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Thin-walled rods of open section in deforming under load do not always follow the same laws which have been established for solid closed section rods. For this reason there has arisen in recent years the practically important problem of the formulation of an acceptable theory of thin-walled rods, in the solution of which fundamental results have been attained in the Soviet Union (V. Z. Vlasov).

This problem is dealt with also in the present paper, which differs from previous work in that the theory of thin-walled rods is constructed without the employment of special assumptions on the basis of a qualitative analysis of the integrals of the equations of the theory of shells. The object of the investigation is to determine approximately the principal stress state in a rod which is loaded along its length by a transverse force R and a system of forces and moments T applied at the end sections. It is assumed that the end sections of the rod are fixed arbitrarily and that the longitudinal edges are free of connections.

In making use of the term "principal stress state" we mean to say that we are not interested in the local stress states arising at the ends and which reduce to zero as the distance from them increases. This assumption lies also at the basis of the theory of solid rods. In the case of thin-walled rods, however, it leads to a more marked deviation from the true conditions because the local stresses reduce to zero less rapidly the less the wall thickness of the rod.

The rod is considered as a long cylindrical shell of arbitrary contour, the thickness of which may vary in the transverse direction. The transverse load R is assumed arbitrarily, but such that similarity is obtained in all cross-sections and the end forces and moments T are varied in a suitable manner by the distributed normal and shear forces. The problem posed is the following: From the system of integrals of the complete system of equations of a shell to separate out those integrals which correspond to the principal stress states of the rods.

*"O teorii tonkostennykh sterzhnei." Prikladnaya Matematika i Mekhanika, Vol. XIII, Nov.-Dec. 1949, pp. 561-596.

There is first of all investigated a particular integral and it is shown that the transverse load in the case, and only in the case, where it varies linearly over the length of the shell-rod may be represented in the form of two components $R = R_1 + R_2$ in such manner that for each of them separately an approximate value of the particular integral may be found by an elementary method; R_1, the part of the total load statically equivalent at each cross-section to the entire load R, gives the particular integral for which on the longitudinal sections only shearing stresses arise, and R_2, the remaining statically self-balanced part of the load in each section, gives the particular integral for which the normal and transverse stresses in the longitudinal sections play the fundamental part.

There are further sought such particular integrals of the homogeneous system in which

(a) The transverse forces and moments on the longitudinal edges may with sufficient accuracy be assumed equal to zero.

(b) The intensity of the stresses in the cross-sections drops at a considerably slower rate than in the case of other integrals.

The class of such particularly slowly damped stress states is found to be wider than that which results from the principle of Saint-Venant because there is added the statically self-balanced solution in which the normal stresses in the cross-section are distributed according to the law of sectorial areas. This stress state does not reduce to zero in a shell type rod that is not too long and must therefore be considered as a principal one. It is in the very presence of this stress state that open section rods differ from solid closed section rods. When the length of the rod-shell exceeds a certain limit, however, a thin-walled rod in the fundamental character of its stress state ceases to differ from a solid rod although of course a difference is maintained in the character and rate of reduction of the local stress states.

These considerations lead to the imposing of restrictions on the upper limit of the length of a thin-walled rod if we wish to consider it as a structure which behaves in a fundamentally different manner from a solid rod. The small rate of damping of the local stress states imposes restrictions also on the lower limit of length of the rod-shell, namely the length must be sufficient for the local stresses to reduce to a sufficient degree at the center sections.

Assuming that both these limitations are satisfied and that the transverse load R varies linearly over the longitudinal direction, a theory of computation of thin-walled rods may be constructed on the assumption that their principal stress state is with a certain accuracy described by a linear combination of the particular integral of a nonhomogeneous system of equations of the rod and very slowly damping solutions of the homogeneous equations. The boundary conditions at the transverse ends of the rod-shell can not in this case be set up at each point but must be replaced by integral conditions. The method resulting from such approach of the computation of short rods leads to practical formulas. Rods of medium length, occupying in the character of their behavior an intermediate position between short and long rods (the latter do not require investigation since they may be considered as solid) were not capable of being investigated to the end.

The method thus obtained of the computation of short thin-walled rods was found to be more complicated than that arrived at by V. Z. Vlasov (reference 1). These two methods approach each other considerably if the equations here presented are simplified by rejecting the components which take into account the effect of shear. Even with this simplification, however, agreement in the computational relations is not complete. Specifically, the equation determining in the theory of V. Z. Vlasov the torsion of a thin-walled rod is not confirmed.

As regards the fundamental assumption made in the theory of Vlasov that the cross-section of the rod maintains its shape, it is not in itself true, nor is there a need for such an assumption. The use of the assumption does not, however, lead to errors in the computations of stresses because the principal stress state is affected only by those deformations for which the cross-section does not vary.

We may remark in conclusion that in the Soviet Union (A. R. Adadurov) and later abroad (Kármán) a theory was worked out of the computation of cylindrical shells, the cross-sectional contour of which can not deform due to the presence of a large number of diaphragms. This structure must be distinguished from a thin-walled rod. For this reason it is not possible from our point of view to agree with G. Y. Dzhanelidze and Y. G. Panovko (reference 2) who consider the theory of A. R. Adadurov as a generalization of the theory of V. Z. Vlasov based on the fact that Adadurov rejected the assumption on the absence of shear. The theory of thin-walled rods is in principle different from the theory of shells with transverse forces because for the former there is sought only the principal stress state while for the latter it is necessary to investigate also the local stress states. It is due to this fact and not to the fact that the assumption on the absence of shear is rejected in the theory of shells with diaphragms that the conditions on the transverse ends are set up at each point. In the present paper, formulas are given for the computation of thin-walled rods with account taken of shear deformations. In view of what was said above, however, they do not agree with the results of A. R. Adadurov.

1. FUNDAMENTAL EQUATIONS

We refer the center of area of the shell to the lines of curvature and introduce the nondimensional parameters

$$\alpha = \frac{\xi}{r} \qquad \beta = \frac{s}{r}$$

where ξ is the distance along the generatrix, s the distance along the director curve, and r the mean radius of curvature of the cross-section of the cylinder. The complete systems of equations defining the elastic equilibrium of the shell can then, in the notation of Love (reference 3), be written as:

A. The Equations of Equilibrium

$$\frac{\partial T_1}{\partial \alpha} - \frac{\partial S_2}{\partial \beta} + rX = 0 \qquad \frac{\partial G_1}{\partial \alpha} + \frac{\partial H_2}{\partial \beta} - rN_1 = 0$$

$$\frac{\partial}{\partial \alpha}\left(S_1 + \frac{H_2}{rR}\right) + \frac{\partial T_2}{\partial \beta} - \left(\frac{N_2}{R} + \frac{\partial}{\partial \alpha}\frac{H_2}{rR}\right) + rY = 0 \qquad \frac{\partial H_1}{\partial \alpha} - \frac{\partial G_2}{\partial \beta} + rN_2 = 0$$

$$\frac{T_2}{R} + \frac{\partial N_1}{\partial \alpha} + \frac{\partial N_2}{\partial \beta} + rZ = 0 \qquad S_1 + S_2 + \frac{H_2}{rR} = 0$$

where R is a nondimensional magnitude equal to the ratio of the radius of curvature of the cross-section R_2 to its mean value r. (For convenience of presentation the second equation in these relations is presented in an unusual form by the formal introduction of the torsional moment H_2.)

B. The Equations of Continuity of the Deformations

$$\frac{\partial \kappa_2}{\partial \alpha} - \frac{\partial \tau}{\partial \beta} = 0 \qquad -\frac{\partial \tau}{\partial \alpha} + \frac{\partial \kappa_1}{\partial \beta} - \left(\frac{\zeta_1}{R} + \frac{\partial}{\partial \alpha}\frac{\gamma}{2rR}\right) = 0 \qquad \frac{\kappa_1}{R} - \frac{\partial \zeta_2}{\partial \alpha} + \frac{\partial \zeta_1}{\partial \beta} = 0$$

$$\frac{\partial \epsilon_2}{\partial \alpha} - \frac{\partial}{\partial \beta}\frac{\gamma}{2} + r\zeta_2 = 0 \qquad \frac{\partial}{\partial \alpha}\frac{\gamma}{2} - \frac{\partial \epsilon_1}{\partial \beta} + r\zeta_1 = 0$$

(The shear deformation we shall denote, in contrast to Love, by γ.)

The number of continuity equations has, by introducing the magnitudes ζ_1, ζ_2 been increased to five, although actually they should be three in number, since they are derived from the Codacci-Gauss equations. These relations are reduced to the usual form if ζ_1 and ζ_2 are eliminated from the first three equations with the aid of the following two. It is in such form of a variation of the Codacci-Gauss relations that they have first been derived by us for the general case (reference 4).

C. Elasticity Relations

$$2Eh\eta\epsilon_1 = T_1 - \sigma T_2 \qquad 2Eh\eta\epsilon_2 = T_2 - \sigma T_1 \tag{1.1}$$

$$2Eh\eta\frac{\gamma}{2} = -(1 + \sigma)S_2 \tag{1.2}$$

$$G_1 = -\frac{h^2}{3\left(1 - \sigma^2\right)} 2Eh\eta^3(\kappa_1 + \sigma\kappa_2), \quad G_2 = -\frac{h^2}{3\left(1 - \sigma^2\right)} 2Eh\eta^3(\kappa_2 + \sigma\kappa_1) \tag{1.3}$$

$$H_1 = -H_2 = \frac{h^2}{3(1 + \sigma)} 2Eh\eta^3\tau \tag{1.4}$$

where $2h\eta(\beta)$ denotes the variable thickness of the wall of the shell, $2h$ being a constant equal to the mean thickness.

In the elasticity relations the force S_1 does not enter; it is assumed that it is connected with S_2 and H_2 by the six equations of equilibrium.

D. Geometrical Relations

$$\epsilon_1 = \frac{1}{r}\frac{\partial u}{\partial \alpha} \qquad \epsilon_2 = \frac{1}{r}\left(\frac{\partial v}{\partial \beta} - \frac{w}{R}\right) \qquad \gamma = \frac{1}{r}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right) \tag{1.5}$$

$$\kappa_1 = \frac{1}{r^2}\frac{\partial^2 w}{\partial \alpha^2} \qquad \kappa_2 = \frac{1}{r^2}\left(\frac{\partial^2 w}{\partial \beta^2} + \frac{\partial}{\partial \beta}\frac{v}{R}\right) \qquad \tau = \frac{1}{r^2}\frac{\partial}{\partial \alpha}\left(\frac{\partial w}{\partial \beta} + \frac{v}{R}\right) \tag{1.6}$$

2. AUXILIARY RELATIONS

In order not to interrupt the discussion in what follows, we shall introduce certain transformations.

1. If the components of the tangential deformation ϵ_1, ϵ_2, γ are assumed known, there is no difficulty in determining the forces, moments, and components of the flexural deformation κ_1, κ_2, τ: with the aid of the three last equations of equilibrium, the continuity equations, and the elasticity relations, we obtain

$$\left.\begin{aligned}
&T_1 = \frac{2Eh\eta}{1 - \sigma^2}\left(\epsilon_1 + \sigma\epsilon_2\right) \qquad T_2 = \frac{2Eh\eta}{1 - \sigma^2}\left(\epsilon_2 + \sigma\epsilon_1\right) \qquad S_2 = -\frac{2Eh\eta}{(1 + \sigma)}\frac{\gamma}{2}\\
&r\kappa_1 = R\frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial\beta}\frac{\gamma}{2} - \frac{\partial\epsilon_2}{\partial\alpha}\right) - R\frac{\partial}{\partial\beta}\left(\frac{\partial\epsilon_1}{\partial\beta} - \frac{\partial}{\partial\alpha}\frac{\gamma}{2}\right)\\
&r\tau = -\int L\left(\frac{\partial\epsilon_1}{\partial\beta}\right)d\alpha + L(\gamma) - \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}R\epsilon_2\\
&r\kappa_2 = -\int d\alpha\int\frac{\partial}{\partial\beta}L\left(\frac{\partial\epsilon_1}{\partial\beta}\right)d\alpha + \int\frac{\partial}{\partial\beta}L(\gamma)d\alpha - \frac{\partial^2}{\partial\beta^2}R\epsilon_2\\
&\frac{1}{r}G_1 = -\frac{h^2\eta^3}{3\left(1 - \sigma^2\right)r^2}\left\{R\frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial\beta}\frac{\gamma^*}{2} - \frac{\partial\epsilon_2^*}{\partial\alpha}\right) - R\frac{\partial}{\partial\beta}\left(\frac{\partial\epsilon_1^*}{\partial\beta} - \frac{\partial}{\partial\alpha}\frac{\gamma^*}{2}\right) + \right.\\
&\qquad\left.\sigma\left[-\int d\alpha\int\frac{\partial}{\partial\beta}L\left(\frac{\partial\epsilon_1^*}{\partial\beta}\right)d\alpha + \int\frac{\partial}{\partial\beta}L(\gamma^*)d\alpha - \frac{\partial^2}{\partial\beta^2}R\epsilon_2^*\right]\right\}\\
&\frac{1}{r}G_2 = -\frac{h^2\eta^3}{3\left(1 - \sigma^2\right)r^2}\left\{\left[-\int d\alpha\int\frac{\partial}{\partial\beta}L\left(\frac{\partial\epsilon_1^*}{\partial\beta}\right)d\alpha + \int\frac{\partial}{\partial\beta}L(\gamma^*)d\alpha - \frac{\partial^2}{\partial\beta^2}R\epsilon_2^*\right] + \right.\\
&\qquad\left.\sigma\left[R\frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial\beta}\frac{\gamma^*}{2} - \frac{\partial\epsilon_2^*}{\partial\alpha}\right) - R\frac{\partial}{\partial\beta}\left(\frac{\partial\epsilon_1^*}{\partial\beta} - \frac{\partial}{\partial\alpha}\frac{\gamma^*}{2}\right)\right]\right\}\\
&\frac{1}{r}H_1 = -\frac{1}{r}H_2 = \frac{h^2\eta^3}{3(1 + \sigma)r^2}\left\{-\int L\left(\frac{\partial\epsilon_1^*}{\partial\beta}\right)d\alpha + L(\gamma^*) - \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}R\epsilon_2^*\right\}
\end{aligned}\right\} \quad (2.1)$$

