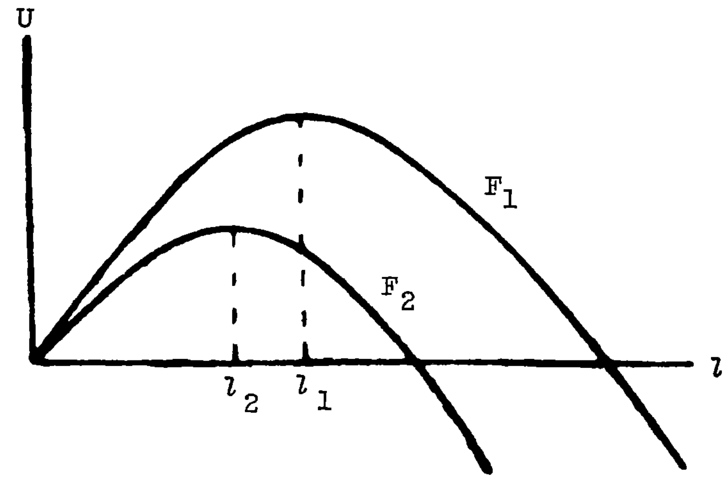


Figure 1.

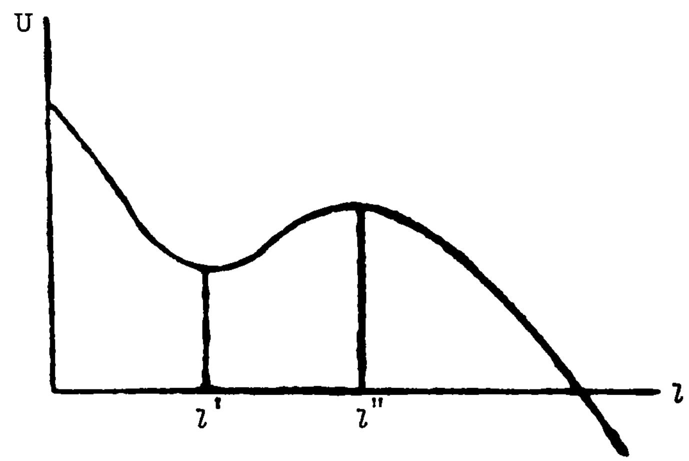


Figure 2.

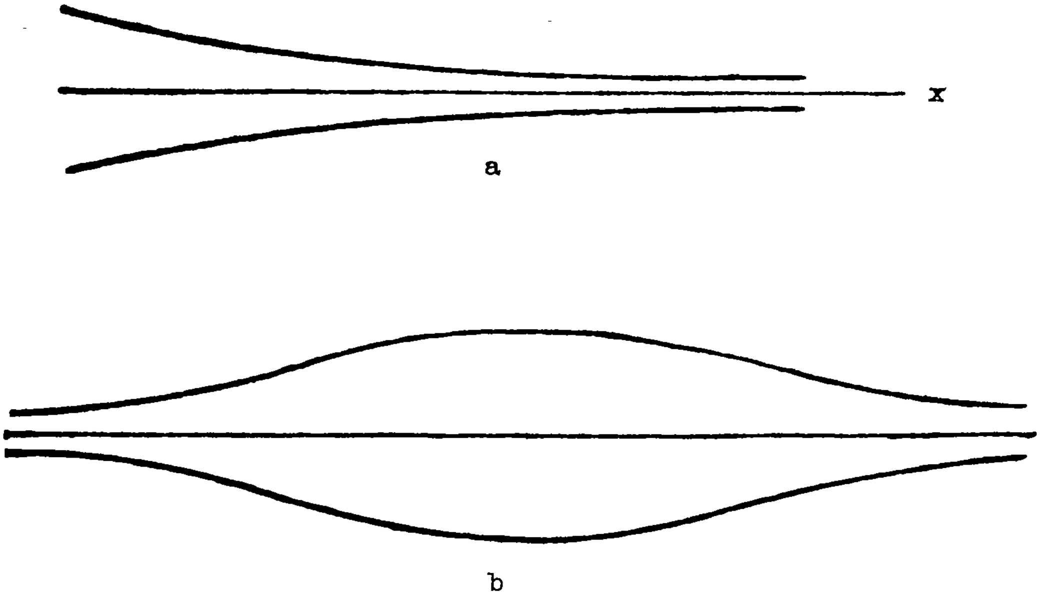
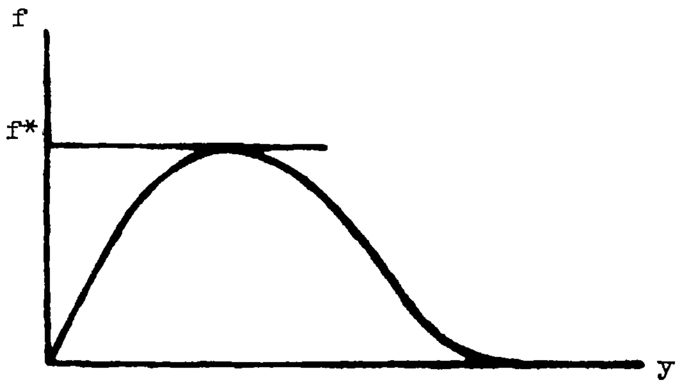


