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# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

## TECHNICAL MEMORANDUM 1362

## ON THE SOUND FIELD OF A POINT-SHAPED SOUND SOURCE IN UNIFORM TRANSLATORY MOTION

By H. Hönl

Translation of "Über das Schallfeld einer gleichförmig-translatorisch bewegten punkförmigen Schallquelle." Annalen der Physik, issue 5, vol. 43, 1943.

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# ON THE SOUND FIELD OF A POINT-SHAPED SOUND SOURCE IN UNIFORM TRANSLATORY MOTION*[1]

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## INTRODUCTION AND SUMMARY[2]

The following report will treat rigorously the excitation of sound by a point-shaped sound source (pulsator) in uniform translatory motion at subsonic or supersonic velocity through a medium at rest for the two- and three-dimensional case (two- and three-dimensional problem).

In qualitative respect, the phenomena of sound propagation excited by a moving sound source are well known. If the construction of surfaces of constant phase is based on Huyghens' principle such that the propagation in the medium at rest of the elementary waves emanating from the sound source is regarded as independent of the momentary state of motion of the sound source, characteristic traits of the sound propagation may be understood even on the basis of simple geometric constructions. If, for instance, the motion of the sound source visualized as point-shaped is uniform translatory along the negative x-axis of a three-dimensional Cartesian coordinate system, the sound propagation for a sound source moved at subsonic or, respectively, supersonic velocity is represented in the known manner by figure 1; according to construction, surfaces of constant phase would correspond to the circles plotted. For the motion of the source at supersonic velocity, it is, above all, characteristic that the excitation of sound always remains limited to the interior of a cone - the Mach cone - the (semi) opening angle $\alpha$ of which results directly from the construction of the elementary waves (one has $\sin \alpha = c/U$ when $c$ denotes the sonic velocity, $U > c$ the velocity of the sound source). From the physical point of view, however, there arises the question how far the construction of the phase surfaces according to the principle of Huyghens' elementary waves can be applied

*"Über das Schallfeld einer gleichförmig-translatorisch bewegten punkförmigen Schallquelle." Annalen der Physik, issue 5, vol. 43, 1943, pp. 437-464.

[1]Dedicated to Privy Councillor A. Sommerfeld for his 75th anniversary on December 5, 1943.

[2]The present investigation was written within the scope of my work for the Kaiser-Wilhelm Institute for Flow Research, Göttingen.

rigorously, also the problem of the exact amplitudes or intensities, respectively, of the sound field. It will be shown on the basis of the rigorous solution of the plane and spatial problem that, on the whole, the qualitative description of the sound field according to the principle of elementary waves is justified, but that in particular, especially in case of supersonic velocity, characteristic deviations from the elementary construction become evident.

The mathematical theory of sound propagation will be based on the linear propagation equation for the medium at rest

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0 \tag{1}$$

$\varphi$ may signify therein for instance the pressure disturbance. It would appear rather obvious to utilize the acoustic optical analogies for the problem of the sound source in uniform translatory motion and, accordingly, to apply to (1) a Lorentz transformation where then the sonic velocity $c$ instead of the velocity of light would appear in the transformation formulas as critical velocity.[3] However, this method would seem somewhat artificial for the present problem since the transformation formulas of the space and time quantities, differently from optics, would have no physical significance and their application thus would be a merely formal methodical expedient; finally, in case of supersonic velocity, one would depart even further from the physical starting point, due to the different relativity formulations.[4]

We shall therefore adopt below another direct method for solution of the problem, making use of the method of the Fourier integral. Therein it is expedient to use a coordinate system with respect to which the sound source is at rest at the origin of the coordinate system. Our problem then is, except for a Galileo transformation, obviously identical with the sound propagation about a sound source in an oncoming flow of subsonic or supersonic velocity, respectively, assumed at rest at the coordinate origin. If the oncoming flow has positive x-direction corresponding

[3]It is known that historically the investigation of the propagation equation (1) for the first time gave occasion for setting up linear transformation formulas for the space and time quantities which are closely related to those of the special relativity theory. Compare W. Voigt, "Über das Dopplersche Prinzip," Göttingen Nachrichten, 1887, p. 41; see also W. Pauli, Enzykl. der Math. Wissensch, vol. 19, p. 543.

[4]For the subsonic region, H. G. Küssner, Luftfahrtforsch, vol. 17, p. 370, 1940, derived exact formulas for the sound field of a moving sound source which agree with the results of this report in an interesting manner by application of a "Lorentz transformation."

to figure 1, the equation of propagation to be solved follows from equation (1) simply by replacement of the time operator $\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$ where U is the free-stream velocity. We thus obtain from equation (1)

$$\Delta\varphi - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\varphi = 0 \tag{1'}$$

If we limit ourselves immediately to the case of a sound excitation periodic in time with the frequency $\omega$ and put accordingly

$$\varphi = u(x,y,z)e^{-i\omega t} \tag{2}$$

the time factor may be split off and we obtain from equation (1')

$$\Delta u - \frac{1}{c^2}\left(-i\omega + U\frac{\partial}{\partial x}\right)^2 u = 0 \tag{3}$$

Equation (3) is to be fulfilled in the entire space with exception of the coordinate origin at which the sound source is located and of possible occurring singular lines or surfaces (delimitation of the Mach cone or, for the two-dimensional problem, of the Mach wedge).

If one sets up an expression for the solution for u in the form of a multiple (two or threefold) Fourier integral one has the advantage that the coefficients of the representation may be readily given as continuous functions in the Fourier space due to the linear character of the differential equation (3); likewise, it is easy to determine simultaneously the coefficients of the Fourier representation in such a manner as corresponds to the presence of a "point-shaped" sound source at the coordinate origin or, for the "plane" problem, of a source distribution visualized as rectilinear. It may appear at first as a difficulty of this method that the integral representations for the solution u thus obtained due to the integrand becoming infinite on characteristic surfaces or lines of the real Fourier space (for the spatial or plane problem) are subject to indeterminate conditions and without special stipulations regarding the course of the integration paths in the complex domain have no unique meaning. The physical viewpoint, which here as in other propagation problems enforces uniqueness, is that aside from the conditions named which u must satisfy, a radiation condition must be fulfilled in infinite space.[5] Actually, it is precisely the original indeterminateness of the integral which confers the freedom of adjusting the solution of equation (3) to the natural physical conditions of the problem (suitably manipulating the integration paths in the complex domain) and of determining it, in this manner, uniquely.

## 1. APPLICATION OF THE METHOD OF THE FOURIER INTEGRAL

We search for solutions of the propagation equation (3) which correspond to the standard point source located at the coordinate origin. We disregard for the time being fulfillment of a radiation condition. We leave the dimension number $n$ indeterminate at first, and put subsequently $n = 2$ or $n = 3$, respectively.

In order to arrive at solutions of this type in a physically and mathematically unobjectionable manner, one will find it expedient to start, instead of starting from equation (3), from the inhomogeneous differential equation

$$\Delta_n u - \frac{1}{c^2}\left(- i\omega + U\frac{\partial}{\partial x_1}\right)^2 u = f\left(x_1, x_2 \ . \ . \ .\right) \tag{3a}$$

in which $f\left(x_1, x_2 \ . \ . \ .\right)$ signifies a source distribution prescribed at first arbitrarily (vanishing sufficiently rapidly at infinity). Let the "unit source" isolated at the zero point now be interpreted as the limit of a sequence of continuous source distributions $f$. Let us therefore first visualize continuous positive functions $f$ which have a large maximum at the zero point of the coordinate system (point O) and are, for the rest, to be subject to the condition

$$\int f \, dS = 1 \tag{4}$$

$\left(dS = dx_1 \, dx_2 \ . \ . \ . \ dx_n\right)$; $f$ is to depend on a parameter in such a manner that, to the same extent to which this parameter tends toward a

[5] Compare A. Sommerfeld, "Die Greensche Funktion der Schwingungsgleichung." Jahresber. d. DMV. 21, p. 309, 1912. Compare also Frank-Mises, "Die Differential- u. Integralgleichungen d. Physik." Braunschweig 1935, vol. 2, p. 803.

limiting value, $f$ tends toward zero everywhere with exception of the point O. Thus, one has in the lines for the characteristic "delta function"

$$f = 0 \text{ for } P \neq 0 \qquad \int f \, dS_0 = 1 \text{ for } P = 0 \tag{4a}$$

where the integral may be restricted to an arbitrarily small neighborhood of O. The formulation of the functions $f$ according to equation (4) was chosen for reasons of simplicity. This standardization suggests itself physically since there results from equations (3a) and (4) for a sound source at rest, thus $U = 0$, with application of Green's theorem

$$\int \frac{\partial u}{\partial n} \, d\sigma_0 = 1 \tag{5}$$

$\left(d\sigma_0\right.$ surface element of a sphere of arbitrary smallness surrounding the point O, derivation with respect to the outward directed normal$\left.\right)$, in accordance with the definition of the unit source.

In order to make the generality of the method for integration of equation (3a) in question stand out clearly, one replaces this differential equation by

$$L[u] \equiv L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots\right)u = f\left(x_1, x_2, \ldots\right) \tag{6}$$

where $L$ is assumed to be an arbitrary <u>linear</u> differential expression in $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots$ with <u>constant</u> coefficients of otherwise arbitrary order and dimension number. One now makes for $f$ and $u$ the statement of a Fourier integral

$$f = \iint \ldots \int A\left(a_1, a_2 \ldots\right) e^{i\left(\alpha_1 x_1 + \alpha_2 x_2 + \ldots\right)} da_1 \, da_2 \ldots da_n \tag{7a}$$

$$u = \iint \ldots \int B\left(a_1, a_2 \ldots\right) e^{i\left(\alpha_1 x_1 + \alpha_2 x_2 + \ldots\right)} da_1 \, da_2 \ldots da_n \tag{7b}$$

Substitution of equation (7b) in $L[u]$ then results

$$L[u] = L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots\right) \iint \ldots \int B\left(a_1, a_2 \ldots\right)$$

$$e^{i\left(\alpha_1 x_1 + \alpha_2 x_2 + \ldots\right)} da_1 \, da_2 \ldots da_n$$

$$= \iint \ldots \int B\left(a_1, a_2 \ldots\right) L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots\right)$$

$$e^{i\left(\alpha_1 x_1 + \alpha_2 x_2 + \ldots\right)} da_1 \, da_2 \ldots da_n$$

and since due to the linearity of $L$

$$L\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}, \ldots\right) e^{i\left(\alpha_1 x_1 + \alpha_2 x_2 \cdots\right)} =$$

$$L\left(i\alpha_1, i\alpha_2, \ldots\right) e^{i\left(\alpha_1 \dot{x}_1 = \alpha_2 x_2 + \ldots\right)} \tag{8}$$

there results from equation (6) by comparison of coefficients

$$B\left(a_1, a_2, \ldots\right) = \frac{A\left(a_1, a_2, \ldots\right)}{L\left(i\alpha_1, i\alpha_2, \ldots\right)} \tag{9}$$

It now remains to determine $A\left(a_1, a_2, \ldots\right)$ corresponding to the conditions (4) and (4a). From equation (7a) there follows first by Fourier inversion

$$A\left(a_1, a_2 \ldots\right) = (2\pi)^{-n} \iint \ldots \int f e^{-i\left(\alpha_2 x_1 + \alpha_2 x_2 + \ldots\right)} dS$$

and thus in the limiting case (equation (4a))

$$A = (2\pi)^{-n} \tag{10}$$

Substitution of equations (9) and (10) in equation (7b) finally yields the desired solution

$$u = (2\pi)^{-n} \iint \ldots \int \frac{e^{i(\alpha_1 x_1 + \alpha_2 x_2 + \ldots)}}{L(i\alpha_1, i\alpha_2, \ldots)} \, d\alpha_1 \, d\alpha_2 \ldots d\alpha_n \qquad (11)$$

The integral (11) is identical with the method of N. Zeilon[6] for preparation of solutions of linear and homogeneous differential equations with constant coefficients for a prescribed pole-type singularity of the solution. The basic idea of Zeilon's motivation of the integral formula (11) also is the utilization of the identity (8). However, whereas Zeilon gives first consideration to the requirement of representing the solutions of equation (6) prescribed arbitrary source distribution of $f$ by means of the "fundamental integral," this integral results in our method more directly from Fourier's integral representation of the functions $f$ and $u$ and subsequent limiting process. The method may be regarded as mathematically strict if a sequence of suitable analytic functions $f$ has been selected and the representation of $f$ and $u$ does not encounter any difficulties before the application of the limiting process. The limiting process to the final formula (11) may then be performed without hesitation.

