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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1308

ON MOTION OF FLUID IN BOUNDARY LAYER NEAR LINE OF INTERSECTION OF TWO PLANES

By L. G. Loitsianskii and V. P. Bolshakov

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"O Dvizhenii Zhidkosti v Pogranichnom Sloe Vblizi Linii Peresechenia Dvukh Ploskostei." Rep. No. 279, CAHI, 1936.

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ON MOTION OF FLUID IN BOUNDARY LAYER NEAR LINE OF INTERSECTION OF TWO PLANES*

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SUMMARY

In the paper "The Mutual Interference of Boundary Layers," the authors investigated the problem of the interference of two planes intersecting at right angles on the boundary layers formed by the motion of fluid along the line of intersection of these planes.

In the present paper, the results of the preceding one are generalized to the case of planes intersecting at any angle. The motion of a fluid in an angle less than 180° is discussed and the enlargement of the boundary layers near the line of intersection of the planes, the limits of the interference effects of the boundary layers, and the corrections on the drag are determined. All computations are conducted by the Kármán-Pohlhausen method for laminar and turbulent boundary layers. The results are reduced to tabulated form.

INTRODUCTION

The problem of the interaction of the boundary layers formed in the dihedral angle between two thin plates parallel to the intersection of the plates was apparently first proposed in reference 1 (for the case of a right angle). In the discussion of reference 1, the need for a generalization of results obtained for the case of the right angle to the case of any angle less than two right angles (180°) was pointed out. The present paper is concerned with this problem and its solution. Although the limits of application for the approximate method of the finite-thickness layer previously used are retained, the problem of the interaction of the boundary layers near the intersection of a dihedral angle of any magnitude from 0° to 180° is solved herein. For a laminar layer, a first and a second approximation are given and also, for a check, a sixth approximation (in the terminology of Pohlhausen). It is

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shown that the sixth approximation differs comparatively little from the second. In conclusion, the case of a turbulent boundary layer is considered with the assumption of the validity of the '1/7' power law for the velocity profiles. As in reference 1, all computations can be carried through to the end, although the procedure is somewhat more cumbersome. The limits of interference of the layers and the correction on the drag due to the interference effect are determined.

1. DERIVATION OF FUNDAMENTAL INTEGRAL CONDITION

Consider the flow of a fluid, approaching from infinity with velocity V, in the dihedral oblique angle θ between two plates of the same finite length x along the flow and infinite in the transverse direction (fig. 1). The Y- and Z-axes are taken along the leading edges of the plates and the X-axis along the flow parallel to the line of intersection of the plates. In the oblique system of coordinates thus obtained, the distribution of the velocities in the boundary layer may be given in the same manner as for the case of the rectangular system of coordinates for the flow in a right angle.

If each plate worked independently of the other, there would be formed on it a layer of thickness δ_0, which is a function only of x and the profile of the longitudinal velocities

$$u_0 = u_0(x, z, \sin\theta)$$

or

$$u_0 = u_0(x, y, \sin\theta)$$

depending on whether the boundary layer is considered to lie in the plane XOY or XOZ.

Because of the retarding effect of one of the plates on the other near the line of intersection, the layers must not be considered as in the plane problem. The layers become three dimensional and the thickness will now be a function of two variables:

$$\delta = \delta(x,y) \quad \text{on the plate containing the Y-axis}$$

$$\delta = \delta(x,z) \quad \text{on the plate containing the Z-axis}$$

The component of velocity parallel to the X-axis, which will be denoted by u without the subscript 0, will be a function of the three variables; that is,

$$u = u(x,y,z)$$

By the conditions of the problem, under the basic assumption of the concept of a finite region of influence of the viscosity, this function must become zero on the surface of the plates, a constant at the outer limit of the boundary layer, and, particularly, at a finite distance from the line of intersection of the plates, must become the velocity distribution $u_0(x,z, \sin\theta)$ or $u_0(x,y, \sin\theta)$, which corresponds to the isolated plate with boundary layer undisturbed by the adjacent plate. The boundary separating the region disturbed by the adjacent plate from the undisturbed region, which corresponds to the isolated plate, shall, for briefness, be denoted as the "interference boundary" of the layers. The equation of this boundary in the planes XOY and XOZ will be

$$h_0 = h_0(x)$$

A section of the boundary layer cut by the plane $x = \xi$ is shown in figure 2. Inasmuch as the coordinates of the system YOZ are oblique, the equation of the boundary layer in this section for the undisturbed region for $y \geqq y_2(\xi)$, where $y_2(\xi) = h_0(\xi) - z_1(\xi)\cos\theta = h_0(\xi) - \delta_0(\xi)\cot\theta$, will have the form

$$z = z_1(\xi) = \frac{\delta_0(\xi)}{\sin\theta} \tag{1}$$

Equation (1) holds true both for $\theta < \frac{\pi}{2}$ and for $\theta > \frac{\pi}{2}$. The non-coincidence for $\theta \neq \frac{\pi}{2}$ of the interference boundary $h_0(\xi)$ and the coordinate $y_0(\xi)$ should be noted, from which starting point the ordinate z of the outer limit of the boundary layer becomes and remains constant, independent of y. All that has been stated about the YOX planes also remains true, of course, for the ZOX plane (because of the symmetry $y_1(\xi) = z_1(\xi)$ and $y_2(\xi) = z_2(\xi)$, which should be remembered in the following development).

For the present, the question as to the equation of the boundary layer in the section $x = \xi$ will be ignored. The fundamental integral condition of the problem will now be set up. For this purpose, as in the work previously cited, the momentum theorem is applied for a tube of flow formed by the coordinate planes, by the surface of the lines of flow passing through the edge of the boundary layer at section $x = x$, and, finally, by the surfaces of the lines of flow passing through the perpendiculars to the plates located at the points $y = h_0(x)$ and $z = h_0(x)$, where the part of the flow tube considered is between the sections $x = x$ and the section located upstream of the flow at a sufficient distance from the section $x = 0$, that is, from the entry of the fluid on the plates. Then, as is known,

$$\iint_{(\sigma)} \rho(V-u)\, u\, d\sigma = W \tag{2}$$

where W is the drag of the plates applied to the segment of the flow tube considered and σ is the section of the boundary layer cut by a plane perpendicular to the X-axis at distance x from the origin 0.

The double integral on the left may be expressed in the following manner (fig. 2):

$$\iint_{(\sigma)} (V-u)\, u\, d\sigma = \sin\theta \left\{ \int_0^{y_1} dy \int_0^{y_2} \rho(V-u)\, u\, dz + \int_{y_1}^{y_2} dy \int_0^{Z} \rho(V-u)\, u\, dz + 2\int_{y_2}^{y_2+z_1\cos\theta} dy \int_0^{\frac{y_2+z_1\cos\theta-y}{\cos\theta}} \rho(V-u)\, u\, dz \right. \tag{3}$$

where Z, the ordinate of the outer edge of the boundary layer, is, as yet, an unknown function of y and x if the angle θ is considered as a parameter maintaining a constant value for the given problem. The first two integrals have a very obvious origin; the presence of the last integral is due to the obliqueness of the coordinates and the differences in the direction between the ordinate $z_1(x)$ and the thickness of the layer $\delta_0(x)$, which makes it necessary to take the integration over the two areas of the triangles shown hatched in figure 2.