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$$
\left.
\begin{aligned}
N_1 &= -\frac{h^2\eta^3}{3\left(1-\sigma^2\right)r^2}\left\{R\,\frac{\partial^2}{\partial\alpha^2}\left(\frac{\partial}{\partial\beta}\,\frac{\gamma^*}{2}-\frac{\partial\epsilon_2^*}{\partial\alpha}\right)-R\,\frac{\partial^2}{\partial\alpha\,\partial\beta}\left(\frac{\partial\epsilon_1^*}{\partial\beta}-\frac{\partial}{\partial\alpha}\,\frac{\gamma^*}{2}\right)+\right.\\
&\qquad\left.\left[-\int\frac{\partial}{\partial\beta}\,L\left(\frac{\partial\epsilon_1^*}{\partial\beta}\right)d\alpha+\frac{\partial}{\partial\beta}\,L(\gamma^*)-\frac{\partial^3}{\partial\alpha\,\partial\beta^2}\,R\epsilon_2^*\right]\right\}\\
N_2 &= -\frac{h^2}{3\left(1-\sigma^2\right)r^2}\left\{\left[-\int d\alpha\int\frac{\partial}{\partial\beta}\,\eta^3\,\frac{\partial}{\partial\beta}\,L\left(\frac{\partial\epsilon_1^*}{\partial\beta}\right)d\alpha+\int\frac{\partial}{\partial\beta}\,\eta^3\,\frac{\partial}{\partial\beta}\,L(\gamma^*)\,d\alpha-\right.\right.\\
&\qquad\left.\frac{\partial}{\partial\beta}\,\eta^3\,\frac{\partial^2}{\partial\beta^2}\,R\epsilon_2^*\right]+\sigma\,\frac{\partial}{\partial\beta}\,\eta^3\left[R\,\frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial\beta}\,\frac{\gamma^*}{2}-\frac{\partial\epsilon_2^*}{\partial\alpha}\right)-R\,\frac{\partial}{\partial\beta}\left(\frac{\partial\epsilon_1^*}{\partial\beta}-\frac{\partial}{\partial\alpha}\,\frac{\gamma^*}{2}\right)\right]+\\
&\qquad\left.\eta^3(1-\sigma)\left[-L\left(\frac{\partial\epsilon_1^*}{\partial\beta}\right)+\frac{\partial}{\partial\alpha}\,L(\gamma^*)-\frac{\partial^2}{\partial\alpha^2}\,\frac{\partial}{\partial\beta}\,R\epsilon_2^*\right]\right\}
\end{aligned}
\right\}\quad(2.1)
$$

In the above equations (2.1) there has been set for briefness

$$\epsilon_1^* = 2Eh\epsilon_1 \qquad \epsilon_2^* = 2Eh\epsilon_2 \qquad \gamma^* = 2Eh\gamma \tag{2.2}$$

and by L we understand the linear differential operator

$$L = \frac{\partial}{\partial\beta}\,R\,\frac{\partial}{\partial\beta} + \frac{1}{R} \tag{2.3}$$

2. In what follows a large part will be played by the linear differential equation

$$\frac{\partial}{\partial\beta}\,L\left(\frac{\partial p}{\partial\beta}\right) = 0 \tag{2.4}$$

where L is defined by equation (2.3). The solution of equation (2.4) is based on the fact that if in the operator L a change is made in the independent variables by the formula

$$d\chi = \frac{d\beta}{R}$$

it assumes the form:

$$L = \frac{1}{R}\left(\frac{d^2}{d\chi^2} + 1\right) \tag{2.5}$$

Hence the general integral of equation (2.4) will be

$$p(\alpha,\beta) = A_1(\alpha) + A_2(\alpha)\frac{x}{r} + A_3(\alpha)\frac{y}{r} + A_4(\alpha)\frac{\omega}{r^2} \tag{2.6}$$

where

$$x = r\int \cos\chi \, d\beta \qquad y = r\int \sin\chi \, d\beta \qquad \omega = r\int (x\sin\chi - y\cos\chi)d\beta$$

and A_1, A_2, A_3, A_4 are arbitrary functions of integration. The geometrical meaning of the magnitudes χ, x, y, ω is obvious. If we refer the cross-section of the shell to a cartesian system of coordinates OXY, then χ may be interpreted as the angle between the tangent to the contour of the cross-section and the X-axis; x and y will be equal to the abscissa and ordinate of the point considered and ω represents the so-called sectorial area, that is, the area of the sector bounded by the contour of the cross-section and two rays issuing from an arbitrary point taken outside the contour (sectorial center). These two straight lines determine on the contour of the cross-section two points, one of which (the origin from which the computation is made) may be arbitrarily chosen.

We introduce the notation

$$2hr\int_0^{\beta_0} x\eta \, d\beta = S_y \qquad 2hr\int_0^{\beta_0} y\eta \, d\beta = S_x \qquad 2hr\int_0^{\beta_0} \omega\eta \, d\beta = S_\omega$$

$$2hr\int_0^{\beta_0} x^2\eta \, d\beta = I_y \qquad 2hr\int_0^{\beta_0} y^2\eta \, d\beta = I_x$$

$$2hr\int_0^{\beta_0} \eta \, d\beta = F \qquad 2hr\int_0^{\beta_0} \omega^2\eta \, d\beta = I_\omega$$

$$2hr\int_0^{\beta_0} xy\eta \, d\beta = I_{xy} \qquad 2hr\int_0^{\beta_0} x\omega\eta \, d\beta = I_{x\omega} \qquad 2hr\int_0^{\beta_0} y\omega\eta \, d\beta = I_{y\omega}$$

(2.7)

Here and in what follows $\beta = 0$ and $\beta = \beta_0$ are the equations of two straight edges of the shell.

The physical meaning of the magnitudes introduced is the following: F is the cross-sectional area of the shell, S_x, S_y, S_ω are the static moments relative to the axes X and Y and the sectorial static moment, I_x, I_y, I_ω are the moments of inertia relative to the axes X and Y and the sectorial moment of inertia, and I_{xy} is the polar moment of inertia relative to the axes XY; the magnitudes $I_{x\omega}$, $I_{y\omega}$ need not be given any special name; for us it is of importance only that under certain conditions they become zero.

To a transfer of the origin of coordinates and a rotation of the axes of the cartesian system of coordinates, and to a change of the sectorial center and origin of sectorial areas, there correspond various changes of the arbitrary functions of integration A_1, A_2, A_3, A_4 in equation (2.6). These may be made use of in order to transform in the required manner the solution (2.6).

The further equations assume the most compact form if we make the assumptions that (a) the origin of coordinates OXY coincides with the center of gravity of the cross-section of the shell; (b) the coordinate axes coincide with the principal directions of the cross-sections; (c) the flexural center is taken as the sectorial center; (d) the point from which the sectorial areas computed coincides with the sectorial zero point.

From conditions (a) and (b) there follow the well known relations:

$$S_x = S_y = I_{xy} = 0 \tag{2.8}$$

Conditions (c) and (d), as shown in the monograph (reference 1), are equivalent to the relations

$$I_{x\omega} = I_{y\omega} = S_\omega = 0 \tag{2.9}$$

Writing equations (2.8) and (2.9) in explicit form we obtain

$$\left.\begin{aligned}\int_0^{\beta_0} x\eta\ d\beta = \int_0^{\beta_0} y\eta\ d\beta = \int_0^{\beta_0} \omega\eta\ d\beta = 0 \\ \int_0^{\beta_0} xy\eta\ d\beta = \int_0^{\beta_0} x\omega\eta\ d\beta = \int_0^{\beta_0} y\omega\eta\ d\beta = 0\end{aligned}\right\} \tag{2.10}$$

This means that the form of the integral (2.6) thus chosen is characterized by the fact that it is a linear combination of four particular integrals 1, x, y, ω having the property of being mutually orthogonal with weight η.

3. Let us consider the problem of finding stress states of the cylindrical shell such that the forces, moments, and deformations depend only on the variable β. Neglecting in the equations of equilibrium and continuity of deformations the terms containing derivatives with respect to α and adjoining to the obtained equations the elasticity relations we obtain the following system of equations

$$\frac{\partial T_2}{\partial \beta} - \frac{N_2}{R} = -rY \qquad \frac{T_2}{R} + \frac{\partial N_2}{\partial \beta} = -rZ \qquad -\frac{\partial G_2}{\partial \beta} + rN_2 = 0 \tag{2.11}$$

$$\frac{\partial \kappa_1}{\partial \beta} - \frac{\zeta_1}{R} = 0 \qquad \frac{\kappa_1}{R} + \frac{\partial \zeta_1}{\partial \beta} = 0 \qquad -\frac{\partial \epsilon_1}{\partial \beta} + r\zeta_1 = 0 \tag{2.12}$$

$$\frac{\partial S_2}{\partial \beta} = rX \tag{2.13}$$

$$\frac{\partial \tau}{\partial \beta} = 0 \tag{2.14}$$

$$T_1 = \frac{2Eh\eta}{1 - \sigma^2}(\epsilon_1 + \sigma\epsilon_2) \qquad T_2 = \frac{2Eh\eta}{1 - \sigma^2}(\epsilon_2 + \sigma\epsilon_1) \qquad 2Eh\eta\,\frac{\gamma}{2} = -(1 + \sigma)S_2$$

$$G_1 = -\frac{2Eh^3\eta^3}{3(1 - \sigma^2)}(\kappa_1 + \sigma\kappa_2) \qquad G_2 = -\frac{2Eh^3\eta^3}{3(1 - \sigma^2)}(\kappa_2 + \sigma\kappa_1)$$

$$H_1 = -H_2 = \frac{2Eh^3\eta^3}{3(1 + \sigma)}\,\tau$$

$$S_1 + S_2 = -\frac{H_2}{rR} \qquad N_1 = \frac{1}{r}\frac{\partial H_2}{\partial \beta} \qquad \zeta_2 = \frac{1}{r}\frac{\partial}{\partial \beta}\frac{\gamma}{2} \tag{2.15}$$

The integration of the complete system of equations is in the given case naturally broken up into the integration of the subsystems (2.11), (2.12), (2.13), (2.14) by which the magnitudes (T_2, N_2, G_2), (κ_1, ϵ_1, ζ_1), S_2, τ are determined. The remaining unknowns are obtained with the aid of equations (2.15) whose integration is not for the present required.

Equations (2.11) possess a simple physical meaning. They represent the static equations of an arch of unit width cut out from the shell by two cross-sections.

By eliminating the forces T_2 and N_2 from this system we obtain the equation

$$L\left(\frac{\partial G_2}{\partial \beta}\right) + r^2\left(\frac{\partial RZ}{\partial \beta} - Y\right) = 0$$

whose integration was considered above. Thus

$$G_2 = A_1 + A_2 \frac{x}{r} + A_3 \frac{y}{r} + G_2^* \qquad N_2 = \frac{1}{r}\frac{\partial G_2}{\partial \beta} = A_2 \frac{\cos \chi}{r} + A_3 \frac{\sin \chi}{r} + N_2^*$$

$$T_2 = -R\frac{\partial N_2}{\partial \beta} - rRZ = A_2 \frac{\sin \chi}{r} - A_3 \frac{\cos \chi}{r} + T_2^* \qquad (2.16)$$

where G_2^*, T_2^*, N_2^* are particular integrals of the system (2.11).

The structure of the left sides of equations (2.12) accurately repeats the structure of the left sides of equations (2.11). By analogy we therefore obtain

$$\epsilon_1 = B_1 + B_2 \frac{x}{r} + B_3 \frac{y}{r} \qquad \zeta_1 = B_2 \frac{\cos \chi}{r} + B_3 \frac{\sin \chi}{r}$$

$$\chi_1 = B_2 \frac{\sin \chi}{r} - B_3 \frac{\cos \chi}{r} \qquad (2.17)$$

Equations (2.13) and (2.14) give

$$S_2 = r\int X \, d\beta + C_1 \qquad \tau = C_2$$

The remaining unknowns are determined from equations (2.15) without integration so that the complete solution depends on eight constants.

3. THE PARTICULAR INTEGRAL DUE TO A TRANSVERSE LOAD

Let the rod be acted on by a transverse load R similar over all cross-sections such that its components are of the form

$$X = 0 \qquad Y = \xi(\alpha)p(\beta) \qquad Z = \xi(\alpha)q(\beta) \tag{3.1}$$

(p and q represent the components of the external load at the cross-section where $\xi = 1$ and in what follows it will always be assumed that this section is $\alpha = 0$).

We shall choose the law of load distribution such that the particular integral of the equation of the rod may with sufficient accuracy be obtained in the simplest form, that is, by the zero moment theory, with the values of the forces and moments at the longitudinal edges negligibly small or equal to zero.

The static equations of the zero moment theory can be obtained, as is known, by taking the first, second, third, and sixth equations of equilibrium and setting the moments and shearing forces in them equal to zero. We shall proceed somewhat differently, namely we shall set in the second and third equations

$$\frac{N_2}{R} + \frac{\partial}{\partial\alpha}\frac{H_2}{rR} = 0 \qquad \frac{\partial N_1}{\partial\alpha} + \frac{\partial N_2}{\partial\beta} = 0$$

and assume T_1, T_2, $S_1 + H_2/rR$, and S_2 as unknown. For these magnitudes there is obtained a complete system of four equations with four unknowns, the integration of which gives

$$\left.\begin{aligned}
T_1 &= +r\int d\alpha\int\frac{\partial}{\partial\beta}\left(Y - \frac{\partial RZ}{\partial\beta}\right)d\alpha = -r\,\frac{\partial}{\partial\beta}\left[\frac{\partial}{\partial\beta}(Rq) - p\right]\int d\alpha\int\xi\,d\alpha \\
S_1 + \frac{H_2}{rR} &= -r\int\left(Y - \frac{\partial RZ}{\partial\beta}\right)d\alpha = +r\left[\frac{\partial}{\partial\beta}(Rq) - p\right]\int\xi\,d\alpha \\
S_2 &= +r\int\left(Y - \frac{\partial RZ}{\partial\beta}\right)d\alpha = -r\left[\frac{\partial}{\partial\beta}(Rq) - p\right]\int\xi\,d\alpha \\
T_2 &= -rRZ = -rRq\,\xi
\end{aligned}\right\} \tag{3.2}$$

In order that this equation satisfy the condition of absence of the normal forces T_2 at the longitudinal edges we shall choose the load distribution law in the cross-section such that $q(\beta)$ passes through zero for $\beta = 0$ and $\beta = \beta_0$.

