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ON ROTATIONAL CONICAL FLOW

By Carlo Ferrari

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SUMMARY

The author determines some general properties of isoenergetic rotational conical fields. For such fields, provided the physical parameters of the fluid flow are known on a conical reference surface Σ, it being understood that they satisfy certain imposed conditions, it is shown how to construct the hodographs in the various meridional semiplanes, as the envelope of either the tangents to the hodographs or of the osculatory circles.

ANALYSIS

1. A method for determining the field of flow about a cone of revolution, the axis of which is aligned with the direction of the impinging supersonic stream, which is taken to have a uniform velocity distribution sufficiently far ahead of the body before the conical field is created, was developed by Busemann (reference 1) at an early date. Several years later the present author (reference 2) extended Busemann's procedure to cover the case of a cone of any shape whatsoever situated in the flow, so that its axis was at any arbitrary finite angle of attack; that investigation was confined, however, solely to irrotational conditions. With proper alterations, nevertheless, this treatment of the yawed arbitrary-shaped conical surface can be applied to the case of rotational flows. The purpose of the present investigation is just to give the relationships which permit one to draw the hodographs for the flow, in reference to the various meridional planes through the body axis, in the case where the motion is defined by a rotational field of conical flow with arbitrary specification of the cone shape.

2. Upon the mere assumption of isoenergetic flow in a perfect fluid, one may write the equations of motion in the form (reference 3):

$$\text{curl } \vec{V} \times \vec{V} = \frac{C^2}{\gamma R} \text{ grad } S$$

$$\text{div } \left[\left(V_l{}^2 - V^2 \right)^{\frac{1}{\gamma-1}} \vec{V} \right] = 0 \tag{1}$$

wherein R is the universal gas constant, γ is the adiabatic exponent, V_l is the limiting velocity obtained when the flow expands to a vacuum, C is the velocity of sound, S is the entropy, and $\vec{V}$ is the fluid velocity.

Upon employment of a spherical coordinate system (r,θ,φ), as depicted in figure 1, the components of $\dfrac{\text{curl } \vec{V}}{V_l}$ are taken as w_r, w_φ, and w_θ; where w_r is the radial component, w_φ is the component lying in the meridional plane and normal to the radius vector, while w_θ is the component that is perpendicular to the meridional plane. Likewise, the corresponding components of $\vec{V}/V_l$ are denoted by v_r, v_φ, and v_θ. Between these components there subsist the following relationships

$$\frac{1}{\sin \varphi} \left[\frac{\partial v_\varphi}{\partial \theta} - \frac{\partial (v_\theta \sin \varphi)}{\partial \varphi} \right] = rw_r$$

$$\frac{\partial (rv_\theta)}{\partial r} - \frac{1}{\sin \varphi} \frac{\partial v_r}{\partial \theta} = rw_\varphi \tag{2}$$

$$\frac{\partial v_r}{\partial \varphi} - \frac{\partial (rv_\varphi)}{\partial r} = rw_\theta$$

Based on the assumption that the flow is conical, the following scalar equations are derived in a straightforward way from the first of the equations (1):

$$w_\theta v_\varphi - w_\varphi v_\theta = 0$$

$$- v_\varphi w_r + w_\varphi v_r = \frac{c^2}{\gamma R} \frac{1}{r \sin \varphi} \frac{\partial s}{\partial \theta} \tag{3}$$

$$v_\theta w_r - v_r w_\theta = \frac{1}{r} \frac{c^2}{\gamma R} \frac{\partial s}{\partial \varphi}$$

wherein $c^2 = C^2/V_l{}^2$.

The set of equations (3) are not independent of each other as is evident from consideration of equation (1) directly, but they are interrelated through the expression

$$\vec{V} \times \text{grad } S = 0 \tag{4}$$

which, for a conical field, becomes

$$\frac{\partial S}{\partial \varphi} = - \frac{v_\theta}{v_\varphi} \frac{1}{\sin \varphi} \frac{\partial S}{\partial \theta} \tag{4'}$$

