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**STABILITY OF THE CYLINDRICAL SHELL
OF VARIABLE CURVATURE**

By K. Marguerre

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STABILITY OF THE CYLINDRICAL SHELL
OF VARIABLE CURVATURE*

By K. Marguerre

Abstract: The report is a first attempt to devise a calculation method for representing the buckling behavior of cylindrical shells of variable curvature. The problem occurs, for instance, in dimensioning wing noses, the stability behavior of which is decisively influenced by the variability of curvature. The calculation is made possible by simplifying the stability equations (permissible for the shell of small curvature) and by assuming that the curvature $1/R$ as a function of the arc length s can be represented by a very few Fourier terms. We evaluated the formulas for the special case of an ellipse-like half oval with an axis ratio $1/3 \leq \epsilon \leq 1$ under compression in longitudinal direction, shear, and a combination of shear and compression. However, the results can also be applied approximately to an unsymmetrical oval-shell segment under compression, shear, and bending so that the numerical values contained in the diagrams 10 to 12 represent directly dimensioning data for the wing nose.

- Outline:
1. INTRODUCTION
 2. THE STABILITY EQUATIONS OF THE CYLINDRICAL SHELL. THEIR SOLUTION IN THE SPECIAL CASE OF THE COMPRESSED SHELL OF CONSTANT CURVATURE
 3. THE BUCKLING DETERMINANT OF THE COMPRESSED SHELL OF VARIABLE CURVATURE
 4. DETERMINATION OF THE CYLINDRICAL FORM FROM THE VARIATION OF CURVATURE
 5. COMPUTATIONAL EVALUATION OF THE BUCKLING DETERMINANT IN THE CASE OF COMPRESSION
 6. THE CLOSED CYLINDER UNDER PURE SHEAR
 7. THE BUCKLING CONDITIONS FOR THE CYLINDRICAL SEGMENT UNDER SHEAR LOAD; ITS SOLUTION IN FIRST APPROXIMATION

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8. CRITICAL SHEAR LOAD FOR THE HALF CYLINDER OF ARBITRARY CURVATURE
9. THE WING NOSE UNDER COMBINED SHEAR AND COMPRESSIVE LOAD
10. VALIDITY LIMITS - UNSYMMETRICAL OVALS UNDER BENDING - SUMMARY

1. INTRODUCTION

So far, only the shell stability theory for sphere and circular cylinder has been developed to include formulas for practical application. Recently, the designs in airplane construction drew attention also to other shell types; however, due to the great mathematical difficulties opposing a stability theory of complicated shell forms, it was attempted here to approximate actual shells by circular cylinders. Thus it is possible, for instance, to calculate (with good approximation) a monocoque fuselage stiffened by stringers as if it were joined together from a large number of circular-cylindrical strips (shell segments). For the calculation of wing skins, one may use an approximation of the buckling formulas for the short circular-cylinder shell; the respective theory may be regarded as completed by Kromm's¹ investigations. Cases exist, however, where the theory of the circular cylinder does not suffice. A wing nose, for instance, is usually developed as a sort of half oval which is supported by a strong spar on the open side (fig. 1). The stability behavior of such a shell is decisively affected by the variability of the curvature. It is not permissible to determine the critical compressive load as in the case of the circular-cylindrical shell from a "mean value" of the curvature, since the weakening effect on the shell of regions of small curvature very considerably exceeds the stiffening effect of regions of larger curvature. (For rough calculation, it is even advisable to regard, in case of wide shells under compression, the minimum curvature alone as decisive for the buckling limit.)

The mathematical difficulties opposing the theoretical investigation of cylindrical shells of variable curvature exceed those arising in the case of the shell of constant curvature; however, on the other hand, they are not of fundamental character for the infinitely long shell, the curvature variation of which over the arc length s may be represented by a sine series. The present investigation gives the general theory of such shells and evaluates the calculation results in particular for the

¹Kromm, A.: Knickfestigkeit gekrümmter Plattenstreifen unter Schub- und Druckkräften (Resistance to Buckling of a Curved Plate Strip Under Shear and Compression). Jahrbuch 1940 der deutschen Luftfahrtforschung, p. 832.

wing nose in the form of a symmetrical half oval. The following loads are considered: compression in axial direction of the cylindrical shell, shear (torsion), and the combination of both.