## 2. CLASSIFICATION OF THE TYPICAL CASES[7]

We return to the initial equation (1') and, with the use of $L$, make in it corresponding to equations (2) and (8) the replacements

$$\frac{\partial}{\partial t} \dashrightarrow -i\omega \qquad \frac{\partial}{\partial x} \longrightarrow i\alpha \qquad \frac{\partial}{\partial y} \dashrightarrow i\beta \qquad \frac{\partial}{\partial z} \longrightarrow i\gamma$$

$$\Delta = \Delta_n \longrightarrow \begin{cases} -\left(\alpha^2 + \beta^2\right) & \text{for } n = 2 \\ -\left(\alpha^2 + \beta^2 + \gamma^2\right) & \text{for } n = 3 \end{cases}$$

(We write furthermore $x$, $y$, $z$, and $\alpha$, $\beta$, $\gamma$ instead of $x_i$ and $\alpha_i$.) Then the fundamental solution $u = F$ becomes according to equation (11):

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[6]N. Zeilon, Arkiv för Matematik 6. 1911, 9. 1913/14; compare Frank-Mises, Bd. I. S. 862ff.

[7]NACA editor's note: The original German version of this document has this section as number 3, but this is believed to be a typographical error and has been changed to number 2 to provide consecutive numbers for this translation.

(a) For the plane problem (n = 2)

$$F = -\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{e^{i(\alpha x+\beta y)}}{\Omega_2}\,d\alpha\,d\beta \tag{12}$$

with

$$\Omega_2 = \left(1 - M^2\right)\alpha^2 + \beta^2 + 2kM\alpha - k^2 \tag{12a}$$

(b) For the spatial problem (n = 3)

$$F = -\frac{1}{(2\pi)^3}\iiint_{-\infty}^{+\infty} - \frac{e^{i(\alpha x+\beta y+\gamma z)}}{\Omega_3}\,d\alpha\,d\beta\,d\gamma \tag{13}$$

with

$$\Omega_3 = \left(1 - M^2\right)\alpha^2 + \beta^2 + \gamma^2 + 2kM\alpha - k^2 \tag{13a}$$

if we introduce instead of $\omega$ and $U$ the propagation parameter $k$ and the dimensionless Mach number $M$

$$k = \frac{\omega}{c} \qquad M = \frac{U}{c} \tag{14}$$

We consider first the plane case. Obviously the integrand of the fundamental solution (12) becomes infinite if the Fourier coefficients $\alpha$, $\beta$ lie on the conic section $\Omega_2 = 0$ of the real Fourier plane. The equation of the characteristic conic section may be written, according to equation (12a)

$$\left(1 - M^2\right)^2\left(\alpha + \frac{kM}{1 - M^2}\right)^2 + \left(1 - M^2\right)\beta^2 = k^2 \tag{15}$$

This equation shows that $\Omega_2 = 0$ represents an ellipse (in the limiting case $M = 0$ a circle), parabola, or hyperbola, according to whether $M < 1$, $= 1$, or $> 1$. Correspondingly, the differential equation (3) is of elliptic, parabolic, or hyperbolic type. Figure 2 represents a number of conic sections $\Omega_2 = 0$ for various values of the parameter M. All conic sections pass through the points $\alpha = 0$, $\beta = \pm k$, and are mirrored reflections with respect to the $\alpha$-axis.

For the spatial problem, $\beta^2$ on the left side of equation (15) is to be replaced, in conformity with equation (13a), by $\beta^2 + \gamma^2$. The surfaces $\Omega_3 = 0$ are generated by rotating the conic sections of figure 2 about the $\alpha$-axis and thus represent a sphere $(M = 0)$, ellipsoids of revolution $(0 < M < 1)$, a paraboloid of revolution $(M = 1)$, or bi-sheeted hyperboloids of revolution $(M > 1)$.

Following, both the plane and spatial problem will be treated separately for the elliptic, the parabolic, and the hyperbolic case. The elliptic case may be completely traced back to the special case $M = 0$. The parabolic and hyperbolic cases, however, require special considerations depending on the various connections regarding the characteristic curves $\Omega_2 = 0$ and surfaces $\Omega_3 = 0$, and on the stipulated behavior of the solution at infinity. In all cases, the selection of the integration paths is the decisive factor.

## 3. THE ELLIPTIC CASE (SUBSONIC REGION)

### (a) Plane Problem

In the two dimensional elliptic case, $M < 1$, equation (15) represents an ellipse. If we write its equation in the normal form

$$\frac{(\alpha - \alpha_0)^2}{a^2} + \frac{\beta^2}{b^2} = 1 \tag{16}$$

we have

$$a = \frac{k}{1 - M^2} \qquad b = \frac{k}{\sqrt{1 - M^2}} \qquad \alpha_0 = -\frac{kM}{1 - M^2} \tag{16a}$$

If we furthermore make an affine transformation in the $\alpha,\beta$ plane, as well as in the x,y plane, by putting

$$\alpha' = \sqrt{1 - M^2}(\alpha - \alpha_0) \qquad \beta' = \beta \qquad \xi = \frac{x}{\sqrt{1 - M^2}} \qquad \eta = y \tag{17}$$

the fundamental integral (12), (12a) assumes the form

$$F = -\frac{1}{(2\pi)^2}\frac{e^{i\alpha_0 x}}{\sqrt{1-M^2}}\iint_{-\infty}^{+\infty}\frac{e^{i(\alpha'\xi+\beta'\eta)}}{\alpha'^2+\beta'^2-\kappa^2}\,d\alpha'\,d\beta' \tag{18}$$

where $\kappa$ had been equated

$$\kappa = \frac{k}{\sqrt{1-M^2}} \tag{17a}$$

Except for the factor $\frac{e^{i\alpha_0 x}}{\sqrt{1-M^2}}$, the integral appearing in equation (18) as a function of the variables $\xi,\eta$ is therefore the same which would result for the case $M = 0$ with the propagation parameter $k = \kappa$. This last integral, however, represents in the known manner, a cylindric wave outgoing from the coordinate origin in $\xi,\eta$, since the equation[8]

$$\frac{1}{4i}H_0^{(1)}(\kappa\rho) = -\frac{1}{(2\pi)^2}\iint_{-\infty}^{+\infty}\frac{e^{i(\lambda\xi+\mu\eta)}}{\lambda^2+\mu^2-\kappa^2}\,d\lambda\,d\mu \qquad \rho = \sqrt{\xi^2+\eta^2} \tag{19}$$

is valid; therein $H_0^{(1)}$ signifies the first Hankel function of zero order if (which will be discussed in more detail later on) the infinite integration paths, with respect to $\lambda$ and $\mu$ provided in equation (19), are conducted on suitable paths in the complex $\lambda$- and $\mu$-plane. (Otherwise an incoming or standing cylindric wave would result.) If we regard the result of equation (19) for the present as prescribed, we obtain, if we substitute equations (16a), (17), and (19) in equation (18)

$$F = \frac{1}{4i}\frac{e^{-i\frac{M}{1-M^2}kx}}{\sqrt{1-M^2}}H_0^{(1)}\left(\frac{k}{\sqrt{1-M^2}}\sqrt{\frac{x^2}{1-M^2}+y^2}\right) \tag{20}$$

[8] Cf. A. Sommerfeld, elsewhere.

Turning to the discussion of the terminal formula (20), we remark first that the Hankel function $H_0^{(1)}$ for large values of its argument $\kappa\rho$ shows asymptotic behavior

$$H_0^{(1)}(\kappa\rho) \longrightarrow \sqrt{\frac{2}{\pi\kappa\rho}}\, e^{i\left(\kappa\rho - \frac{\pi}{4}\right)} \tag{21}$$

The factor $H_0^{(1)}(\kappa\rho)$ in equation (20) corresponds, therefore, taken by itself with consideration of the time dependency selected, compare equation (2), to an outgoing cylindric wave in the $\xi,\eta$ plane. The space-time dependency of the phase is largely modified by occurrence of the factor

$$\exp\left(- i \frac{M}{1 - M^2} kx\right)$$

The phase $\Phi$ thus becomes, again asymptotically, according to equation (20)

$$\Phi = - \frac{M}{1 - M^2} kx + \frac{k}{\sqrt{1 - M^2}}\sqrt{\frac{x^2}{1 - M^2} + y^2} - \frac{\pi}{4} - \omega t \tag{22}$$

If we consider, for instance, the propagation of the special phase surface $\Phi = - \frac{\pi}{4}$, an elementary conversion of equation (22) yields

$$\left(x - \frac{\omega M}{k} t\right)^2 + y^2 = \left(\frac{\omega t}{k}\right)^2$$

or, with consideration of equation (14),

$$(x - Ut)^2 + y^2 = c^2t^2 \tag{23}$$

This simple result signifies that the surfaces of constant phase are propagated, in relation to the flowing medium, asymptotically at the normal sonic velocity $c$, whereas their center is carried along at the velocity $U$ of the flow, just as to be expected on the basis of Huyghens' construction of the elementary waves. Going beyond the asymtotic agreement with the geometric construction of the elementary waves, our rigorous formula (20) remains correct up to arbitrary proximity to the sound source at any rate as long as the idealizations leading to the wave equation (1) ("small" amplitudes, "point-shaped" sound source) hold true, thus also in a region where Huyghens' principle would fail.

It is interesting to investigate in addition to the phase the behavior of the amplitude. According to equations (20) and (21), the decrease of the amplitude takes place asymptotically as $1/\sqrt{\rho}$. The surfaces of constant amplitude are therefore cylindric surfaces of elliptic cross-section, the axis of which (parallel z) lies in the sounds once visualized as rectilinear. The decrease of the amplitude, and therewith also of the intensity, takes place more rapidly in the $\pm x$ direction than in the $\pm y$ direction. Here, different from the phase, the direction of the flow is never one-sided.