From the second of equations (1), upon use of the hypothesis that the flow is conical, and by taking into account the relationships expressed by equations (2) and (4'), one then obtains that

$$\left(1 - \frac{v_\varphi{}^2}{c^2}\right)\left(v_r + \frac{\partial v_\varphi}{\partial \varphi}\right) = \frac{2}{\sin \varphi} \frac{v_\varphi v_\theta}{c^2} \frac{\partial v_\varphi}{\partial \theta} - \cot \varphi \; v_\varphi \left(1 - \frac{v_\theta{}^2}{c^2}\right) -$$

$$\left(1 - \frac{v_\theta{}^2}{c^2}\right)\left(v_r + \frac{1}{\sin \varphi} \frac{\partial v_\theta}{\partial \theta}\right) - \frac{v_\varphi}{\gamma R} \frac{\partial S}{\partial \varphi} - \frac{v_\varphi v_r}{c^2} r w_\theta \tag{5}$$

This expression differs only by the presence of the rotationality terms from the analogous relationship derived in reference 2 previously mentioned.

3. By means of equations (3), (4), and (4') one may deduce some interesting properties of conical fields. Let it be assumed that one of the stream surfaces of the flow is conical (this will be the case for a field of flow arising by the action of a uniform supersonic stream impinging from any direction whatsoever upon a conical-shaped obstacle); it will be convenient to designate this surface as Σ_c. Let the versor of any arbitrary general one of the generatrices of the conical surface Σ_c be denoted by $\vec{r}$, then grad $S \times \vec{r} = 0$. On the other hand, if the versor of the tangent to any streamline whatsoever that is traced upon the surface of the cone Σ_c is denoted by $\vec{\tau}$ then it is true, in addition, that grad $S \times \vec{\tau} = 0$. It is evident, therefore, that grad S is perpendicular to the surface Σ_c; that is, the above-described conical surface is a surface of constant entropy.

Besides, let it be assumed that the conical flow is symmetric with respect to the meridional plane $\theta = \pm 90°$ (this will be the case already mentioned for the field of flow about a conical-shaped obstacle). At all the points of this plane it is true that $v_\theta = 0$. One then deduces, upon the basis of the first of equations (3), that $w_\theta = 0$ provided that v_φ is not zero everywhere. On account of this, and through utilizing the third of equations (3) it follows that $\frac{\partial S}{\partial \varphi} = 0$. Thus even the meridional plane of symmetry for the conical field is itself a constant entropy surface, and at this plane the flow is irrotational as is easily deduced upon taking cognizance of equations (2).

In conjunction with the result obtained above one can derive from this latter fact that, for the case of flow about a conical obstacle, the shock wave in the two semiplanes $\theta = 90°$ and $\theta = -90°$ must produce the same change in direction of the stream velocities; and so the tangents to the trace of the shock wave in these semiplanes are symmetrically inclined with respect to the undisturbed stream velocity vector. Now let us consider an obstacle in the form of a right circular cone. Upon the surface of this cone it is true that $\frac{\partial S}{\partial \theta} = v_\varphi = 0$, and therefore one gets that $w_\varphi = 0$. Thus the following relationship results

$$v_\theta = \frac{1}{\sin \varphi} \frac{\partial v_r}{\partial \theta} \tag{6}$$

On the cone one can always express the v_r values as a periodic function of θ, and thus $v_r = B_0 + \Sigma\, B_n \sin n\theta$. It follows that on the cone's surface the peripherial velocity component is given by

$$v_\theta = \frac{1}{\sin \varphi}\, \Sigma\, B_n n \cos n\theta$$

just as in the case of irrotational flow.

Now, if we let the angle of incidence of the axis of the cone be denoted by β, then the expression for v_r becomes simply

$$v_r = B_0 + B_1 \sin \theta$$

if only terms of the order of magnitude of β are taken into account. Since this is true, then because B_1 is proportional to β, the B_n coefficients have to be at least as small as β^2.

The relationship given by equation (6) may be generalized for the case of a cone of any shape whatsoever. It is assumed for this purpose that the cone's surface is divided up by a network of orthogonal coordinate lines σ_1 and σ_2 (r = const. and φ = const., respectively). The former of which are the intersections of the spheres with radius r upon the cone under consideration, while the latter are the generatrices of the cone itself. At an arbitrary general point P on the cone the length of the linear element $d\sigma_1$ can be written as: $d\sigma_1 = rh_1 (\theta)\, d\theta$ while the length of the linear element $d\sigma_2$ along the line σ_2 is given by: $d\sigma_2 = dr$.