2. THE STABILITY EQUATIONS OF THE CYLINDRICAL SHELL. THEIR
SOLUTION IN THE SPECIAL CASE OF THE COMPRESSED
SHELL OF CONSTANT CURVATURE

The shell segment we are going to investigate is an intermediate between the flat plate and the true shell; it owes part of its load capacity to the reinforcement by the longitudinal-edge stiffenings (like the flat plate), part, however, also to the stiffening produced by the curvature (the curvature prevents a bending without stretching of the median surface). Thus the stability equations of the cylinder theory are buckling differential equations of the shell segment; however, their solutions have to be adapted to the boundary conditions at the edges $s = \text{constant}$.

The two stability equations of the thin cylindrical shell are²

$$\left. \begin{aligned} \Delta \Delta \phi - \frac{E}{R} w_{xx} &= 0 \\ \frac{Et^2}{12(1-\nu^2)} \Delta \Delta w + \frac{1}{R} \phi_{xx} &= -\sigma w_{xx} + 2\tau w_{xy} \end{aligned} \right\} (2.1)$$

where ϕ is the stress function from which one obtains three "membrane" stresses additionally originating in buckling, according to

$$\frac{N_x}{t} = \frac{\partial^2 \phi}{\partial s^2} \equiv \phi_{ss}, \quad \frac{N_{xs}}{t} = -\phi_{xs}, \quad \frac{N_s}{t} = \phi_{xx};$$

σ is the critical compressive stress (hence the minus sign in equation (2.1)), τ the critical shearing stress; the remaining designations may be found in figure 2 which represents a shell segment. The indices

²Compare for instance K. Marguerre: Zur Theorie der gekrümmten Platte. . . , Jahrbuch 1939 der deutschen Luftfahrtforschung, p. 1413.

appearing with ϕ and w indicate the respective derivatives with respect to x or s . The equations each differ from the respective disk or plate equation by the $\frac{1}{R}$ -term. By precisely this term they are coupled with one another, and this is the reason for the mathematical difficulty of the shell problem. The equations (2.1) result from the very complicated "exact" shell equations³ by radical simplification.⁴ These simplifications are physically justified under the presupposition that the shell either has only very slight curvature⁵ ($R \gg U$) or at least is, in buckling, subdivided into so many waves that the wave length in the circumferential direction is small compared to the radius of curvature. If this presupposition is not satisfied, the result is to a certain extent problematical; however, the numerous careful investigations concerning the buckling of a plate of constant curvature⁶ have shown that the errors arising from the simplification of the equations are not of a type to endanger the technical usefulness of the results.

In this report we consider only shells that are so long that the effect of end constraint on the buckling load can be neglected. Then the stresses and displacements are purely periodical in x , and for $\tau = 0$, in particular, directly proportional to $\sin \frac{\pi x}{l}$, with the wave length l , which is small compared to the shell length L , remaining open at first. If we put

$$\phi = \bar{\phi} \sin \frac{\pi x}{l}, \quad w = \bar{w} \sin \frac{\pi x}{l},$$

the partial equations (2.1) are transformed into total equations (bars now signify derivatives with respect to s)

$$\bar{\phi}'''' - 2 \frac{\pi^2}{l^2} \bar{\phi}'' + \frac{\pi^4}{l^4} \bar{\phi} + \frac{E\pi^2}{Rl^2} \bar{w} = 0 \quad (2.1')$$

³Flügge: Statik und Dynamik der Schalen (Statics and dynamics of shells), Berlin 1934, p. 191.

⁴A few notes on this can be found in K. Marguerre's "Der Einfluss der Lagerungsbedingungen . . ." (The influence of the arrangement conditions . . .). Jahrbuch 1940 der deutschen Luftfahrtforschung, p. 867.

⁵A. Kromm: Die Stabilitätsgrenze eines gekrümmten Plattenstreifens, (The stability limit of a curved plate strip), Luftf.-Forschg. 1938, p. 517 ff.