Finally, the limiting case lines $k \longrightarrow 0$ will be considered. It corresponds to $\omega = 0$, thus to a static pressure disturbance, as is caused by the flow against a thin rod normally to the flow direction. According to equation (12a) $\Omega_2$ becomes therein

$$\Omega_2 = \left(1 - M^2\right)\alpha^2 + \beta^2 \tag{24}$$

therewith

$$F = -\frac{1}{(2\pi)^2}\iint_{-\infty}^{+\infty}\frac{e^{i(\alpha x+\beta y)}}{\Omega_2}\,d\alpha\,d\beta$$

$$= -\frac{1}{(2\pi)^2}\iint_{-\infty}^{+\infty}\frac{e^{i\left(\alpha'\frac{x}{\sqrt{1-M^2}}+\beta' y\right)}}{\alpha'^2+\beta'^2}\frac{d\alpha'\,d\beta'}{\sqrt{1-M^2}}$$

and after the known transformation

$$F = \frac{1}{2\pi\sqrt{1 - M^2}}\ln\sqrt{\frac{x^2}{1 - M^2} + y^2} \tag{25}$$

Here also the decrease of the pressure disturbance in the $\pm x$ direction takes place more rapidly than in the $\pm y$ direction by the factor $\dfrac{1}{\sqrt{1 - M^2}}$. The characteristic appearance of the argument $\rho = \sqrt{\dfrac{x^2}{1 - M^2} + y^2}$ instead of $\sqrt{x^2 + y^2}$ for $M = 0$ is usually denoted as Prandtl's rule.

(b) Spatial Problem

All results derived for the plane elliptic case may be transferred directly to the spatial problem. One obtains from equations (13) and (13a), maintaining the substitutions (equation (17)) supplemented by $\gamma' = \gamma$, $\zeta = z$,

$$F = -\frac{1}{(2\pi)^3} \frac{e^{i\alpha_0 x}}{\sqrt{1 - M^2}} \iiint_{-\infty}^{+\infty} \frac{e^{i(\alpha'\xi+\beta'\eta+\gamma'\zeta)}}{\alpha'^2 + \beta'^2 + \gamma'^2 - \kappa^2} d\alpha' \, d\beta' \, d\gamma' \tag{26}$$

(with $\alpha_0$ and $\kappa$ having the same values given in equations (16a) and (17a).

If one, furthermore, makes use of the Fourier representation of the spherical wave outgoing from the zero point $\xi = \eta = \zeta = 0$

$$\left.\begin{aligned} \frac{1}{4\pi} \frac{e^{i\kappa\rho}}{\rho} &= \frac{-1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} \frac{e^{i(\lambda\xi+\mu\eta+\nu\zeta)}}{\lambda^2 + \mu^2 + \nu^2 - \kappa^2} d\lambda \, d\mu \, d\nu \\ \rho &= \sqrt{\xi^2 + \eta^2 + \zeta^2} \end{aligned}\right\} \tag{27}$$

(postponing intended remarks on the integration paths), there results from equations (26) and (27), with consideration of equations (16a) and (18a), the terminal formula

$$F = -\frac{1}{4\pi} \frac{e^{-i\frac{M}{1-M^2}kx}}{\sqrt{1 - M^2}} \frac{\exp\left\{i \dfrac{k}{\sqrt{1 - M^2}} \sqrt{\dfrac{x^2}{1 - M^2} + y^2 + z^2}\right\}}{\sqrt{\dfrac{x^2}{1 - M^2} + y^2 + z^2}} \tag{28}$$

The discussion is analogous to that for the plane problem. The surfaces of constant phase are spherical surfaces which are propagated in relation to the flowing medium at the velocity $c$, while their center is simultaneously carried along. This is rigorously valid up to arbitrary nearness to the sound source. The surfaces of constant amplitude and intensity, respectively, are the surfaces $\rho$ = const; the intensity decreases as $1/\rho^2$, thus more rapidly in the $\pm x$-direction than in the $\pm y$- and $\pm z$-direction.

(c) For the sake of completeness, a short supplementary remark should be added to the proof of the formulas (19) and (27) for the Fourier representation of the cylindrical and spherical wave. Regarding detailed treatment of these problems, we refer to A. Sommerfeld, elsewhere.

Let, for the plane case, the double integral (19) be designated

$$G_2 = -\frac{1}{(2\pi)^2}\iint_{-\infty}^{+\infty}\frac{e^{i(\lambda\xi+\mu\eta)}}{\lambda^2+\mu^2-\kappa^2}\,d\lambda\,d\mu \tag{19}$$

Obviously, this integral has at first not yet a unique meaning, since the integrand becomes infinite on the circle $\lambda^2 + \mu^2 = x^2$ in the real $\lambda,\mu$ plane. If one introduces in this plane, and likewise in the x,y plane, polar coordinates

$$\lambda = \sigma\cos\psi \qquad \mu = \sigma\sin\psi \qquad \lambda^2 + \mu^2 = \sigma^2$$

$$\xi = \rho\cos\varphi \qquad \eta = \rho\sin\varphi \qquad \xi^2 + \eta^2 = \rho^2$$

and visualizes the integration with respect to the azimuth $\psi$ before the integration with respect to $\rho$, one may write equation (19), with consideration of the known integral representation of the Bessel function of zero order

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\cos\chi}\,d\chi = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\cos(\psi-\varphi)}\,d\psi \tag{29}$$

after a simple transformation, in the form

$$G_2 = -\frac{1}{2\pi}\int_0^{\infty}\frac{J_0(\sigma\rho)}{\sigma^2-\kappa^2}\,\sigma\,d\sigma \tag{19a}$$

In equation (19a) the indeterminateness of the integral is not yet eliminated due to the integrand's becoming infinite for $\sigma = \kappa$. The integral (19a) represents a cylindrical wave extending to infinity, only when the integration path is transferred to the negative-imaginary complex $\sigma$-plane, leaving the point $\sigma = \kappa$ to the left in the manner characterized in figure 3. The correctness of the contention is known to result immediately from the decomposition of the Bessel function $J_0$ into the two Hankel functions $H_0^{(1)}$ and $H_0^{(2)}$

$$J_0(\sigma\rho) = \frac{1}{2}\left\{H_0^{(1)}(\sigma\rho) + H_0^{(2)}(\sigma\rho)\right\} \tag{30}$$

of different asymptotic behavior at infinity of the complex $\sigma$-plane, and from the deformation of the integration paths indicated in figure 3. The integral (19) may then be reduced to the residue of the constituent part of the integrand stemming from the first summand in equation (30) for $\sigma = \kappa$ and yields

$$G_2 = \frac{1}{4i} H_0^{(1)}(\kappa\rho) \tag{19b}$$

Because of the asymptotic behavior of $H_0^{(1)}$ for large real values of the argument (compare equation (21)), equation (19b) actually corresponds to an outgoing (divergent) wave. Had the integration path been transferred from equation (19a) to the positive-imaginary complex $\sigma$-plane, an incoming (convergent) cylindric wave would have resulted, corresponding to the asymptotic behavior of $H_0^{(2)}$. Completely analogous relations prevail for the spatial problem $(G_3)$.

The behavior of the wave outgoing from the zero point divergent toward infinity may be comprised according to Sommerfeld into an analytical condition, which is denoted as radiation condition. It reads for the plane and for the spatial problem, respectively

$$\lim_{\rho \to \infty} \sqrt{\rho}\left(\frac{\partial G_2}{\partial \rho} - i\kappa G_2\right) = 0 \quad \text{or, respectively} \quad \lim_{\rho \to \infty} \rho\left(\frac{\partial G_3}{\partial \rho} - i\kappa G_3\right) = 0 \tag{31}$$

on the infinitely distant boundary of the plane and spatial region, respectively. If $G$ is interpreted as Green's function of the infinite domain, the addition of a condition of the type (31) together with the known remaining conditions for $G$ is sufficient for uniquely fixing this function[9].

In contrast to the cylindrical and spherical wave $G_2$ and $G_3$ propagated from the zero point isotropically in all directions (compare equations (19) and (27)), one deals in the general elliptic case according to equations (20) and (28) with unsymmetric radiation. We omit formulating for this more general case a radiation condition for infinity in analogy to equation (31) and are content with having reduced the general elliptic case $0 < M < 1$ to the isotropic problem $M = 0$. The difference between outgoing and incoming wave is therefore brought about solely by the selection of the integration path in the general elliptic case as well. The possibility of reducing the sound propagation in case of subsonic approach flow to isotropic sound propagation is physically understandable since the propagation of a sound wave is modified by the existence of a flow $U < c$ but, basically, not essentially changed; however, the relations become completely different if we now turn to the hyperbolic and parabolic case.

## 4. THE HYPERBOLIC CASE $M > 1$ (SUPERSONIC REGION)

### (a) Plane Problem

In the hyperbolic case, the characteristic conical section $\Omega_2 = 0$ is a hyperbola. If one writes its equation in the normal form

$$\frac{(\alpha - \alpha_0)^2}{a'^2} - \frac{\beta^2}{b'^2} = 1 \tag{32}$$

one has according to equation (15)

$$a' = \frac{k}{M^2 - 1} \qquad b' = \frac{k}{\sqrt{M^2 - 1}} \qquad \alpha_0 = \frac{kM}{M^2 - 1} \tag{32a}$$

By virtue of the affine transformation

$$\lambda = \sqrt{M^2 - 1}\,(\alpha - \alpha_0) \qquad \mu = \beta \qquad \xi = \frac{x}{\sqrt{M^2 - 1}} \qquad \eta = y \tag{33}$$

[9]Compare also W. Magnus, Jahresberichte d. DMV, vol. 52, p. 177, 1943.

the fundamental integral (12) becomes

$$F = +\frac{1}{(2\pi)^2}\frac{e^{i\alpha_0 x}}{\sqrt{M^2 - 1}}\iint_{-\infty}^{+\infty}\frac{e^{i(\lambda\xi+\mu\eta)}}{\lambda^2 - \mu^2 - \kappa^2}\,d\lambda\,d\mu \tag{34}$$

with

$$\kappa = \frac{k}{\sqrt{M^2 - 1}} \tag{34a}$$

The position of the characteristic hyperbola

$$\Omega_2 = -\lambda^2 + \mu^2 + \kappa^2 = 0$$

in the $\lambda,\mu$ plane obviously suggests performance of the integration over $\lambda$ before that over $\mu$ (compare fig. 4). The integration over $\lambda$ then leads first to the integral

$$\int_{-\infty}^{+\infty}\frac{e^{i\lambda\xi}}{\left(\lambda + \sqrt{\mu^2 + \kappa^2}\right)\left(\lambda - \sqrt{\mu^2 + \kappa^2}\right)}\,d\lambda \tag{35}$$

the integrand of which has a pole each at $\lambda = \pm\sqrt{\mu^2 + \kappa^2}$. In order to eliminate the indeterminateness of the integral, the integration path leading from $-\infty$ to $+\infty$ has to be suitably detoured into the complex $\lambda$-plane. The type of transfer of the integration path again can be selected only on the basis of physical considerations. Since the sound wave approaches here supersonic velocity, the sound excitation must be required to disappear for $x < 0$. Since in equation (35) $x$ is replaced by $\xi$, it is therefore a requirement that the integral (35) yields zero for negative values of $\xi$. Except for continuous deformations in which no singularities are transversed, the integration path is therewith fixed; one must make a detour into the negative-imaginary complex $\lambda$-plane, leaving both poles to the left. If $\xi$ is positive, the integration path (compare fig. 5) may be replaced by a path in the infinitely distant positive-imaginary $\lambda$-plane and one circling, in positive sense, each of the poles at $\pm\sqrt{\mu^2 + \kappa^2}$. For negative $\xi$, in contrast, the shifting to the infinitely distant negative-imaginary $\lambda$-plane results in zero as it should. One obtains thus by formation of residues

$$\int_{-\infty}^{+\infty} \cdot \cdot \cdot \, d\lambda = \begin{cases} -2\pi \dfrac{\sin \xi\sqrt{\mu^2 + \kappa^2}}{\sqrt{\mu^2 + \kappa^2}} & \text{for } \xi > 0 \\ 0 & \text{for } \xi < 0 \end{cases} \tag{35a}$$