The drag W, as is easily seen from figure 3, is determined by the integral

$$W = 2\int_0^x\left[\int_0^{h_0(\xi)} \tau\,dy + \int_{h_0(\xi)}^{h(x)} \tau_0\,dy\right]d\xi = 2\int_0^x\left[\int_0^{y_2(\xi)+z_1(\xi)\cos\theta} \tau\,dy + \int_{y_2(\xi)+z_1(\xi)\cos\theta}^{y_2(x)+z_1(x)\cos\theta} \tau_0\,dy\right]d\xi \qquad (4)$$

where τ denotes the friction stress in the region S disturbed by the adjacent plate and τ_0 denotes the corresponding stress in the undisturbed region S_0. The boundary between the regions S and S_0 is, of course, the boundary of interference of the boundary layers.

Combining equations (3) and (4) yields, in general form, the integral condition of the problem:

$$\sin\theta\left[\int_0^{y_1} dy\int_0^{z_2} \rho(V-u)\,u\,dz + \int_{y_1}^{v_2} dy\int_0^{Z} \rho(V-u)\,u\,dz + 2\int_{y_2}^{y_2+z_1\cos\theta} dy\int_0^{\frac{y_2+z_1\cos\theta-y}{\cos\theta}} \rho(V-u)\,u\,dz\right]$$

$$= 2\int_0^x\left[\int_0^{y_2(\xi)+z_1(\xi)\cos\theta} \tau\,dy + \int_{y_2(\xi)+z_1(\xi)\cos\theta}^{y_2(x)+z_1(x)\cos\theta} \tau_0\,dy\right]d\xi \qquad (5)$$

Equation (5) is a generalization of the integral condition derived in the previously cited paper for the case of a right-angled dihedral $(\theta = \frac{\pi}{2})$. The limits of integration are as yet unknown functions of the arguments. In order to determine the form of these functions, it is first necessary that the shape of the boundary layer be given.

2. SHAPE OF CROSS SECTION OF BOUNDARY LAYER; VELOCITY DIAGRAMS;

SOLUTION OF PROBLEM BY FIRST APPROXIMATION

The section of the boundary layer on the plane $x = \xi$ (fig. 2) is considered and it is assumed that the equation of the curve defining the edge of the boundary layer is

$$\left.\begin{aligned} f(y,z) &= a(\xi,\theta) \\ y_1(\xi) \leqq y &\leqq y_2(\xi) \\ z_1(\xi) \leqq z &\leqq z_2(\xi) \end{aligned}\right\} \tag{6}$$

The form of the function $f(y,z)$ cannot be determined unless the additional assumption is made as to the similarity of the approximate velocity diagrams in the different planes $x = \xi$. The curve is sought in the form

$$\frac{u}{V} = \varphi\left[\frac{f(y,z)}{a(\xi,\theta)}\right] \tag{7}$$

where the function $\varphi(t)$ is subjected to the conditions

$$\left.\begin{aligned} \varphi(0) &= 0 \\ \varphi(1) &= 1 \end{aligned}\right\} \tag{8}$$

and the function $f(y,z)$ is subjected to the conditions

$$f(0,z) = f(y,0) = 0 \tag{9}$$

Then the diagram of velocities of equation (7) will evidently satisfy the end conditions of the problem at the walls and at the edge of the boundary layer of equation (6). Setting

$$y = l(\xi,\theta)\ y'$$

$$z = l(\xi,\theta)\ z'$$

where $l(\xi,\theta)$ is a certain length characteristic for the section $x = \xi$ and y' and z' are nondimensional magnitudes, yields equation (7) in the form

$$\frac{u}{V} = \varphi\left[\frac{f(l \cdot y', l \cdot z')}{a(\xi,\theta)}\right]$$

From the condition of similarity of the velocity diagrams, the right side of this equation must not contain ξ, which can be the case if $f(y,z)$ is a homogeneous function of the n^{th} degree, so that

$$\frac{u}{V} = \varphi\left[\frac{l^n(\xi,\theta)}{a(\xi,\theta)} f(y',z')\right]$$

where

$$\frac{l^n(\xi,\theta)}{a(\xi,\theta)} = \text{constant} \tag{10}$$

the constant depending, of course, on θ. It is easy to see that the degree of homogeneity should be equal to 2 $(n = 2)$ and the function $f(y,z)$ must simply be equal to the product of the variables yz because otherwise, with conditions (9) satisfied, the derivatives of the velocity with respect to the coordinates would become zero at the walls and would not give any friction. From equation (6), the following equation of the edge of the boundary layer at the section $x = \xi$ is obtained:

$$yz = a(\xi,\theta) \tag{11}$$

where, from equation (10), for example, choosing the coordinate $y_1(\xi) = z_1(\xi)$ of the undisturbed layer for the characteristic length $l(\xi,\theta)$ yields

$$a(\xi,\theta) = k(\theta)\ z_1^{\ 2}(\xi) = \frac{k(\theta)}{\sin^2\theta}\ \delta_0^{\ 2}(\xi) \tag{12}$$

where $k(\theta)$ is a certain nondimensional constant parametrically dependent on the angle θ.

It is now easy to obtain the magnitudes $y_2(\xi)$, $h_0(\xi)$, and the function $Z(y,\xi,\theta)$ for given angle θ. From equations (11) and (12),

$$y_2(\xi) = \frac{a(\xi,\theta)}{z_1(\xi)} = k(\theta)\ z_1(\xi) = \frac{k(\theta)}{\sin\theta}\ \delta_0(\xi) \tag{13}$$

where $\delta_0(\xi)$ is a known function of ξ independent of the angle θ and determined by solving the problem of the boundary layer of the isolated plate.

From the definition of the interference boundary $h_0(\xi)$ given in equation (1),

$$h_0(\xi,\theta) = y_2(\xi) + z_1(\xi)\cos\theta = \left[k(\theta) + \cos\theta\right]z_1(\xi) = \frac{k(\theta) + \cos\theta}{\sin\theta}\,\delta_0(\xi) \tag{14}$$

Finally, from equation (11), there directly follows that

$$Z(y,\xi,\theta) = \frac{a(\xi,\theta)}{y} = \frac{k(\theta)}{\sin^2\theta} \cdot \frac{\delta_0^{\,2}(\xi)}{y} \tag{15}$$

It is easily verified that, for $y = y_2(\xi)$, Z becomes $z_1(\xi)$. The velocity diagram in the disturbed region of the boundary layer will therefore be smoothly joined with the velocity diagram in the undisturbed region, that is, with the diagram of the isolated plate.