Let us evaluate the accuracy of the solution (3.2). For this purpose, assuming T_1, T_2, S_2 given by equations (3.2) we determine with the aid of the elasticity relations and the relations (2.1) the moments and shearing forces. There may then be found also the expressions

$$Y' = \frac{N_2}{R} + \frac{\partial}{\partial\alpha}\frac{H_2}{rR} \qquad Z' = -\frac{\partial N_1}{\partial\alpha} - \frac{\partial N_2}{\partial\beta}$$

which were assumed above to be equal to zero.

If Y' and Z' are assumed as the components of a certain fictitious load R', it may be stated that for a shell subject to the action of the load $R + R'$ solution (3.2) will be an accurate particular integral. This makes possible a rough estimate of the error of the solution (3.2) where it is necessary to remember that a thin-walled rod is a long shell for which the relative length $l = L/r$ (L is the absolute length of the shell) is a large magnitude.

From this it follows that the maximum values of the integrals with respect to α of the function $\xi(\alpha)$ will, generally speaking, considerably exceed the values of the original functions. Thus, for example, in the most unfavorable from this viewpoint, but practically most important case, when ξ = const

$$\sup \int \ldots \int \xi(\alpha)\, d\alpha^n = \frac{l^n}{n!} \sup \xi(\alpha) \tag{3.3}$$

Let us therefore write down the equations expressing the moments and the shearing forces, arranging the terms in descending order of integration. Starting from equation (3.2) and making use of the relations of elasticity and the auxiliary equations (2.1), we obtain

$$\frac{1}{r}G_1 = -\frac{h^2}{3\left(1-\sigma^2\right)r^2}\eta^3\left\{-\sigma\frac{\partial}{\partial\beta}L\frac{\partial}{\partial\beta}\frac{1}{\eta}\left(\frac{\partial p}{\partial\beta}-\frac{\partial^2 Rq}{\partial\beta^2}\right)\int d\alpha\int d\alpha\int d\alpha\int\xi\,d\alpha - \right.$$

$$\sigma^2\frac{\partial}{\partial\beta}L\frac{\partial}{\partial\beta}\frac{Rq}{\eta}\int d\alpha\int\xi\,d\alpha - \left[2\sigma(1+\sigma)\frac{\partial}{\partial\beta}L\frac{1}{\eta} - \sigma^2\frac{\partial^2}{\partial\beta^2}\frac{R}{\eta}\frac{\partial}{\partial\beta} + \right.$$

$$\left.\left.R\frac{\partial^2}{\partial\beta^2}\frac{1}{\eta}\frac{\partial}{\partial\beta}\right]\left(p-\frac{\partial Rq}{\partial\beta}\right)\int d\alpha\int\xi\,d\alpha + \ldots\right\}$$

$$\frac{1}{r}G_2 = -\frac{h^2}{3\left(1-\sigma^2\right)r^2}\eta^3\left\{-\frac{\partial}{\partial\beta}L\frac{\partial}{\partial\beta}\frac{1}{\eta}\left(\frac{\partial p}{\partial\beta}-\frac{\partial^2 Rq}{\partial^2\beta}\right)\int d\alpha\int d\alpha\int d\alpha\int\xi\,d\alpha - \right.$$

$$\sigma\frac{\partial}{\partial\beta}L\frac{\partial}{\partial\beta}\frac{R}{\eta}q\int d\alpha\int\xi\,d\alpha - \left[2(1+\sigma)\frac{\partial}{\partial\beta}L\frac{1}{\eta} - \sigma\frac{\partial^2}{\partial\beta^2}\frac{R}{\eta}\frac{\partial}{\partial\beta} + \right.$$

$$\left.\left.\sigma R\frac{\partial^2}{\partial\beta^2}\frac{1}{\eta}\frac{\partial}{\partial\beta}\right]\left(p-\frac{\partial Rq}{\partial\beta}\right)\int d\alpha\int\xi\,d\alpha + \ldots\right\}$$

$$\frac{1}{r}\frac{\partial H_1}{\partial\alpha} = -\frac{1}{r}\frac{\partial H_2}{\partial\alpha} = \frac{h^2\eta^3}{3(1+\sigma)r^2}\left\{-L\frac{\partial}{\partial\beta}\frac{1}{\eta}\frac{\partial}{\partial\beta}\left(p-\frac{\partial Rq}{\partial\beta}\right)\int d\alpha\int\xi\,d\alpha + \ldots\right\}$$

$$\frac{\partial N_1}{\partial\alpha} = \frac{h^2\eta^3}{3\left(1-\sigma^2\right)r}\left\{\left[\frac{\partial}{\partial\beta}L\frac{\partial}{\partial\beta}\frac{1}{\eta}\frac{\partial}{\partial\beta}\left(p-\frac{\partial Rq}{\partial\beta}\right)\int d\alpha\int\xi\,d\alpha + \ldots\right\}\right.$$

$$N_2 = \frac{h^2}{3\left(1-\sigma^2\right)r^2}\left\{\frac{\partial}{\partial\beta}\eta^3\frac{\partial}{\partial\beta}L\frac{\partial}{\partial\beta}\frac{1}{\eta}\frac{\partial}{\partial\beta}\left(p-\frac{\partial Rq}{\partial\beta}\right)\int d\alpha\int d\alpha\int d\alpha\int\xi\,d\alpha + \right.$$

$$\frac{\partial}{\partial\beta}\eta^3\frac{\partial}{\partial\beta}L\frac{\partial}{\partial\beta}\frac{Rq}{\eta}\int d\alpha\int\xi\,d\alpha + \left[-\sigma\frac{\partial}{\partial\beta}\eta^3\frac{\partial^2}{\partial\beta^2}\frac{R}{\eta}\frac{\partial}{\partial\beta} + \sigma\frac{\partial}{\partial\beta}\eta^3 R\frac{\partial^2}{\partial\beta^2}\frac{1}{\eta}\frac{\partial}{\partial\beta} + \right.$$

$$\left.\left.2(1+\sigma)\frac{\partial}{\partial\beta}\eta^3\frac{\partial}{\partial\beta}L\frac{1}{\eta} - (1-\sigma)\eta^3 L\frac{\partial}{\partial\beta}\frac{1}{\eta}\frac{\partial}{\partial\beta}\right]\left(p-\frac{\partial Rq}{\partial\beta}\right)\int d\alpha\int\xi\,d\alpha + \ldots\right\}$$

(3.4)

The dots in this equation indicate terms not required for what follows that do not contain integrals of ξ with respect to α and in place of H_1, H_2 and N_1 their derivatives with respect to α are given since only the latter are required for our purpose.

Upon examination of these equations we see that the maximum values of the components of the fictitious loads Y' and Z' will decrease with the thickness of the shell as h^2/r^2 and, if the least favorable case (3.3) is taken, will increase with increase of its relative length roughly as $l^4/4!$. Hence, if the coefficients in front of the integrals of greatest multiplicity are neither accurately nor approximately equal to zero, the error of the solution (3.2) can be small only in the case that

$$\frac{h^2}{r^2}\frac{l^4}{4!} \ll 1$$

which for such a long shell as a thin-walled rod fails entirely to correspond to actual conditions.

The zero moment theory as a method of determining the particular integrals in rods is therefore in the general case not suitable. We shall now seek transverse loads R_1 for which these defects of the zero moment theory are eliminated. This can occur only when the coefficients of the integrals of maximum multiplicity in the expressions for the components of the fictitious load become zero. It is easily seen that the required result will be obtained if

$$\frac{\partial}{\partial\beta} L \frac{\partial}{\partial\beta} \frac{1}{\eta}\left(\frac{\partial p}{\partial\beta} - \frac{\partial^2 Rq}{\partial\beta^2}\right) = 0$$

or, what is equivalent,

$$\frac{\partial}{\partial\beta} L \frac{\partial}{\partial\beta} \frac{1}{\eta} \frac{\partial}{\partial\beta}\left(Y - \frac{\partial RZ}{\partial\alpha}\right) = 0 \tag{3.5}$$

At the same time the coefficients of the integrals of maximum multiplicity vanish not only in the expressions for the components of the fictitious load, which increases the accuracy of the solution (3.2), but also in the expressions for all the moments and shearing forces, which brings about the approximate satisfying of the condition of the absence of stresses on the straight edges of the shell.

Integrating with the aid of the results of equation (3.5), section 2, we obtain

$$Y - \frac{\partial RZ}{\partial \beta} = \xi(\alpha)\left[C_1\int_0^\beta \eta\ d\beta + \frac{C_2}{r}\int_0^\beta x\eta\ d\beta + \frac{C_3}{r}\int_0^\beta y\eta\ d\beta + \frac{C_4}{r^2}\int_0^\beta \omega\eta\ d\beta + C_5\right] \tag{3.6}$$

In order that the straight edges be free from shearing forces it is necessary, as is shown by (3.2), to put

$$\left[Y - \frac{\partial RZ}{\partial \beta}\right]_{\beta=0} = \left[Y - \frac{\partial RZ}{\partial \beta}\right]_{\beta=\beta_0} = 0$$

This, by virtue of the conditions of orthogonality (2.10), gives $C_1 = C_5 = 0$.

There remain three constants which can be chosen such that the required load R_1 at each cross-section of the shell is statically equivalent to the given load R. We have

$$\int_0^{\beta_0}\left(Y - \frac{\partial RZ}{\partial \beta}\right)dx = \xi(\alpha)P_x \qquad \int_0^{\beta_0}\left(Y - \frac{\partial RZ}{\partial \beta}\right)dy = \xi(\alpha)P_y$$

$$\int_0^{\beta_0}\left(Y - \frac{\partial RZ}{\partial \beta}\right)d\omega = \xi(\alpha)M \tag{3.7}$$

where P_x, P_y, and M are the forces per unit length acting in the direction of the axes X and Y and the torsional moment to which the load R is reduced at the cross-section $\alpha = 0$.

Substituting in equation (3.7) the value $Y - \partial RZ/\partial\beta$ from equation (3.6) we obtain for example

$$P_x = \int_0^{\beta_0}\left(Y - \frac{\partial RZ}{\partial \beta}\right)dx = \frac{C_2}{r}\int_0^{\beta_0} dx\int_0^\beta x\eta\ d\beta + \frac{C_3}{r}\int_0^{\beta_0} dx\int_0^\beta y\eta\ d\beta + \frac{C_4}{r^2}\int_0^{\beta_0} dx\int_0^\beta \omega\eta\ d\beta$$

Integration by parts gives

$$P_x = \left[\frac{C_2}{r} x \int_0^\beta x\eta \, d\beta + \frac{C_3}{r} x \int_0^\beta y\eta \, d\beta + \frac{C_4}{r^2} x \int_0^\beta \omega\eta \, d\beta\right]_{\beta=0}^{\beta=\beta_0} -$$

$$\frac{C_2}{r} \int_0^{\beta_0} x^2\eta \, d\beta - \frac{C_3}{r} \int_0^{\beta_0} xy\eta \, d\beta - \frac{C_4}{r^2} \int_0^{\beta_0} x\omega\eta \, d\beta$$

Because of equations (2.10) there remains on the right side only the coefficient of C_2, whence for C_2, and similarly for C_3 and C_4, we have

$$C_2 = -\frac{2hr^2 P_x}{I_y} \qquad C_3 = -\frac{2hr^2 P_y}{I_x} \qquad C_4 = -\frac{2hr^3 M}{I_\omega}$$

where I_x, I_y, I_ω are the moments of inertia determined by (2.7). Hence

$$Y - \frac{\partial RZ}{\partial \beta} = -\xi(\alpha)\left[\frac{2hr}{I_y} P_x \int_0^\beta x\eta \, d\beta + \frac{2hr}{I_x} P_y \int_0^\beta y\eta \, d\beta + \frac{2hr}{I_\omega} M \int_0^\beta \omega\eta \, d\beta\right] \tag{3.8}$$

Equations (3.2) assume the form

$$S_1 + \frac{H_2}{rR} = -S_2 = 2hr^2\left[\frac{P_x}{I_y}\int_0^\beta x\eta \, d\beta + \frac{P_y}{I_x}\int_0^\beta y\eta \, d\beta + \frac{M}{I_\omega}\int_0^\beta \omega\eta \, d\beta\right]\int \xi \, d\alpha$$

$$T_1 = -2hr^2\eta\left[\frac{P_x}{I_y} x + \frac{P_y}{I_x} y + \frac{M}{I_\omega} \omega\right]\int d\alpha \int \xi \, d\alpha \qquad T_2 = -RZ \tag{3.9}$$

The elementary solution constructed in this manner corresponds not to the given load R but to the load R_1 statically equivalent to it at each cross-section. The shape of the load R_1 is not completely determined. Between its two components Y and Z there has been established only the relation (3.8). In order to proceed further we evaluate the accuracy of the solution (3.9). For this it is necessary to turn to equations (3.4) in which, on account of the restrictions imposed on the functions $p(\beta)$ and $q(\beta)$, it is necessary to assume all the coefficients in front of the quadruple integrals of ξ equal to zero. Repeating the considerations applied in evaluating the accuracy of the solution (3.2) we arrive at the result that the integral (3.9) will have an acceptable accuracy if

$$\frac{h^2}{r^2}\frac{l^2}{2!} \ll 1$$

An inequality of this type, as we shall see below, is characteristic of short rods but cannot be used for rods of medium size.

By a suitable choice of the remaining as yet undetermined component Z a further increase in accuracy may be attained. For this purpose it is necessary that that part of the fictitious load which depends on the double integrals of $\xi(\alpha)$ be self-balanced at each cross-section. This principal part of the fictitious load may then be added to the self-balanced part of the load R which we must still consider. We shall not dwell on the mathematical details of this operation. We shall only remark that, however, the component Z of the load R_1 is chosen in the elementary solution (3.9), the fundamental forces T_1, S_1, and S_2 for the thin-walled rod remain the same. For short rods it is not necessary to render more accurate the particular integral and we can therefore set $Z = 0$.