Therewith the double integral occurring in equation (34) may be reduced, disregarding the factor $-2\pi$, for $\xi > 0$ to the simple integral

$$K = \int_{-\infty}^{+\infty} e^{i\mu\eta} \frac{\sin \xi\sqrt{\mu^2 + \kappa^2}}{\sqrt{\mu^2 + \kappa^2}} d\mu \tag{36}$$

Its calculation is achieved in the following manner. If one puts

$$\mu = \kappa \sinh \vartheta \qquad \sqrt{\mu^2 + \kappa^2} = \kappa \cosh \vartheta \qquad \frac{d\mu}{\sqrt{\mu^2 + \kappa^2}} = d\vartheta$$

$K$ becomes

$$K = \frac{1}{2i} \int_{-\infty}^{+\infty} \left\{ e^{i\kappa(\xi \cosh \vartheta + \eta \sinh \vartheta)} - e^{-i\kappa(\xi \cosh \vartheta - \eta \sinh \vartheta)} \right\} d\vartheta$$

If one substitutes furthermore

$$\xi = \tau \cosh \chi \qquad \eta = \tau \sinh \chi \qquad \xi^2 - \eta^2 = \tau^2 \tag{37}$$

$K$ may, with consideration of the addition theorem

$$\cosh \vartheta \cosh \chi \pm \sinh \vartheta \sinh \chi = \cosh (\vartheta \pm \chi)$$

also be written

$$K = \frac{1}{2i} \int_{-\infty}^{+\infty} \left\{ e^{i\kappa\tau \cosh (\vartheta + \chi)} - e^{-i\kappa\tau \cosh (\vartheta - \chi)} \right\} d\vartheta \tag{36a}$$

One now uses for the Hankel functions $H_0^{(1)}$ and $H_0^{(2)}$ an integral representation going back to Heine[10]

$$H_0^{(1)}(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} e^{iz\cosh\vartheta}\,d\vartheta \qquad H_0^{(2)}(z) = -\frac{1}{\pi i}\int_{-\infty}^{+\infty} e^{-iz\cosh\vartheta}\,d\vartheta \tag{38}$$

Since the appearance of $\pm\chi$ beside $\vartheta$ in equation (36a) is obviously insignificant, because of the integration limits lying at infinity, one obtains therewith

$$K = \frac{1}{2i}\pi i\left\{H_0^{(1)}(\kappa\tau) + H_0^{(2)}(\kappa\tau)\right\} = \pi J_0(\kappa\tau) \tag{39}$$

Through the substitution, equation (37), evidently only a part of the entire $\xi,\eta$-plane is covered; for positive $\tau$'s, only the region characterized in figure 6 by I and cross-hatching, respectively. That $K$ shows for negative values of $\tau$ in the region II, according to equation (36a), also values different from zero, is insignificant, since the integral (36) has been defined originally only for positive values of $\xi$. It is shown, furthermore, that the integral (36) results in zero

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[10] Heine's integral representations for $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ are

$$H_n^{(1)}(z) = (-i)^{n+1}\frac{2}{\pi}\int_0^{\infty} e^{iz\cosh\vartheta}\cosh n\vartheta\,d\vartheta$$

$$H_n^{(2)}(z) = i^{n+1}\frac{2}{\pi}\int_0^{\infty} e^{-iz\cosh\vartheta}\cosh n\vartheta\,d\vartheta$$

with the first representation, for arbitrary $n$, being valid only for the upper (positive-imaginary), the second only for the lower (negative-imaginary) $z$ half-plane. Specially for $n = 0$ both representations are valid also on the real $z$ axis, which fact is made use of in the text. Compare, for instance, R. Weyrich, "Die Zylinderfunktionen und ihre Anwendungen", B. G. Teubner, 1937, p. 30.

in the regions III and IV. If one puts for this purpose

$$\xi = \tau' \sinh \psi \qquad \eta = \tau' \cosh \psi \qquad \eta^2 - \xi^2 = \tau'^2 \tag{37a}$$

($\tau' > 0$ in III, $\tau' < 0$ in IV), one obtains from equation (36), with consideration of

$$\sinh \vartheta \cosh \psi \pm \cosh \vartheta \sinh \psi = \sinh (\vartheta \pm \psi)$$

actually

$$K = \frac{1}{2i} \int_{-\infty}^{+\infty} \left\{ e^{i\kappa\tau' \sinh(\vartheta+\psi)} - e^{i\kappa\tau' \sinh(\vartheta-\psi)} \right\} d\vartheta = 0 \tag{39a}$$

The function $K$, defined by the integral (36), thus represents a discontinuous function which assumes values different from zero, only in the regions I and II, but disappears in III and IV.

Finally, one substitutes the expression for $K$ in $F$, equation (34). If the factors are combined, there results

$$F = \frac{1}{(2\pi)^2} \frac{e^{i\alpha_0 x}}{\sqrt{M^2 - 1}} (-2\pi K)$$

and hence after insertion of equations (37), (39), (32a), and (34a)

$$\left. \begin{aligned} & F = -\frac{1}{2} \frac{e^{i\frac{M}{M^3-1}kx}}{\sqrt{M^2 - 1}} J_0 \left\{ \frac{k}{\sqrt{M^2 - 1}} \sqrt{\frac{x^2}{M^2 - 1} - y^2} \right\} \\ & \text{for } |y| \leqq \frac{x}{\sqrt{M^2 - 1}} \text{ and } x > 0; \text{ otherwise } F = 0 \end{aligned} \right\} \tag{40}$$

The analytical result, that the sound propagation is limited to a conical region situated symmetrically to the x-axis, including the positive x-axis, has the simple physical significance: the appearance of the Mach angle $\alpha$. In fact, there results for the boundary of the propagation range for the sound excitation from Huyghens' elementary-wave construction immediately (compare fig. 7)

$$\sin a = \frac{c}{U} = \frac{1}{M} \qquad \tan a = \frac{|y|}{x} = \frac{1}{\sqrt{M^2 - 1}} \tag{41}$$

The wave system represented by equation (40) appears, in contrast to the relations in the elliptic case, as a standing-wave system with fixed nodal lines (Bessel function $J_0$) which is modulated by a progressing wave (exponential factor). One can explain this behavior in detail in the following manner. If one splits the Bessel function $J_0$ in equation (40), corresponding to equation (39), into the two Hankel functions $H_0^{(1)}$ and $H_0^{(2)}$ and uses their asymptotic representations[11] for sufficiently large values of the argument

$$\kappa\tau = \kappa\sqrt{\xi^2 - \eta^2} = \frac{k}{\sqrt{M^2 - 1}}\sqrt{\frac{x^2}{M^2 - 1} - y^2}$$

there results

$$Fe^{-i\omega t} \sim \frac{1}{\sqrt{\kappa\tau}} e^{i\frac{M}{M^2-1}kx} \left\{ e^{i\left(\frac{k}{\sqrt{M^2-1}}\sqrt{\frac{x^2}{M^2-1} - y^2} - \frac{\pi}{4}\right)} + e^{-i\left(\frac{k}{\sqrt{M^3-1}}\sqrt{\frac{x^2}{M^2-1} - y^2} - \frac{\pi}{4}\right)} \right\} e^{-i\omega t} \tag{40a}$$

and thus the space-time dependence of the phase becomes

$$\Phi_{1,2} = \frac{M}{M^2 - 1} kx \pm \frac{k}{\sqrt{M^2 - 1}}\sqrt{\frac{x^2}{M^2 - 1} - y^2} \mp \frac{\pi}{4} - \omega t \tag{42}$$

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[11] Compare, for instance, R. Weyrich, elsewhere, p. 46; compare also equation (21a) in the text.

with the indices 1, 2, and the upper and lower sign referring, respectively, to the origin in the Hankel functions $H_0^{(1)}$ and $H_0^{(2)}$. If one now considers for instance the propagation of the phases $\Phi_{1,2} = \mp \frac{\pi}{4}$, an elementary calculation yields just as in the elliptic case

$$(x - Ut)^2 + y^2 = c^2t^2 \tag{43}$$

The lines (or surfaces) with the constant phase values $\Phi_1 = -\frac{\pi}{4}$ and $\Phi_2 = +\frac{\pi}{4}$ fill, therefore, at any rate sectors of the same circle propagating at the velocity $c$, the center of which is propagated along the positive x-axis at the velocity $U > c$. It has to be noted that the argument $\kappa\tau$ of the Bessel or Hankel functions must always be positive. Hence, there results that, viewed by an observer located at large positive x, the first summand (in equation (40a)) represents a convex wave outgoing at a velocity $w_1 > U$, the second summand, a concave wave outgoing at a velocity $w_2 < U$ (fig. 8). By superposition of the two outgoing partial waves, the entire wave system originates. Thus, one may speak in the hyperbolic case of directed radiation.

It should be stressed that the circles moving away, which result after construction of the elementary waves, do no longer turn out to be curves of constant phase (for rigorous consideration). Rather, a phase shift $\Phi_1 - \Phi_2 = -\frac{\pi}{2}$ exists between the front and rear part of these curves (in the sense of the direction of propagation). The gradual phase change occurs in the neighborhood of the straight line bounding the Mach propagation range where the asymptotic representation of the Hankel functions is no longer sufficient (indicated by cross-hatching in fig. 9). These deviations from the elementary construction, which only the exact theory of the propagation phenomenon can disclose, are comparable to those occurring in the theory of the refraction of wave systems.

According to equation (40), the amplitude is constant on hyperbola branches $\kappa\tau$ = const, and decreases, according to equation (40a), like $1/\sqrt{\kappa\tau}$ toward the interior of the Mach region. On the straight lines bounding the propagation range themselves, there occurs a finite jump of F (pressure drop, compression shock) which is the larger the more closely U approaches the sonic velocity c.