The component, in the direction of the normal to the cone at the point P, of the curl is

$$w_n = \frac{1}{rh_1}\, \frac{\partial}{\partial r}\left(v_1 h_1 r\right) - \frac{\partial v_2}{\partial \theta}$$

where v_1 and v_2 are now the velocity components in the direction of σ_1 and σ_2, respectively. On the other hand, upon referring to the first of equations (1), it is still true that $w_n = 0$, and on account of this it is evident that:

$$v_1 = \frac{1}{h_1} \frac{\partial v_2}{\partial \theta} \tag{6'}$$

If the component of velocity in the direction of the radius vector is expressed as a periodic function of θ, as is still permissible, then equation (6') immediately furnishes the means of obtaining the corresponding expression for v_1.

4. It is now easy to determine how to continue the construction of the flow field downstream of a given conical reference surface, Σ, upon which the physical conditions of the flow are assumed known. Let the equation of the conical reference surface, Σ, be given by

$$\varphi = \varphi(\theta) \tag{7}$$

From the relationship

$$\left(\frac{dS}{d\theta}\right)_\Sigma = \frac{\partial S}{\partial \theta} + \dot{\varphi} \frac{\partial S}{\partial \varphi}$$

wherein

$$\dot{\varphi} = \frac{d\varphi}{d\theta}$$

one obtains

$$\frac{\partial S}{\partial \varphi} = v_\theta \frac{\left(\frac{dS}{d\theta}\right)_\Sigma}{v_\theta \dot{\varphi} - v_\varphi \sin \varphi} \tag{8}$$

provided φ is expressed as a function of θ as in equation (7). The above relationship allows one to calculate $\frac{\partial S}{\partial \varphi}$ at the points of the surface Σ.

In like manner, by use of the equation

$$\left(\frac{\partial v_\varphi}{\partial \theta}\right)_\Sigma = \frac{\partial v_\varphi}{\partial \theta} + \dot{\varphi} \frac{\partial v_\varphi}{\partial \varphi}$$

and by setting

$$\frac{\partial v_\phi}{\partial \phi} + v_r = R_1 \qquad (9)$$

it is found that

$$\frac{\partial v_\phi}{\partial \theta} = \left(\frac{dv_\phi}{d\theta}\right)_\Sigma - \dot\phi R_1 + \dot\phi v_r = A_1 - \dot\phi R_1 \qquad (10)$$

wherein the A_1 are calculated at the points of the conical surface through means of the relation

$$A_1 = \left(\frac{dv_\phi}{d\theta}\right)_\Sigma + \dot\phi v_r \qquad (10')$$

Finally, it is found (the intermediate steps are omitted, and just the final result presented) that:

$$\frac{\partial v_\theta}{\partial \theta} = A_2 + \frac{\dot\phi^2}{\sin \phi} R_1$$

and

$$r w_\theta = R_2 = \frac{\partial v_r}{\partial \phi} - v_\phi = \frac{v_\theta^2}{v_\phi} - \frac{v_\theta}{v_\phi} \frac{1}{\sin \phi} A_3 \qquad (11)$$

where the quantities A_2 and A_3 are the expressions:

$$A_2 = \left(\frac{dv_\theta}{d\theta}\right)_\Sigma - \frac{\dot\phi}{\sin \phi} \left(-v_\theta \cos \phi + A_1 - \frac{c^2}{\gamma R} \frac{1}{v_\theta} \frac{\partial S}{\partial \phi} \sin \phi\right.$$

$$\left. - \frac{v_r}{v_\phi} v_\theta \sin \phi + \frac{v_r}{v_\phi} A_3\right) \qquad (11')$$

$$A_3 = \frac{\partial v_r}{\partial \theta} = \frac{v_\varphi \left(\frac{dv_r}{d\theta}\right)_\Sigma - \dot{\varphi}\left(v_\varphi{}^2 + v_\theta{}^2\right)}{v_\varphi \sin \varphi - \dot{\varphi}\, v_\theta} \sin \varphi \qquad (11'')$$

and their values are known on the reference conical surface, Σ, and thus so also is the value of R_2.

