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MOTION OF A CYLINDER UNDER THE SURFACE OF A HEAVY FLUID

By L. N. Sretensky

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MOTION OF A CYLINDER UNDER THE SURFACE OF A HEAVY FLUID*

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1. INTRODUCTION

The present work on the theory of the motion of a solid body in a fluid having a free surface consists of two parts.

In the first part (sections 2 to 8), general equations are given for the determination of the flow of a heavy fluid of infinite depth about a submerged circular cylinder. The problem of the motion of a cylinder under the surface of a heavy fluid presents considerable difficulties in its solution. These difficulties were first pointed out by Kelvin. A solution is given herein for the simplest part of the problem of Kelvin, namely, setting up the equations of the problem and obtaining certain approximations of their solution. The approximate solution obtained replaces the moving circular cylinder by a certain vortex.

T. H. Havelock (reference 1) in a recent paper considers the problem of Kelvin under the same general assumptions as are herein considered, but gives a more advanced approximate solution.

In the second part of the paper (sections 9 and 10), the cylinder is replaced by a dipole of a certain strength, and an equation is set up for the computation of the wave resistance of a circular cylinder moving in a fluid of finite depth.

2. DERIVATION OF BOUNDARY CONDITIONS OF PROBLEM

In order to study the problem of the motion of a cylinder under the surface of a fluid, a system of Cartesian coordinates XOY is introduced. The OX-axis is placed along the undisturbed surface of the liquid with its positive direction coinciding with the velocity c of

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*"Dvizhenie tsilindra pod poverkhnostyu tyazheloi zhidkosti." CAHI Rep. No. 346, 1938, pp. 3-27.

the flow at infinity, and the OY-axis is taken vertically upward. The radius of the cylinder is denoted by a; and the ordinate of its center by h. The origin of the XOY system is chosen directly above the center of the cross section of the cylinder.

The fluid moves with the velocity potential Φ and the stream function ψ. The disturbance (due to the presence of the cylinder) of the horizontal projection of the fundamental flow velocity c is denoted by u; the disturbance of the vertical projection, by v. If c is the horizontal velocity of the flow at infinity, the following expression can be obtained for the projection of the total velocity v of the particles of the fluid:

$$u+c,\ v \tag{1}$$

The equation of Bernoulli can now be written, if the motion is assumed to be steady:

$$\frac{p}{\delta}=C-gy-\frac{1}{2}V^2 \tag{2}$$

where δ is the mass density of the fluid.

Along the free surface the pressure has the constant value p_0. Equation (2) is applied to the particles of the free surface lying far ahead of the cylinder, that is, to those for which $x = -\infty$. For $x = -\infty$, $V = c$ and $y = 0$; therefore,

$$\frac{p_0}{\delta}=C-\frac{1}{2}c^2$$

Equation (2) can then be written

$$\frac{p-p_0}{\delta}=-gy-\frac{1}{2}(V^2-c^2)$$

But

$$V^2=(u+c)^2+v^2$$

hence

$$\frac{p-p_0}{\delta}=-gy-\frac{1}{2}[u^2+v^2+2cu] \tag{3}$$

The increments u and v are now assumed to be so small that their squares may be neglected. With this assumption, the accurate equation (3) becomes the approximate equation

$$\frac{p-p_0}{\delta}=-gy-cu$$

When this equation is applied to the free surface,

$$y = -\frac{c}{g} u \tag{4}$$

This equation can be used to determine the shape of the disturbed surface of the fluid.

The fact that the velocity of the particles moving along the surface is directed along the tangent to the surface is now taken into consideration:

$$\frac{dy}{dx} = \frac{v}{u + c}$$

This relation may be represented in the form

$$\frac{dy}{dx} = \frac{v}{c}\left[1 - \frac{u}{c} + \frac{u^2}{c^2} - \ldots\right]$$

If the terms of the second and higher order smallness are rejected,

$$\frac{dy}{dx} = \frac{v}{c} \tag{5}$$

Eliminating the ordinate y by combining equations (4) and (5) yields

$$\frac{\partial u}{\partial x} = -\frac{g}{c^2} v \tag{6}$$

By introducing the potential $\varphi(x,y)$ and the stream function $\psi(x,y)$ of the disturbed velocities, this condition may be given another form:

$$u = -\frac{\partial \varphi}{\partial x} = -\frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \varphi}{\partial y} = \frac{\partial \psi}{\partial x} \tag{7}$$

The functions φ and ψ are harmonic conjugate functions. Substituting expressions (7) in condition (6) yields

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{g}{c^2} \frac{\partial \psi}{\partial x} \tag{8}$$

A second simplification of the problem is now introduced by requiring that y be replaced by zero. Equation (8) will then be satisfied along the axis of abscissas. Integrating along this axis gives

$$\frac{\partial\psi}{\partial y}=\frac{g}{c^2}\psi \tag{9}$$

where the constant of integration is zero from the consideration that the zero value of the stream function Ψ was ascribed to the surface, and moreover, for $x = -\infty$, $u = -\partial\psi/\partial y = 0$.

Condition (9) is the first boundary condition of the problem. The conditions on the surface of the cylinder will now be discussed. Let the cylinder be washed by the streamline $\Psi = \alpha$. Since the relation between Ψ and ψ is given by

$$\Psi = cy + \psi$$

the condition $\Psi = \alpha$ on the cylinder will take the form

$$\psi = \alpha + cy$$

where $\alpha > 0$.

The determination of the motion of the fluid thus depends on the integration of the Laplace equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

for the following boundary conditions:

$$\frac{\partial\psi}{\partial y}=\frac{g}{c^2}\psi \qquad \text{for } y=0 \tag{I}$$

$$\psi=\alpha+cy \qquad \text{for } x^2+(y+h)^2=a^2 \tag{II}$$

3. TRANSFORMATION OF FLOW REGION INTO CIRCULAR RING

In order to obtain the integral of the Laplace equation corresponding to conditions (I) and (II), a conformal mapping of the complex variable plane $z = x + iz$ on the plane $\zeta = \xi + i\eta$ is carried out by setting

$$\zeta = \frac{z - \lambda i}{z + \lambda i} \tag{1}$$

where

$$\lambda = -\sqrt{h^2 - a^2} < 0 \tag{2}$$

When the complex variable z is varied along the OX-axis from $-\infty$ to $+\infty$ the point ζ describes, starting from the point $\zeta = 1$, the circle $\xi^2 + \eta^2 = 1$ in the clockwise direction.

If $z = -hi + ae^{\gamma i}$ is set next in formula (1) and the angle γ is varied from zero to 2π, the surface of the cylinder will correspond to the circle $\xi^2 + \eta^2 = \rho^2$ of the plane of the complex variable ζ. The radius ρ of this circle is determined in terms of a and h by the formula

$$\rho = \frac{h - \sqrt{h^2 - a^2}}{a} \tag{3}$$

or

$$\rho = \frac{h + \lambda}{a} \tag{3'}$$

The number ρ is evidently less than one; $h > a$.

The region occupied by the flow is thus transformed into the space enclosed between the two concentric circles $|\zeta| = 1$ and $|\xi| = \rho$.

The numbers λ and ρ introduced previously have simple geometrical meanings: λ is the length of the tangent drawn from the origin of coordinates to the cylinder, and ρ is the tangent of the quarter angle Ω subtended by the cylinder at the origin.

From formula (1) z is determined:

$$z = \lambda i \frac{1 + \zeta}{1 - \zeta} \tag{4}$$

and from this value are found the values of x and y in terms of the argument ω and the modulus r of the complex number ζ:

$$x = -\frac{2\lambda r \sin\omega}{1 - 2r\cos\omega + r^2}$$

$$y = \frac{\lambda(1 - r^2)}{1 - 2r\cos\omega + r^2} \tag{5}$$

$$r\sin\omega = -\frac{2\lambda x}{x^2 + (y+\lambda)^2}$$

$$r\cos\omega = \frac{x^2 + y^2 - \lambda^2}{x^2 + (y+\lambda)^2} \tag{5'}$$

The transformation of boundary conditions (I) and (II) of the problem follow.