(b) Spatial Problem

In the transition from the plane to the spatial hyperbolic case there results from equations (13) and (13a) in analogy to equation (34) the triple integral

$$F = \frac{1}{(2\pi)^3} \frac{e^{i\alpha_0 x}}{\sqrt{M^2 - 1}} \iiint_{-\infty}^{+\infty} \frac{e^{i(\lambda\xi+\mu\eta+\nu\zeta)}}{\lambda^2 - \mu^2 - \nu^2 - \kappa^2} d\lambda \, d\mu \, d\nu \tag{44}$$

with the same significance of $\alpha_0$, $\xi$, . . ., $\lambda$, . . . as in equations (32a) and (33); furthermore, we put for reasons of symmetry $\gamma = \nu$, $z = \zeta$. The integration over $\lambda$ corresponds exactly to that of equation (35); likewise, the integration path is to be selected in the same manner as sub a), with the same motivation as in the case of the plane problem. One obtains therewith

$$\int_{-\infty}^{+\infty} \frac{e^{i\gamma\xi}}{\lambda^2 - \mu^2 - \nu^2 - \kappa^2} d\lambda = \begin{cases} -2\pi \dfrac{\sin \xi\sqrt{\mu^2 + \nu^2 + \kappa^2}}{\sqrt{\mu^2 + \nu^2 + \kappa^2}} & \text{for } \xi > 0 \\ 0 & \text{for } \xi < 0 \end{cases}$$

If one now introduces in the $\mu,\nu$ plane, as well as in the $\eta,\zeta$ plane, polar coordinates by putting

$$\mu = \sigma \cos \psi \qquad \nu = \sigma \sin \psi \qquad \mu^2 + \nu^2 = \sigma^2$$

$$\eta = \rho \cos \varphi \qquad \zeta = \rho \sin \varphi \qquad \eta^2 + \zeta^2 = \rho^2$$

$F$ becomes

$$F = \frac{-1}{(2\pi)^2} \frac{e^{i\alpha_0 x}}{\sqrt{M^2 - 1}} \int_0^{\infty} \int_0^{2\alpha} e^{i\rho\sigma \cos (\psi-\varphi)} \frac{\sin \xi \sqrt{\sigma^2 + \kappa^2}}{\sqrt{\sigma^2 + \kappa^2}} \sigma \, d\sigma \, d\psi$$

If one now again makes use of the integral representation of the Bessel function $J_0$ (equation (29)), integrating first with respect to $\psi$, one obtains further

$$F = -\frac{e^{i\alpha_0 x}}{\sqrt{M^2 - 1}} \frac{1}{2\pi} \int_0^\infty J_0(\rho\sigma) \frac{\sin \xi\sqrt{\sigma^2 + \kappa^2}}{\sqrt{\sigma^2 + \kappa^2}} \sigma \, d\sigma \tag{45}$$

The remaining, purely mathematical problem consists in the evaluation of the integral in equation (45). We shall solve this problem in the mathematical appendix (section 6, integrals with respect to cylinder functions) from general viewpoints. Here we should like to remark that the integration occurring in equation (45) is easily performed for the special cases $\rho = 0$ and $\kappa = 0$. For $\rho = 0$ one obtains, if one substitutes $\sqrt{\sigma^2 + \kappa^2} = \omega$ the expression which is, at first, indefinite

$$J_{\rho=0} = \int_0^\infty \frac{\sin \xi\sqrt{\sigma^2 + \kappa^2}}{\sqrt{\sigma^2 + \kappa^2}} \sigma \, d\sigma = \int_\kappa^\infty \sin(\omega\xi) \, d\omega \tag{46}$$

If one notes that in the integrand of equation (45) the (at first omitted) factor $J_0(\rho\sigma)$ has, due to its oscillatory character, a convergence-enforcing effect, and that for us only the limiting case of this integral for $\rho \to 0$ is of interest, one obtains therewith (for instance analytically, by introducing in the integrand of equation (46a) a convergence-enforcing factor $e^{-\beta\omega}$ and finally making the transition to the limits $\beta \to 0$) in a unique manner.

$$J_{\rho=0} = \frac{\cos \kappa\xi}{\xi} \tag{47a}$$

On the other hand, the integral originating for $\kappa = 0$ from equation (45)

$$J_{\kappa=0} = \int_0^\infty J_0(\rho\sigma) \sin(\rho\xi) \, d\sigma$$

may be easily calculated according to the residuum method, with introduction of the integral representation of $J_0$ and interchange of the integration sequences; the integral represents the discontinuous function[11]

[12]Cf. G. N. Watson, Theory of Bessel Functions, I. Aufl., s. 405, equation (6).

$$J_{\kappa=0} = \begin{cases} \dfrac{1}{\sqrt{\xi^2 - \rho^2}} & \text{for} \quad |\xi| > \rho(>0) \\ 0 & \text{for} \quad |\xi| < \rho \end{cases} \tag{47b}$$

Comparison of equations (47a) and (47b) with the corresponding formulas for the plane hyperbolic problem suggests generalization to arbitrary $\rho$ and $\kappa$

$$J = \int_0^\infty J_0(\rho\sigma) \frac{\sin \xi\sqrt{\sigma^2 + \kappa^2}}{\sqrt{\sigma^2 + \kappa^2}} \sigma \, d\sigma = \begin{cases} \dfrac{\cos \kappa\sqrt{\xi^2 - \rho^2}}{\sqrt{\xi^2 - \rho^2}} & \text{for} \quad |\xi| > \rho(>0) \\ 0 & \text{for} \quad |\xi| > \rho \end{cases} \tag{48}$$

In section 6 it will be proved that the performed generalization is actually justified.

By inserting equation (48) in equation (45), one may immediately go over to the final formula for the spatial hyperbolic case; one obtains

$$\left.\begin{aligned} &F = -\frac{1}{2\pi} \frac{e^{i\frac{M}{M^2-1}kx}}{\sqrt{M^2 - 1}} \frac{\cos\left\{\dfrac{k}{\sqrt{M^2 - 1}}\sqrt{\dfrac{x^2}{M^2 - 1} - y^2 - z^2}\right\}}{\sqrt{\dfrac{x^2}{M^2 - 1} - y^2 - z^2}} \\ &\text{for} \quad |y| \geqq \frac{\rho}{\sqrt{M^2 - 1}} \quad \text{and} \quad x > 0; \text{ otherwise} \quad F = 0 \end{aligned}\right\} \tag{49}$$

The discussion of the basic solution (49) in the spatial case is in all points corresponding to that for the plane problem. The wave propagation is restricted to the Mach cone. If one splits the standing wave system characterized by the factor $\cos\left\{\ldots\right\}$, modulated by the preceding complex phase factor again into its complex parts

$$\cos\left\{\kappa\sqrt{\xi^2 - \rho^2}\right\} = \frac{1}{2}\left(e^{i\kappa\sqrt{\xi^2-\rho^2}} + e^{-ix\sqrt{\xi^2-\rho^2}}\right)$$

the combination of the first summand with the preceding phase factor (plane wave) results after the transformation indicated above in an outgoing convex spherical wave, the combination of the second summand with the preceding phase factor, in contrast, in an outgoing concave spherical wave. Altogether there results again directed radiation. Differently from the plane problem, however, here the phase on a propagated spherical surface

$$\left\{x - U\,(t - \tau)\right\}^2 + y^2 + z^2 = c^2\,(t - \tau)^2 \tag{50}$$

is exact constant, not only asymptotically constant, since here the phase of the two partial systems is exactly defined by decomposition of the $\cos\left\{\ldots\right\}$ and obviously no phase shift occurs between the spherical zones pertaining to the partial systems. Accordingly, the Huyghens' construction of the elementary waves (insofar as it is to contain only a statement on the surfaces of constant phase) up to arbitrary proximity to the Mach cone is here justified.

The amplitude is constant on the hyperboloidal surfaces $\frac{x^2}{M^2 - 1} - y^2 - z^2 = \text{const}$ and becomes infinite approaching the Mach cone. On the x-axis $(y = z = 0)$, the amplitude decreases like $1/x$ the intensity thus like $1/x^2$. The occurrence of amplitudes which increase with approach to the Mach cone beyond all limits indicates that, rigorously speaking, the validity range of linear theory has been exceeded. In fact, the Mach cone represents a compression shock for a more exact representation of which the nonlinear hydrodynamic and thermodynamic equations must be used.

## 5. THE PARABOLIC LIMITING CASE: M = 1

### (a) Plane Problem

The quantity $\Omega_2$ here becomes according to equation (12a)

$$\Omega_2 = 2k\alpha + \beta^2 - k^2 \tag{51}$$

and after substitution of $\alpha = \alpha' + k/2$

$$\Omega_2 = 2k\alpha' + \beta^2 \tag{51a}$$

The characteristic equation $\Omega_2 = 0$ thus represents in the $\alpha',\beta$ plane a parabola which passes through the coordinate origin $\alpha' = 0$, $\beta = 0$ and is opened toward the negative $\alpha'$-axis. The fundamental solution F, equation (12), may be written

$$F = -\frac{1}{(2\pi)^2} e^{i\frac{k}{2}x} \int_{-\infty}^{+\infty} d\beta \; e^{i\beta y} \int_{-\infty}^{+\infty} d\alpha' \; \frac{e^{i\alpha' x}}{2k\alpha' + \beta^2} \tag{52}$$

The integration over $\alpha'$ may now easily be carried out according to the residuum method. Here again the choice of the integration path is decisive. We select it in such a manner that we integrate in the complex $\alpha'$-plane from $-\infty$ to $+\infty$ on leaving the pole of the integrand at $\alpha' = -\frac{\beta^2}{2k}$ to the left. Then we may replace the integration path, for positive values of x, by a circling of the pole $\alpha' = -\frac{\beta^2}{2k}$ in positive sense whereas for negative values of x, there results zero. Thus

$$\int_{-\infty}^{+\infty} \frac{e^{i\alpha' x}}{2k\alpha' + \beta^2} d\alpha' = \begin{cases} \frac{2\pi i}{2k} e^{-i\frac{\beta^2}{2k}x} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{53}$$

This result justifies subsequently our choice of the integration path. In fact, it is to be expected according to the principle of the elementary waves that for a sound source approached by a flow at the velocity $U = c$ no sound excitation is brought about for $x < 0$ since the sound propagation is carried along by the flow at the velocity c in the direction of the positive x-axis.

Substitution of equation (53) into equation (52) now yields for $x > 0$

$$F = -\frac{\pi i}{(2\pi)^2 k} e^{i\frac{k}{2}x} \int_{-\infty}^{+\infty} e^{i\left(\beta y - \frac{\beta^2}{2k}x\right)} d\beta \tag{52a}$$

By quadratic supplement of the exponent, one readily finds hence

$$F = \frac{-i}{4\pi k}\sqrt{\frac{2k}{x}}\; e^{i\frac{k}{2}\frac{x^2+y^2}{x}} \int_{-\infty}^{+\infty} e^{-i\sigma^2} d\sigma \tag{52b}$$

The integral appearing here is well known as "Fresnel's Integral"

$$\int_{-\infty}^{+\infty} e^{-i\sigma^2}\, d\sigma = \sqrt{\frac{\pi}{2}}\,(1 - i) = \sqrt{\pi}\, e^{-i\frac{\pi}{4}}$$

Therewith, one finally obtains

$$F = \begin{cases} \dfrac{e^{-i\frac{3\pi}{4}}}{2\sqrt{2\pi}} \dfrac{e^{i\frac{k}{2}\frac{x^2+y^2}{x}}}{\sqrt{kx}} & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \tag{54}$$

First, one considers again the curves (or surfaces, respectively) of constant phase. If one inserts in the expression for the phase

$$\Phi = \frac{k}{2}\frac{x^2 + y^2}{x} - \frac{3\pi}{4} - \omega t$$

$$\Phi + \frac{3\pi}{4} = \Phi' = \omega\tau$$

there results because of $k = \omega/c$ immediately

$$\left\{x - c\,(t - \tau)\right\}^2 + y^2 = c^2\,(t - \tau)^2 \tag{55}$$

Thus, the surfaces $\Phi$ = const ($\tau$ = const) form, as is to be expected according to the construction of the elementary waves, a system of circles which are tangent at the coordinate origin $x = y = 0$ and the centers of which fill the positive x-axis (fig. 10). The propagation of the phase surfaces in the medium takes place to all sides and up to arbitrary proximity to the sound source at the velocity $c$. On the other hand, it is noteworthy that the decrease of the amplitude over the entire front occurs independently of $y$ with $1/\sqrt{x}$. With the approach of positive x-values to $x = 0$ the amplitude increases beyond all limits; for $x < 0$ it is zero.