When the equation of the boundary layer is found, the required velocity profile is obtained from equation (7):

$$\frac{u}{V} = \varphi\left(\frac{yz}{a(\xi,\theta)}\right) \tag{16}$$

or from equation (15):

$$\frac{u}{V} = \varphi\left(\frac{z}{Z(y,\xi,\theta)}\right) \tag{17}$$

If Y denotes the variable ordinate of the edge of the boundary layer, that is, the magnitude that, from equation (11), satisfies the equation

$$Y \cdot Z = a(\xi,\theta)$$

then

$$\frac{u}{V} = \varphi\left(\frac{y}{Y(z,\xi,\theta)}\right) \tag{17'}$$

It is important to note that equations (16), (17), and (17') are true not only for those values of y and z that satisfy inequalities (6):

$$y_1(\xi,\theta) \leqq y \leqq y_2(\xi,\theta)$$

$$z_1(\xi,\theta) \leqq z \leqq z_2(\xi,\theta)$$

but in the entire range of interest:

$$0 \leqq y \leqq y_2(\xi,\theta)$$

$$0 \leqq z \leqq z_2(\xi,\theta)$$

When the velocities of the points located in the rhomb

$$0 \leqq y \leqq z_1(\xi,\theta)$$

$$0 \leqq z \leqq z_1(\xi,\theta)$$

are considered, the boundary layer for these points is as though infinite, but the velocities are determined from computation on the hyperbolic edge of the boundary layer.

The velocity profile has thus been determined and the edge of the boundary layer is known. Substituting the values of u and Z in the integral condition (5) yields an ordinary differential equation with one unknown $a(\xi,\theta)$ inasmuch as all the magnitudes involved, including the friction at the wall

$$\tau = \mu \cdot \frac{1}{\sin\theta} \cdot \left(\frac{\partial u}{\partial z}\right)_{z=0}$$

$$\tau_0 = \mu \cdot \frac{1}{\sin\theta}\left(\frac{\partial u_0}{\partial z}\right)_{z=0}$$

are expressed in terms of this function.

As is easily seen, however, the differential equation reduces to a simple equation in finite form for determining the coefficient $k(\theta)$ appearing in equation (12). In order to obtain this single unknown coefficient, certain boundary conditions are assigned for the function $\varphi(t)$ and its derivatives, as in the classical Kármán-Pohlhausen method.

A consideration of the first approximation is the first step. The function $\varphi(t)$ is subjected only to the conditions

$$\varphi(0) = 0$$

$$\varphi(1) = 1$$

that is, the velocity $u(x,y,z)$ becomes zero at the walls and 1 at the outer limit of the boundary layer, which leads to the profile

$$\frac{u}{V} = \frac{yz}{a(x,\theta)} \tag{19}$$

For substitution in the integral condition, the integrals that enter this condition are first computed:

$$\int_0^{y_1} dy \int_0^{z_2} u\,dz = \frac{V}{a(x,\theta)} \int_0^{y_1} y\,dy \int_0^{z_2} z\,dz = \frac{V}{a(x,\theta)} \frac{y_1{}^2 z_2{}^2}{4} = \frac{1}{4}\, a(x,\theta)\, V$$

$$\int_{y_1}^{y_2} dy \int_0^{Z} u\,dz = \int_{y_1}^{y_2} dy \int_0^{y} u\,dz$$

$$= \frac{V}{a(x,\theta)} \int_{y_1}^{y_2} \frac{a^2(x,\theta)}{2y}\,dy = \frac{1}{2}\, Va(x,\theta)\, \log_e \frac{a(x,\theta)}{y_1{}^2}$$

$$2\int_{y_2}^{y_2+z_1 \cos\theta} dy \int_0^{\frac{y_2+z_1 \cos\theta - y}{\cos\theta}} u\,dz$$

$$= 2\,\frac{V}{a(x,\theta)} \int_{y_2}^{h_0(x)} y\,dy \int_0^{\frac{h_0(x)-y}{\cos\theta}} z\,dz$$

$$= \frac{V}{a(x_1\theta)\cos^2\theta} \int_{y_2}^{h_0(x)} \left[h_0{}^2(x) - 2h_0(x)\,y + y^2\right] y\,dy$$

$$= \frac{V}{a(x,\theta)} \left[\frac{1}{3}\, a(x,\theta)\, v_1{}^2 \cos\theta + \frac{1}{12}\, y_1{}^2 \cos^2\theta\right]$$

In a similar manner, the integrals of the square of the velocity are computed. The development will be limited to computing the integrals just given and the remaining ones are written out in ready form:

$$\int_0^{y_1} dy \int_0^{z_2} u^2\, dz = \frac{1}{9} V^2 a(x,\theta)$$

$$\int_{y_1}^{y_2} dy \int_0^{\frac{a(x_1)\theta}{y}} u^2\, dz = \frac{1}{3} V^2 a(x,\theta) \log_e \frac{a(x,\theta)}{y_1{}^2}$$

$$2\int_{y_2}^{h_0(x)} dy \int_0^{\frac{h_0(x)-y}{\cos\theta}} u^2\, dz = V^2\left[\frac{1}{6} y_1{}^2 \cos\theta + \frac{1}{15} \frac{y_1{}^4 \cos^2\theta}{a(x,\theta)} + \frac{1}{90} \frac{y_1{}^6 \cos^3\theta}{a^2(x,\theta)}\right]$$

Thus, the left side of the integral condition reduces to the form

$$\iint_{(\sigma)} \rho(V-u)\, u\, d\sigma = \rho V^2 \sin\theta\left[\frac{5}{36} a(x,\theta) + \frac{1}{6} a(x,\theta) \log_e \frac{a(x,\theta)}{y_1{}^2} + \frac{1}{6} y_1{}^2 \cos\theta + \frac{1}{60} \frac{y_1{}^4 \cos^2\theta}{a(x,\theta)} - \frac{1}{90} \frac{y_1{}^6 \cos^3\theta}{a^2(x,\theta)}\right]$$

The computation of the right side of integral condition (5) reduces to

$$\int_0^x \left[\int_0^{h_0(\xi)} \tau\, ay + \int_{h_0(\xi)}^{h_0(x)} \tau_0\, dy\right] d\xi = \frac{2}{\sin\theta}\int_0^x \left[\int_0^{h_0(\xi)} \mu V \frac{y}{a(\xi,\theta)}\, dy + \right.$$

$$\left.\int_{h_0(\xi)}^{h_0(x)} \mu V \frac{dy}{y_1(\xi)}\right] d\xi = \frac{\mu V}{\sin\theta}\left\{\cos^2\theta \int_0^x \frac{y_1^{\,2}(\xi)}{a(\xi,\theta)}\, d\xi + \right.$$

$$\left. 2\,\frac{a(x,\theta)}{y_1(x)}\int_0^x \frac{d\xi}{y_1(\xi)} + 2y_1(x)\cos\theta \int_0^x \frac{d\xi}{y_1(\xi)} - \int_0^x \frac{a(\xi,\theta)}{y_1^{\,2}(\xi)}\, d\xi \right\}$$