Condition (II) will be considered first. Formula (5) shows that this condition may be written as

$$[\psi]_{r=\rho} = \alpha + \frac{c\lambda(1-\rho^2)}{1 - 2\rho\cos\omega + \rho^2} \tag{II'}$$

In transforming condition (I) it is noted that from formula (5) there follows for $y = 0$

$$dy = -\frac{\lambda dr}{1 - \cos\omega}$$

from which condition (I) assumes the following form:

$$\left[\frac{\partial\psi}{\partial r} = \frac{\mu\psi}{1 - \cos\omega}\right]_{r=1} \tag{I'}$$

where

$$\mu = -\frac{g\lambda}{c^2} > 0$$

Conditions (II') and (I') may be given another form by introducing the complex stream function:

$$w = \varphi + i\psi \tag{6}$$

The conditions then assume the following forms:

$$\text{Imag.}\left[2\mu w + (1-\zeta)^2 \frac{dw}{d\zeta}\right] = 0 \quad \text{for } |\zeta| = 1 \qquad (\text{I}'')$$

$$\text{Imag.}(w - cz) = \alpha \quad \text{for } |\zeta| = \rho \qquad (\text{II}'')$$

The function w is holomorphic at all points of the ring $\rho < |\zeta| < 1$, but it will be assumed that the flow around the cylinder has a definite circulation. The function w will therefore not be single-valued in the ring under consideration.

In order to obtain the required function $w(\zeta)$, the form of the following infinite series will be used:

$$w = q \ln \zeta + \sum_{m=-\infty}^{+\infty} p_m \zeta^m \qquad (7)$$

with the undetermined coefficients p_m and q.

4. COMPUTATION OF COEFFICIENTS OF SERIES REPRESENTING FUNCTION $w(\zeta)$

First considered are the relations obtained when condition (II'') is required to be satisfied by the function $w(\zeta)$.

In the notation

$$p_m = A_m + B_m i$$

$$q = -\frac{\Gamma}{2\pi i} \qquad (8)$$

the coefficients A_m and B_m are real numbers, and Γ is the circulation of the flow about the cylinder.

Condition (II'') then easily transforms into

$$\frac{\Gamma}{2\pi}\ln\rho+\sum_{m=-\infty}^{+\infty}\rho^m\left(B_m\cos m\omega+A_m\sin m\omega\right)-\lambda cR\left(1+2\zeta+2\zeta^2+\ .\ .\ .\ \right)=\alpha$$

from which the following relations are obtained:

$$\frac{\Gamma}{2\pi}\ln\rho+B_0=\lambda c+\alpha \tag{9}$$

$$\left.\begin{aligned}\rho^m B_m+\frac{B_{-m}}{\rho^m}&=2\lambda c\rho^m\\ \rho^m A_m-\frac{A_{-m}}{\rho^m}&=0\end{aligned}\right\}m=1,\ 2,\ 3... \tag{10}$$

Condition (I'') is now considered. When the following formulas are used

$$\text{Imag. }[2\mu w]=2\mu B_0+2\mu\sum_{m=1}^{+\infty}(B_m+B_{-m})\cos m\omega+(A_m-A_{-m})\sin m\omega$$

$$\text{Imag. }\left[(1-\zeta)^2\frac{dw}{d\zeta}\right]=-\frac{\Gamma}{\pi}+\text{Imag}\cdot\frac{\Gamma i}{2\pi}\left(\zeta+\frac{1}{\zeta}\right)+$$

$$\text{Imag.}\sum_{m=-\infty}^{+\infty}\left\{(m+1)p_{m+1}-2mp_m+(m-1)p_{m-1}\right\}\zeta^m$$

no difficulty is encountered in representing the condition

$$\text{Imag.}\left[2\mu w+(1-\zeta)^2\frac{dw}{d\zeta}\right]=0$$

in the form of the following trigonometric series:

$$2\mu B_0+2\mu\sum_{m=1}^{\infty}\left\{(B_m+B_{-m})\cos m\omega+(A_m-A_{-m})\sin m\omega\right\}-$$

$$\frac{\Gamma}{\pi}+\frac{\Gamma}{\pi}\cos\omega+\sum_{m=-\infty}^{+\infty}\left\{\left[(m+1)B_{m+1}-2mB_m+(m-1)B_{m-1}\right]\cos m\omega+\right.$$

$$\left.[(m+1)A_{m+1}-2mA_m+(m-1)A_{m-1}]\sin m\omega\right\}=0$$

This equation gives a series of relations between the coefficients A_m and B_m:

$$B_1 - B_0 = \frac{1}{\pi}\Gamma - 2\mu B_0 \tag{11}$$

$$\mu(B_1 + B_{-1}) = (B_1 - B_{-1}) - (B_2 - B_{-2}) - \frac{\Gamma}{2\pi} \tag{12}$$

$$\mu(B_m + B_{-m}) = m(B_m - B_{-m}) - \frac{m+1}{2}(B_{m+1} - B_{-(m+1)}) - \frac{m-1}{2}(B_{m-1} - B_{-(m-1)}) \tag{13}$$

$$[m = 2, 4, 6 \ldots]$$

$$\mu(A_m - A_{-m}) = m(A_m + A_{-m}) - \frac{m+1}{2}(A_{m+1} + A_{-(m+1)}) - \frac{m-1}{2}(A_{m-1} + A_{-(m-1)}) \tag{14}$$

$$[m = 1, 2, 3, 4, 5, \ldots]$$

The last of these recurrent relations will be discussed first. From recurrent relation (10)

$$A_{-m} = \rho^{2m} A_m$$

as a result of which relation (14) may be written

$$\frac{m-1}{2}(1 + \rho^{2m-2}) A_{m-1} + [\mu(1 - \rho^{2m}) - m(1 + \rho^{2m})] A_m + \frac{m+1}{2}(1 + \rho^{2m+2}) A_{m+1} = 0$$

$$[m = 1, 2, 3, \ldots]$$

For briefness,

$$\frac{m}{2}(1 + \rho^{2m}) A_m = k_m$$

$$\frac{2\mu}{m} \cdot \frac{1 - \rho^{2m}}{1 + \rho^{2m}} - 2 = s_m \tag{15}$$

With these notations the obtained relation can be rewritten in the following form:

$$k_{m-1} + s_m k_m + k_{m-1} = 0 \qquad [m = 1, 2, 3, \ldots] \tag{16}$$

The coefficient A_0 is a certain finite number so that $k_0 = 0$.

Relation (13) is now considered. Eliminating from this relation the numbers B with negative indices and making use of relation (10) yield

$$\frac{m-1}{2}(1+\rho^{2m-2})B_{m-1} + [\mu(1-\rho^{2m}) - m(1+\rho^{2m})]B_m +$$

$$\frac{m+1}{2}(1+\rho^{2m+2})B_{m+1} = 2\lambda c\rho^{2m}\left[\frac{m-1}{2\rho^2} + \frac{m+1}{2}\rho^2 - m - \mu\right]$$

For brevity,

$$\frac{m}{2}(1+\rho^{2m})B_m = \varkappa_m \tag{17}$$

is written to obtain for $\varkappa_m$ the following relation:

$$\varkappa_{m-1} + s_m \varkappa_m + \varkappa_{m+1} = 2\lambda c\rho^{2m}\left[\frac{m-1}{2\rho^2} + \frac{m+1}{2}\rho^2 - m - \mu\right] \qquad [m = 2, 3, 4, 5, \ldots] \tag{18}$$

If

$$\varkappa_0 = \frac{\Gamma}{2\pi} \tag{19}$$

relation (18) for $m = 1$ gives condition (12).

The system of recurrent relations (16) and (18) is thus obtained for the coefficients

$$A_1, A_2, A_3, \ldots$$

$$B_1, B_2, B_3, \ldots$$

To these relations there must be added relations (9) and (11).

5. INVESTIGATION OF RECURRENT RELATIONS BETWEEN NUMBERS k_m

The relation

$$k_{m-1}+s_m k_m+k_{m+1}=0 \qquad [m=1,\,2,\,3,\,\ldots] \tag{16}$$

may be used to set up two functional relations between the two functions

$$\left.\begin{aligned} F(\zeta) &= \sum_{m=1}^{\infty} A_m \zeta^m \\ \sigma(\zeta) &= \sum_{m=1}^{\infty} k_m \zeta^m \end{aligned}\right\} \tag{20}$$

of the complex variable ζ.