(b) The Spatial Problem

$\Omega_3$ becomes according to equation (13a) and with replacement of $\alpha$ by $\alpha'$ as in the plane case

$$\Omega_3 = 2k\alpha + \beta^2 + \gamma^2 - k^2 = 2k\alpha' + \beta^2 + \gamma^2 \tag{56}$$

therewith the fundamental integral (13), if one visualizes the integration over $\alpha'$ carried out in a manner exactly corresponding to equations (52) and (53)

$$\left.\begin{aligned} F &= -\frac{1}{(2\pi)^3} e^{i\frac{k}{2}x} \iint_{-\infty}^{+\infty} d\beta\, d\gamma\, e^{i(\beta y+\gamma z)} \int_{-\infty}^{+\infty} d\alpha' \frac{e^{i\alpha' x}}{2k\alpha' + \beta^2 + \gamma^2} \\ &= \begin{cases} -\dfrac{\pi i}{(2\pi)^3 k} e^{i\frac{k}{2}x} \displaystyle\iint_{-\infty}^{+\infty} e^{i\left(\beta y+\gamma z - \frac{\beta^2+\gamma^2}{2k}x\right)} d\beta\, d\gamma & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \end{aligned}\right\} \tag{57}$$

If one uses furthermore polar coordinates in the $\beta,\gamma$ plane just as in the $y,z$ plane

$$\left.\begin{aligned} \beta &= \sigma \cos \psi & \gamma &= \sigma \sin \psi & \beta^2 + \gamma^2 &= \sigma^2 \\ y &= \rho \cos \varphi & z &= \rho \sin \varphi & y^2 + z^2 &= \rho^2 \end{aligned}\right\} \tag{58}$$

one may write the integral (57) for $x > 0$

$$F = -\frac{\pi i}{(2\pi)^3 k} e^{i\frac{k}{2}x} \int_0^{\infty}\int_0^{2\pi} e^{i\sigma\rho \cos (\psi-\varphi) - i\frac{\sigma^2}{2k}x}\, \sigma\, d\sigma\, d\psi \tag{57a}$$

First performing the integration over $\psi$, one obtains with consideration of the integral representation of the Bessel function $J_0$ equation (29), from equation (57a)

$$F = -\frac{i}{4\pi k} e^{i\frac{k}{2}x} \int_0^\infty J_0(\rho\sigma) e^{-i\frac{x}{2k}\sigma^2} \sigma \, d\sigma \tag{57b}$$

The integral which appears here is of the type of the integral named after H. Weber[12]

$$\int_0^\infty J_0(at)\, e^{-b^2t^2}\, t\, dt = \frac{1}{2b^2} e^{-\frac{a^2}{4b^2}}$$

and may therefore be reduced to an elementary function; for $a = \rho$, $b^2 = i\,\frac{x}{2k}$[13]

there results

$$\int_0^\infty J_0(\rho\sigma)\, e^{-i\frac{x}{2k}\sigma^2}\, \sigma\, d\sigma = \frac{k}{ix} e^{i\frac{k}{2}\frac{\rho^2}{x}}$$

and finally

$$F = \begin{cases} -\dfrac{1}{4\pi} \dfrac{e^{i\frac{k}{2}\frac{x^2+\rho^2}{x}}}{x} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{59}$$

The structure of this formula is perfectly analogous to that of equation (54) for the plane problem. The surfaces of constant phase are spherical surfaces which are all tangent at the point $x = \rho = 0$ and the centers of which again fill the positive x-axis. The amplitude is constant on the planes $x = \text{const}$ (thus independent of $\rho$) and, corresponding to the intensity relations for the spatial problem, decreases for positive $x$ as $1/x$.

Furthermore, the analogy between equations (54) and (59) and the basic solution (19b) of the plane problem (compare also the asymptotic representation (equation (21a)) and of the corresponding spatial problem

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[13]G. N. Watson, Theory of Bessel-Functions, Cambridge 1922, p. 392.

[14]Convergence of the integral exists as long as $|\arg b| \leq \frac{\pi}{4}$.

for the case $M = 0$ of disappearing approach flow is remarkable. Finally, it should be noted that the formulas (54) and (58) as limiting cases $M \to 1$ can be represented only from the hyperbolic case $M > 1$, not the elliptic $M < 1$.

## 6. MATHEMATICAL APPENDIX

### Integrals Over Cylinder Functions

It remains to append the proof for the integral formula (48). Since this proof requires a few more general considerations, we shall, in connection with it, derive a few more related integral formulas.

We start from Heine-Schafheitlin's integral representation (38) of the Hankel functions $H_0^{(1)}$ and $H_0^{(2)}$ which we write

$$\int_{-\infty}^{+\infty} e^{iz\cosh\vartheta}\,d\vartheta = i\pi H_0^{(1)}(z) \qquad \int_{-\infty}^{+\infty} e^{-iz\cosh\vartheta}\,d\vartheta = -i\pi H_0^{(2)}(z) \tag{38}$$

We now consider the integrals

$$K_1 = \int_{+\infty}^{-\infty} \frac{e^{+i\sqrt{\mu^2+\kappa^2}\xi}}{\sqrt{\mu^2+\kappa^2}} e^{+i\mu\eta}\,d\mu \qquad K_2 = \int_{+\infty}^{-\infty} \frac{e^{-i\sqrt{\mu^2+\kappa^2}\xi}}{\sqrt{\mu^2+\kappa^2}} e^{+i\mu\eta}\,d\mu \tag{60a, b}$$

As on page 18 we substitute

$$\left.\begin{array}{lll} \sqrt{\mu^2+\kappa^2} = \kappa\cosh\vartheta & \mu = \kappa\sinh\vartheta & \dfrac{d\mu}{\sqrt{\mu^2+\kappa^2}} = d\vartheta \\ \xi = \tau\cosh\chi & \eta = \tau\sinh\chi & \xi^2 - \eta^2 = \tau^2 \end{array}\right\} \tag{37}$$

and thus obtain, according to the addition theorem of the hyperbolic functions and to the integral formulas (38)

$$K_1 = \int_{-\infty}^{+\infty} e^{+i\kappa\tau\cosh(\vartheta+\chi)}\,d\vartheta = i\pi H_0^{(1)}\left(\kappa\sqrt{\xi^2-\eta^2}\right) \tag{61a}$$

$$K_2 = \int_{-\infty}^{+\infty} e^{-i\kappa\tau \cosh(\vartheta - \chi)} \, d\vartheta = -i\pi H_0^{(2)}\left(\kappa\sqrt{\xi^2 - \eta^2}\right) \tag{61b}$$

both formulas are valid for the range of representation of the coordinates $\xi$, $\eta$ by the equation (37) ($\chi$ real), that is, for $|\xi| \geqq |\eta|$. In exactly the same manner as on page 20, one proves that $K_1$ and $K_2$ disappear for $|\xi| < |\eta|$. The integrals (60a) and (60b) thus represent discontinuous functions which assume values different from zero only in the regions I and II of figure 8, but disappear in the regions III and IV. Thus we have

$$\int_{-\infty}^{+\infty} \frac{e^{+i\sqrt{\mu^2+\kappa^2}\xi}}{\sqrt{\mu^2 + \kappa^2}} e^{i\mu\eta} \, d\mu = \begin{cases} i\pi H_0^{(1)}\left(\kappa\sqrt{\xi^2 - \eta^2}\right) & \text{for } |\xi| \geqq |\eta| \\ 0 & \text{for } |\xi| < |\eta| \end{cases} \tag{62a}$$

$$\int_{-\infty}^{+\infty} \frac{e^{-i\sqrt{\mu^2+\kappa^2}\xi}}{\sqrt{\mu^2 + \kappa^2}} e^{i\mu\eta} \, d\mu = \begin{cases} -i\pi H_0^{(2)}\left(\kappa\sqrt{\xi^2 - \eta^2}\right) & \text{for } |\xi| \geqq |\eta| \\ 0 & \text{for } |\xi| < |\eta| \end{cases} \tag{62b}$$

In order to indicate the discontinuous character of the integrals $K_1$ and $K_2$ in the notation, we shall write below instead of the right sides of equations (62a) and (62b) abbreviatedly $i\pi\bar{H}_0^{(1)}\left(\kappa\sqrt{\xi^2 - \eta^2}\right)$ and $-i\pi\bar{H}_0^{(2)}\left(\kappa\sqrt{\xi^2 - \eta^2}\right)$, indicating by a bar above the symbol that the Hankel functions are to be applied only for $|\xi| \geqq |\eta|$, however, zero.

We now use a new integration variable $\nu$ in equations (62a) and (62b) by putting

$$+\sqrt{\mu^2 + \kappa^2} = \nu \geqq \kappa \qquad \mu = \pm\sqrt{\nu^2 - \kappa^2} \gtreqless 0$$

We thus obtain

$$K_1 = \int_{-\infty}^{+\infty} \ldots d\mu = \int_{-\infty}^{0} \ldots d\mu + \int_{0}^{+} \ldots d\mu$$

$$= -\int_{+\infty}^{\kappa} \frac{e^{-i\sqrt{\nu^2-\kappa^2}\eta}}{\sqrt{\nu^2 - \kappa^2}} e^{i\nu\xi}\, d\nu + \int_{\kappa}^{+\infty} \frac{e^{+i\sqrt{\nu^2-\kappa^2}\eta}}{\sqrt{\nu^2 - \kappa^2}} e^{i\nu\xi}\, d\nu$$

or else

$$K_1 = 2\int_{\kappa}^{\infty} \frac{\cos \eta\sqrt{\nu^2 - \kappa^2}}{\sqrt{\nu^2 - \kappa^2}} e^{+i\nu\xi}\, d\nu = i\pi\overline{H}_0^{(1)}\left(\kappa\sqrt{\xi^2 - \eta^2}\right) \tag{63a}$$

In exact analogy

$$K_2 = 2\int_{\kappa}^{\infty} \frac{\cos \eta\sqrt{\nu^2 - \kappa^2}}{\sqrt{\nu^2 - \kappa^2}} e^{-i\nu\xi}\, d\nu = -i\pi\overline{H}_0^{(2)}\left(\kappa\sqrt{\xi^2 - \eta^2}\right) \tag{63b}$$

By addition and subtraction of equations (63a) and (63b) there results with consideration of the mutual connection of the cylinder functions

$$J_0(z) = \frac{1}{2}\left\{H_0^{(1)}(z) + H_0^{(2)}(z)\right\} \qquad N_0(z) = \frac{1}{2i}\left\{H_0^{(1)}(z) - H_0^{(2)}(z)\right\}$$