Thus the first half of the problem of the construction of a particular integral has been solved, namely from the general arbitary load R there is separated out the statically unbalanced part R_1 for which the elementary solution (3.9) is given. The second half of the problem consists in the investigation of the statically balanced part of the load $R_2 = R - R_1$.

Let the shell be acted upon at each cross-section by the balanced load R_2 with components of the form (3.1). It is assumed that it is known, that is, we have already chosen in a definite manner the load R_1 from the considerations given above. We shall seek to obtain those conditions under which every transverse strip of unit width of the shell works as an arch, as a result of which the stresses in the cross-sections will have a secondary character as compared with the stresses arising in the longitudinal sections.

It is easily seen that the second, third, and fifth equations of equilibrium of the cylindrical shell go over into the statical equations of an arch if, and only if, in the first of them the terms with derivatives with respect to the variable α are rejected, that is, if

$$\frac{\partial N_1}{\partial \alpha} = \frac{\partial S_1}{\partial \alpha} = \frac{\partial H_1}{\partial \alpha} = 0 \tag{3.10}$$

We then arrive at the equations (2.11) from which the transverse forces and moments arising in the shell can be determined. According to equation (2.16) an integral of this system will be

$$G_2 = A_1 + A_2 \frac{x}{r} + A_3 \frac{y}{r} + G_2^* \qquad N_2 = A_2 \frac{\cos \chi}{r} + A_3 \frac{\sin \chi}{r} + N_2^*$$

$$T_2 = +A_2 \frac{\sin \chi}{r} - A_3 \frac{\cos \chi}{r} + T_2^* \tag{3.11}$$

where it is now necessary to assume that A_1, A_2, A_3 are functions of the variable α since we do not now make use of the assumption that T_2, G_2, N_2 do not depend on α.

Since the operator Z does not contain the variable α explicitly, its particular integral may be obtained in the same form in which the components of the external load are given, that is

$$G_2^* = \xi(\alpha) g_2^*(\beta) \qquad N_2^* = \xi(\alpha) n_2^*(\beta) \qquad T_2^* = \xi(\alpha) t_2^*(\beta) \tag{3.12}$$

The arbitrary functions of integration A_1, A_2, A_3 entering the general integral determine the boundary values of the forces and moments T_2, N_2, G_2. These magnitudes may be considered as the end transverse load and, therefore, imposing on it the condition that it preserve similarity at all cross sections we obtain

$$A_1 = a_1 \xi(\alpha) \qquad A_2 = a_2 \xi(\alpha) \qquad A_3 = a_3 \xi(\alpha) \tag{3.13}$$

where a_1, a_2, a_3 are constants.

Since the arch strip is not acted upon by other than the external surface and end loads, it is evident that for any choice of the constants a_1, a_2, a_3 the system of external loads, end forces, and moments will be in equilibrium at any cross section. We thus arrive at a certain generalization of the concept of a self-balanced load in which there is now included the end as well as the surface forces.

From equations (3.11), (3.12), (3.13) there follows

$$G_2 = \xi(\alpha) g_2(\beta) \qquad N_2 = \xi(\alpha) n_2(\beta) \qquad T_2 = \xi(\alpha) t_2(\beta) \tag{3.14}$$

On the basis of these relations it is possible to show that $\xi(\alpha)$ must be a linear function of the variable α. From the elasticity relations, from the sixth equation of equilibrium, and equations (3.10) we have

$$\frac{\partial \gamma}{\partial \alpha} = \frac{\partial \tau}{\partial \alpha} = \frac{\partial H_2}{\partial \alpha} = 0 \tag{3.15}$$

Hence differentiating with respect to α the first equation of continuity of the deformations and the fourth equilibrium equation we obtain

$$\frac{\partial^2 \kappa_2}{\partial \alpha^2} = \frac{\partial^2 G_1}{\partial \alpha^2} = 0$$

From the elasticity relations there is then obtained

$$\frac{\partial^2 \kappa_1}{\partial \alpha^2} = \frac{\partial^2 G_2}{\partial \alpha^2} = 0$$

whence on account of equation (3.14)

$$\frac{\partial^2 \xi}{\partial \alpha^2} = 0$$

Thus in a thin-walled rod each cross section under a self-balanced load can work as an arch only in the case where the load varies linearly over the length of the rod.

This condition is not only necessary but also sufficient, that is, for such load the problem of determining the forces, moments, and deformations in such a manner that all equations of equilibrium and continuity of deformations and all elasticity relations are satisfied may be solved completely.

In order to prove this we shall start from the equations of continuity of deformations. We shall differentiate the fourth of these equations with respect to α. If we take into account equation (3.15) this gives

$$\frac{\partial \zeta_2}{\partial \alpha} = 0$$

The second, third, and fifth continuity equations then assume the form

$$\frac{\partial \kappa_1}{\partial \beta} = \frac{\zeta_1}{R} \qquad \frac{\kappa_1}{R} = -\frac{\partial \zeta_1}{\partial \beta} \qquad \frac{\partial \epsilon_1}{\partial \beta} = r\zeta_1 \tag{3.16}$$

We arrived at the system (2.12) integrated in the preceding section. Hence according to (2.17) we have

$$\epsilon_1 = B_1 + B_2 \frac{x}{r} + B_3 \frac{y}{r} \qquad \zeta_1 = B_2 \frac{\cos \chi}{r} + B_3 \frac{\sin \chi}{r}$$

$$\kappa_1 = B_2 \frac{\sin \chi}{r} - B_3 \frac{\cos \chi}{r} \tag{3.17}$$

Since in deriving (3.17) it was not assumed that ζ_1, κ_1, and ϵ_1 were functions only of β, it is necessary to assume that B_1, B_2, and B_3 are functions of α. In order not to violate the previously obtained relations they must be chosen linear.

We may now assume as known two groups of magnitudes: T_2, N_2, G_2 uniquely determined by equations (2.11) and the boundary conditions on the straight edges, and ϵ_1, ζ_1, κ_1 determined with an accuracy up to three linear functions of α. The remaining desired magnitudes can be obtained in an elementary fashion with the aid of the elasticity relations, the first and sixth equations of equilibrium and the first equation of continuity of deformations. We obtain

$$T_1 = 2Eh\eta\epsilon_1 + \sigma T_2 \qquad S_2 = 2Eh \int_0^\beta \eta \frac{\partial \epsilon_1}{\partial \alpha} d\beta + \sigma \int_0^\beta \frac{\partial T_2}{\partial \alpha} d\beta + B_4'$$

$$H_1 = -H_2 = -(1 - \sigma)\eta^3 \int_0^\beta \frac{1}{\eta^3} \frac{\partial G_2}{\partial \alpha} d\beta - \sigma \frac{2Eh^3\eta^3}{3(1 + \sigma)} \int_0^\beta \frac{\partial \kappa_1}{\partial \alpha} d\beta + B_5'$$

$$S_1 = -S_2 - \frac{H_2}{rR} \qquad G_1 = -\frac{2Eh^3}{3} \eta^3 \kappa_1 + \sigma G_2 \tag{3.18}$$

where B_4' and B_5' are constants.

There remain undetermined eight arbitrary constants: two in each of the linear functions B_1, B_2, B_3 and in addition B_4' and B_5'. These constants can be disposed of in such manner that no longitudinal forces are applied at the straight edges and the forces in the cross-sections constitute a self-balanced system of forces, that is, so that the required particular integral is not subjected to stress states corresponding to tension, bending, and torsion of the shell by external forces applied to its transverse ends. Since the shell as a whole is in equilibrium because the equations of equilibrium are satisfied, it is sufficient to require that the system of forces and moments applied to the cross-section $\alpha = 0$ be self-balanced. This gives six relations. Adding the condition

$$S_2\Big|_{\beta=0} = 0 \qquad S_2\Big|_{\beta=\beta_0} = 0$$

we obtain eight equations for the eight arbitrary constants.

If the transverse self-balanced load in the longitudinal direction is not a linear function, it can give rise to considerable stresses in the transverse sections (in comparison with the stresses in the longitudinal sections).

When the load in the longitudinal direction has a broken line character, the shell must be divided into parts and the obtained results applied to each of them. It should be remembered, however, that in the neighborhood of sections where the load is discontinuous the stress state will be impaired by the effect of the coupling conditions of the parts of the shell so that such sections should not be too frequent.

4. RATE OF DAMPING OF THE INTEGRALS OF THE HOMOGENEOUS EQUATIONS

The object of the present section is to seek to obtain particular integrals of the homogeneous equations of a cylindrical shell which correspond to the stress states with minimum damping (increase) with respect to the variable α. The equations of an arbitrary cylindrical shell do not contain variable coefficients with respect to the parameter α and this system can be reduced to a single equation. In practice such an operation is very complicated but it is not difficult to see, without going into explanations, that the form of this equation will be

$$\frac{\partial^8\Phi}{\partial\alpha^8} + L_2\left(\frac{\partial^6\Phi}{\partial\alpha^6}\right) + L_4\left(\frac{\partial^4\Phi}{\partial\alpha^4}\right) + n\frac{r^2}{h^2}\frac{\partial^4\Phi}{\partial\alpha^4} + L_6\left(\frac{\partial^2\Phi}{\partial\alpha^2}\right) + L_8(\Phi) = 0 \qquad (4.1)$$

where the L_i terms are the linear differential operators relative to the variable β, the order of which is equal to their index, n is a known function of β, Φ is a function of the stress in terms of which the forces and moments are expressed with the aid of certain differential operations. The operators L_i and the coefficient n may depend on h/r but for $h/r \longrightarrow 0$ they remain restricted.

We shall determine the slowly decreasing (increasing) function Φ in the direction of the variable α by the condition that its derivative with respect to α is considerably less than the function itself. Mathematically this may be expressed by the equation

$$\frac{\partial \Phi}{\partial \alpha} = k\psi(\alpha, \beta)\Phi$$

where k is such a small number that the absolute values of the function $k\psi(\alpha,\beta)$ and its derivatives are much smaller than unity.

Integrating this equation we obtain

$$\Phi = \chi(\beta)\exp \int k\psi \, d\alpha$$

Then

$$\frac{\partial^2 \Phi}{\partial \alpha^2} = \left(k^2\psi^2 + k\frac{\partial \psi}{\partial \alpha}\right)\Phi$$

For $\partial\psi/\partial x \neq 0$ and for sufficiently small k we may write approximately the equation

$$\frac{\partial^2 \Phi}{\partial \alpha^8} \approx k\frac{\partial \psi}{\partial \alpha}\Phi$$

which shows that the first derivative will not be a slowly decreasing (increasing) function. But the derivatives of Φ with respect to α enter the expressions for the forces and moments and in order that the latter be slowly decreasing (increasing) functions it is necessary to impose the requirement

$$\frac{\partial \psi}{\partial \alpha} = 0 \quad \text{or} \quad \psi = \psi(\beta)$$

It follows that of all solutions of equation (4.1) it is sufficient to consider the integrals of the form

$$\Phi = e^{k\alpha}\varphi(\beta) \tag{4.2}$$

since only by these will the slowing decreasing (increasing) stress states be determined. For φ there is then obtained the equation

$$k^8\varphi + k^6L_2(\varphi) + k^4L_4(\varphi) + n(\beta)\frac{r^2}{h^2}k^4\varphi + k^2L_6(\varphi) + L_8(\varphi) = 0 \tag{4.3}$$

The constant number k is left arbitrary. This constant will determine the character of the decrease of the required magnitudes with respect to the variable α.

For a given k, equation (4.3) determines eight linearly independent solutions for φ. This means that a given character of the decrease (increase) can take place only if in the transverse directions the required magnitudes are distributed in one of eight definite ways.

Since we are considering thin shells it should be remembered that h/r is small. Making use of this fact we focus our attention on the asymmetrical properties of the integrals of equation (4.3), that is, on the properties for h/r as small as we please.

We define a new magnitude κ by the equation:

$$k = \left(\frac{h}{r}\right)^{\kappa}$$

and shall call it the damping coefficient. Equation (4.3) then assumes the form

$$\left(\frac{h}{r}\right)^{8\kappa}\varphi + \left(\frac{h}{r}\right)^{6\kappa}L_2(\varphi) + \left(\frac{h}{r}\right)^{4\kappa}L_4(\varphi) + n(\beta)\left(\frac{h}{r}\right)^{4\kappa-2}\varphi + \left(\frac{h}{r}\right)^{2\kappa}L_6(\varphi) + L_8(\varphi) = 0 \tag{4.4}$$

It is necessary further to consider separately three cases:

(a) $\kappa < 0$ (b) $0 \leqslant \kappa \leqslant 1/2$ (c) $\kappa > 1/2$

For $\kappa < 0$ there will be a rapid decrease (increase) which will be more intense the thinner the shell. This stress state (end effect) is of no interest to us.

In cases (b) and (c) the intensity of the decrease (increase) will decrease with decreasing thickness of the shell. The integrals of equation (4.4) for $h/r \to 0$ will asymptotically approach in case (b) the integrals of the equation

$$L_8(\varphi) + n(\beta)\left(\frac{h}{r}\right)^{4\kappa-2} \varphi = 0 \tag{4.5}$$

and in case (c) the integrals of the equation

$$L_8(\varphi) = 0 \tag{4.6}$$

The principal difference between these two cases lies in the circumstance that as long as κ falls in the interval (b) the law of change of the required forces over the cross-section depends on κ whereas when κ falls in the interval (c) the law stops changing. The physical meaning of this fact will become clear if we agree on how to delimit the local and principal stress states.

In the problem of the computation of thin-walled rods there enters only the determination of the stresses at a sufficient distance from the ends. The stress states which damp out without reaching this zone must be considered as local end stresses and ignored in the computation. From this point of view it is possible for each rod-shell, when its dimensions are given, to determine a number κ_1 such that:

(a) The integral of equation (4.4) for $\kappa \leqslant \kappa_1$ will give the local stress states;

(b) the integrals of equation (4.4) for $\kappa > \kappa_1$ will give the principal stress state which is the problem to be solved.

It is evident that the number κ_1 for a given relative thickness of the shell h/r and for given requirements as to the accuracy of the computation is determined by the length of the shell. In this connection shells may be divided as regards their length into two classes: (1) shells for which $\kappa_1 \leqslant \frac{1}{2}$, (2) shells for which $\kappa_1 > \frac{1}{2}$.