⁶We name in particular:

Flügge (footnote 3)

Timoshenko: Theory of Elastic Stability, Chapter IX

Kromm (footnote 1, p. 2)

Donnell: Thin Shell Theory. Proc. of the 5th Inter. Congr. of Appl. Mech., Cambridge 1938, p. 66.

$$\frac{Et^2}{12(1 - \nu^2)} \left(\bar{w}'''' - 2 \frac{\pi^2}{l^2} \bar{w}'' + \frac{\pi^4}{l^4} \bar{w} \right) - \sigma \frac{\pi^2}{l^2} \bar{w} - \frac{1}{R} \frac{\pi^2}{l^2} \bar{\phi} = 0 \quad (2.1')$$

The equations (2.1') are linear and homogeneous in the unknowns \bar{w} and $\bar{\phi}$ and contain the variable quantity $1/R$; they can be converted into a system of an infinite number of equations for the coefficients of a Fourier expression if $1/R$ (or R) can be represented by a Fourier polynomial, and if the support at the longitudinal edges $s = 0, s = U$ is such that by a Fourier expression the boundary conditions can, term by term, be satisfied.

We consider first the cylinder of constant curvature because in this simple special case the train of thought leading to the solution is outlined more clearly than in the general case; simultaneously, we have the opportunity of introducing the abbreviations which are expedient for the later calculation in an easily surveyable form.

If we put

$$\left. \begin{aligned} \bar{w} &= \sum_{n=1}^{\infty} a_n \sin n \frac{\pi s}{U} \\ \bar{\phi} &= \sum_{n=1}^{\infty} A_n \sin n \frac{\pi s}{U} \end{aligned} \right\} \quad (2.2)$$

two equation systems linear and homogeneous in a_n and A_n are formed from (2.1') which with the abbreviation

$$U/l = \beta \quad (2.3)$$

may be written

$$\left. \begin{aligned} \frac{\pi^2}{U^2} (\beta^2 + n^2)^2 A_n + \frac{E}{R} \beta^2 a_n &= 0 \\ \frac{Et^2}{12(1 - \nu^2)} \frac{\pi^2}{U^2} (\beta^2 + n^2)^2 a_n - \sigma \beta^2 a_n - \frac{1}{R} \beta^2 A_n &= 0 \end{aligned} \right\} \quad (2.4)$$

For $1/R = \text{constant} = 1/R_0$, only one n occurs in any equation, so that the system is broken down into pairs of equations only; calculating A_n from the first equation and introducing it into the second, we obtain

$$\left(\frac{(\beta^2 + n^2)^2}{\beta^2} - \frac{12(1 - \nu^2)U^2}{\pi^2 E t^2} \sigma + \frac{12(1 - \nu^2)}{\pi^4} \frac{U^4}{R_0^2 t^2} \frac{\beta^2}{(\beta^2 + n^2)^2} \right) a_n = 0$$

The vanishing of the factor of a_n characterizes the critical state. With the abbreviations

$$\left. \begin{aligned} \sigma^* &= \frac{E}{1 - \nu^2} \frac{\pi^2 t^2}{3 U^2} \\ \omega &= \frac{12(1 - \nu^2)}{\pi^4} \frac{U^4}{R_0^2 t^2} \end{aligned} \right\} \quad (2.5)$$

the equation for buckling (by which the eigen-value σ is determined) reads

$$\frac{(\beta^2 + n^2)^2}{\beta^2} + \omega \frac{\beta^2}{(\beta^2 + n^2)} - 4 \frac{\sigma}{\sigma^*} = 0 \quad (2.6)$$