($J_0$ Bessel function, $N_0$ Neumann function of zero order), then further

$$\int_{\kappa}^{\infty} \frac{\cos \eta\sqrt{\nu^2 - \kappa^2}}{\sqrt{\nu^2 - \kappa^2}} \cos \nu\xi\, d\nu = -\frac{\pi}{2}\overline{N}_0\left(\kappa\sqrt{\xi^2 - \eta^2}\right) \tag{64a}$$

$$\int_{\kappa}^{\infty} \frac{\cos \eta\sqrt{\nu^2 - \kappa^2}}{\sqrt{\nu^2 - \kappa^2}} \sin \nu\xi\, d\nu = +\frac{\pi}{2}\overline{J}_0\left(\kappa\sqrt{\xi^2 - \eta^2}\right) \tag{64b}$$

(the bar over the symbol having the same significance as above). By application of Fourier's inversion formulas

$$g_1(u) = \int_0^\infty f_1(t) \cos ut \, dt \qquad f_1(t) = \frac{2}{\pi}\int_0^\infty g_1(u) \cos ut \, du$$

$$g_2(u) = \int_0^\infty f_2(t) \sin ut \, dt \qquad f_2(t) = \frac{2}{\pi}\int_0^\infty g_2(u) \sin ut \, du$$

one now obtains from equations (64a) and (64b)

$$\int_{|\eta|}^{+\infty} \overline{N}_0\left(\kappa\sqrt{\xi^2 - \eta^2}\right) \cos \nu\xi \, d\xi = \begin{cases} -\dfrac{\cos \eta\sqrt{\nu^2 - \kappa^2}}{\sqrt{\nu^2 - \kappa^2}} & \text{for} \quad |\nu| \geqq |\kappa| \\ 0 & \text{for} \quad |\nu| < |\kappa| \end{cases} \tag{65a}$$

$$\int_{|\eta|}^{+\infty} \overline{J}_0\left(\kappa\sqrt{\xi^2 - \eta^2}\right) \sin \nu\xi \, d\xi = \begin{cases} +\dfrac{\cos \eta\sqrt{\nu^2 - \kappa^2}}{\sqrt{\nu^2 - \kappa^2}} & \text{for} \quad |\nu| \geqq |\kappa| \\ 0 & \text{for} \quad |\nu| < |\kappa| \end{cases} \tag{65b}$$

If we finally put

$$+\sqrt{\xi^2 - \eta^2} = \tau \qquad \xi = +\sqrt{\tau^2 + \eta^2} \qquad d\xi = \frac{\tau \, d\tau}{\sqrt{\tau^2 + \eta^2}}$$

the integral formulas (65a) and (65b) assume the form

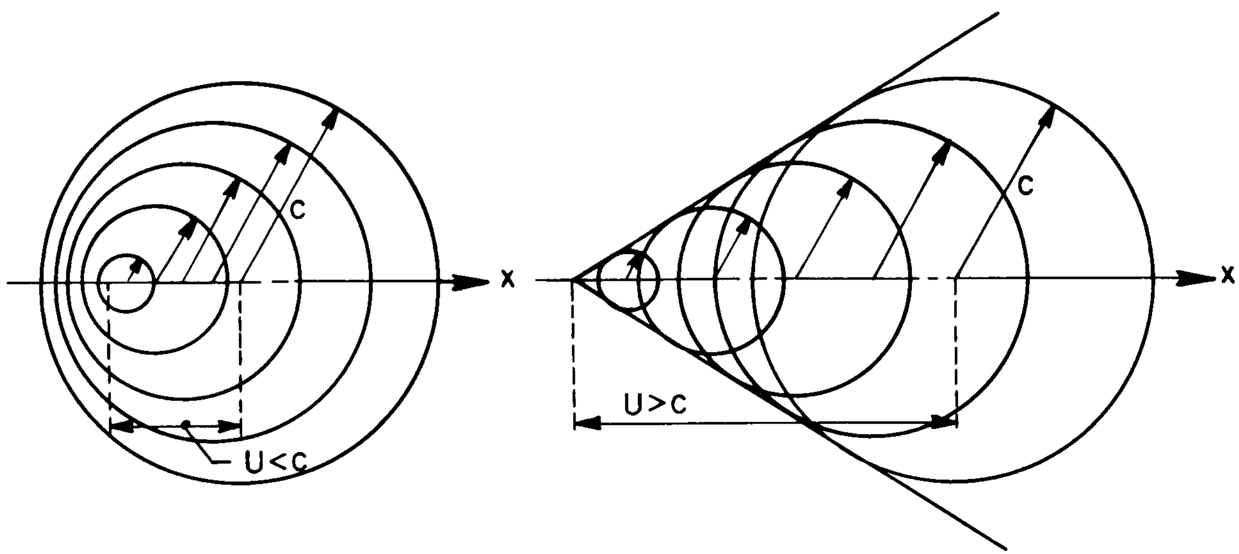
$$\left.\begin{aligned} \int_0^\infty J_0(\kappa\tau) \frac{\sin \nu\sqrt{\tau^2 + \eta^2}}{\sqrt{\tau^2 + \eta^2}} \tau \, d\tau &= -\int_0^\infty N_0(\kappa\tau) \frac{\cos \nu\sqrt{\tau^2 + \eta^2}}{\sqrt{\tau^2 + \eta^2}} \tau \, d\tau \\ &= \begin{cases} \dfrac{\cos \eta\sqrt{\nu^2 - \kappa^2}}{\sqrt{\nu^2 - \kappa^2}} & \text{for} \quad |\nu| \geqq \kappa > 0 \\ 0 & \text{for} \quad |\nu| < \kappa \end{cases} \end{aligned}\right\} \tag{66}$$

The first of the integral formulas (66) is identical with equation (48) as one recognizes immediately if one makes in the designation the replacements

$$\kappa \longrightarrow \rho \qquad \nu \longrightarrow \xi \qquad \tau \longrightarrow \sigma \qquad \eta \longrightarrow \kappa$$

quod erat demonstrandum.

Translation by Mary L. Mahler
National Advisory Committee
for Aeronautics




(a) Subsonic velocity.

(b) Supersonic velocity.

Figure 1.- Propagation of Huyghens' elementary waves for a source moved at subsonic velocity and at supersonic velocity.

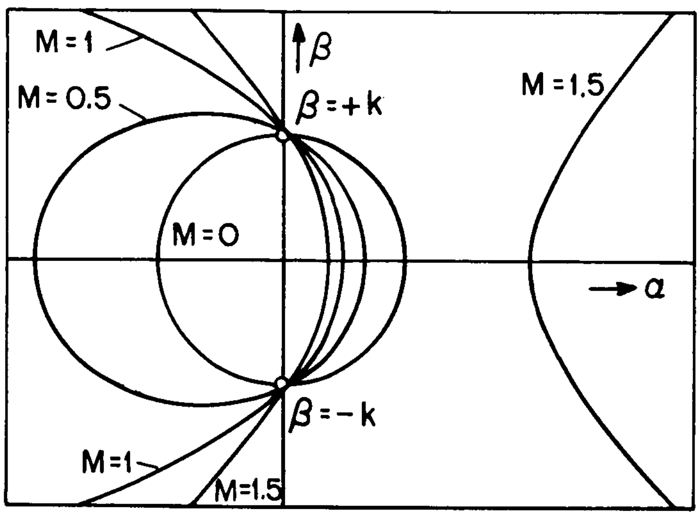



Figure 2.- The characteristic conic section $\Omega_2(\alpha,\beta) = 0$ for various values of the Mach number $M$.

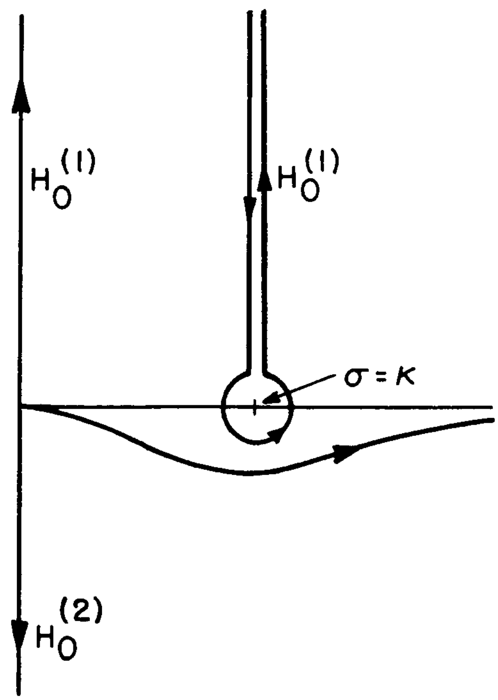



Figure 3.- Complex $\sigma$-plane. Integration paths for the constituents of the integrand of (19a) corresponding to $J_0$, $H_0^{(1)}$, and $H_0^{(2)}$.

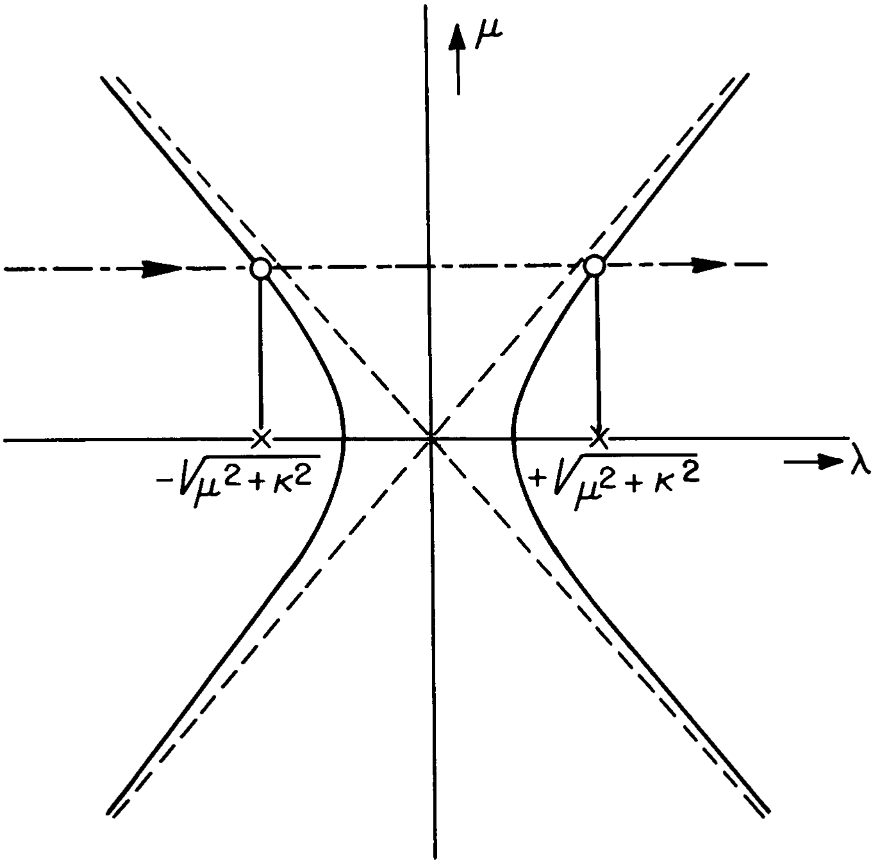



Figure 4.- Position of the characteristic hyperbola in the $\lambda$, $\mu$-plane.