The difference between them consists in the fact that in shells of the second kind the number of principal stress states differing among each other by the law of variation over the cross-section does not exceed eight whereas for shells of the first kind there can be an infinite number of such general stress states.

The theory of solid rods operates with a finite number of general stress states differing from one another in the law of stress distribution over the cross-sections (the stress states corresponding to bending, elongation, and torsion of the rod). The theory of thin-walled rods, in generalizing the theory of solid rods, may be constructed evidently only for shells of the second kind. Shells of the first kind must be referred to the type of long shells, to the computation of which there may be applied the method of V. Z. Vlasov (reference 5) or V. V. Novozhilov (reference 6). This determines the lower limit for the relative length of a thin-walled rod. The shell may be considered as a rod only in the case where

$$e^{-\frac{1}{2}kl} \ll 1 \quad \text{for} \quad k = \left(\frac{h}{r}\right)^{\frac{1}{2}}$$

If this inequality is not satisfied the law of distribution of forces and moments of the principal stress state over the cross-section will be determined by equation (4.5). Space does not permit dwelling on the problem of how the principal stress state is altered in this case. We shall therefore restrict ourselves to the statement that the above inequality estimates the minimum length of the shell to which the theory of thin-walled rods is entirely applicable without any distortions.

We shall turn to the investigation of the stress states with the damping coefficient $\kappa > 1/2$ and consider first the limiting case $\kappa = \infty$. The function Φ then degenerates into a function which is linear in the variable α, and the corresponding stress state will likewise be linear in α (it is assumed that all the required magnitudes are determined in terms of Φ without the aid of integration). Such stress state linear with respect to α has already been obtained in section 3 in considering the transverse load. It is determined by equations (3.11), (3.17), and (3.18) in which the load terms, indicated by asterisks, must be taken equal to zero. In these relations the arbitrary elements are:

(a) the functions A_1, A_2, A_3 linear in α in the expressions for T_2, N_2, G_2

(b) the functions B_1, B_2, B_3 linear in α in the expressions for ϵ_1, ζ_1, κ_1

(c) the constants B_4' and B_5' entering, besides the above linear functions, in the expressions for S_2, H_1, H_2.

In section 3 it was shown that the functions A_1, A_2, A_3 correspond to the shell loaded by forces and moments along the longitudinal edges. This type of stress state may be referred to a particular integral and it may be assumed that $A_1 = A_2 = A_3 = 0$. From this it follows that $T_2 = G_2 = N_2 = 0$. If, assuming $T_2 = 0$, we require additionally that there be no shearing forces S_2 at the end of the shell we obtain with the aid of equations (3.17) and (3.18)

$$S_2\Big|_{\beta=0} = B_4' = 0$$

$$S_2\Big|_{\beta=\beta_0} = 2Eh\left\{\frac{dB_1}{d\alpha}\int_0^{\beta_0} \eta \, d\beta + \frac{dB_2}{d\alpha}\int_0^{\beta_0} \frac{x}{r} \eta \, d\beta + \frac{dB_3}{d\alpha}\int_0^{\beta_0} \frac{y}{r} \eta \, d\beta\right\} + B_4' = 0$$

or, on account of the orthogonality conditions (2.10),

$$B_4' = \frac{dB_1}{d\alpha} = B_1' = 0$$

The constant B_5' gives a stress state in which only the torsional moments and the shearing force S_1 will be different from zero. To this, within the limits of accuracy of the theory of shells there corresponds the Saint-Venant torsion of a thin-walled rod. There remain arbitrary the functions B_1, B_2, B_3. Equations (3.17) and (3.18) show that the stress state determined by them is characterized by the fact that the normal stresses in the cross-sections will be subject to the so-called law of the plane, that is, the shell behaves like a rod subjected to the action of bending and tension.

It follows that sufficiently long thin-walled rods do not in that case differ in their behavior from solid rods and it is therefore necessary to focus our attention on rods which differ along their length both from shells and from solid rods.

In order to carry out this analysis we turn our attention to the fact that equation (4.6) or, what is equivalent, equation

$$L_8(\Phi) = 0 \tag{4.7}$$

may be obtained in a purely formal manner on assuming that Φ is a function only of β although we shall also study such integrals of equation (4.7) which depend on α. With this assumption we arrive at those results which were obtained in section 2 in seeking to obtain a stress state of the shell which did not depend on α.

It may be asserted that among the slowly damping stress states of a cylindrical shell there are

(a) those in which the magnitudes T_2, N_2, G_2 are determined by the system (2.11) if we put in it $Y = Z = 0$;

(b) those in which κ_1, ζ_1, ϵ_1 are determined by the system (2.12);

(c) those in which the magnitude S_2 is determined by equation (2.13) if we put in it $X = 0$;

(d) those in which the magnitude τ is determined by equation (2.14).

The physical sense of the stress states of the types (a) and (c) is clear. The first of them corresponds to the loading of the shell by transverse forces and moments along the straight edges. As they have already been considered in studying the particular integrals of the transverse loads they are of no interest. The stress states of the type (c) correspond to the loading of the shell by longitudinal forces along the straight edges. These likewise need not be considered since in what follows they will automatically be included in stress states of the type (b).

There remain to be investigated the nondamping integrals of the type (b) and (d). Only these may be found suitable for the construction of a theory of thin-walled rods, that is, they give stress states in which the normal stresses in the cross-sections are essentially greater than in the longitudinal sections. For this, in particular, it is necessary that

$$T_1 \gg T_2 \tag{4.8}$$

We shall make the assumption that condition (4.8) actually holds for integrals of the type of (b) and (d). This assumption was proven in section 5.

If equation (4.8) is taken into account there follows from relations (1.1)

$$T_1 = 2Eh\eta\epsilon_1 \qquad \epsilon_2 = -\sigma\epsilon_1 \tag{4.9}$$

On the other hand, making use of the first equation of equilibrium and the elasticity relation (1.2), we obtain

$$S_2 = \int_0^{\beta_0} \frac{\partial T_1}{\partial \alpha} d\beta \qquad 2Eh\eta \frac{\gamma}{2} = -(1 + \sigma) \int_0^{\beta_0} \frac{\partial T_1}{\partial \alpha} d\beta \tag{4.10}$$

Since we are immediately interested only in the slowly damping integrals it is necessary to assume that each differentiation with respect to α leads to an essential decrease of the differentiated function. From this it follows that

$$\frac{\partial}{\partial \alpha} \frac{\gamma}{2} \ll \epsilon_1 \qquad \frac{\partial \epsilon_2}{\partial \alpha} \ll \epsilon_1$$

and if the fourth equation of continuity is taken into account it is necessary to add to these inequalities the inequality

$$\frac{\partial \zeta_2}{\partial \alpha} \ll \epsilon_1$$

Rejecting in the second, third, and fourth equations of continuity of deformations the magnitudes which are small by comparison with ϵ_1 we obtain

$$\frac{\partial \kappa_1}{\partial \beta} - \frac{\zeta_1}{R} = \frac{\partial \tau}{\partial \alpha} \qquad \frac{\kappa_1}{R} + \frac{\partial \zeta_1}{\partial \beta} = 0 \qquad \frac{\partial \epsilon_1}{\partial \beta} = r\zeta_1 \tag{4.11}$$

This system embraces both the integral of the type (b), which is obtained when one considers $\partial\tau/\partial\alpha = 0$, and the integrals of the type (d) which are obtained if one assumes

$$\tau = C(\alpha) \qquad \frac{\partial \tau}{\partial \alpha} = C'(\alpha)$$

Eliminating from equations (4.11) the unknowns κ_1 and ζ_1 we obtain for determining ϵ_1 the equation

$$L\left(\frac{\partial \epsilon_1}{\partial \beta}\right) = -rC'(\alpha) \quad \text{or} \quad \frac{\partial}{\partial \beta} L\left(\frac{\partial \epsilon_1}{\partial \beta}\right) = 0 \tag{4.12}$$

5. ELEMENTARY SOLUTIONS OF THE HOMOGENEOUS EQUATIONS

The fundamental result of the preceding section is the derivation of equation (4.12) by which the character of the change of deformation ϵ_1 in slowing damped stress states of a cylindrical shell of arbitrary section is determined. Integrating this equation we obtain

$$\epsilon_1^* = 2Eh\epsilon_1 = A_1 + A_2 \frac{x}{r} + A_3 \frac{y}{r} + A_4 \frac{\omega}{r^2} \tag{5.1}$$

where A_1, A_2, A_3, A_4 are functions of α.

With the object of interpreting the meaning of the functions A_1, A_2, A_3, A_4 let us investigate in greater detail the corresponding stress state. We shall maintain the above mentioned assumption that the normal stresses in the cross-sections exceed in absolute value the normal stresses in the longitudinal sections so that in particular inequality (4.8) is satisfied and we shall bear in mind that all the required magnitudes decrease slowly in the longitudinal direction and as a result of this they decrease on differentiation and increase on integration with respect to α.

The deformation ϵ_1 is determined by equation (5.1). We shall express the required magnitudes in terms of it. On the basis of assumption (4.8) we shall assume that T_2 may be neglected by comparison with T_1. From the elasticity relations we then have

$$T_1 = \eta\epsilon_1^* = A_1\eta + A_2 \frac{x}{r} \eta + A_3 \frac{y}{r} \eta + A_4 \frac{\omega}{r^2} \eta \qquad \epsilon_2 = -\sigma\epsilon_1 \tag{5.2}$$

The shearing force S_2 and the deformation ω are determined from the first equation of equilibrium and the elasticity relation (1.2)

$$S_2 = \int \frac{\partial \epsilon_1^*}{\partial \alpha} \eta \, d\beta + A_5' \qquad \gamma = -\frac{2(1 + \sigma)}{\eta} \int \frac{\partial \epsilon_1}{\partial \alpha} \eta \, d\beta - \frac{1 + \sigma}{Eh\eta} A_5' \tag{5.3}$$

where A_5' is an arbitrary function of integration depending only on α and decreasing, as also A_1, A_2, A_3, A_4, on differentiation. It is easily seen that with the aid of the function A_5' we include in our consideration the stress state of the type (c) omitted in section 4.

Further, if the components of the tangential deformation ϵ_1, ϵ_2, γ are known, it is possible with the aid of the relations (2.1) to express the remaining unknown functions.

In these equations which we shall make use of for estimating the order of magnitude of the forces and moments special attention must be paid to the components involving integration with respect to α because they will increase together with the length of the shell. In connection with this we observe that since $\epsilon_1{}^*$ satisfies equation (4.13) and ω depends only on the derivatives of $\epsilon_1{}^*$ and A_5 with respect to α the integrals are retained only in the expressions for H_1 and H_2. Hence equations (2.1) may be briefly written as follows:

$$\frac{1}{r} G_1 = \frac{h^2}{3(1 - \sigma^2)r^2} g_1 \qquad \frac{1}{r} G_2 = \frac{h^2}{3(1 - \sigma^2)r^2} g_2$$

$$\frac{1}{r} H_1 = -\frac{1}{r} H_2 = -\frac{h^2\eta^3}{3(1 + \sigma)r^2}\int L\left(\frac{\partial \epsilon_1{}^*}{\partial \beta}\right)d\alpha + \frac{h^2}{3(1 - \sigma^2)r^2}\frac{\partial h_{12}}{\partial \alpha}$$

$$= -\frac{h^2\eta^3}{3(1 + \sigma)r^2}\int A_4 d\alpha + \frac{h^2}{3(1 - \sigma^2)r^2}\frac{\partial h_{12}}{\partial \alpha}$$

$$N_1 = \frac{h^2}{3(1 - \sigma^2)r^2}\frac{\partial n_1}{\partial \alpha} \qquad N_2 = \frac{h^2\eta^3}{3(1 + \sigma)r^2} A_4 + \frac{h^2}{3(1 - \sigma^2)r^2} n_2 \qquad (5.4)$$

where g_1, g_2, h_{12}, n_1, n_2 denote certain functionals of $\epsilon_1{}^*$ which can be written out easily if desired; it is important to note that they are magnitudes of the same order as $\epsilon_1{}^*$.

The latter unknowns T_2 and S_1 are obtained from the third and sixth equations of equilibrium:

$$T_2 = -\frac{h^2}{3(1 + \sigma)r^2} R \frac{\partial \eta^3}{\partial \beta} A_4 + \frac{h^2}{3(1 - \sigma^2)r^2} t_2 \qquad S_1 + \frac{H_2}{rR} = -S_2 \qquad (5.5)$$

Equations (5.4) and (5.5) show that the investigated solutions actually give greater normal stresses in the cross sections than in the longitudinal, as was postulated above.

The auxiliary relations (2.1) are derived without using the first three equations of equilibrium. In determining S_2 and T_2 it was necessary to resort to the first and third of these equations. Hence only the second equilibrium equation remained untouched. Substitution in it of the obtained results gives

$$\frac{\partial}{\partial \alpha}\left(S_1 + \frac{H_2}{rR}\right) + \frac{\partial T_2}{\partial \beta} - \left(\frac{N_2}{R} + \frac{\partial}{\partial \alpha}\frac{H_2}{rR}\right) = Y^* \tag{5.6}$$

$$Y^* = -\int_0^{\beta} \frac{\partial^2 \epsilon_1{}^*}{\partial \alpha^2} \eta \, d\beta - A_5{}'' - \frac{h^2}{3(1 + \sigma)r^2}\left(\frac{\partial}{\partial \beta} R \frac{\partial \eta^3}{\partial \beta} + \frac{2}{R} \eta^3\right)A_4 + \frac{h^2}{3(1 - \sigma^2)r^2}\left(\frac{\partial t_2}{\partial \beta} - \frac{n_2}{R} - \frac{1}{R}\frac{\partial^2 h_{12}}{\partial \alpha^2}\right) \tag{5.7}$$

The magnitude Y^* may evidently be considered as a component of a certain transverse tangential load which must be applied to the shell in order that the required stress state could exist in it. The arbitrary functions A_1, A_2, A_3, A_4 must be chosen such that this fictitious load has the minimum effect on the magnitude and intensity of the stresses in the cross-sections. On the basis of physical considerations it may be stated that for this it is necessary to impose two conditions, namely, that the fictitious load be self-balanced in all cross-sections and that Y^* pass through zero on the straight edges of the shell.