First of all

$$F(\zeta)=2\sum_{m=1}^{\infty}\frac{k_m}{1+\rho^{2m}}\cdot\frac{\zeta^m}{m}$$

or

$$F(\zeta)=2\int_0^{\zeta}\sum_{m=1}^{\infty}\frac{k_m}{1+\rho^{2m}}\,\zeta^{m-1}\cdot d\zeta$$

The new function $S(\zeta)$ is introduced

$$S(\zeta)=\sum_{m=1}^{\infty}\frac{k_m}{1+\rho^{2m}}\,\zeta^m \tag{21}$$

from which

$$F(\zeta)=2\int_0^{\zeta}S(\zeta)\cdot\frac{d\zeta}{\zeta} \tag{22}$$

The function $S(\zeta)$ is connected with the function $\sigma(\zeta)$ by the following obvious relation:

$$S(\zeta)+S(\rho^2\zeta)=\sigma(\zeta) \tag{23}$$

Together with this relation there is still another obtained from the following considerations. Replacing k_m on the right side of equation (20) by

$$-s_{m-1}k_{m-1}-k_{m-2}$$

yields

$$\sigma(\zeta)=-\sum_{m=2}^{\infty}s_{m-1}k_{m-1}\zeta^m-\sum_{m=3}^{\infty}k_{m-2}\zeta^m+k_1\zeta$$

or

$$(1+\zeta^2)\sigma(\zeta)=k_1\zeta-\zeta\sum_{m=2}^{\infty}s_{m-1}k_{m-1}\zeta^{m-1} \tag{24}$$

Transforming the infinite sum on the right side gives

$$\sum_{m=2}^{\infty}s_{m-1}k_{m-1}\zeta^{m-1}=2\mu\sum_{m=2}^{\infty}\frac{1-\rho^{2m-2}}{1+\rho^{2m-2}}k_{m-1}\frac{\zeta^{m-1}}{m-1}-2\sigma(\zeta)$$

or

$$\sum_{m=2}^{\infty}s_{m-1}k_{m-1}\zeta^{m-1}=2\mu\int_0^{\zeta}\sum_{m=2}^{\infty}\frac{1-\rho^{2m-2}}{1+\rho^{2m-2}}k_{m-1}\zeta^{m-1}\frac{d\zeta}{\zeta}-2\sigma(\zeta)$$

The function under the integral may be expressed as follows in terms of the functions $\sigma(\zeta)$ and $S(\zeta)$:

$$\sum_{m=2}^{\infty}\frac{1-\rho^{2m-2}}{1+\rho^{2m-2}}k_{m-1}\zeta^{m-1}=2S(\zeta)-\sigma(\zeta)$$

hence

$$\sum_{m=2}^{\infty} s_{m-1} k_{m-1} \zeta^{m-1} = 2\mu \int_0^{\zeta} \frac{2S(\zeta) - \sigma(\zeta)}{\zeta} d\zeta - 2\sigma(\zeta)$$

Substituting in equation (24) the obtained value of the infinite sum produces a new relation between the functions $S(\zeta)$ and $\sigma(\zeta)$:

$$(1-\zeta)^2 \sigma(\zeta) = k_1 \zeta - 2\mu\zeta \int_0^{\zeta} \frac{2S(\zeta) - \sigma(\zeta)}{\zeta} d\zeta$$

Rewriting this relation in the differential form,

$$\frac{d}{d\zeta}\left[\frac{(1-\zeta)^2 \sigma(\zeta)}{\zeta}\right] = -2\mu \cdot \frac{2S(\zeta) - \sigma(\zeta)}{\zeta} \tag{25}$$

and adding to this differential equation the functional equation

$$S(\zeta) + S(\rho^2 \zeta) = \sigma(\zeta) \tag{23}$$

furnish the two necessary equations for the determination of the functions $\sigma(\zeta)$ and $S(\zeta)$. The function $S(\zeta)$ being determined from these equations, the function $F(\zeta)$ may be obtained with the aid of relation (22) or the relation

$$F(\zeta) = \frac{1}{2\mu}\left| k_1 - \frac{(1-\zeta)^2}{\zeta} \sigma(\zeta) \right| + \int_0^{\zeta} \frac{\sigma(\zeta)}{\zeta} d\zeta \tag{22'}$$

The function

$$F_1(\zeta) = \sum_{m=1}^{\infty} A_{-m} \zeta^m$$

may be represented through the function $F(\zeta)$ as follows:

$$F_1(\zeta) = F\left(\frac{\rho^2}{\zeta}\right)$$

The complex stream function (7) is represented thus:

$$w = w_1 + iw_2 \tag{26}$$

Setting

$$w_1 = \sum_{m=-\infty}^{+\infty} A_m \zeta^m \tag{27}$$

and

$$w_2 = \frac{\Gamma}{2\pi} \ln\zeta + \sum_{m=-\infty}^{+\infty} B_m \zeta^m \tag{28}$$

the function $w_1(\zeta)$ may be constructed from the function $F(\zeta)$ by the formula

$$w_1(\zeta) = A_0 + F(\zeta) + F\left(\frac{\rho^2}{\zeta}\right) \tag{27'}$$

or

$$w_1(\zeta) = A_0 + \sum_{m=1}^{\infty} A_m \left(\zeta^m + \frac{\rho^{2m}}{\zeta^m}\right)$$

In the variable z this function may be written

$$w_1(z) = A_0 + \sum_{m=1}^{\infty} A_m \cdot \frac{(z - \lambda i)^{2m} + \rho^{2m}(z + \lambda i)^{2m}}{(z^2 + \lambda^2)^m}$$

The function w_2 must now be investigated.

6. INVESTIGATION OF RECURRENT RELATIONS BETWEEN NUMBERS $\varkappa_m$

In the previously obtained recurrent relation

$$\varkappa_{m-1} + s_m \varkappa_m + \varkappa_{m+1} = 2\lambda c \rho^{2m} \left[\frac{m-1}{2\rho^2} + \frac{m+1}{2} \rho^2 - m - \mu \right]$$

$$\varkappa_0 = \frac{\Gamma}{2\pi} \qquad [m = 1, 2, 3, \ldots] \tag{29}$$

the three functions of the complex variable ζ may be considered:

$$F'(\zeta) = \sum_{m=1}^{\infty} B_m \zeta^m$$

$$\sigma'(\zeta) = \sum_{m=1}^{\infty} \varkappa_m \zeta^m$$

$$S'(\zeta) = \sum_{m=1}^{\infty} \frac{\varkappa_m}{1+\rho^{2m}} \zeta^m$$

Between these functions there exist the following two easily obtained relations:

$$S'(\zeta) + S'(\rho^2 \zeta) = \sigma'(\zeta) \tag{30}$$

$$F'(\zeta) = 2 \int_0^{\zeta} S'(\zeta) \cdot \frac{d\zeta}{\zeta} \tag{31}$$

Equation (30) gives the first relation between the functions $S'(\zeta)$ and $\sigma'(\zeta)$; for obtaining the second relation, the recurrent relations (29) are used as follows:

$$\sigma'(\zeta) = -\sum_{m=2}^{\infty} s_{m-1} \varkappa_{m-1} \zeta^m - \sum_{m=2}^{\infty} \varkappa_{m-2} \zeta^m + \varkappa_1 \zeta + L(\zeta)$$

or

$$L(\zeta) = 2\lambda c \sum_{m=1}^{\infty} \rho^{2m} \left[\frac{m-1}{2\rho^2} + \frac{m+1}{2} \rho^2 - m - \mu \right] \zeta^{m+1} \tag{32}$$

Transforming this relation gives

$$(1+\zeta^2)\,\sigma'(\zeta)=-\zeta\sum_{m=2}^{\infty}s_{m-1}\varkappa_{m-1}\zeta^{m-1}-\varkappa_0\zeta^2+\varkappa_1\zeta+L(\zeta)$$

But

$$\sum_{m=2}^{\infty}s_{m-1}\varkappa_{m-1}\zeta^{m-1}=2\mu\int_0^{\zeta}\frac{2S'(\zeta)-\sigma'(\zeta)}{\zeta}\,d\zeta-2\sigma'(\zeta)$$