With the abbreviations

$$\begin{aligned} \frac{1}{k_n} &= \frac{1}{\sqrt{\omega}} \frac{(\beta^2 + n^2)^2}{\beta^2} = \frac{1}{\sqrt{\omega}} \left(\beta^2 + 2n^2 + \frac{n^4}{\beta^2} \right) \\ \lambda &= 4 \frac{\sigma}{\sigma^*} \frac{1}{\sqrt{\omega}} = \sqrt{12(1 - \nu^2)} \frac{\sigma}{E t} \end{aligned} \quad (2.5')$$

it is written in the still more compressed form

$$\frac{1}{k_n} + k_n - \lambda = 0 \quad (2.6')$$

In equation (2.6) the wave length is still open in the x-direction (that is, $\beta = \frac{U}{l}$). It adjusts itself so that σ (or λ) assumes the minimum value. Thus, there is, in addition to equation (2.6) or (2.6') the condition $\frac{d\sigma}{d\beta} = 0$ or

$$\frac{d\lambda}{d\beta^2} = \frac{d\lambda}{d(1/k_n)} \frac{d(1/k_n)}{d\beta^2} = \left(1 - k_n^2 \right) \left(1 - \frac{n^4}{\beta^4} \right) = 0 \quad (2.7)$$

From equations (2.6) and (2.7) it follows that

$$\beta = n \quad \sigma = \sigma^* \left(n^2 + \frac{\omega}{16n^2} \right) \quad \text{for } \sqrt{\omega} \leq 4n^2$$

$$\left. \begin{aligned} \frac{(\beta^2 + n^2)^2}{\beta^2} = \sqrt{\omega} \quad \sigma = \frac{1}{2} \sigma^* \sqrt{\omega} \\ \text{or respectively } K_n = 1 \quad \lambda = 2 \end{aligned} \right\} \quad \text{for } \sqrt{\omega} \geq 4n^2 \quad (2.8)^7$$

The simple expression (2.2) is usable only if at the longitudinal edges $s = 0$, $s = U$ the equations

$$w = 0, \quad w_{SS} = 0, \quad N_S \equiv t\phi_{XX} = 0, \quad Eu \equiv \int (\phi_{SS} - \nu\phi_{XX}) dx = 0 \quad (2.9)$$

are prescribed as boundary conditions, that is, if the edge supports are developed in such a manner that they offer a very large resistance to a displacement of the sheet in radial and axial direction, a very small resistance to a displacement in hoop direction and to a torsion. For the further calculation, we shall assume the boundary conditions (2.9) to be satisfied. For the circular cylinder, a modification of the boundary condition for the stresses and displacements in longitudinal and hoop direction (thus relative to ϕ) can be shown to have almost no effect on the magnitude of critical load.⁸ The same applies, with high probability, to the cylinder of variable curvature. For if the edge terms are arranged in the more strongly curved region, they are practically without any influence because in this region the shell buckles into many small waves so that it does not matter at what point exactly the node lines are enforced; and if the stiffenings (as in the case of the wing nose) support the region of small curvature, it does become important whether the shell is simply supported or clamped (w -conditions), but it is quite unimportant what happens in the two

⁷If one sets in the second σ -formula the notations abbreviated in equation (2.5), one obtains with $\nu = 0.3$ the well-known cylinder formula for buckling $\sigma = \frac{0.6Et}{R}$.

⁸K. Marguerre; compare figure 6 of footnote 4, page 4.

other directions, because for a flat shell (similar to a plate) large radial displacements w produce only very small hoop and axial displacements v and u . Moreover, the more important⁹ of the two assumptions, $N_s = 0$ (bending soft supports in hoop direction), lies on the safe side.

3. THE BUCKLING DETERMINANT OF THE COMPRESSED SHELL OF VARIABLE CURVATURE

If the curvature $1/R$ can be represented by a Fourier polynomial of r^* terms¹⁰

$$\begin{aligned} \frac{1}{R} &= \frac{1}{R_0} \left(1 + 2\zeta_2 \cos \frac{2\pi s}{U} + 2\zeta_4 \cos \frac{4\pi s}{U} + \dots + 2\zeta_{2r^*} \cos \frac{2r^*\pi s}{U} \right) \\ &= \frac{1}{R_0} \left(1 + \sum_{r=1}^{r^*} 2\zeta_{2r} \cos \frac{2r\pi s}{U} \right) \end{aligned} \quad (3.1)$$