In violating the first condition in the cross-sections there unavoidably arise equilibrating stresses. In violating the second condition, as equations (3.2) show, there are applied to the straight edges tangential forces which give rise to balanced stresses in the cross-sections.

The self-equilibrium of the fictitious load is mathematically expressed by the equations

$$\int_0^{\beta_0} Y^* \, dx = \int_0^{\beta_0} Y^* \, dy = \int_0^{\beta_0} Y^* \, d\omega = 0$$

These relations may be transformed by integration by parts. Taking into account the fact that the fictitious load passes through zero on the straight edges of the shell, that is,

$$Y^*\Big|_{\beta=0} = 0 \qquad Y^*\Big|_{\beta=\beta_0} = 0 \tag{5.8}$$

we obtain

$$\int_0^{\beta_0} \frac{\partial Y^*}{\partial \beta} x \, d\beta = \int_0^{\beta_0} \frac{\partial Y^*}{\partial \beta} y \, d\beta = \int_0^{\beta_0} \frac{\partial Y^*}{\partial \beta} \omega \, d\beta = 0 \tag{5.9}$$

The five relations (5.8) and (5.9) give a system of equations from which A_1, A_2, . . ., A_5 may be determined. We have

$$\frac{\partial Y^*}{\partial \beta} = -\eta\left(A_1'' + A_2'' \frac{x}{r} + A_3'' \frac{y}{r} + A_4'' \frac{\omega}{r^2}\right) - \frac{h^2}{3(1 + \sigma)r^2} A_4 \frac{\partial}{\partial \beta}\left(\frac{\partial}{\partial \beta} R \frac{\partial \eta^3}{\partial \beta} + \frac{2}{R} \eta^3\right) + \frac{h^2}{3\left(1 - \sigma^2\right)r^2} K$$

where

$$K = \frac{\partial}{\partial \beta}\left(\frac{\partial t_2}{\partial \beta} - \frac{n_2}{R} - \frac{1}{R} \frac{\partial^2 h_{12}}{\partial \alpha^2}\right)$$

The magnitude K represents a linear combination of A_i and a certain number of their derivatives with respect to α. With this in mind and recalling that the system of functions $(1, \ x, \ y, \ \omega)$ is a mutually orthogonal one with weight η and making use of the notation (2.7) the above system of equations may be written in the form

$$-\frac{I_y}{2hr^2} A_2'' - \frac{h^2}{3\left(1 + \sigma\right)r^2} \int_0^{\beta_0} x \frac{\partial}{\partial \beta}\left[\frac{\partial}{\partial \beta} R \frac{\partial \eta^3}{\partial \beta} + \frac{2}{R} \eta^3\right] d\beta \, A_4 + \frac{h^2}{3\left(1 - \sigma^2\right)r^2}\left[\sum_{k=1}^{5} a_{ik}A_k + . . .\right] = 0$$

Continued on next page

$$-\frac{I_x}{2hr^2} A_3'' - \frac{h^2}{3(1+\sigma)r^2}\int_0^{\beta_0} y \frac{\partial}{\partial\beta}\left[\frac{\partial}{\partial\beta} R \frac{\partial\eta^3}{\partial\beta} + \frac{2}{R}\eta^3\right] d\beta\, A_4 +$$

$$\frac{h^2}{3\left(1-\sigma^2\right)r^2}\left[\sum_{k=1}^{5} a_{2k}A_k + \ldots\right] = 0$$

$$-\frac{I_\omega}{2hr^3} A_4'' - \frac{h^2}{3(1+\sigma)r^2}\int_0^{\beta_0} \omega \frac{\partial}{\partial\beta}\left[\frac{\partial}{\partial\beta} R \frac{\partial\eta^3}{\partial\beta} + \frac{2}{R}\eta^3\right] d\beta\, A_4 +$$

$$\frac{h^2}{3\left(1-\sigma^2\right)r^2}\left[\sum_{k=1}^{5} a_{3k}A_k + \ldots\right] = 0$$

$$-A_5'' - \frac{h^2}{3(1+\sigma)r^2}\left[\frac{\partial}{\partial\beta} R \frac{\partial\eta^3}{\partial\beta} + \frac{2}{R}\eta^3\right]_{\beta=0} A_4 +$$

$$\frac{h^2}{3\left(1-\sigma^2\right)r^2}\left[\sum_{k=1}^{5} a_{4k}A_k + \ldots\right] = 0$$

$$-\frac{F}{2hr} A_1'' - \frac{h^2}{3(1+\sigma)r^2}\left[\frac{\partial}{\partial\beta} R \frac{\partial\eta^3}{\partial\beta} + \frac{2}{R}\eta^3\right]_{\beta=\beta_0} A_4 - A_5'' -$$

$$\frac{h^2}{3\left(1-\sigma^2\right)r^2}\left[\sum_{k=1}^{5} a_{5k}A_k + \ldots\right] = 0$$

(5.10)

The dots represent the terms containing the derivatives of A_i; they play a secondary part in comparison with A_i since the latter by assumption decrease on differentiation.

Assuming it possible to reject these nonwritten out terms we arrive at a system of linear equations with constant coefficients of the tenth order. Its particular integrals will be functions of the form

$$\exp\left(\pm p_i \frac{h}{r} \alpha\right) \qquad (i = 1, 2, \ldots, 5)$$

The magnitudes p_i which may be considered positive are entirely determined by the form of the cross-section of the shell and do not depend on the ratio h/r.

We shall dwell on those solutions of the system (5.10) which are obtained if the lower signs are chosen in the argument of the exponential function. They will evidently correspond to stress states with die down with increasing α.

On the shape of the cross-section of the shell there will to a certain extent depend also the rate of damping of these solutions with respect to the variable α. It is possible only to make the general statement that the rate will decrease together with h/r. This is the fundamental property of the slowly damping integrals (in the terminology of the preceding section these integrals have the damping coefficient $\kappa = 1$).

Shells in which slowly damping stress states are to be considered as the principal stress states will be denoted as rods of medium length. Mathematically a rod of medium length is determined by the condition

$$\frac{1}{a} \leq \exp\left[\pm \bar{p} \frac{h}{r} \frac{l}{2}\right] \leq a \tag{5.11}$$

where a is a number which does not differ greatly from 1 and $\bar{p}$ is the greatest of the magnitudes p_i. This condition guarantees that the solutions of the equation (5.10) and therefore their corresponding stress states are not local, that is, are not damped out toward the center sections of the shell.

In the preceding section it was shown that starting with a sufficiently large relative length the shell, in the character of its principal stress state, ceases to differ from a solid rod. Such a shell we shall call a long rod. Evidently it does not require special consideration. In the interval between short and long rods there must be a rod of medium length which is characterized in that the principal stress state is transitional from the one considered in this section to the one which takes place in the solid rod. The solutions of type (b) do not here undergo essential changes because they correspond to the cases where the stresses in the cross-sections of the shell obey the law of the plane and such solutions hold also for $\kappa = \infty$. The limiting transition reduces to the case where the exponential law of change of the required magnitudes in the longitudinal direction degenerates into a linear law. In contrast to this the solution of the type (d) changes radically. This solution goes over evidently into that integral of the limiting system (at $\kappa = \infty$) of equations to which corresponds the Saint-Venant torsion of the shell

$$rRS_1 = H_1 = -H_2 = \text{const}$$

It will be shown below that in a short rod H_1/r and H_2/r in absolute values are less than S_1 and S_2. Hence it may be said that in a rod of medium length there occurs a transition from the principal stress state in which the magnitudes $|H_1|/r$ and $|H_2|/r$ are less than $|S_1|$ and $|S_2|$ to the principal stress state in which $|H_1|/r$ and $|H_2|/r$ are commensurable with $|S_1|$.

The construction of a theory of thin-walled rods of medium length and the establishing of the corresponding limits of the relative length of the shell is based on the necessity for a more detailed qualitative investigation of the system (5.10). It may be remarked that in the theory proposed by V. Z. Vlasov (reference 1) the problem of shells of medium length, if expressed in our terminology, was solved by replacing the system (5.10) by the approximate equations:

$$A_1'' = A_2'' = A_3'' = 0 \qquad \frac{EI_\omega}{r^2} A_4'' - GI_d A_4 = 0 \tag{5.12}$$

where the latter relation was obtained from the condition of equilibrium of the moments of the shearing forces S_1 (the first component of the left side) and of the torsional moments H_1 (second component). This evidently presupposes that $|H_1|/r$ and $|S_1|$ are comparable in magnitude as should be the case in a rod of medium length.

In addition to the fact that the passage from (5.10) to (5.12) is not clear, this point in the theory of Vlasov is doubtful because on the one hand he proposes the comparability of $|H_1|/r$ and $|S_1|$ and on the other hand there is assumed the law of conjugate tangential forces which under these conditions stands in evident contradiction to the six equations of equilibrium.

The further discussion refers to the theory of short rods. Their relative length is restricted by the inequality (5.11) and depends not only on h/r but to some extent also on the geometrical shape of the cross-section since the magnitude $\overline{p}$ enters in (5.11).

The solution of the system (5.10) for such a rod may be simplified if the inequality (5.11) is somewhat strengthened and $\overline{p}h/r$ taken to be so small that in the interval $(0,l)$ the function $\exp(\overline{p}\alpha h/r)$ may with sufficient accuracy be approximated by a linear function, that is, if inequality (5.11) is replaced by the inequality:

$$\frac{\overline{p}^2 h^2 l^2}{2r^2} \ll 1$$

The system (5.10) then reduces to the form

$$A_1'' = A_2'' = A_3'' = A_4'' = A_5'' = 0 \tag{5.13}$$

and its integral will be

$$A_i = \alpha a_i + b \qquad (i = 1,2,3,4,5)$$

In correspondence with equations (5.2) and (5.3) the forces T_1 and S_2 are then expressed by the equations

$$T_1 = \alpha \frac{\partial f_1}{\partial \beta} + f_2 \qquad S_2 = f_1 \tag{5.14}$$

where f_1, f_2 are functions of the variable β having the form:

$$f_1 = a_1 \int_0^\beta \eta \, d\beta + a_2 \int_0^\beta \frac{x}{r} \eta \, d\beta + a_3 \int_0^\beta \frac{y}{r} \eta \, d\beta + a_4 \int_0^\beta \frac{\omega}{r^2} \eta \, d\beta + a_5$$

$$f_2 = b_1 \eta + b_2 \frac{x}{r} \eta + b_3 \frac{y}{r} \eta + b_4 \frac{\omega}{r^2} \eta \tag{5.15}$$

For the remaining forces and moments we have equations (5.4) and (5.5). The right hand sides of these relations contain small factors h^2/r^2 and the magnitudes g_1, g_2, h_{12}, n_1, n_2, t_2 which on account of equation (5.13) will vary linearly with α, as also the force T_1.

It may therefore be stated that the stresses due to the bending moments, the shearing forces and the normal forces T_2 will be considerably less than the stresses due to T_1. As follows, however, from equation (5.4), the torsion moments for $A_4 \neq 0$ will not, generally speaking, be subject to this rule. In fact, if in equation (5.4) the second order of magnitude $\partial h_{12}/\partial\alpha$ is rejected and the assumption made that all a_i and b_i, with the exception of a_4, are equal to zero and $\sigma(H_1)$ and $\sigma(T_1)$, the stresses due to the torsional moments and the normal forces compared, there is obtained

$$\frac{\sigma(H_1)}{\sigma(T_1)} = \pm\eta \frac{r^2}{\omega} \frac{h}{r} \frac{\alpha}{2}$$

This magnitude will not always be negligibly small even for a short rod. From this the conclusion follows that a short thin-walled rod behaves like a shell in which there arises a zero-moment stress state, supplemented by the presence of torsional moments while bending moments are absent.

In connection with this it is necessary to verify whether it is possible to make use of the law of conjugate tangential forces. For this purpose, under the same assumptions as above, we compute the above mentioned magnitudes and form their ratio. We have

$$\frac{H_1}{rS_1} = \frac{h^2\eta^3\alpha^2}{6(1 + \sigma)r^2 \int \frac{\omega}{r^2} \eta \, d\beta}$$

This magnitude is considerably smaller than unity provided that $\bar{p}$ and $\int \omega\eta \, d\beta/r^2$ are not too small so that the law of conjugate tangential forces in short rods is satisfied.

Let us return to the equations for the tangential forces and replace relation (5.5) by the condition of conjugate tangential forces:

$$S_1 + S_2 = 0$$

With the aid of equations (5.14) and (5.15) we then obtain

$$T_1 = \alpha\left[a_1 + a_2 \frac{x}{r} + a_3 \frac{y}{r} + a_4 \frac{\omega}{r^2}\right]\eta + \left[b_1 + b_2 \frac{x}{r} + b_3 \frac{y}{r} + b_4 \frac{\omega}{r^2}\right]\eta$$

$$S_1 = -S_2 = -a_1 \int_0^\beta \eta \, d\beta - a_2 \int_0^\beta \frac{x}{r} \eta \, d\beta - a_3 \int_0^\beta \frac{y}{r} \eta \, d\beta -$$

$$a_4 \int_0^\beta \frac{\omega}{r^2} \eta \, d\beta - a_5$$

We see that the stress in the cross-sections of a thin-walled rod depends on nine constants. Of these a_1 and a_5 determine the values of the force S_2 at the straight edges of the shell, since on account of the relations of orthogonality (2.10) the coefficients of a_2, a_3, a_4 in the expression for S_2 become zero for $\beta = 0$ and $\beta = \beta_0$. Assuming for simplicity that there are no longitudinal forces at the straight edges we must set

$$a_1 = a_5 = 0$$

The five constants a_2, a_3, b_1, b_2, b_3 have a simple physical meaning: They determine those stress states in a shell for which it behaves like a solid rod, namely, b_1 gives the tension, b_2, b_3 give the pure bending by the end moments relative to the principal axes Y and X respectively; a_2, a_3 give the bending by the transverse forces directed respectively along the principal axes Y and X. The two remaining constants a_4 and b_4 correspond to the stress states different from those which arise in solid rods, namely, a_4 gives the torsion which must be distinguished from the Saint-Venant torsion, since it is brought about by the shearing forces and not by the torsional moments; b_4 gives the stress state which we shall call sectorial, thereby emphasizing that in it, as also in the above mentioned torsion, the normal stresses are distributed according to the law of sectorial areas.