(by the computations of the preceding section). Hence

$$(1-\zeta)^2\sigma'(\zeta)=-2\mu\zeta\int_0^{\zeta}\frac{2S'(\zeta)-\sigma'(\zeta)}{\zeta}\,d\zeta-\varkappa_0\zeta^2+\varkappa_1\zeta+L(\zeta)$$

from which is obtained

$$\frac{d}{d\zeta}\left[\frac{(1-\zeta)^2\sigma'(\zeta)}{\zeta}\right]=-2\mu\frac{2S'(\zeta)-\sigma'(\zeta)}{\zeta}-\varkappa_0+\frac{d}{d\zeta}\left[\frac{L(\zeta)}{\zeta}\right] \tag{33}$$

The function $L(\zeta)$ may be represented in finite form:

$$\frac{1}{2\lambda c}\frac{L(\zeta)}{\zeta}=\frac{\rho^2(\zeta-1)^2}{2(1-\rho^2\zeta)^2}-\frac{\rho^2}{2}-\frac{\mu\rho^2\zeta}{1-\rho^2\zeta}$$

whence

$$\frac{1}{2\lambda c}\frac{d}{d\zeta}\frac{L(\zeta)}{\zeta}=\frac{\rho^2(1-\rho^2)(\zeta-1)}{(1-\rho^2\zeta)^3}-\frac{\rho^2\mu}{(1-\rho^2\zeta)^2}$$

Substituting this value of the derivative $\frac{d}{d\zeta}\frac{L(\zeta)}{\zeta}$ in equation (33) yields

$$\frac{d}{d\zeta}\left|\frac{(1-\zeta)^2\sigma'(\zeta)}{\zeta}\right|=-2\mu\frac{2S'(\zeta)-\sigma'(\zeta)}{\zeta}-\frac{\Gamma}{2\pi}+2\lambda c\rho^2\left|\frac{(1-\rho^2)(\zeta-1)}{(1-\rho^2\zeta)^3}-\frac{\mu}{(1-\rho^2\zeta)^2}\right| \tag{34}$$

This equation together with equation (30) may serve for the determination of the functions $\sigma'(\zeta)$ and $S'(\zeta)$. The function $S'(\zeta)$ being found, $F'(\zeta)$ may be computed by formula (31).

The function w_2 of the preceding section may be written as follows:

$$w_2(\zeta) = B_0 + \frac{\Gamma}{2\pi}\ln\zeta + \sum_{m=1}^{\infty} B_m \zeta^m + \sum_{m=1}^{\infty} B_{-m}\zeta^{-m} \tag{35}$$

The second sum may be transformed into

$$\sum_{m=1}^{\infty} B_{-m}\zeta^{-m} = -\sum_{m=1}^{\infty} B_m\left(\frac{\rho^2}{\zeta}\right)^m + 2\lambda c\sum_{m=1}^{\infty}\rho^{2m}\zeta^{-m}$$

or

$$\sum_{m=1}^{\infty} B_{-m}\zeta^{-m} = -F'\left(\frac{\rho^2}{\zeta}\right) + \frac{2c\lambda\rho^2}{\zeta - \rho^2}$$

Thus

$$w_2(\zeta) = B_0 + \frac{\Gamma}{2\pi}\ln\zeta + \frac{2\lambda c\rho^2}{\zeta - \rho^2} + F'(\zeta) - F'\left(\frac{\rho^2}{\zeta}\right) \tag{36}$$

or

$$w_2(\zeta) = B_0 + \frac{\Gamma}{2\pi}\ln\zeta + \frac{2\lambda c\rho^2}{\zeta - \rho^2} + \sum_{m=1}^{\infty} B_m\left(\zeta^m - \frac{\rho^{2m}}{\zeta^m}\right) \tag{37}$$

In the variable z this expression may be written

$$w_2(z) = B_0 + \frac{\Gamma}{2\pi}\ln\frac{z - \lambda i}{z + \lambda i} + \frac{2\lambda c\rho^2(z + \lambda i)}{(1 - \rho^2)z - \lambda i(1 + \rho^2)} + \sum_{m=1}^{\infty} B_m \cdot \frac{(z - \lambda i)^{2m} - \rho^{2m}(z + \lambda i)^{2m}}{(z^2 + \lambda^2)^m} \tag{38}$$

7. FORMULA FOR COMPLEX STREAM FUNCTION

On the basis of formulas (27') and (36), the expression for the function $w(\zeta)$ may now be stated

$$w(\zeta) = (A_0 + i B_0) + \frac{\Gamma i}{2\pi} \ln \zeta + \frac{2\lambda c \rho^2}{\zeta - \rho^2} i + \left[F(\zeta) + F\left(\frac{\rho^2}{\zeta}\right)\right] + i\left[F'(\zeta) - F'\left(\frac{\rho^2}{\zeta}\right)\right] \tag{39}$$

or

$$w(\zeta) = (A_0 + i B_0) + \frac{\Gamma i}{2\pi} \ln \zeta + \frac{2\lambda c \rho^2}{\zeta - \rho^2} i + \sum_{m=1}^{\infty} p_m \zeta^m + \sum_{m=1}^{\infty} \bar{p}_m \left(\frac{\rho^2}{\zeta}\right)^m \tag{39'}$$

The number $\bar{p}_m$ is the conjugate complex of the number

$$p_m = A_m + i B_m$$

Equations (9) and (11) permit the determination of the coefficient B_0 and the number α:

$$B_0 = \frac{B_1 - \frac{\Gamma}{\pi}}{1 - 2\mu}$$

$$\alpha = \frac{\Gamma}{2\pi} \ln \rho + \frac{B_1 - \frac{\Gamma}{\pi}}{1 - 2\mu} - \lambda c \tag{40}$$

The number A_0 remains arbitrary, and the circulation Γ likewise remains arbitrary.

Formula (39') may be given a more symmetrical form by introducing in place of the coefficient p_m new coefficients r_m and $\bar{r}_m$ by the formulas

$$r_m = p_m - i \lambda c$$

$$\bar{r}_m = \bar{p}_m + i \lambda c \tag{41}$$

to obtain

$$w(\zeta) = (A_0 + iB_0) + \frac{\Gamma i}{2\pi} \ln \zeta + \frac{\lambda c \rho^2}{\zeta - \rho^2} + \frac{\lambda c \zeta}{1 - \zeta} +$$

$$\sum_{m=1}^{\infty} r_m \zeta^m + \sum_{m=1}^{\infty} \bar{r}_m \left(\frac{\rho^2}{\zeta} \right)^m \qquad (39'')$$

8. APPROXIMATE SOLUTION OF PROBLEM OF DEEPLY SUBMERGED CYLINDER

The problem of determining the flow of a heavy fluid about a circular cylinder is thus reduced to the solution of the two systems of the functional equations (23), (25), (30), and (34).

Since it is not possible to solve these equations without the aid of infinite series or in a finite combination of elementary functions, an approximate treatment of the equations of the problem is proposed. For this purpose the number ρ^2 must be considered. In the following table are given various values of the ratio a/h and the corresponding values of ρ^2 and ρ^4.

$\frac{a}{h}$	Ω	ρ^2	ρ^4
0.500	60°	0.0718	0.005151
.423	50°	.0492	.002416
.342	40°	.0311	.000966
.259	30°	.0173	.000300
.174	20°	.0077	.000059
.087	10°	.0019	.000004

This table shows that for a ratio a/h less than $0.342 \cong 1/3$, the number ρ^4 does not exceed 0.001, which justifies the rejection of all powers of ρ^2 starting with the second in considering the motion of the stream for $a/h < 1/3$. The results here obtained are in somewhat complicated form; therefore only formulas which are suitable for the condition at which it is permissible to reject the components with ρ^2 are presented. The preceding table shows that this can be done by starting, for example, with $a/h < 1/3$.