(because $\int_0^U \cos \frac{2r\pi s}{U} ds = 0$, $1/R_0$ is therein the arithmetic mean of the curvature), we may, in order to solve equation (2.1'), again start from the Fourier expression (2.2) which term by term satisfies the boundary conditions (2.9). We obtain

$$\left. \begin{aligned} &\frac{\pi^2}{U^2} \sum (\beta^2 + n^2)^2 A_n \sin n \frac{\pi s}{U} + \frac{E}{R} \beta^2 \sum a_n \sin n \frac{\pi s}{U} = 0 \\ &\frac{Et^2}{12(1-\nu^2)} \frac{\pi^2}{U^2} \sum (\beta^2 + n^2)^2 a_n \sin n \frac{\pi s}{U} - \sigma \beta^2 \sum a_n \sin n \frac{\pi s}{U} - \\ &\frac{\beta^2}{R} \sum A_n \sin n \frac{\pi s}{U} = 0 \end{aligned} \right\} (3.2)$$

⁹Only v (not u) is directly coupled with w over the hoop strain $\epsilon_y = \frac{\delta v}{\delta s} + \frac{w}{R}$.

¹⁰The equations of buckling for a similar case: $\frac{1}{R} = \frac{1}{R_0} \sin \frac{\pi s}{U}$ can be found in the yearbook report of the author quoted before. (Compare footnote 4, p. 4.)

whereby $1/R$ has to be substituted according to equation (3.1). On the basis of the trigonometric identity

$$2 \cos \frac{2r\pi s}{U} \sin \frac{n\pi s}{U} = \sin \frac{(n+2r)\pi s}{U} + \sin \frac{(n-2r)\pi s}{U} \quad (3.3)$$

each of the two equations may be written in the form of a simple sine series:

$$\begin{aligned} & \frac{\pi^2}{U^2} \sum_{n=1}^{\infty} (\beta^2 + n^2)^2 A_n \sin \frac{n\pi s}{U} + \frac{E}{R_0} \beta^2 \left[\sum_{n=1}^{\infty} a_n \sin \frac{n\pi s}{U} + \right. \\ & \left. \sum_{r=1}^{r^*} \zeta_{2r} \left(\sum_{n=1}^{\infty} a_n \sin \frac{(n+2r)\pi s}{U} + \sum_{n=1}^{\infty} a_n \sin \frac{(n-2r)\pi s}{U} \right) \right] = 0, \\ & \sum_{n=1}^{\infty} \left(\frac{\pi^2}{U^2} \frac{Et^2}{12(1-\nu^2)} (\beta^2 + n^2)^2 - \beta^2 \sigma \right) a_n \sin \frac{n\pi s}{U} - \frac{1}{R_0} \beta^2 \left[\sum_{n=1}^{\infty} A_n \sin \frac{n\pi s}{U} + \right. \\ & \left. \sum_{r=1}^{r^*} \zeta_{2r} \left(\sum_{n=1}^{\infty} A_n \sin \frac{(n+2r)\pi s}{U} + \sum_{n=1}^{\infty} A_n \sin \frac{(n-2r)\pi s}{U} \right) \right] = 0 \quad (3.4) \end{aligned}$$

By renaming the indices in the two last sums each equation can be made to contain only $\sin \frac{n\pi s}{U}$. It is true that then the first sum begins at $1+2r$ and the last at $1-2r$ which involves a certain inconvenience if one wants to read off in formally surveyable form the equations for the coefficients a_n , A_n which result from the requirement that the factors of $\sin \frac{n\pi s}{U}$ individually have to be zero. The difficulty can be easily avoided; one has

$$\begin{aligned} & \sum_{r=1}^{r^*} \zeta_{2r} \left(\sum_{n=1}^{\infty} a_n \sin \frac{(n+2r)\pi s}{U} + \sum_{\lambda=1}^{\infty} a_n \sin \frac{(n-2r)\pi s}{U} \right) \\ & = \sum_{r=1}^{r^*} \zeta_{2r} \left(\sum_{n=1+2r}^{\infty} a_{n-2r} \sin \frac{n\pi s}{U} + \sum_{n=1-2r}^{\infty} a_{n+2r} \sin \frac{n\pi s}{U} \right) \end{aligned}$$

or, splitting up the second sum with consideration of
 $-\sin(n - 2r)\frac{\pi s}{U} = \sin(2r - n)\frac{\pi s}{U}$,

$$= \sum_{r=1}^{r^*} \zeta_{2r} \left(\sum_{n=1+2r}^{\infty} a_{n-2r} \sin \frac{n\pi s}{U} - a_1 \sin \frac{(2r-1)\pi s}{U} - a_2 \sin \frac{(2r-2)\pi s}{U} - \dots - a_{2r-1} \sin \frac{\pi s}{U} + \sum_{n=1}^{\infty} a_{n+2r} \sin \frac{n\pi s}{U} \right)$$