The latter stress state is in principle a new one. The forces which give rise to it, applied at the end sections of the rod, are self-balanced at each end separately. We introduce the notation

$$r\int_0^{\beta_0} T_1 d\beta = N \qquad r\int_0^{\beta_0} T_1 x\, d\beta = -M_y \qquad r\int_0^{\beta_0} T_1 y\, d\beta = M_x$$

$$r\int_0^{\beta_0} T_1 \omega\, d\beta = B \qquad \int_0^{\beta_0} S_1 dx = Q_x \qquad \int_0^{\beta_0} S_1 dy = Q_y \qquad \int_0^{\beta_0} S_1 d\omega = M_\omega$$

(5.16)

where N, M_x, M_y, Q_x, Q_y are static factors familiar from the theory of solid rods: N is the tensile force, M_x, M_y the principal bending moments, Q_x, Q_y the shearing forces acting along the principal directions. The magnitudes B and M_ω in the theory of solid rods do not enter: B is the flexural-torsional bimoment (a term used by V. Z. Vlasov), a force factor statically equivalent to zero which may be thought of as a pair of moments; M_ω is the torsional moment due to the shearing forces S_1; which, again using the terminology of Vlasov, we shall call flexural-torsional in order to differentiate it from the torsional moment due to the nonuniformity of the distribution of the shearing stresses over the thickness of the wall of the shell.

The magnitudes, introduced by equations (5.16), are connected with the constants a_i and b_k by equations which are derived from the conditions of orthogonality (2.10):

$$N = \frac{F}{2h} b_1 \qquad -M_y = \frac{I_y}{2hr}\left(a_2\alpha + b_2\right)$$

$$M_x = \frac{I_x}{2hr}\left(a_3\alpha + b_3\right) \qquad B = \frac{I_\omega}{2hr^2}\left(a_4\alpha + b_4\right)$$

Representing M_x, M_y, M_ω in the form

$$M_x = M_x^{(0)} + \alpha M_x^{(1)} \qquad M_y = M_y^{(0)} + \alpha M_y^{(1)}$$

$$B = B^{(0)} + \alpha B^{(1)} \qquad (5.16')$$

we may write

$$a_1 = 0 \qquad a_2 = -\frac{2hr}{I_y} M_y^{(1)} \qquad a_3 = \frac{2hr}{I_x} M_x^{(1)}$$

$$a_4 = \frac{2hr^2}{I_\omega} B^{(1)} \qquad a_5 = 0$$

$$b_1 = \frac{2h}{F} N \qquad b_2 = -\frac{2hr}{I_y} M_y^{(0)} \qquad b_3 = \frac{2hr}{I_x} M_x^{(0)} \qquad b_4 = \frac{2hr^2}{I_\omega} B^{(0)} \tag{5.17}$$

On the other hand

$$a_2 = \frac{2hr^2}{I_y} Q_x \qquad a_3 = \frac{2hr^2}{I_x} Q_y \qquad a_4 = \frac{2hr^3}{I_\omega} M_\omega$$

whence

$$M_y^{(1)} = -rQ_x \qquad M_x^{(1)} = rQ_y \qquad B^{(1)} = rM_\omega$$

The first two of these equations express the conditions of the equilibrium of an arbitrary part of the rod taken between the initial section $\alpha = 0$ and another arbitrary section. The third equation establishes a relation between the flexural-torsional bimoment and the flexural-torsional moment. This relation does not, however, follow from abstract statical considerations like the first two but reflects definite properties of cylindrical shells.

6. DISPLACEMENTS OF THIN-WALLED RODS

The displacements of rod-shells may be obtained by making use of the geometric relations

$$\epsilon_1 = \frac{1}{r}\frac{\partial u}{\partial \alpha} \qquad \epsilon_2 = \frac{1}{r}\left(\frac{\partial v}{\partial \beta} - \frac{w}{R}\right) \qquad \gamma = \frac{1}{r}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right)$$

Let T_1 and S_2 be defined by equations (5.14) and $T_2 = 0$, as follows from (5.5) with a certain degree of approximation.

Then

$$2Eh\eta\epsilon_1 = T_1 = 2Eh\eta \frac{1}{r}\frac{\partial u}{\partial \alpha} = \alpha \frac{\partial f_1}{\partial \beta} + f_2$$

$$2Eh\eta\epsilon_2 = -\sigma T_1 = 2Eh\eta \frac{1}{r}\left(\frac{\partial v}{\partial \beta} - \frac{w}{R}\right) = -\sigma\left(\alpha \frac{\partial f_1}{\partial \beta} + f_2\right)$$

$$2Eh\eta \frac{\gamma}{2} = -(1 + \sigma)S_2 = 2Eh\eta \frac{1}{2r}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right) = -(1 + \sigma)f_1$$

Or, if these equations are solved for u, v, w, we obtain

$$\frac{2Eh}{r} u = \frac{1}{\eta}\left(\frac{\alpha^2}{2!}\frac{\partial f_1}{\partial \beta} + \alpha f_2\right) + \frac{2Eh}{r}\psi_1$$

$$\frac{2Eh}{r} v = -\frac{\alpha^3}{3!}\frac{\partial}{\partial \beta}\frac{1}{\eta}\frac{\partial f_1}{\partial \beta} - \frac{\alpha^2}{2!}\frac{\partial}{\partial \beta}\frac{f_2}{\eta} - 2(1 + \sigma)\alpha\frac{f_1}{\eta} - \frac{2Eh}{r}\left(\alpha\frac{\partial \psi_1}{\partial \beta} - \psi_2\right)$$

$$\frac{2Eh}{r} w = -R\left[\frac{\alpha^3}{3!}\frac{\partial^2}{\partial \beta^2}\frac{1}{\eta}\frac{\partial f_1}{\partial \beta} + \frac{\alpha^2}{2!}\frac{\partial^2}{\partial \beta^2}\frac{f_2}{\eta} + 2(1 + \sigma)\alpha\frac{\partial}{\partial \beta}\frac{f_1}{\eta} + \frac{2Eh}{r}\left(\alpha\frac{\partial^2 \psi_1}{\partial \beta^2} - \frac{\partial \psi_1}{\partial \beta}\right)\right] + \sigma R\left(\alpha\frac{1}{\eta}\frac{\partial f_1}{\partial \beta} + \frac{1}{\eta}f_2\right) \quad (6.1)$$

where ψ_1, ψ_2 are arbitrary functions of integration depending only on the magnitudes β.

In order to clarify the geometric meaning of the functions ψ_1 and ψ_2 we put

$$f_1 = f_2 = 0$$

Then

$$u = \psi_1 \qquad v = -\alpha\frac{\partial \psi_1}{\partial \beta} + \psi_2 \qquad \frac{w}{R} = -\alpha\frac{\partial^2 \psi_1}{\partial \beta^2} + \frac{\partial \psi_2}{\partial \beta}$$

$$2Eh\epsilon_1 = 0 \qquad 2Eh\epsilon_2 = 0 \qquad 2Eh\gamma = 0$$

Continued on next page

$$2Eh\kappa_1 = 0 \qquad 2Eh\kappa_2 = -\frac{2Eh}{r}\frac{\partial}{\partial\beta} L\left(\alpha\frac{\partial\psi_1}{\partial\beta} - \psi_2\right)$$

$$2Eh\tau = -\frac{2Eh}{r} L\frac{\partial\psi_1}{\partial\beta} \tag{6.2}$$

This corresponds to an infinitely small bending of the middle surface (to a deformation which maintains the geometry at the middle surface).

The arbitrary factors which are contained in the functions ψ_1, ψ_2 make it possible to satisfy the geometric boundary conditions but in choosing ψ_1 and ψ_2 it is necessary to maintain the correspondence between the geometrical and statical results. This reduces to the requirement that the deformations κ_1 and κ_2, and together with these the bending moments, become zero because in the principal stress state of a short rod the torsional moments must predominate over the bending moments.

Equations (6.2) show that for this it is necessary that the two equations be satisfied:

$$\frac{\partial}{\partial\beta} L\frac{\partial\psi_1}{\partial\beta} = 0 \qquad \frac{\partial}{\partial\beta} L\psi_2 = 0 \tag{6.3}$$

whence

$$\psi_1 = c_1 + c_2\frac{x}{r} + c_3\frac{y}{r} + c_4\frac{\omega}{r^2}$$

$$\psi_2 = \frac{d_1}{r}\frac{\partial x}{d\beta} + \frac{d_2}{r}\frac{dy}{d\beta} + \frac{d_3}{r^2}\frac{d\omega}{d\beta} \tag{6.4}$$

There remain undetermined the seven constants c_i and d_k. Six of them c_1, c_2, c_3, d_1, d_2, d_3 correspond to the displacement of the shell as a rigid whole. This follows from the fact that for $c_4 = 0$ the function ψ_1 determined by the first of equations (6.4) in addition to satisfying the second equation of (6.3) satisfies also the equation

$$L\frac{\partial\psi_1}{\partial\beta} = 0$$

and all the six components of the deformation are equal to zero.

The constant c_4 defines a new geometrical factor, namely, the "deplanation" (term introduced by V. Z. Vlasov). The basis for the introduction of such a term is the fact that for $c_4 \neq 0$ the cross-sections of the rod cease being plane.

The functions ψ_1, ψ_2, if they are given by equations (6.4), determine the deflections of the middle surface (among them are included also the trivial deflections, that is, the motions of the shell as a rigid whole) which are either altogether not accompanied by the appearance of moments or which give rise to only torsional moments.

All other deflections in the construction of a theory of thin-walled rods must be rejected because according to the results of the preceding section they either correspond to local stress states or will necessarily be accompanied by the appearance of forces and moments at the straight edges.

Let us substitute expressions (6.4) for ψ_1 and ψ_2 in equations (6.2) for the tangential displacements u, v. We have

$$u = c_1 + c_2 \frac{x}{r} + c_3 \frac{y}{r} + c_4 \frac{\omega}{r^2}$$

$$v = -\alpha\left[\frac{c_2}{r}\frac{dx}{d\beta} + \frac{c_3}{r}\frac{dy}{d\beta} + \frac{c_4}{r^2}\frac{d\omega}{d\beta}\right] + \frac{d_1}{r}\frac{dx}{d\beta} + \frac{d_2}{r}\frac{dy}{d\beta} + \frac{d_3}{r^2}\frac{d\omega}{d\beta} \qquad (6.5)$$

Making use of the fact that the system of functions (1, x, y, ω) is mutually orthogonal with weight η we obtain

$$c_1 = \frac{2hr}{F}\int_0^{\beta_0} u\eta \, d\beta \qquad c_2 = \frac{2hr^2}{I_y}\int_0^{\beta_0} ux\eta \, d\beta$$

$$c_3 = \frac{2hr^2}{I_x}\int_0^{\beta_0} uy\eta \, d\beta \qquad c_4 = \frac{2hr^2}{I_\omega}\int_0^{\beta_0} u\omega\eta \, d\beta$$

$$-\alpha c_2 + d_1 = \frac{2hr^2}{I_y}\int_0^{\beta_0} x\eta \, d\beta \int_0^{\beta} v \, d\beta \qquad -\alpha c_3 + d_2 = \frac{2hr^2}{I_x}\int_0^{\beta_0} y\eta \, d\beta \int_0^{\beta} v \, d\beta$$

$$-\alpha c_4 + d_3 = \frac{2hr^3}{I_\omega}\int_0^{\beta_0} \omega\eta \, d\beta \int_0^{\beta} v \, d\beta$$

We introduce the notation

$$2hr\int_0^{\beta_0} u\eta \, d\beta = F\zeta \qquad 2hr\int_0^{\beta_0} ux\eta \, d\beta = I_y\vartheta_y$$

$$2hr\int_0^{\beta_0} uy\eta \, d\beta = I_x\vartheta_x \qquad 2hr\int_0^{\beta_0} u\omega\eta \, d\beta = I_\omega\vartheta_\omega$$

$$2hr^2\int_0^{\beta_0} x\eta \, d\beta \int_0^{\beta} v \, d\beta = I_y\xi_x \qquad 2hr^2\int_0^{\beta_0} y\eta \, d\beta \int_0^{\beta} v \, d\beta = I_x\xi_y$$

$$2hr^2\int_0^{\beta_0} \omega\eta \, d\beta \int_0^{\beta} v \, d\beta = I_\omega\theta \tag{6.6}$$

The equation for the tangential displacements may then be written

$$u = \zeta + \vartheta_y x + \vartheta_x y + \vartheta_\omega \omega \qquad v = \frac{\xi_x}{r}\frac{dx}{d\beta} + \frac{\xi_y}{r}\frac{dy}{d\beta} + \frac{\theta}{r}\frac{d\omega}{d\beta} \tag{6.7}$$

From this the geometric meaning of the symbols becomes clear: ζ is the displacement in the direction of the axis of the cylinder, ϑ_x, ϑ_y are the angles of rotation with respect to the principal axes X and Y lying in the plane $\alpha = 0$, ξ_x, ξ_y are the displacements in the direction of the principal axes X and Y, θ is the angle of rotation relative to the axis passing through the center of rotation, and ϑ_ω is the deplanation.

If the shell is subject to some arbitrary bending the concept of its displacement as a rigid whole loses its significance. We shall, however, formally introduce this concept, assuming that from the equations (6.6) from an arbitrary bending of the shell there are separated out its displacements as a rigid whole and the deplanation.

The considerations of this section lead to the result that in constructing a theory of thin-walled rods the two deflections giving the same rigid displacement and deplanation must be considered as equivalent to each other.

7. BOUNDARY CONDITIONS

In the preceding sections, from the general system of integrals of the theory of cylindrical shells, a certain number of elementary integrals were separated out to which it was necessary to restrict oneself in constructing a theory of thin-walled rods. It is therefore natural that under such conditions it cannot be expected that the boundary conditions will be satisfied with all rigor. Certain boundary conditions must be entirely rejected and those which are retained must be put in a weakened form which reduces to the fact that the functional boundary conditions for which the value of the required functions is given at each point of the bounding contour are replaced by the requirements that definite integral relations are satisfied at the boundaries.