According to the first approximation for the isolated plate (according to Pohlhausen),

$$\delta_0(\xi) = \sqrt{\frac{12\nu\xi}{V}}$$

$$y_1(\xi) = \frac{\delta_0(\xi)}{\sin\theta} = \frac{1}{\sin\theta}\sqrt{\frac{12\nu\xi}{V}}$$

$$\int_0^x \frac{d\xi}{y_1(\xi)} = \sin\theta\sqrt{\frac{V}{12\nu}}\int_0^x \frac{d\xi}{\sqrt{\xi}} = \frac{1}{6}\,\frac{V}{\nu}\,\delta_0(x)\sin\theta$$

so that

$$W = \frac{\mu V}{\sin\theta}\left[\cos^2\theta \int_0^x \frac{y_1^{\,2}(\xi)}{a(\xi,\theta)}\, d\xi - \int_0^x \frac{a(\xi,\theta)}{y_1^{\,2}(\xi)}\, d\xi + \right.$$

$$\left. \frac{1}{3}\,\frac{a(x,\theta)}{\nu}\, V \sin^2\theta + \frac{1}{3}\,\delta_0^{\,2}(x)\,\frac{V}{\nu}\cos\theta \right]$$

All these expressions are immediately simplified as soon as equation (12) is considered; then

$$a(\xi,\theta) = \frac{k(\theta)}{\sin^2 \theta} \delta_0^{\ 2}(\xi)$$

After all the simplifications are made, the following equation for determining $k(\theta)$ is obtained:

$$\log_e k(\theta) = \frac{2}{3} + \frac{\cos \theta}{k(\theta)} + \frac{2}{5} \frac{\cos^2 \theta}{k^2(\theta)} + \frac{1}{15} \frac{\cos^3 \theta}{k^3(\theta)} \tag{20}$$

It is readily seen that, for $\theta = \frac{\pi}{2}$, equation (20) becomes

$$\log_e k = \frac{2}{3}$$

$$k = 1.95$$

as given in the previously cited work on the interference of the boundary layers on mutually perpendicular plates.

The explicit dependence of k on θ is not given in equation (20). This relation may be obtained by the following simple device: When

$$\frac{\cos \theta}{k(\theta)} = \zeta \tag{21}$$

equation (20) becomes

$$\cos \theta = \zeta e^{\frac{2}{3} + \zeta + \frac{2}{5} \zeta^2 + \frac{1}{15} \zeta^3} \tag{22}$$

When the values of ζ are given in the interval where the absolute value of the right side of equation (22) will not exceed 1, the corresponding value of θ is obtained, ζ is determined, and then $k(\theta)$ is found from equation (21). It would also be possible, of course, to proceed in another manner: Equation (22) may be rewritten

$$k(\zeta) = e^{\frac{2}{3} + \zeta + \frac{2}{5} \zeta^2 + \frac{1}{15} \zeta^3} \tag{22'}$$

From a given value of ζ, $k(\zeta)$ is obtained and then $\cos \theta$, and so forth. The simplest method is to draw the function

$$\eta = \Psi(\zeta) = \zeta e^{\frac{2}{3} + \zeta + \frac{2}{5}\zeta^2 + \frac{1}{15}\zeta^3} \tag{23}$$

and obtain its intersection with straight lines parallel to the ζ-axis

$$\eta = \cos\,\theta$$

and then to obtain k from ζ.

The values of θ°, $k(\theta)$, and $\zeta(\theta)$ are tabulated in table 1.

The dimensions of the region of interference of the layers are first determined. According to equation (14),

$$h_0(x,\theta) = m(\theta)\ \delta_0(x) \tag{14'}$$

where

$$m = \frac{k + \cos\,\theta}{\sin\,\theta} = \frac{1+\zeta}{\zeta}\cot\,\theta \tag{24}$$

The values of m for different θ can be determined from table 1 The value $\theta = 180^\circ$ is somewhat isolated; for this value of θ the value of m becomes indeterminate. The value of m can easily be obtained, however, by the usual device of analysis:

$$\left(\frac{1+\zeta}{\tan\,\theta}\right)_{\theta=\pi} = \left(\frac{d\zeta}{d\theta}\cos^2\,\theta\right)_{\theta=\pi} = -\left[\frac{\zeta^2\ \Psi^2(\zeta)\ \sin\,\theta}{\Psi(\zeta) + \zeta\ \Psi'(\zeta)}\right]_{\substack{\zeta=-1\\ \theta=1}} = 0$$

so that, for $\theta = 180^\circ$, $\zeta = -1$ and $m = 0$. The values of $m(\theta)$ are given in the last column of table 1.

The correction in the drag of one side of the plate due to interference of the layers may be computed by the equation

$$\Delta W = \int_0^{l}\left[\int_0^{h_0(\xi)} (\tau_0 - \tau)\ dy\right]d\xi \tag{25}$$

inasmuch as the difference in the drag may show up only in the region S (fig. 3). The quantity l is the length of the plate in the direction of the flow.

The integral equation (25) is readily computed and yields

$$\Delta W = \int_0^l \left[\int_0^{y_2(\xi)+z_1(\xi)\cos\theta} (\tau_0 - \tau)\, dy \right] d\xi$$

$$= \frac{\mu V}{\sin\theta} \int_0^l \left\{ \int_0^{y_2(\xi)+z_1(\xi)\cos\theta} \left[\frac{1}{z_1(\xi)} - \frac{y}{a(\xi,\theta)} \right] dy \right\} d\xi$$

$$= \frac{\mu V}{2\sin\theta} \left[k(\theta) - \frac{\cos^2\theta}{k(\theta)} \right] l$$

Finally,

$$\left.\begin{aligned} \Delta W &= p(\theta)\,\mu V l \\ p(\theta) &= \frac{k^2(\theta) - \cos^2\theta}{2k(\theta)\sin\theta} \end{aligned}\right\} \tag{26}$$