Figure 3.



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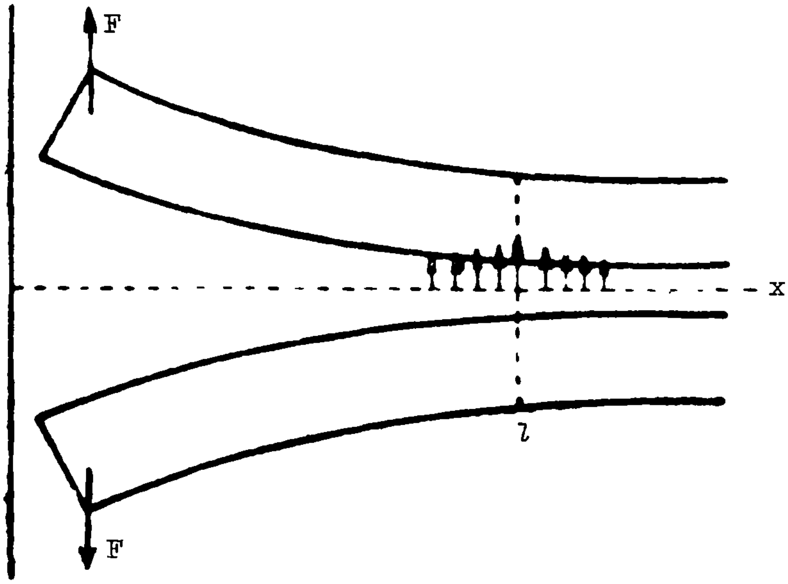

Figure 5.

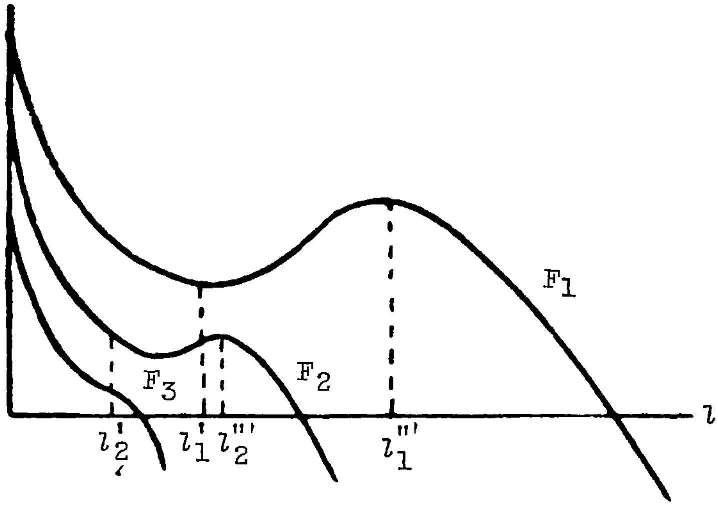

Figure 6.

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