By means of equation (5), therefore, one obtains

$$\left[1 - \frac{v_\varphi{}^2}{c^2} + \frac{2}{\sin \varphi}\frac{v_\theta\, v_\varphi}{c^2}\dot{\varphi} + \left(1 - \frac{v_\theta{}^2}{c^2}\right)\frac{1}{\sin^2 \varphi}\dot{\varphi}^2\right] R_1 =$$

$$\frac{2}{\sin \varphi}\frac{v_\theta v_\varphi}{c^2} A_1 - v_\varphi\left(1 + \frac{v_\theta{}^2}{c^2}\right)\cot \varphi - \left(1 - \frac{v_\theta{}^2}{c^2}\right)\left(v_r + \frac{A_2}{\sin \varphi}\right) -$$

$$\frac{v_\varphi}{\gamma R}\frac{\partial S}{\partial \varphi} - \frac{v_\varphi v_r}{c^2} r w_\theta \qquad (12)$$

Thus it is possible to calculate the values of R_1 at the points of the reference conical surface, Σ. This formula is the natural extension of the analogous relationship already derived in reference 2 in the case of an irrotational flow.

The complete solution of the problem as to how to continue constructing the flow field downstream of the reference surface Σ is thus presented by the formulations given as equations (8), (11), (12), and the additional equation (13), since it is also evident that

$$\frac{\partial v_\theta}{\partial \varphi} = \frac{1}{\sin \varphi}\left[A_1 - \dot{\varphi}R_1 - v_\theta \cos \varphi - \frac{c^2}{\gamma R}\frac{1}{v_\theta}\frac{\partial S}{\partial \varphi}\sin \varphi\right] \qquad (13)$$

5. The graph of the hodograph corresponding to an arbitrary general semiplane, θ = const., is represented in figure 2. Let P be any point whatsoever on this hodograph, and let $\vec{R}$ signify the vector $\vec{R} = R_1\vec{r} - R_2\vec{\tau}$. In addition let $\vec{r}$ again denote the versor that has the sense and direction of the radius vector in the semiplane in question, and let $\vec{\tau}$ now be the versor normal to $\vec{r}$ in the semiplane and oriented in the sense of an increasing φ. With these conventions it follows that $\vec{R} \times \dfrac{d\vec{P}}{d\varphi} = 0$ and therefore it becomes clear that the tangent to the hodograph at the point P is perpendicular to the above-defined vector $\vec{R}$. Thus this tangent is inclined to the direction of $\vec{r}$ by an angle

$$\chi = \tan^{-1} \frac{R_2}{R_1} \tag{14}$$

If the linear element of length along the hodograph is denoted by ds then the absolute value of $ds/d\varphi$ is given by

$$\left|\frac{ds}{d\varphi}\right| = \sqrt{R_1{}^2 + R_2{}^2}$$

Thanks to the formulae developed in the preceding sections the drawing in of the tangent to the hodograph at the point P, situated on this hodograph, is therefore made possible, since everything is known about the point P. If the element of length along the hodograph is calculated as

$$ds = \sqrt{R_1{}^2 + R_2{}^2}\, d\varphi$$

then it is easy to find another neighboring point P'. When used repetitiously in the various semiplanes, this procedure allows one to determine the respective hodographs as the envelope of their tangent lines.

The angle of intersection $d\alpha$ with respect to the direction of the linear element of the hodograph ds is thus determined by

$$d\alpha = d\varphi \left(1 - \frac{dX}{d\varphi}\right) = d\varphi \left(1 - \frac{R_1 \frac{dR_2}{d\varphi} - R_2 \frac{dR_1}{d\varphi}}{R_1^2 + R_2^2}\right)$$

The radius of curvature of the hodograph at the point P just mentioned turns out to be consequently

$$R_c = \frac{\left(R_1^2 + R_2^2\right)^{3/2}}{R_1\left(R_1 - \frac{dR_2}{d\varphi}\right) + R_2\left(R_2 + \frac{dR_1}{d\varphi}\right)} \tag{15}$$

When the point P' is determined in the manner that was described above, in consequence of having available all information about the known point P, it follows that R_1 and R_2 are also known at the point P', and thus the values of $\frac{dR_1}{d\varphi}$ and $\frac{dR_2}{d\varphi}$ are calculable. Through means of equation (15) the radius of curvature R_c is then determined, using these values of $\frac{dR_1}{d\varphi}$ and $\frac{dR_2}{d\varphi}$. The direction of the principal normal as defined by equation (14) is also known, and thus the hodograph can be obtained in every meridional semiplane as the envelope of the respective osculatory circles.