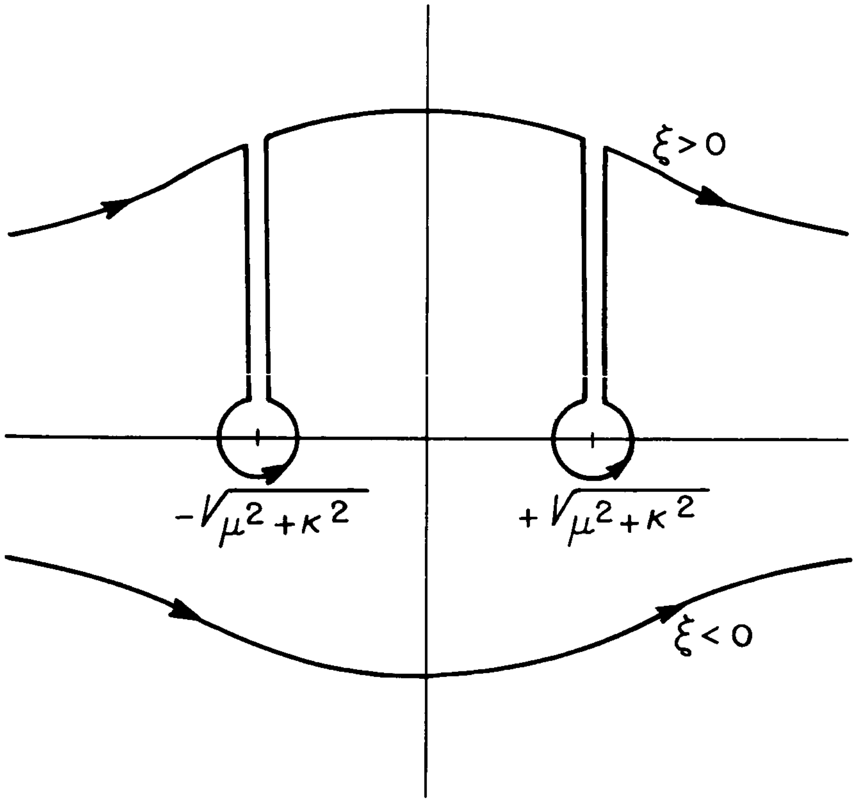



Figure 5.- Integration paths in the complex $\lambda$-plane for $\xi > 0$ and $\xi < 0$.

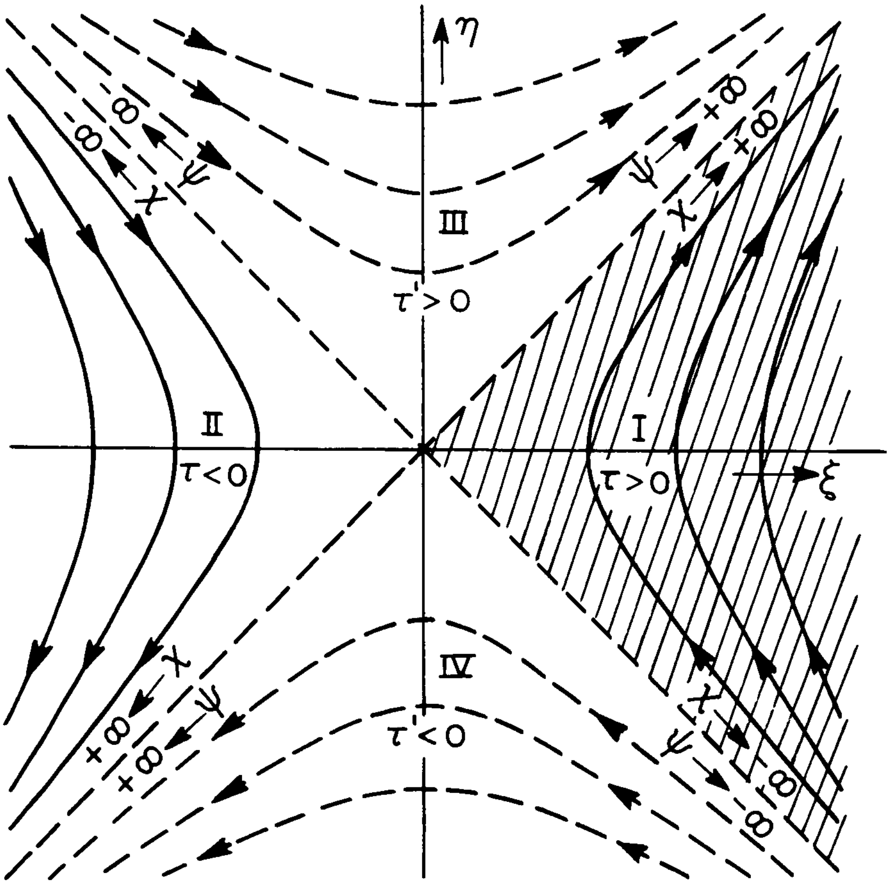



Figure 6.- Division into regions in the $\xi$, $\eta$-plane. Plotted hyperbolas: $\tau$ = const (I and II), $\tau'$ = const (III and IV).

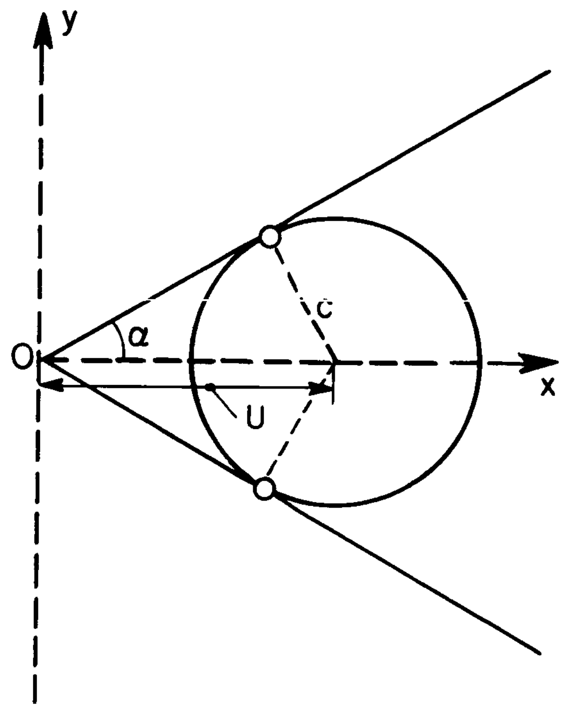



Figure 7.- Construction of the Mach angle.

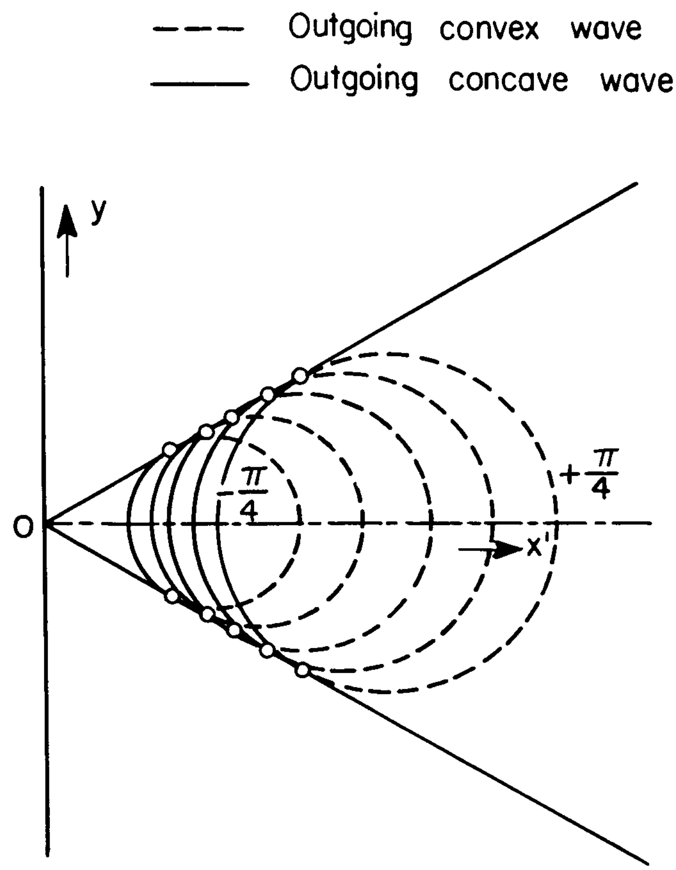



Figure 8.- Splitting of the wave system into two partial systems.

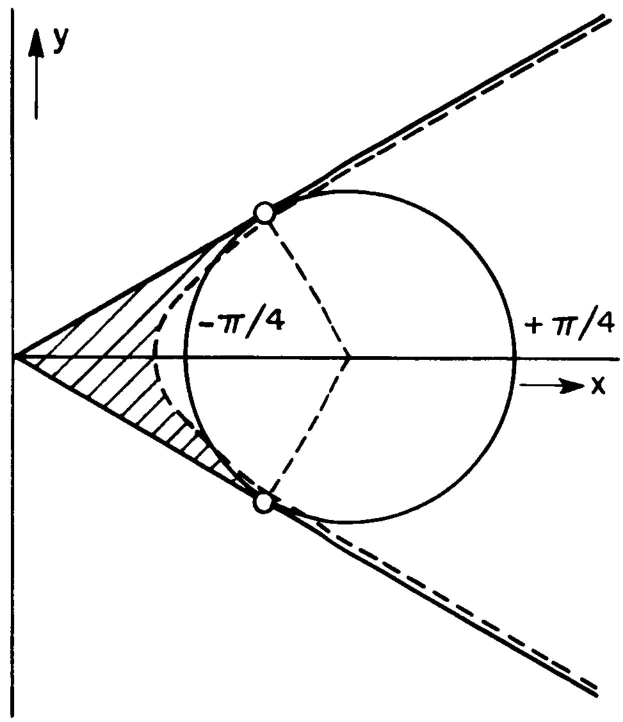



Figure 9.- Region of transition (cross-hatched) from $\Phi = -\pi/4$ to $\Phi = +\pi/4$ for the plane problem.

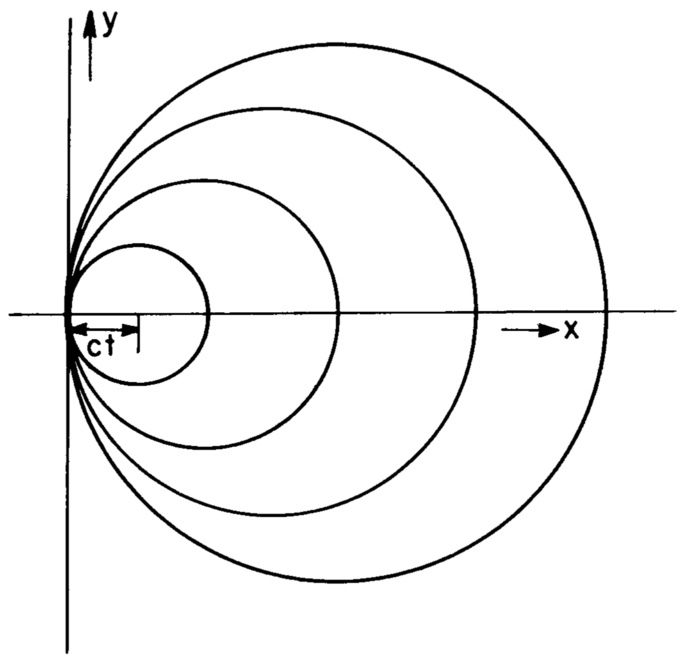



Figure 10.- Propagation of the surfaces (or waves, respectively) $\Phi$ = const.