By rejecting the terms with ρ^2 equations (23) and (25) may be rewritten as follows:

$$S(\zeta) = \sigma(\zeta)$$

$$\frac{d}{d\zeta} \left[\frac{(1 - \zeta)^2 \sigma(\zeta)}{\zeta} \right] = 2\mu \frac{\sigma(\zeta)}{\zeta}$$

from which

$$s'(\zeta) = \frac{\varkappa_1 e^{-\frac{2\mu\zeta}{1-\zeta}}}{(1-\zeta)^2}\zeta$$

The function $F(\zeta)$ is defined

$$F(\zeta) = \frac{\varkappa_1}{\mu}\left[1 - e^{-\frac{2\mu\zeta}{1-\zeta}}\right],$$

from which

$$F\left(\frac{\rho^2}{\zeta}\right) = 0$$

Equations (30) and (34) are now considered. These, with the preceding approximation, may be written as

$$S'(\zeta) = \sigma'(\zeta)$$

$$\frac{d}{d\zeta}\left[\frac{(1-\zeta)^2\sigma'(\zeta)}{\zeta}\right] = -2\mu\frac{\sigma'(\zeta)}{\zeta} - \frac{\Gamma}{2\pi}$$

from which

$$\sigma'(\zeta) = \frac{\zeta}{(1-\zeta)^2} e^{-\frac{2\mu\zeta}{1-\zeta}}\left[\varkappa_1 - \frac{\Gamma}{2\pi}\int_0^\zeta e^{\frac{2\mu\zeta}{1-\zeta}} d\zeta\right]$$

The function $F'(\zeta)$ is now obtained:

$$F'(\zeta) = 2\int_0^\zeta \frac{\sigma'(\zeta)}{\zeta} d\zeta = -\frac{\Gamma}{\pi\mu}\zeta - \frac{1}{\mu}\left[\frac{(1-\zeta)^2\sigma'(\zeta)}{\zeta} - \varkappa_1\right]$$

$$F'\left(\frac{\rho^2}{\zeta}\right) = 0$$

The complex stream function may now be found. For this purpose formula (39) is used to obtain

$$w(\zeta) = (A_0 + iB_0) + \frac{\Gamma i}{2\pi}\ln\zeta + \frac{k_1}{\mu} - \frac{\Gamma i}{\pi\mu}\zeta - \frac{k_1 + \varkappa_1 i}{\mu}e^{\frac{2\mu\zeta}{\zeta - 1}} +$$
$$\frac{\Gamma i}{2\pi\mu}e^{-\frac{2\mu\zeta}{1-\zeta}}\int_0^{\zeta} e^{\frac{2\mu\zeta}{1-\zeta}}d\zeta \tag{41}$$

The constant B_0 may be determined by formula (40). Since $B_1 = 2\varkappa_1$ (by relation (17)),

$$B_0 = \frac{2\varkappa_1 - \frac{\Gamma}{\pi}}{1 - 2\mu}$$

In order to determine the remaining constants k_1 and $\varkappa_1$ the condition of the absence of disturbance of the flow ahead of the cylinder is used. The original variable z in formula (41) is used:

$$w(\zeta) = \left(A_0 + \frac{k_1}{\mu} + iB_0\right) - \frac{\Gamma i}{\pi\mu}\cdot\frac{z - \lambda i}{z + \lambda i} + \frac{\Gamma i}{2\pi}\cdot\ln\frac{z - \lambda i}{z + \lambda i} -$$
$$\frac{k_1 + \varkappa_1 i}{\mu}e^{\mu}\cdot e^{-\frac{gz}{c^2}i} + \frac{\Gamma c^2}{\pi g}e^{-\frac{gz}{c^2}i}\int_{\lambda i}^{z} e^{\frac{gz}{c^2}i}\frac{dz}{(z + \lambda i)^2}$$

Integrating by parts,

$$w(\zeta) = \left(A_0 + \frac{k_1}{\mu} + B_0 i - \frac{\Gamma i}{\pi\mu}\right) + \frac{\Gamma i}{2\pi}\ln\frac{z - \lambda i}{z + \lambda i} - \frac{k_1 + \varkappa_1 i}{\mu}e^{\mu}e^{-\frac{gz}{c^2}i} +$$
$$\frac{\Gamma i}{\pi}e^{-\frac{gz}{c^2}i}\int_{\lambda i}^{z}\frac{e^{\frac{gz}{c^2}i}}{z + \lambda i}dz$$

But

$$\int_{\lambda i}^{z}\frac{e^{\frac{gz}{c^2}i}}{z + \lambda i}dz = \int_{\lambda i}^{-\infty + \lambda i} + \int_{-\infty}^{z}$$

$$= \int_0^{\infty}\frac{\xi\cos\frac{g\xi}{c^2} + 2\lambda\sin\frac{g\xi}{c^2}}{4\lambda^2 + \xi^2}d\xi + \pi i e^{\frac{2g\lambda}{c^2}} + \int_{-\infty}^{z}\frac{e^{\frac{gz}{c^2}i}}{z + \lambda i}dz$$

Hence

$$w(\zeta) = \left(A_0 + \frac{k_1}{\mu} + B_0 i - \frac{\Gamma i}{\pi\mu}\right) + \frac{\Gamma i}{2\pi} \ln \frac{z - \lambda i}{z + \lambda i} +$$

$$\left[-\frac{k_1 + \varkappa_1 i}{\mu} e^{\mu} - \Gamma e^{\frac{2g\lambda}{c^2}} + \frac{\Gamma i}{\pi} \int_0^\infty \frac{\xi \cos \frac{g\xi}{c^2} + 2\lambda \sin \frac{g\xi}{c^2}}{4\lambda^2 + \xi^2} d\xi \right] e^{-\frac{gz}{c^2} i} +$$

$$\frac{\Gamma i}{\pi} e^{-\frac{gz}{c^2} i} \int_\infty^z \frac{e^{\frac{gz}{c^2} i}}{z + \lambda i} dz$$

The condition at which the function Ψ becomes zero for $y = 0$ and $x = -\infty$ will be satisfied if the brackets are equated to zero:

$$k_1 = \mu\Gamma e^{-3\mu}$$

$$\varkappa_1 = \frac{\mu\Gamma}{\pi} e^{-\mu} \int_0^\infty \frac{\xi \cos \frac{g\xi}{c^2} + 2\lambda \sin \frac{g\xi}{c^2}}{4\lambda^2 + \xi^2} d\xi$$

and in addition the imaginary part of the parentheses is equated to zero:

$$B_0 = \frac{\Gamma}{\pi\mu}$$

The number A_0 is arbitrary; the real part of the terms in the parentheses is also equated to zero to give

$$A_0 = -\frac{k_1}{\mu}$$

In this way all the constants have been determined and the complex stream function of the total flow may be written

$$W = -cz - \frac{\Gamma}{2\pi i} \ln \frac{z - \lambda i}{z + \lambda i} - \frac{\Gamma}{\pi i} e^{-\frac{gz}{c^2} i} \int_{-\infty}^{z} \frac{e^{\frac{gz}{c^2} i}}{z + \lambda i} dz \tag{42}$$

Hence for the case here considered of the deep submersion of the cylinder, the flow consists principally of the circulation of the velocity about the cylinder. The sources of different strengths entering the general formula (39") located at the point $z = \lambda i$ begin to be effective only at small depths of submersion of the cylinder.

The vortex representing the cylinder is situated at the point $z = \lambda i$, this point being somewhat displaced with reference to the center of the cylinder.

Now formula (42) is applied to the computation of the pressure of the stream on the cylinder.

Denoting by δ the mass density of the fluid and by X and Y the components of the resultant force yields, by the formula of Chaplygin,

$$Y + iX = -\frac{1}{2}\delta \int \frac{dW}{dz} dW$$

The integral is taken over a closed contour surrounding the cylinder:

$$\frac{dW}{dz} = -c - \frac{\Gamma}{2\pi i}\left(\frac{1}{z+\lambda i} + \frac{1}{z-\lambda i}\right) + \frac{\Gamma g}{\pi c^2} e^{-\frac{gz}{c^2} i} \int_{-\infty}^{z} \frac{e^{\frac{gz}{c^2} i}}{z+\lambda i} dz$$

Substituting the preceding in the formula of Chaplygin and applying the theory of residues yield

$$X = \frac{g\delta\Gamma^2}{c^2} e^{\frac{2g\lambda}{c^2}}$$

$$Y = \delta\Gamma\left(c - \frac{\Gamma}{4\pi\lambda}\right) + \qquad (43)$$

$$\frac{\delta\Gamma^2 g}{\pi c^2} e^{\frac{g\lambda}{c^2}} \left[\int_0^1 \frac{e^{-\frac{g\lambda}{c^2}\zeta}}{1+\zeta} d\zeta + \int_0^\infty \frac{z\cos\frac{gz}{c^2} + \lambda\sin\frac{gz}{c^2}}{z^2+\lambda^2} dz\right]$$

The first formula determines the wave resistance of the cylinder and the second, the lift force.