If L denotes the width of the plate in the direction transverse to the flow and the effect of its free end is not considered, the relative correction due to the interference of the boundary layers may be computed:

$$\frac{\Delta W}{W} = \frac{p(\theta)\,\mu V l}{0.578\, L\sqrt{\mu\rho V^3 l}} = \frac{p(\theta)}{0.578} \frac{l}{L} \sqrt{\frac{\nu}{V l}}$$

or, when the Reynolds number R_l of the plate, which is equal to Vl/ν, is introduced,

$$\frac{\Delta W}{W} = q(\theta) \frac{l}{L} \frac{1}{\sqrt{R_l}} \tag{27}$$

where $q(\theta)$ denotes the relation

$$q(\theta) = \frac{p(\theta)}{0.578}$$

Generally, the correction obtained is extremely insignificant for plates that are long in the transverse direction. If, however, the transverse length L is comparable with the width of the region of the disturbed layer at the end of the plate $x = l$, the correction is not insignificant. Thus, if

$$L = n \cdot h_0(l) = n \cdot m(\theta)\, \delta_0(l) = n \cdot m(\theta) \sqrt{\frac{12 \nu l}{V}}$$

the relative correction will now be equal to

$$\frac{\Delta W}{W} = \frac{q(\theta)}{\sqrt{12}\, m(\theta)} \cdot \frac{1}{n}$$

or

$$\frac{\Delta W}{W} = \left[\frac{s(\theta)}{n}\right] \text{percent} \tag{28}$$

where

$$s = \frac{100\, q(\theta)}{\sqrt{12}\, m(\theta)} \tag{29}$$

All the magnitudes introduced in the preceding equations may be expressed in terms of the previously given parameter ξ:

$$\left.\begin{aligned} p(\zeta) &= \frac{1-\zeta^2}{2\zeta} \cot \theta \\ q(\zeta) &= \frac{1-\zeta^2}{1.156\, \zeta} \cot \theta \\ s(\zeta) &= \frac{1-\zeta}{1.156\sqrt{12}}\, 100 \end{aligned}\right\} \tag{30}$$

In table 1, it is possible to find the values of these magnitudes for different angles θ or the corresponding values of the parameter ζ. The magnitude $s(\theta)$ increases with an increase of the angle θ. Somewhat paradoxical is the value of 50 percent for $s(\theta)$ at $\theta^{\circ} = 180$ and $m = 0$. It is found that the region of interference is equal to zero, and that a relative effect occurs on the drag, which is due to the fact that the width of the region and the absolute correction on the drag simultaneously approach zero. On the other hand, when θ decreases to 0 and $m \rightarrow \infty$, s decreases to 16 percent. Both these cases are limiting cases.

In figure 4, the curves of the relation between ζ and θ are drawn for the first and other approximations and also for the case of turbulent layers.

3. SECOND AND SIXTH APPROXIMATIONS

The following approximations differ from the first only in the boundary conditions that are imposed on the function $\varphi(t)$.

The second approximation corresponds to the assumptions

$$\varphi(0) = 0$$

$$\varphi(1) = 1$$

$$\varphi'(1) = 0$$

The velocity profile for these conditions will have the form

$$u = V\left[2\left(\frac{yz}{a(x,\theta)}\right) - \left(\frac{yz}{a(x,\theta)}\right)^2\right]$$

No new difficulties in principle, as compared with the first approximation, are obtained. Inasmuch as the basic computations have been explained in the first approximation, the results are presented with the same notations.

The magnitudes are all given in table 2.

The final form of the equation for the determination of $k(\theta)$ is

$$\log_e k(\theta) = \frac{41}{120} + \frac{3\cos\theta}{4k(\theta)} + \frac{15}{28}\cdot\frac{\cos^2\theta}{k^2(\theta)} + \frac{2}{21}\cdot\frac{\cos^3\theta}{k^3(\theta)} - \frac{5}{168}\cdot\frac{\cos^4\theta}{k^4(\theta)} + \frac{1}{420}\cdot\frac{\cos^5\theta}{k^4(\theta)} \tag{31}$$

The solution is effected by the same device as in the case of the first approximation.

All computations were also carried out for the sixth approximation, by which is meant the results of the assumption of the following boundary conditions:

$$\varphi(0) = 0 \qquad \varphi(1) = 1$$
$$\varphi''(0) = 0 \qquad \varphi'(1) = 0$$
$$\varphi'''(0) = 0 \qquad \varphi''(1) = 0$$
$$\varphi'''(1) = 0$$

This approximation gives, for the undisturbed layer, the velocity profile

$$u_0 = V\left[2\left(\frac{y}{\delta_0}\right) - 5\left(\frac{y}{\delta_0}\right)^4 + 6\left(\frac{y}{\delta_0}\right)^5 - 2\left(\frac{y}{\delta_0}\right)^6\right]$$

and the following values of the frictional force and layer thickness, which are extremely close to the accurate solution of Blasius:

$$\left.\begin{aligned} \tau_0 &= 0.331\sqrt{\frac{\mu\rho V^3}{x}} \\ \delta_0 &= 6.048\sqrt{\frac{\nu x}{V}} \end{aligned}\right\}$$

It is therefore reasonable to suppose that, for investigating the problem of the interference of the layers, the sixth approximation is considerably more satisfactory than the second.

The results are again collected in table 3.

In figure 4, the curve of ζ against θ is given for the second and sixth approximations.

As before, the equation for the determination of $k(\theta)$ is also given

$$\log_e k = 0.1857 + 0.6370\,\frac{\cos\,\theta}{k} + 0.6107\,\frac{\cos^2\,\theta}{k^2} + 0.1043\,\frac{\cos^3\,\theta}{k^3} -$$

$$0.0524\,\frac{\cos^4\,\theta}{k^4} + 0.0053\,\frac{\cos^5\,\theta}{k^5} + 0.0025\,\frac{\cos^6\,\theta}{k^6} - 0.0001\,\frac{\cos^7\,\theta}{k^7} \tag{32}$$

4. INTERFERENCE OF TURBULENT BOUNDARY LAYERS

For the solution of this concluding part of the problem, the well-known '1/7' power law is applied, which, when the boundary layer is not two dimensional, assumes the form

$$u = V\left[\frac{yz}{a(x,\theta)}\right]^{1/7} \tag{33}$$

As is known, for an infinite plate without interference of the layers, this law gives excellent agreement with experiment; it may be expected that the application of the '1/7' power law to this case will be justified by experiment.

The generalized integral condition (5) is substituted in the velocity-profile equation (33). Unfortunately, the integrals will now not be so easily evaluated because the presence of fractional exponents, particularly in the integrals taken over the hatched areas of the triangles shown in figure 2, strongly complicates the computations and leads to the necessity of taking integrals of binomial differentials. All the computations can be made with a sufficient degree of accuracy, however, by using converging series and integrating them. The left side of the integral condition reduces to the following expression:

$$\begin{aligned}\int_\sigma \rho(V-u)\, u\, d\sigma = \rho V^2\, a(x,\theta) \sin\theta \Bigg\{&0.1607 + 0.0972 \log_e \frac{a(x,\theta)}{y_1{}^2(x)} + \\ &0.1360\, \frac{y_1{}^2 \cos\theta}{a(x,\theta)} - 0.0231\left[\frac{y_1{}^2 \cos\theta}{a(x,\theta)}\right]^2 + \\ &0.0023\left[\frac{y_1{}^2 \cos\theta}{a(x,\theta)}\right]^3 - 0.0004\left[\frac{y_1{}^2 \cos\theta}{a(x,\theta)}\right]^4 + \ldots\Bigg\}\end{aligned} \tag{34}$$

In computing the right side of the integral condition, the Kármán formula is used for the expression of the friction at the wall in the undisturbed region of the layer:

$$\tau_0 = 0.0225\, \rho V^2 \left(\frac{\nu}{V\delta_0}\right)^{1/4} = 0.0225\, \rho V^2 \left(\frac{\nu}{V y_1 \sin\theta}\right)^{1/4} \tag{35}$$