With regard to the static boundary conditions this is attained with the aid of equations (5.16). In place of the condition

$$S_1 = S_{10}$$

where S_{10} is a given function of β it must be required that the shearing forces Q_x, Q_y and the flexural-torsional moment M_ω have the given values. This gives

$$\int_0^{\beta_0} S_1 dx = Q_{x0} \qquad \int_0^{\beta_0} S_1 dy = Q_{y0} \qquad \int_0^{\beta_0} S_1 d\omega = M_{\omega 0}$$

In place of the condition $T_1 = T_{10}$ it must be required that the tensile force N, the bending moments M_x, M_y, and the flexural-torsional bimoment B have given values; this gives

$$r\int_0^{\beta_0} T_1 d\beta = N_0 \qquad r\int_0^{\beta_0} T_1 x\ d\beta = -M_{y0}$$

$$r\int_0^{\beta_0} T_1 y\ d\beta = M_{x0} \qquad r\int_0^{\beta_0} T_1 \omega\ d\beta = B_0$$

The boundary conditions imposed on the moments and transverse forces may be rejected since they influence only the end effect (reference 7). If there is transmitted a nonselfbalanced load in the shell with the aid of shearing forces or moments it is necessary of course to take it into account in computing the magnitudes Q_{x0}, Q_{y0}, $M_{\omega 0}$, N_0, M_{x0}, M_{y0}, B_0.

In setting up the geometrical boundary conditions a similar device may be used.

In place of the boundary condition $u = u_0$ the requirement must be set that ζ, ϑ_x, ϑ_y and ϑ_ω have given values at the end cross-section, that is, according to equations (6.6), that the integral equations be satisfied

$$\frac{2hr}{F} \int_0^{\beta_0} u\eta \, d\beta = \zeta_0 \qquad \frac{2hr}{I_y} \int_0^{\beta_0} ux\eta \, d\beta = \vartheta_y$$

$$\frac{2hr}{I_x} \int_0^{\beta_0} uy\eta \, d\beta = \vartheta_{x0} \qquad \frac{2hr}{I_\omega} \int_0^{\beta_0} u\omega\eta \, d\beta = \vartheta_{\omega 0}$$

Similarly the boundary condition $v = v_0$ is replaced by the three integral relations:

$$\frac{2hr^2}{I_y} \int_0^{\beta_0} x\eta \, d\beta \int_0^{\beta} v \, d\beta = \xi_{x0} \qquad \frac{2hr^2}{I_x} \int_0^{\beta_0} y\eta \, d\beta \int_0^{\beta} v \, d\beta = \xi_{y0}$$

$$\frac{2hr^2}{I_\omega} \int_0^{\beta_0} \omega\eta \, d\beta \int_0^{\beta} v \, d\beta = \theta_0$$

The conditions imposed on the normal displacement and the angle of rotation influence, as a rule, only the end effect and therefore they need not be taken into account. The investigation of the exceptional cases which may hardly be encountered in practice would lead us too far from our subject.

In replacing the functional boundary conditions by integral conditions we replace the actual end load (including in its composition the reactive forces) by another statically equivalent load, which gives, moreover, the same flexural-torsional bimoment. The difference between these two loads produces local stress states not taken into account in the theory of rods.

Within the limits of accuracy of the theory the boundary conditions may be replaced by integral relations and also by other methods. The approach here employed is to be preferred only because of its simplicity and physical clearness.

8. GENERAL EQUATIONS OF THE THEORY OF THIN-WALLED OPEN SECTION RODS OF MEDIUM LENGTH

We now have at our disposal all the necessary data for proceeding to set up the fundamental equations of the theory of thin-walled rods. Let the rod be loaded by an arbitrary transverse surface load and end forces. The particular integral determined by the surface load is expressed by equations (3.9). According to these conditions, in the same manner as in section 6, the displacements may be found. For the tangential displacements we obtain the equations

$$u^* = -\frac{r^3}{E}\left[\frac{P_x}{I_y} x + \frac{P_y}{I_x} y + \frac{M}{I_\omega} \omega\right] \int_0^\alpha d\alpha \int_0^\alpha d\alpha \int_0^\alpha \xi(\alpha)\, d\alpha$$

$$v^* = +\frac{r^3}{E}\left[\frac{P_x}{I_y}\frac{dx}{d\beta} + \frac{P_y}{I_x}\frac{dy}{d\beta} + \frac{M}{I_\omega}\frac{d\omega}{d\beta}\right] \int_0^\alpha d\alpha \int_0^\alpha d\alpha \int_0^\alpha d\alpha \int_0^\alpha \xi(\alpha)\, d\alpha +$$

$$\frac{2(1+\sigma)r^3}{E}\left[\frac{P_x}{I_y}\frac{1}{\eta}\int_0^\beta x\eta\, d\beta + \frac{P_y}{I_x}\frac{1}{\eta}\int_0^\beta y\eta\, d\beta + \right.$$

$$\left.\frac{M}{I_\omega}\frac{1}{\eta}\int_0^\beta \omega\eta\, d\beta\right] \int_0^\alpha d\alpha \int_0^\alpha \xi(\alpha)\, d\alpha \tag{8.1}$$

By adding this particular integral due to the transverse surface load to the displacements (6.1) due to the end forces we obtain the total displacement. Having the total displacement the components of the rigid displacement and the deplanation may be computed, making use of the determination of these magnitudes from equations (6.6). If we also make use of the notations (5.16) and (5.17), the fundamental relations of the theory of thin-walled rods may be represented in the form (in the computations it is necessary to use the equations of section 2)

$$\zeta = \alpha r \frac{N}{EF} + c_1 \tag{8.2}$$

$$\vartheta_y = \frac{\alpha^2}{2!} r^2 \frac{Q_x}{EI_y} - \alpha r \frac{M_y^{(0)}}{EI_y} - r^3 \frac{P_x}{EI_y} \int_0^\alpha d\alpha \int_0^\alpha d\alpha \int_0^\alpha \xi(\alpha) d\alpha + \frac{c_2}{r} \tag{8.3}$$

$$\vartheta_x = \frac{\alpha^2}{2!} r^2 \frac{Q_y}{EI_x} + \alpha r \frac{M_x^{(0)}}{EI_x} - r^3 \frac{P_y}{EI_x} \int_0^\alpha d\alpha \int_0^\alpha d\alpha \int_0^\alpha \xi(\alpha) d\alpha + \frac{c_3}{r} \tag{8.4}$$

$$\vartheta_\omega = \frac{\alpha^2}{2!} r^2 \frac{M_\omega}{EI_\omega} + \alpha r \frac{B^{(0)}}{EI_\omega} - r^3 \frac{M}{EI_\omega} \int_0^\alpha d\alpha \int_0^\alpha d\alpha \int_0^\alpha \xi(\alpha) d\alpha + \frac{c_4}{r^2} \tag{8.5}$$

$$\begin{aligned} \xi_x = &-\frac{\alpha^3}{3!} r^3 \frac{Q_x}{EI_y} + \frac{\alpha^2}{2!} r^2 \frac{M_y^{(0)}}{EI_y} - 2(1 + \sigma)\alpha r^3 \left(\frac{S_{xx}}{I_y} \frac{Q_x}{EI_y} + \frac{S_{xy}}{I_y} \frac{Q_y}{EI_x} + \right. \\ &\left. \frac{S_{x\omega}}{I_y} \frac{M_\omega}{EI_\omega}\right) + r^4 \frac{P_x}{EI_y} \int_0^\alpha d\alpha \int_0^\alpha d\alpha \int_0^\alpha d\alpha \int_0^\alpha \xi(\alpha) d\alpha + \\ &r^4 2(1 + \sigma)\left[\frac{P_x}{EI_y} \frac{S_{xx}}{I_y} + \frac{P_y}{EI_x} \frac{S_{xy}}{I_y} + \frac{M_\omega}{EI_\omega} \frac{S_{x\omega}}{I_y}\right] \int d\alpha \int \xi \, d\alpha - \alpha c_2 + d_1 \end{aligned} \tag{8.6}$$

$$\begin{aligned} \xi_y = &-\frac{\alpha^3}{3!} r^3 \frac{Q_y}{EI_x} + \frac{\alpha^2}{2!} r^2 \frac{M_x^{(0)}}{EI_x} - 2(1 + \sigma)\alpha r^3 \left(\frac{S_{yx}}{I_x} \frac{Q_x}{EI_y} + \frac{S_{yy}}{I_x} \frac{Q_y}{EI_x} + \right. \\ &\left. \frac{S_{y\omega}}{I_x} \frac{M_\omega}{EI_\omega}\right) + r^4 \frac{P_y}{EI_x} \int_0^\alpha d\alpha \int_0^\alpha d\alpha \int_0^\alpha d\alpha \int_0^\alpha \xi(\alpha) d\alpha + \\ &r^4 2(1 + \sigma)\left(\frac{P_x}{EI_y} \frac{S_{yx}}{I_x} + \frac{P_y}{EI_x} \frac{S_{yy}}{I_x} + \frac{M}{EI_\omega} \frac{S_{y\omega}}{I_x}\right) \int d\alpha \int \xi \, d\alpha - \alpha c_3 + d_2 \end{aligned} \tag{8.7}$$

$$\theta = -\frac{\alpha^3}{3!} r^3 \frac{M_\omega}{EI_\omega} - \frac{\alpha^2}{2!} r^2 \frac{B(0)}{EI_\omega} - 2(1 + \sigma)\alpha r^3\left(\frac{S_{\omega x}}{I_\omega} \frac{Q_x}{EI_y} + \frac{S_{\omega x}}{I_\omega} \frac{Q_y}{EI_x} + \right.$$

$$\left.\frac{S_{\omega\omega}}{I_\omega} \frac{M_\omega}{EI_\omega}\right) - r^4 \frac{M}{EI_\omega} \int_0^\alpha d\alpha \int_0^\alpha d\alpha \int_0^\alpha d\alpha \int_0^\alpha \xi(\alpha) d\alpha +$$

$$r^4 2(1 + \sigma)\left[\frac{M}{EI_x} \frac{S_{\omega\omega}}{I_\omega} + \frac{P_x}{EI_y} \frac{S_{\omega x}}{I_\omega} + \frac{P_y}{I_x} \frac{S_{\omega y}}{I_\omega}\right] \int_0^\alpha d\alpha \int_0^\alpha \xi\, d\alpha - \alpha \frac{c_4}{r} + \frac{d_3}{r} \tag{8.8}$$

In addition to the already known notations there were also used above the new ones

$$S_{AB} = 2hr \int_0^{\beta_0} A\eta \, d\beta \int_0^\beta \frac{d\beta}{\eta} \int_0^\beta \beta\eta \, d\beta \tag{8.9}$$

where A and B are to be replaced by any combination of x, y, ω.

Equations (8.2), (8.3), (8.4) in no way differ from the relations of the theory of solid rods. The first of them connects the longitudinal displacement ζ with the normal force N. The two last equations establish the relation between the angles of rotation ϑ_x, ϑ_y on the one hand and the shearing forces Q_x, Q_y, the bending moments $M_y^{(0)}$, $M_x^{(0)}$ and the surface transverse load on the other. If all the magnitudes S_{AB} were equal to zero equations (8.6) and (8.7) would transform into the relations of the theory of solid rods. They would then express the deflections ξ_α, ξ_β by the shearing forces, the bending moments, and the transverse load.

The difference of the theory of open thin-walled rods from the theory of solid rods consists, in the first place, in the presence of terms depending on S_{AB}; in the second place, in the theory of thin-walled rods it is necessary to take into account a new static factor, namely the bimoment; in the third place the torsion of a thin-walled rod is produced by a flexural-torsional moment statically equivalent to the torsional moment of the Saint-Venant theory of torsion but having an essentially different origin. A very important consideration is that the deplanation ϑ_ω and the angle of torsion θ are both connected with the flexural-torsional moment and the bimoment by equations (8.5) and (8.8) (having no analogue in the theory of solid rods). It follows that ϑ_ω and θ are closely connected with each other.

It is easy to see that the magnitudes S_{AB} will not figure in the fundamental equations (8.2) to (8.8) if the elasticity relation (1.3) is replaced by the requirement of the absence of shear $\gamma = 0$. The terms containing the magnitudes S_{AB} thus take into account the effect of the shear. These terms, generally speaking, play a secondary role since they contain the variable α to lower degrees than the fundamental components. The question, however, of the order of the errors taking the shear into account requires further investigation.

If we make the assumption

$$\gamma = 0$$

the solution of concrete problems is considerably simplified. In this case the relations (8.2) to (8.8) will almost entirely agree with the equations derived by V. Z. Vlasov. Only equation (8.8) will be different. The corresponding equation of Vlasov is more complicated and as has been shown above the increase in accuracy was obtained by a formally contradictory device.

In concluding this section we shall make one more remark. The particular integral (3.9) corresponds to a certain fictitious load R_1 statically equivalent at each cross-section of the shell to the true load R. The difference between the true and fictitious loads is the self-balanced load R_2, the action of which was considered in the second part of section 3. It was shown that by a corresponding choice of the arbitrary constants of integration for this load (if it varies linearly with α) a particular integral may be specified which in the cross-sections gives only the forces statically equivalent to zero. In the theory of thin-walled rods, however, they cannot be ignored. More accurately, it is necessary to separate out and take into account those forces which vary according to the law of sectorial areas. This can be done with the aid of equations (3.18) according to which the force T_1 has the form:

$$T_1 = 2Eh\eta\epsilon_1 + \sigma T_2$$

and ϵ_1 in turn may be written

$$\epsilon_1 = B_1 + B_2 \frac{x}{r} + B_3 \frac{y}{r}$$

The functions B_1, B_2, B_3 are so chosen that T_1 is self-balanced, that is, the integral equations are satisfied

$$r\int_0^{\beta_0} T_1 d\beta = N = 0 \qquad r\int_0^{\beta_0} T_1 x\ d\beta = -M_y = 0$$

$$r\int_0^{\beta_0} T_1 y\ d\beta = M_x = 0$$

The bimoment B^* will be different from zero and may be computed:

$$B^* = r\int_0^{\beta_0} T_1 \omega\ d\beta = \sigma r\int T_2 \omega\ d\beta \tag{8.10}$$

The force T_2 entering the above equation is obtained from the computation of a curved strip of unit width cut out from the shell and loaded by the load R_2. The bimoment B^* must be added in equations (8.2) to (8.8) to the bimoment $B^{(0)}$.

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