For $\lambda = -\infty$ the second formula gives the theorem of Joukowski.

By formula (4), section 2, it is now possible to find the form of the free surface of the fluid. The equation of the surface of the fluid may be written in terms of the function W in the following form:

$$y = \frac{c}{g} \operatorname{Reel} \left(\frac{dW}{dz} + c \right)_{z=x+0i}$$

Making use of formula (42),

$$y = \frac{\Gamma}{\pi c} \int_{-\infty}^{x} \frac{\xi \cos \frac{g}{c^2}(x-\xi) - \lambda \sin \frac{g}{c^2}(x-\xi)}{\xi^2 + \lambda^2} d\xi$$

or

$$\frac{\pi c}{\Gamma} y = \cos \frac{gx}{c^2} \int_{-\infty}^{x} \frac{\xi \cos \frac{g\xi}{c^2} + \lambda \sin \frac{g\xi}{c^2}}{\xi^2 + \lambda^2} d\xi +$$

$$\sin \frac{gx}{c^2} \int_{-\infty}^{x} \frac{\xi \sin \frac{g\xi}{c^2} - \lambda \cos \frac{g\xi}{c^2}}{\xi^2 + \lambda^2} d\xi$$

Making x approach ∞ gives

$$y = \frac{2\Gamma}{c} e^{\frac{g\lambda}{c^2}} \sin \frac{gx}{c^2}$$

Hence, far behind the cylinder the surface of the fluid carries steady waves of length $L = 2\pi c^2/g$.

The amplitude of these waves is

$$A = \frac{2\Gamma}{c} e^{\frac{g\lambda}{c^2}}$$

For determining the wave resistance, energy considerations are applied. The wave resistance R is determined in terms of the energy E of the steady wave by the formula

$$R = \frac{c - U}{c} \cdot E$$

where U is the group velocity, and

$$U = \frac{1}{2} c$$

The energy E is determined by the formula

$$E = \frac{1}{2} g\delta A^2$$

Making use of all these results formula (43) is again found:

$$R = \frac{g\delta\Gamma^2}{c^2} e^{\frac{2g\lambda}{c^2}}$$

9. MOTION OF CYLINDER UNDER SURFACE OF FLUID OF FINITE DEPTH

Under consideration is a circular cylinder of radius R moving with constant velocity c under the surface of a fluid of depth H. The problem is proposed of finding its wave resistance as a function of the submersion depth h of the center of the cylinder. This problem is solved under the assumptions of Lamb; that is, the magnitude h/R is assumed to be small and the entire cylinder is replaced by a dipole of moment $-2\pi cR^2$.

The motion of the fluid is studied in relation to a system of axes of coordinates x and y displaced uniformly together with the cylinder. The velocity of the approaching flow at infinity is c. By $\Phi(x,y)$ is denoted the potential of the wave velocities. The harmonic function $\Phi(x,y)$ satisfies for y = - H the condition

$$\frac{\partial \Phi}{\partial y} = 0 \tag{1}$$

Along the free boundary $y = 0$ the function $\Phi(x,y)$ satisfies the following condition:

$$\frac{\partial^2 \Phi}{\partial x^2} + \mu \frac{\partial \Phi}{\partial x} + \frac{g}{c^2} \frac{\partial \Phi}{\partial y} = 0 \qquad (2)$$

The expression $\mu > 0$ is the Rayleigh coefficient of the dissipating forces.

Near the point $y = -h$, that is, near the center of the cylinder, the function $\Phi(x,y)$ on the basis of the above considerations must have the following form:

$$\Phi = -\frac{cR^2 x}{x^2 + (y + h)^2} + \ldots$$

Then

$$\Phi_1 = -\frac{cR^2 x}{x^2 + (y + h)^2} \qquad (3)$$

and

$$\Phi_2 = +\frac{cR^2 x}{x^2 + (y - h)^2} \qquad (4)$$

where Φ_2 is the velocity potential of a certain fictitious dipole situated at the point $y = h$.

In place of the function Φ a new function $\varphi(x,y)$ is introduced, which is valid for the entire region occupied by the fluid, setting

$$\Phi = \Phi_1 + \Phi_2 + \varphi \qquad (5)$$

The harmonic function $\varphi(x,y)$ satisfies the following conditions on the boundary of the fluid:

$$\frac{\partial \varphi}{\partial y} = -\frac{\partial \Phi_1}{\partial y} - \frac{\partial \Phi_2}{\partial y} \text{ for } y = -H \qquad (1')$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \mu \frac{\partial \varphi}{\partial x} + \frac{g}{c^2} \frac{\partial \varphi}{\partial y} = -\frac{2g}{c^2} \frac{\partial \Phi_2}{\partial y} \text{ for } y = 0 \qquad (2')$$

The function $\varphi(x,y)$ is obtained in the form of the following definite integral:

$$\int_0^{\infty}(A\cos kx + C\sin kx)\cosh ky\,dk + \int_0^{\infty}(B\cos kx + D\sin kx)\sinh ky\,dk \qquad (6)$$

where A, B, C, and D are unknown functions of the variable parameter k. To determine these functions condition (2') is first considered:

$$\left[\frac{\partial \Phi_2}{\partial y}\right]_{y=0} = \frac{2\cosh R^2 x}{(x^2+h^2)^2} = cR^2\int_0^{\infty} ke^{-hk}\sin kx\,dk$$

Substituting this result and (6) in condition (2') yields

$$\int_0^{\infty}\left[\left(-k^2A + \mu kC + \frac{gk}{c^2}B\right)\cos kx + \left(-k^2C - \mu kA + \frac{gk}{c^2}D\right)\sin kx\right]dk$$

$$= -\frac{2g}{c}R^2\int_0^{\infty} ke^{-hk}\sin kx\,dk$$

This relation gives the following two equations:

$$\left.\begin{aligned} kA - \mu C - \frac{g}{c^2}B &= 0 \\ \mu A + kC - \frac{g}{c^2}D &= \frac{2g}{c}R^2 e^{-hk} \end{aligned}\right\} \qquad (7)$$

Employing condition (1') gives

$$\left[\frac{\partial \Phi_1}{\partial y}\right]_{y=-H} = -\frac{2cR^2x(H-h)}{[x^2+(H-h)^2]^2} = -cR^2\int_0^{\infty} ke^{-(H-h)k}\sin kx\,dk$$

$$\left[\frac{\partial \Phi}{\partial y}\right]_{y=-H} = \frac{2cR^2x(H+h)}{[x^2+(H+h)^2]^2} = +cR^2\int_0^{\infty} ke^{-(H+h)k}\sin kx\,dk$$

Applying the results to condition (1') yields

$$\int_0^\infty \left[k(B\cosh kH - A\sinh kH)\cos kx + k(D\cosh kH - C\sinh kH)\sin kx \right] dk$$

$$= 2cR^2 \int_0^\infty ke^{-Hk}\sinh kh \sin kx\, dk$$

from which

$$B\cosh kH - A\sinh kH = 0$$

$$D\cosh kH - C\sinh kH = 2cR^2e^{-Hk}\sinh kh$$

or

$$B = A\tanh kH$$

$$D = C\tanh kH + 2cR^2e^{-Hk}\frac{\sinh kh}{\cosh kH} \tag{8}$$

These equations together with equations (7) permit determining the unknown functions A, B, C, and D.