In the disturbed part of the layer, the analogous formula

$$\tau = 0.0225\ \rho V^2\left(\frac{\nu}{V\delta}\right)^{1/4} = 0.0225\ \rho V^2\left(\frac{\nu}{VZ\ \sin\ \theta}\right)^{1/4}$$

$$= 0.0225\ \rho V^2\left(\frac{\nu \cdot y}{V\ a(x,\theta)\ \sin\ \theta}\right)^{1/4} \tag{36}$$

is assumed. These values are substituted, as in the previous sections, in the formula for the resistance of the walls forming part of the boundary of a tube of flow. Then,

$$W = 2\int_0^x\left[\int_0^{h_0(\xi)} \tau\ dy + \int_{h_0(\xi)}^{h_0(x)} \tau_0\ dy\right]d\xi = 0.1943\ \rho V^2\ \delta_0(x)\left[\frac{a(x,\theta)}{y_1(x)} + \right.$$

$$\left. y_1(x)\ \cos\ \theta\right] + \rho\ \frac{V^{\frac{1}{4}}\ \nu^{\frac{1}{4}}}{\sin^{\frac{1}{4}}\ \theta}\left\{- 0.0090\int_0^x \frac{a(\xi,\theta)\ d\xi}{y_1^{\frac{5}{4}}} + \right.$$

$$0.0056\int_0^x \frac{y_1^{\frac{11}{4}}(\xi)\ \cos^2\ \theta}{a(\xi,\theta)}\ d\xi - 0.0014\int_0^x \frac{y_1^{\frac{19}{4}}(\xi)\ \cos^3\ \theta}{a^2(\xi,\theta)}\ d\zeta +$$

$$0.0006\int_0^x \frac{y_1^{\frac{27}{4}}(\xi)\ \cos^4\ \theta}{a^3(\xi,\theta)}\ d\xi - 0.0003\int_0^x \frac{y_1^{\frac{35}{4}}(\xi)\ \cos^5\ \theta}{a^4(\xi,\theta)}\ d\zeta +$$

$$0.0002\int_0^x \frac{y_1^{\frac{43}{4}}(\xi)\ \cos^6\ \theta}{a^5(\xi,\theta)}\ d\zeta + \ .\ .\ . \tag{37}$$

The Kármán formula was then used:

$$\delta_0(x) = 0.370\left(\frac{\nu}{Vx}\right)^{1/5} x$$

When it is known (see equation (12)) in advance that the solution of the problem will have the form

$$a(x,\theta) = k(\theta)\, y_1^{\,2}(x) = k(\theta)\, \frac{\delta_0^{\,2}(x)}{\sin^2 \theta} \tag{38}$$

this value is substituted in equations (34) and (37); then, after some simplifications, the following solution is obtained:

$$\left.\begin{aligned}
&\iint_\sigma \rho(V-u)\, u\, d\sigma = \rho\, \frac{V^2\delta_0^{\,2}(x)}{\sin\theta}\Bigl(0.1607\, k + 0.0972\, k \log_e k + 0.1360 \cos\theta - {} \\
&\quad 0.0230\, \frac{\cos^2\theta}{k} + 0.0023\, \frac{\cos^3\theta}{k^2} - 0.0004\, \frac{\cos^4\theta}{k^3} + 0.0001\, \frac{\cos^5\theta}{k^4} + \ldots\Bigr) \\
&W = \rho\, \frac{V^2\delta_0^{\,2}(x)}{\sin\theta}\Bigl\{0.1949\, k + 0.1949 \cos\theta - 0.0195\, k + 0.0122\, \frac{\cos^2\theta}{k} - {} \\
&\quad 0.00305\, \frac{\cos^3\theta}{k^2} + 0.00133\, \frac{\cos^4\theta}{k^3} + 0.00073\, \frac{\cos^5\theta}{k^4} + \ldots\Bigr\}
\end{aligned}\right\}$$

Equating these two expressions yields the following equation for determining $k(\theta)$:

$$\log_e k = 0.1518 + 0.6056\, \frac{\cos\theta}{k} + 0.3632\, \frac{\cos^2\theta}{k^2} - 0.0556\, \frac{\cos^3\theta}{k^3} + 0.0175\, \frac{\cos^4\theta}{k^4} - 0.0082\, \frac{\cos^5\theta}{k^5} + 0.0051\, \frac{\cos^6\theta}{k^6} + \ldots$$

As in the preceding sections, the change of variables is made:

$$\frac{\cos\theta}{k(\theta)} = \zeta(\theta) \tag{39}$$

The transcendental equation then becomes

$$\cos\theta = \zeta \exp\left(0.1518 + 0.6056\,\zeta + 0.3632\,\zeta^2 - 0.0556\,\zeta^3 + 0.0175\,\zeta^4 - 0.0082\,\zeta^5 + \ldots\right) \tag{40}$$

This equation is easily solved by the tabular method, where

$$0^\circ < \theta < 180^\circ \qquad 0.5540 > \zeta > -1.0000$$

The corresponding values of θ, $k(\theta)$, and $\zeta(\theta)$ are given in table 4. The further investigation in no way differs from that for laminar layers. The values of the coefficients characterizing the boundary of the region of interference and the corrections on the drag are given in table 4 for various values of θ, with all the computations ommitted, in the notation previously used.

In connection with the formulas of turbulent friction, the coefficients q and p are determined by the formulas:

$$\Delta W = p \cdot \rho\, V^2 l^2 \left(\frac{\nu}{Vl}\right)^{2/5} \tag{41}$$

and

$$\frac{\Delta W}{W} = g \cdot \frac{l}{L}\left(\frac{\nu}{Vl}\right)^{1/5} \tag{41'}$$

The dependence of p on ζ is determined by the following series:

$$p = \frac{1}{\zeta}\cot\theta\left(0.00133 - 0.00083\,\zeta^2 + 0.00021\,\zeta^3 - 0.00009\,\zeta^4 + 0.00005\,\zeta^5 - 0.00003\,\zeta^6 + \ldots\right) \tag{42}$$

$$q = \frac{p}{0.036} \tag{43}$$

The dependence of ζ on θ is given for the case of turbulent layers in figure 4. The boundary of the region of interference (the coefficient $m(\theta)$) in the case of the turbulent layers differs little from the corresponding boundary of the laminar layers, according to the sixth approximation; whereas the relative correction on the drag s in percent is several times less in the turbulent case.

CONCLUSION

The results obtained have a readily understandable form. In general, the effect of the interference of the layers on the drag of the plates is insignificant. The effect assumes an appreciable value only in the case where the plates in the dimensions transverse to the flow become comparable with the width of the region of disturbed boundary layer. Moreover, interference plays a large part in the motion of fluids through small dihedral angles. Thus, for example, in the motion near the intersection of a dihedral angle of about 10°, the region of interference exceeds by 16 times the thickness of the layer at the given section. At smaller angles, the phenomenon is still more marked.