6. The results obtained here for the determination of the conical field of flow in the case of rotational motion are employed in an exactly analogous way as described in reference 2 when making a numerical application. In general it will be convenient to assume as the reference surface, Σ, upon which the initial values are taken to be known, the conical surface of the shock wave. The required information about the shock wave may be obtained in first approximation by utilization of the hypothesis that the flow is irrotational.

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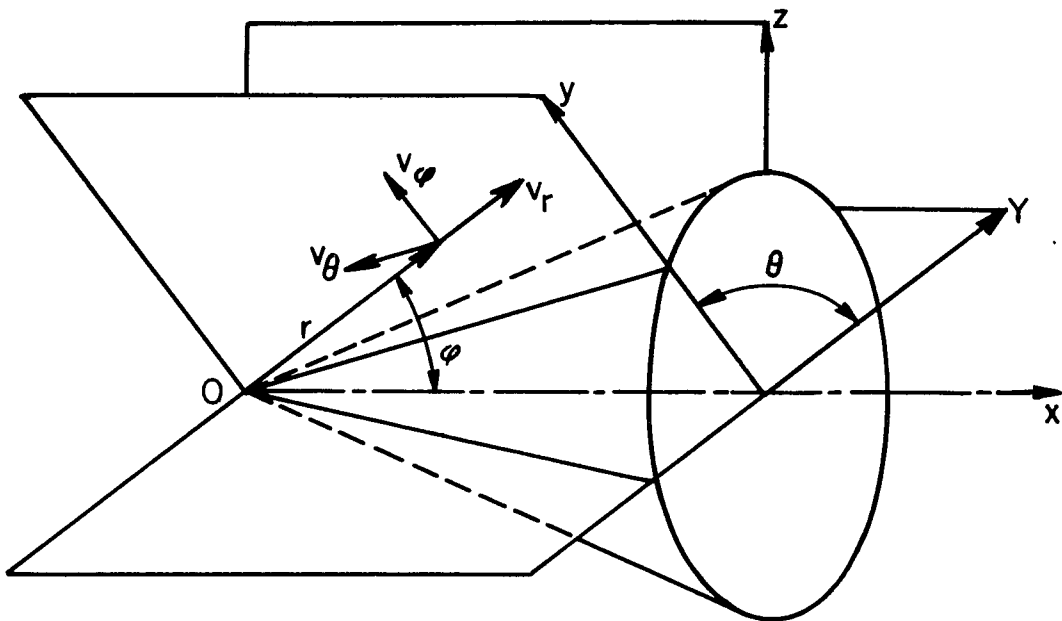   ser. 6, fasc. 2, Feb. 1936.

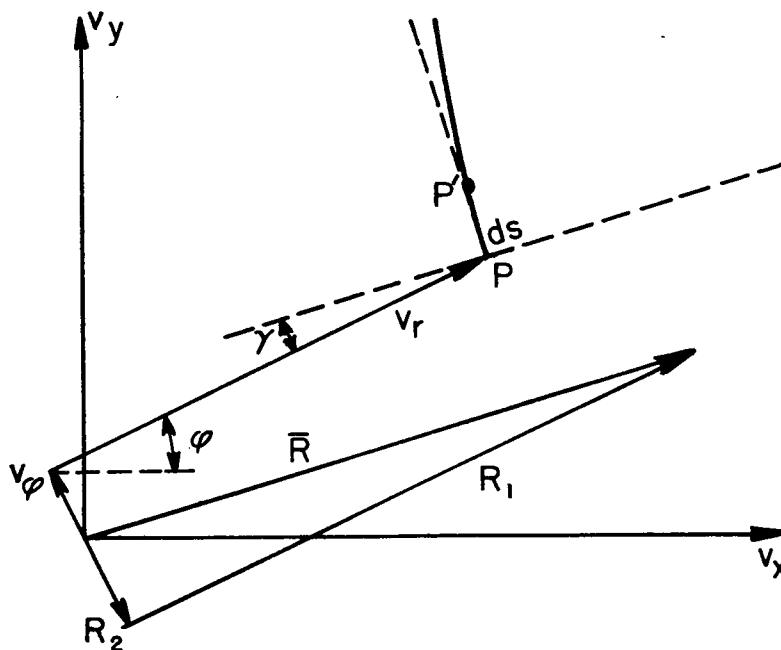Figure 1.- Coordinate and vectorial orientation in the spherical coordinate system.

Figure 2.- Construction in the hodograph plane for finding velocities at P' when they are known at P.