From equations (7) and (8) the two equations for A and C are found:

$$\left(k - \frac{g}{c^2}\tanh kH\right)A - \mu C = 0$$

$$\mu A + \left(k - \frac{g}{c^2}\tanh kH\right)D = \frac{2g}{c}R^2\cdot\frac{\cosh k(H-h)}{\cosh kH}$$

Solving these equations determines A and C

$$A = \frac{\frac{2g}{c} R^2 \frac{\cosh k(H-h)}{\cosh kH}}{\mu^2 + \left(k - \frac{g}{c^2}\tanh kH\right)^2} \cdot \mu$$

$$C = \frac{\frac{2g}{c} R^2 \frac{\cosh k(H-h)}{\cosh kH}\left(k - \frac{g}{c^2}\tanh kH\right)}{\mu^2 + \left(k - \frac{g}{c^2}\tanh kH\right)^2}$$

Equations (8) then permit finding B and D. In this way the function φ is determined, which makes it possible to obtain the velocity potential Φ. The complex potential of the absolute velocities must be found:

$$w = \Phi + i\psi$$

First

$$\Phi_1 = \text{Reel}\left(-\frac{cR^2}{z + hi}\right)$$

$$\Phi_2 = \text{Reel}\left(\frac{cR^2}{z - hi}\right)$$

$$\varphi = \text{Reel}\int_0^\infty \left[(A + Di)\cos kz + (C - Bi)\sin kz\right] dk$$

where $z = x + iy$. Hence

$$w = \frac{2cR^2hi}{z^2 + h^2} + \int_0^\infty \left[(A + Di)\cos kz + (C - Bi)\sin kz\right] dk$$

The complex potential W of the relative velocities is obtained by adding to the right side the term -cz. In terms of this potential the pressure of the stream on the cylinder is obtained by the formula of Chaplygin:

$$X - iY = \frac{1}{2}\rho i \int \left(\frac{dW}{dz}\right)^2 dz$$

where the integral is taken over a contour containing the dipole z = - ih.

10. COMPUTATION OF WAVE RESISTANCE

The problem of finding wave resistance is restricted only to the computation of X, and therefore only to the imaginary part of the integral of the formula of Chaplygin:

$$X = -\frac{1}{2}\rho \,\mathrm{Imag.} \int \left(\frac{dW}{dz}\right)^2 dz \tag{9}$$

The function W may be written

$$W = -\frac{cR^2}{z+hi} + G(z)$$

The function G(z) is holomorphic about the point z = - hi:

$$G(z) = -cz + \frac{cR^2}{z+hi} + \int_0^{\infty} [(A+Di)\cos kz + (C-Bi)\sin kz]\, dk \tag{10}$$

From formula (9) is obtained

$$\frac{dW}{dz} = \frac{cR^2}{(z+hi)^2} + G'(z)$$

from which

$$\int \left(\frac{dW}{dz}\right)^2 dz = \int \frac{2cR^2}{(z+hi)^2} G'(z)\, dz = 4\pi icR^2G''(-ih)$$

Therefore

$$X = -2\pi R^2\delta c \cdot \text{Reel}\, G''(-ih)]$$

Making use of formula (10) the real part of the second derivative of the function G(z) at z = - ih is determined:

$$\text{Reel}\,[G''(-ih)] = -\int_0^\infty k^2(A \cosh kh - B \sinh kh)\, dk$$

from which

$$X = 2\pi R^2\delta c \int_0^\infty k^2(A \cosh kh - B \sinh kh)\, dk$$

But, as was found in formula (8), B = A tanh kh. Hence

$$X = 2\pi R^2\delta c \int_0^\infty k^2 A \frac{\cosh k(H-h)}{\cosh kH}\, dk$$

Replacing A by its value gives

$$X = 4\pi\delta g R^4\mu \int_0^\infty \frac{\cosh^2 k(H-h)}{\cosh^2 kH} \cdot \frac{k^2 dk}{\mu^2 + (k - \frac{g}{c^2} \tanh kH)^2}$$

which is the formula for the wave resistance in the presence of dissipative Rayleigh forces. Freed from these forces

$$L = \lim_{\mu=0} \int_0^\infty \mu \cdot \frac{\cosh^2 k(H-h)}{\cosh^2 kH} \cdot \frac{k^2 dk}{\mu^2 + \left(k - \frac{g}{c^2}\tanh kH\right)^2}$$

A new variable of integration ξ is introduced, setting $kH = \xi$ and with the following notations:

$$\frac{gH}{c^2} = \varkappa$$

$$\frac{h}{H} = \alpha$$

$$\mu H = \mu_1$$

In this notation L is rewritten

$$L = \frac{1}{H^2} \lim_{\mu_1=0} \int_0^\infty \mu_1 \cdot \frac{\cosh^2 (1-\alpha)\xi}{\cosh^2 \xi} \cdot \frac{\xi^2 d\xi}{\mu_1^2 + (\xi - \varkappa \tanh \xi)^2} \tag{11}$$

If the number $\varkappa$ is less than unity

$$c > \sqrt{gH}$$

then $L = 0$ and the wave resistance is equal to zero. When $\varkappa > 1$ the roots of the equation in ξ are considered:

$$\mu_1^2 + (\xi - \varkappa \tanh \xi)^2 = 0 \tag{12}$$

This equation may be resolved into the two following equations:

$$\xi = \varkappa \tanh \xi + \mu_1 i$$

$$\xi = \varkappa \tanh \xi - \mu_1 i$$

Only the first of these equations will be considered. In view of the fact that $\varkappa > 1$, the equation $\xi = \varkappa \tanh \xi$ has one real root which is denoted by ξ_0. Equation (11) has one root $\overline{\xi}$, approaching ξ_0 as μ_1 approaches zero; $\xi_0 = \lim_{\mu=0} \overline{\xi}$. By applying these notations, the left side of equation (12) is expanded in a series about the point $\overline{\xi}$:

$$\mu_1^2 + (\xi - \varkappa \tanh \xi)^2 = [\xi - \varkappa \tanh \xi - \mu_1 i][\xi - \varkappa \tanh \xi + \mu_1 i]$$

$$= 2\mu_1 i \frac{d}{d\xi}[\xi - \varkappa \tanh \xi - \mu_1 i]_{\xi = \bar{\xi}} \cdot (\xi - \bar{\xi}) + \ldots \tag{13}$$

$$= 2\mu_1 i \left(1 - \frac{\varkappa}{\cosh^2 \bar{\xi}}\right)(\xi - \bar{\xi}) + \ldots$$

The integral (11) may be represented as the sum of a certain contour integral taken along the path OIK_∞ and a residue multiplied by $2\pi i$ of the function under the integral relative to the point $\xi = \bar{\xi}$:

$$L = \frac{1}{H^2} \lim_{\mu_1 = 0} \int_{(OIK)} + \lim_{\mu_1 = 0} \frac{2\pi i}{H^2} \operatorname{Rés}_{\xi = \bar{\xi}} \left[\mu_1 \frac{\cosh^2(1-\alpha)\xi}{\cosh^2 \xi} \cdot \frac{\xi^2}{\mu_1^2 + (\xi - \varkappa \tanh \xi)^2} \right]$$

The first component on the right side gives zero in the limit. From series (13) the limit of the second component can likewise easily be found. The following expression is finally found:

$$L = \frac{\pi}{H^2} \cdot \frac{\xi_0^2 \cosh^2(1-\alpha)\xi_0}{\cosh^2 \xi_0 - \varkappa}$$

For the expression for X,

$$X = \frac{4\pi^2 \rho g R^4}{H^2} \cdot \frac{\xi_0^2 \cosh^2(1-\alpha)\xi_0}{\cosh^2 \xi_0 - \varkappa} \tag{14}$$

is now obtained. This is the formula for the computation of the wave resistance for velocities c less than the critical velocity $\sqrt{gH}$. The value ξ_0 is the real root of the equation

$$\xi_0 = \varkappa \tanh \xi_0 \tag{15}$$

From formula (14) is readily obtained the formula of Lamb for the wave resistance of a cylinder moving under the surface of a fluid of infinite depth. When H is very large, the root ξ_0 of equation (15) is $\xi_0 = \varkappa$, from which

$$\frac{\xi_0^2 \cosh^2(1-\alpha)\xi_0}{\cosh^2\xi_0 - \varkappa} = \varkappa^2 e^{-2\alpha\varkappa}$$

Substituting this expression in formula (14), the formula of Lamb is obtained:

$$X_{\infty} = 4\pi^2 \delta g^3 \left(\frac{R}{c}\right)^4 e^{-\frac{2gh}{c^2}}$$