All the conclusions of the present and preceding papers require experimental check.

By agreement with the Central Aero-Hydrodynamical Institute, the aerodynamic laboratory of the Leningrad Industrial Institute is undertaking an experimental investigation of the phenomenon of the interference of boundary layers. It is proposed, through use of the method of microtunnels, to observe directly the distortion in the velocity profiles, and so forth, of the phenomenon.

The present work was carried out at the Aerodynamic Laboratory of the Leningrad Industrial Institute.

Translated by S. Reiss,
National Advisory Committee
for Aeronautics

REFERENCE

1. Loiziansky, L. G.: Interference of Boundary Layers. CAHI Rep. No. 249, (Moskow), 1936.

TABLE 1

θ (deg)	$k(\theta)$	$\zeta(\theta)$	$m(\theta)$	$p(\theta)$	$q(\theta)$	$s(\theta)$ (percent)
0	2.894	0.3455	∞	∞	∞	16.3
10	2.880	.3419	22.257	7.324	12.671	16.4
20	2.840	.3309	11.315	3.697	6.396	16.7
30	2.773	.3123	7.278	2.503	4.330	17.2
40	2.682	.2856	5.364	1.916	3.315	17.8
50	2.567	.2504	4.190	1.570	2.716	18.7
60	2.434	.2054	3.388	1.346	2.329	19.8
70	2.283	.1498	2.793	1.187	2.054	21.2
80	2.120	.0819	2.329	1.069	1.849	22.9
90	1.948	0	1.948	.974	1.685	25.0
100	1.773	-.0980	1.624	.892	1.543	27.4
110	1.601	-.2136	1.340	.813	1.407	30.3
120	1.442	-.3467	1.088	.732	1.266	33.6
130	1.300	-.4945	.858	.641	1.109	37.3
140	1.185	-.6464	.6 2	.536	.927	41.0
150	1.099	-.7880	.466	.417	.721	44.7
160	1.042	-.9018	.301	.285	.493	47.3
170	1.010	-.9750	.144	.142	.246	49.3
180	1.000	-1.0000	0	0	0	50.0

TABLE 2

θ (deg)	$k(\theta)$	$\zeta(\theta)$	$m(\theta)$	$p(\theta)$	$q(\theta)$	$s(\theta)$ (percent)
0	2.217	0.4511	∞	∞	∞	13.7
10	2.205	.4466	18.360	10.162	13.921	13.8
20	2.168	.4334	9.090	5.148	7.052	14.2
30	2.108	.4108	5.948	3.584	4.910	14.7
40	2.026	.3781	4.343	2.702	3.701	15.6
50	1.925	.3339	3.352	2.233	3.059	16.7
60	1.808	.2765	2.665	1.928	2.641	18.1
70	1.678	.2038	2.150	1.712	2.345	19.9
80	1.542	.1126	1.742	1.546	2.118	22.2
90	1.407	0	1.407	1.407	1.927	25.0
100	1.284	-0.1353	1.127	1.279	1.752	28.4
110	1.182	-0.2893	.894	1.153	1.579	32.2
120	1.108	-0.4513	.702	1.018	1.395	36.3
130	1.060	-0.6064	.544	.875	1.199	40.2
140	1.032	-0.7422	.414	.721	.988	43.6
150	1.016	-0.8524	.300	.556	.762	46.3
160	1.007	-0.9332	.196	.379	.519	48.3
170	1.002	-0.9832	.097	.192	.263	49.6
180	1.000	-1.0000	0	0	0	50.0

TABLE 3

θ (deg)	$k(\theta)$	$\zeta(\theta)$	$m(\theta)$	$p(\theta)$	$q(\theta)$	$s(\theta)$ (percent)
0	1.969	0.5079	∞	∞	∞	12.3
10	1.957	.5032	16.940	8.416	12.713	12.4
20	1.921	.4892	8.365	4.272	6.453	12.8
30	1.863	.4648	5.458	2.921	4.412	13.4
40	1.783	.4296	3.965	2.262	3.417	14.3
50	1.685	.3815	3.039	1.879	2.838	15.5
60	1.572	.3181	2.393	1.631	2.464	17.0
70	1.450	.2359	1.907	1.457	2.201	19.1
80	1.327	.1309	1.524	1.324	2.000	21.7
90	1.204	0	1.204	1.204	1.819	25.0
100	1.106	-0.1571	.946	1.094	1.653	28.9
110	1.038	-0.3295	.741	.985	1.480	33.2
120	1.004	-0.4982	.582	.887	1.340	37.5
130	1.000	-0.6428	.466	.766	1.157	41.1
140	1.000	-0.7660	.364	.643	.971	44.2
150	1.000	-0.8660	.268	.500	.755	46.7
160	1.000	-0.9397	.176	.342	.517	48.5
170	1.000	-0.9848	.088	.174	.263	49.6
180	1.000	-1.0000	0	0	0	50.0

TABLE 4

θ (deg)	$k(\theta)$	$\zeta(\theta)$	$m(\theta)$	$p(\theta)$	$q(\theta)$	$s(\theta)$ (percent)
0	1.805	0.5540	∞	∞	∞	5.35
10	1.796	.5483	16.010	0.0115	0.3189	5.38
20	1.767	.5318	7.915	.0058	.1614	5.51
30	1.720	.5035	5.172	.0039	.1094	5.73
40	1.654	.4631	3.765	.0030	.0839	6.03
50	1.577	.4076	2.898	.0025	.0689	6.43
60	1.485	.3367	2.292	.0021	.0594	7.00
70	1.382	.2475	1.835	.0019	.0525	7.73
80	1.273	.1364	1.469	.0017	.0472	8.68
90	1.164	0	1.164	.0015	.0431	10.00
100	1.065	-0.1631	.905	.0014	.0394	11.76
110	1.000	-0.3420	.700	.0013	.0364	14.05
120	1.000	-0.5000	.577	.0012	.0325	15.22
130	1.000	-0.6428	.466	.0010	.0280	16.24
140	1.000	-0.7660	.364	.0008	.0230	17.08
150	1.000	-0.8660	.268	.0006	.0178	17.95
160	1.000	-0.9397	.176	.0004	.0122	18.73
170	1.000	-0.9848	.088	.0002	.0064	19.65
180	1.000	-1.000	0	0	0	19.97

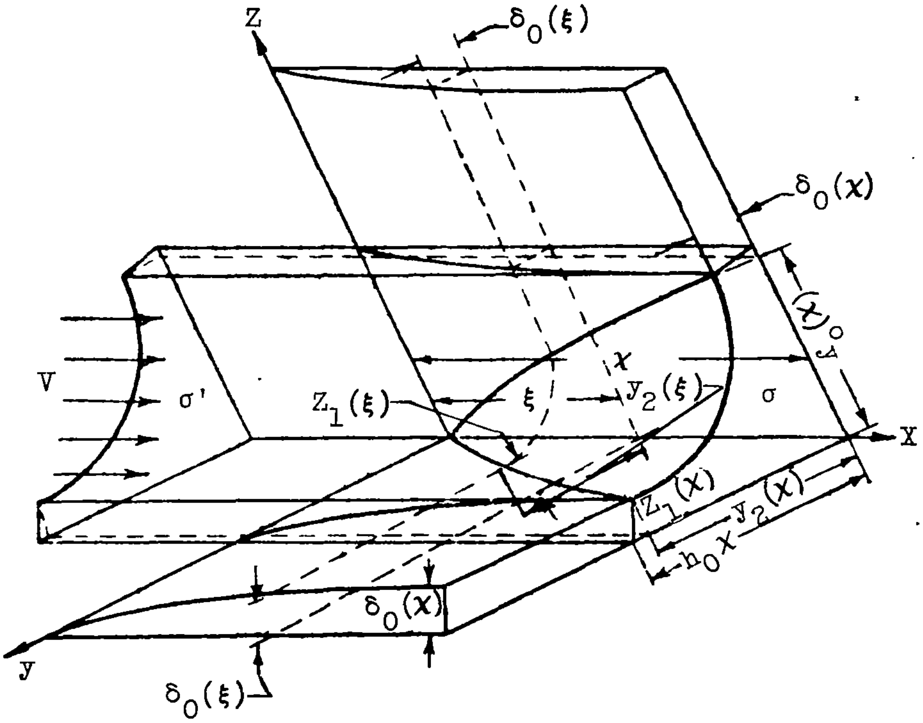

Figure 1.

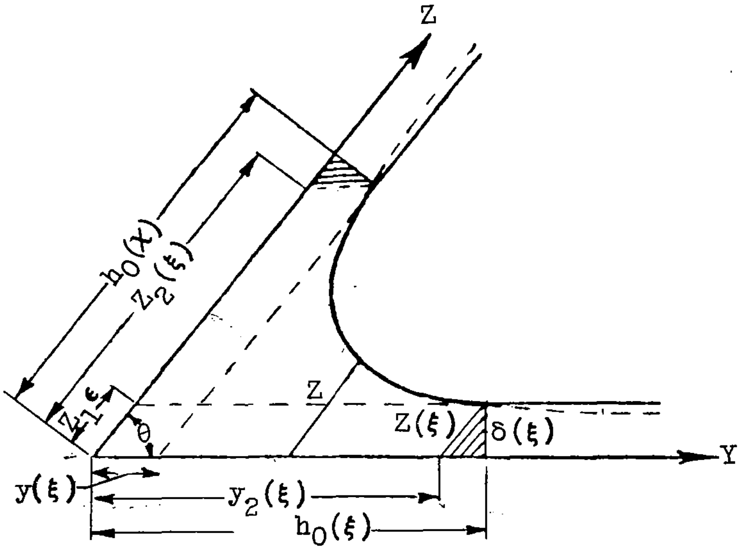

Figure 2.

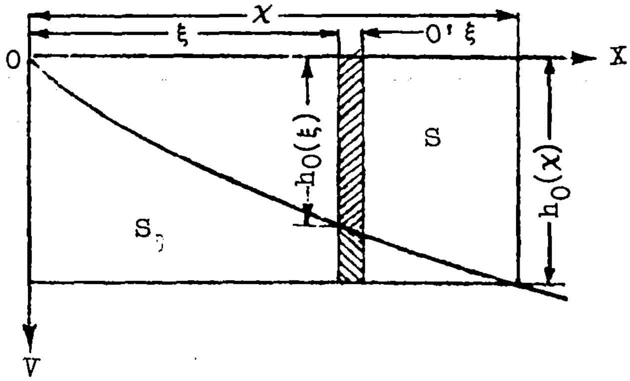

Figure 3.

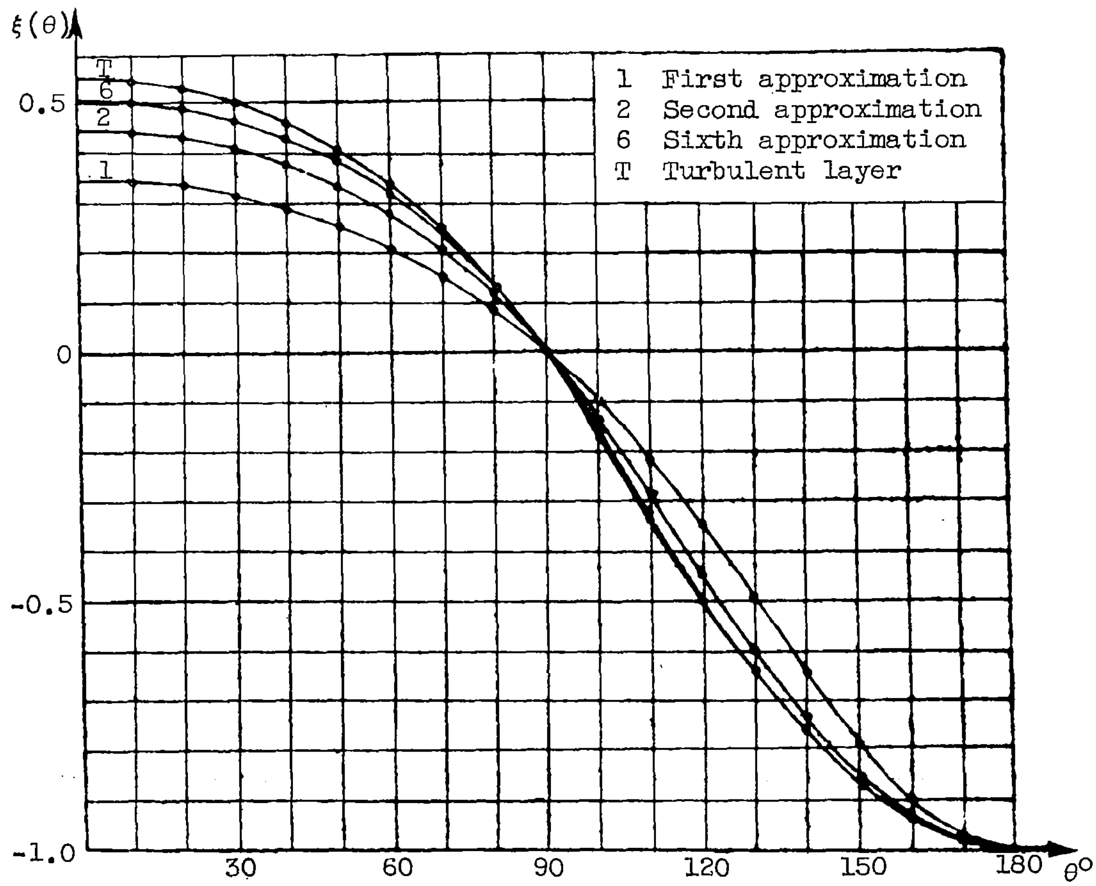

Figure 4.