Now formula (14) is investigated. In place of ξ_0 is written ξ. In the second factor on the right side $\varkappa$ is replaced by $\xi/\tanh\xi$ to obtain

$$X = \frac{8\pi^2 \delta g R^4}{H^2} \cdot \frac{\xi^2 \tanh\xi \cdot \cosh^2(1-\alpha)\xi}{\sinh 2\xi - 2\xi} \tag{16}$$

The parameter $\varkappa$ varied from 1 to ∞ so that ξ will vary from zero to infinity. For $\xi = 0$ the second factor on the right side of this formula has a value equal to 3/4. It will be shown that this value is a minimum or maximum. Denoting by A the investigated factor gives

$$A = \frac{1}{2} \cdot \frac{\xi^2 \sinh 2\xi}{\sinh 2\xi - 2\xi} \cdot \left| \frac{\cosh(1-\alpha)\xi}{\cosh\xi} \right|^2$$

Then

$$\frac{1}{2} \cdot \frac{\xi^2 \sinh 2\xi}{\sinh 2\xi - 2\xi} = \frac{3}{4}\left(1 + \frac{7}{15}\xi^2 + \ldots\right)$$

$$\left| \frac{\cosh(1-\alpha)\xi}{\cosh\xi} \right|^2 = 1 - \alpha(2-\alpha)\xi^2 + \ldots$$

from which

$$A = \frac{3}{4}\left[1 + \left\{\frac{7}{15} - \alpha(2-\alpha)\right\}\xi^2 + \ldots\right]$$

which shows that

$$\left(\frac{dA}{d\xi}\right)_0 = 0$$

$$\left(\frac{d^2A}{d\xi^2}\right)_0 = \frac{3}{2}\left[\frac{7}{15} - \alpha(2-\alpha)\right]$$

Thus for $\xi = 0$ the wave resistance will be an extreme value. It will be a minimum if $\alpha < 1 - \sqrt{8/15}$; for larger values of α the wave resistance is a maximum.

For $\xi = \infty$ the factor A becomes zero. This investigation explains the general features of the dependence of the wave resistance on the velocity.

11. CONCLUSIONS

Formula (16) obtained in section 10 for the wave resistance of a cylinder of radius R moving under the surface of a fluid of finite depth H led to the following conclusions with regard to the change in the wave resistance with velocity of motion and with depth h of its submersion:

The magnitude

$$X = [illegible]$$

was computed for a series of values of $\alpha = h/H$:

$$\alpha = \frac{1}{8}, \frac{3}{16}, \frac{1}{4}, 1 - \sqrt{\frac{8}{15}}; 0.3; 0.36; 0.4; 0.5; \frac{5}{8}; \frac{3}{4}; \frac{7}{8}; \frac{15}{16}$$

for the Froude number $f = c/\sqrt{gH}$ varying between zero and 1. The results of the numerical computations are given in figure 1.

The curves of this diagram show a considerable wave resistance for small depths of submersion of the cylinder. With increased submersion of the cylinder the wave resistance drops sharply for most of the values of the Froude number.

For the value of the parameter $\alpha = 1 - \sqrt{8/15}$, the character of the maximum or minimum changes at the point $f = 1$. For values of

$\alpha < 1 - \sqrt{8/15}$ the curves of the wave resistance have a sharply formed maximum. With α increasing from zero to $1 - \sqrt{8/15}$ this maximum decreases. The presence of this maximum shows up also on the curves corresponding to values of α somewhat less than the number $1 - \sqrt{8/15}$; the curves of wave resistance for α between $1 - \sqrt{8/15}$ and 0.3 (approximately) have two peaks, one at the point $f = 1$ and the other near the point $f = 0.55$. For values of $\alpha > 0.3$ the wave resistance increases monotonically with increasing velocity; the rate of increase of the wave resistance is considerable for Froude numbers f near the critical number $f = 1$.

In regard to the problem of the first part of the paper (sections 2 to 8), the results may be described as follows: When the motion of a circular cylinder under the surface of a fluid is studied, the cylinder may be replaced by a vortex if the ratio of the radius of the cylinder to the depth of submersion of its center is less than 1/3 (see table). In the composition of the flow and the lift force the principal part is played by the circulation of the velocity about the cylinder. However, by considering the general solution of the problem the terms obtained by Lamb and Havelock in their work are found. The effect of these terms is appreciable only for small depths of submersion of the cylinder; for larger depths the circulation of the stream velocity plays the fundamental part.

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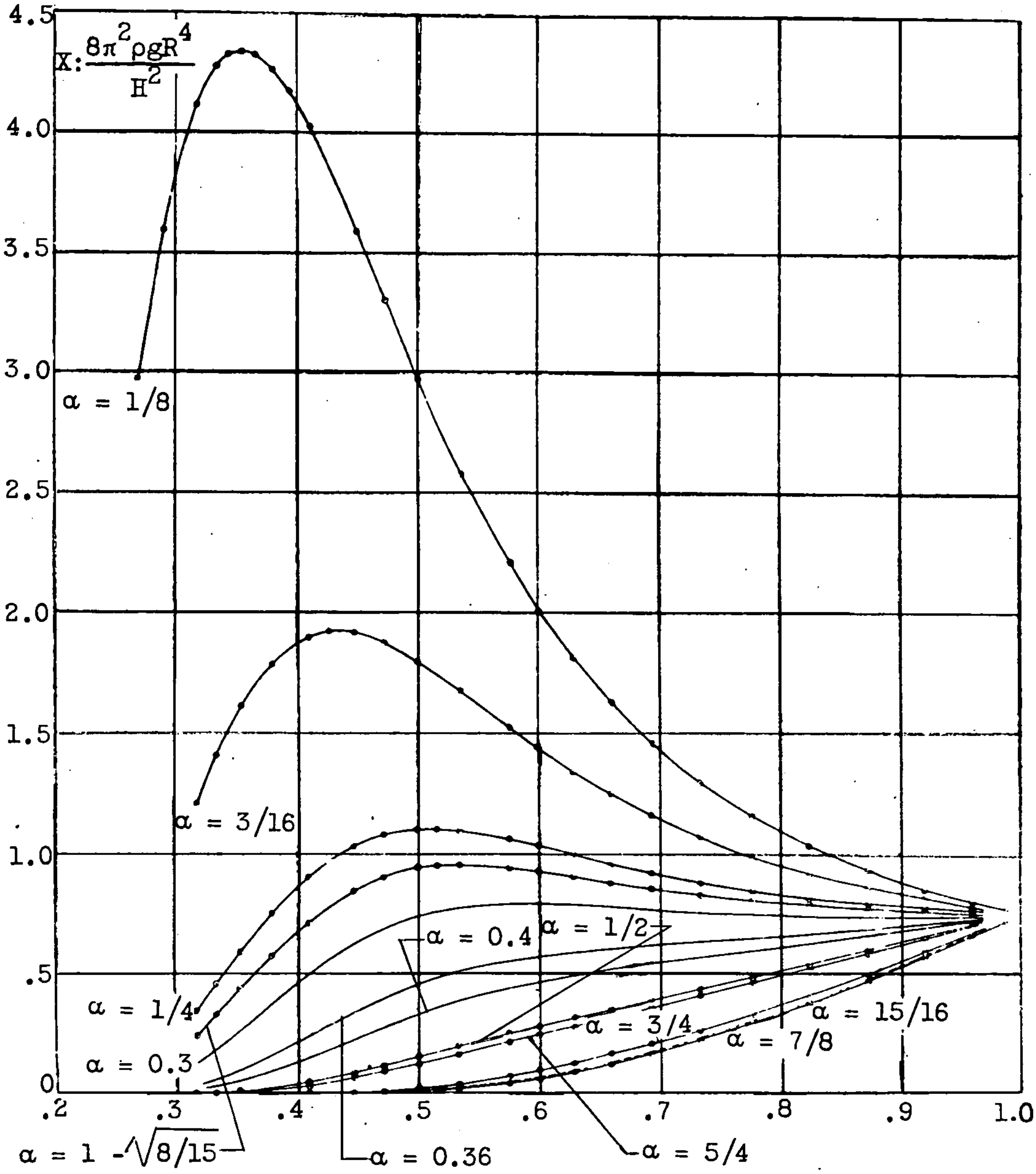

Figure 1.

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