One can include the fully written terms in the first sum (which then starts with 1 instead of $1 + 2r$), if one stipulates that by a_{k-2r} (with negative index) the coefficient $-a_{2r-k}$ (with positive index) must be understood, and puts $a_0 = 0$. If one makes such a stipulation, all sums begin with 1, and (3.4) then reads simply

$$\left. \begin{aligned} \sum_{n=1}^{\infty} \left\{ \frac{\pi^2}{U^2} (\beta^2 + n^2)^2 A_n + \frac{E}{R_0} \beta^2 \left[a_n + \sum_{r=1}^{r^*} \zeta_{2r} (a_{n-2r} + a_{n+2r}) \right] \right\} \sin \frac{n\pi s}{U} = 0 \\ \sum_{\lambda=1} \left\{ \left(\frac{\pi^2}{U^2} \frac{Et^2}{12(1-\nu^2)} (\beta^2 + n^2)^2 - \sigma\beta^2 \right) a_n - \frac{1}{R_0} \beta^2 \left[A_n + \sum_{r=1}^{r^*} \zeta_{2r} (A_{n-2r} + A_{n+2r}) \right] \right\} \sin \frac{n\pi s}{U} = 0 \end{aligned} \right\} (3.5)$$

Since the left sides of equations (3.5) for all values of the variables are to disappear, the curved braces each must be zero; thus one has the two equation systems for determination of the coefficients a_n , A_n of the expression (2.2) which, because of

$$A_{-m} = -A_m, \quad a_{-m} = -a_m, \quad (A_0 = a_0 = 0) \quad (3.6)$$

are valid for all n

$$n = 1, 2, 3, \dots, \infty$$

The first system does not contain the eigen-value σ , moreover only one A_n appears in each equation; if we solve for A_n we obtain

$$A_n = -\frac{E}{R_0} \frac{U^2}{\pi^2} \frac{\beta^2}{(\beta^2 + n^2)^2} \left(a_n + \sum_{r=1}^{r^*} \zeta_{2r} (a_{n-2r} + a_{n+2r}) \right) \quad (3.7)$$

Because of (3.6), this equation is valid also for $n \leq 0$; one can therefore (without restriction for n) immediately substitute in the second system, and thus obtain one infinite equation system for determination of the unknown a_n which, with the abbreviations (2.5'), can be written in the form

$$\begin{aligned} & (1/k_n - \lambda) a_n + \left[k_n \left(a_n + \sum_{r=1}^{r^*} \zeta_{2r} (a_{n-2r} + a_{n+2r}) \right) \right] + \zeta_2 \left[k_{n-2} (a_{n-2} + \right. \\ & \left. \sum_{r=1}^{r^*} \zeta_{2r} (a_{n-2-2r} + a_{n-2+2r}) \right) + k_{n+2} (a_{n+2} + \sum_{r=1}^{r^*} \zeta_{2r} (a_{n+2-2r} + \\ & a_{n+2+2r})) \right] + \zeta_4 \left[k_{n-4} (a_{n-4} + \sum_{r=1}^{r^*} \zeta_{2r} (a_{n-4-2r} + a_{n-4+2r})) + \right. \\ & \left. k_{n+4} (a_{n+4} + \sum_{r=1}^{r^*} \zeta_{2r} (a_{n+4-2r} + a_{n+4+2r})) \right] + \zeta_6 [\dots] + \dots = 0 \end{aligned} \quad (3.8)$$

The determinant Δ of the homogeneous system (3.8) is the buckling determinant; $\Delta = 0$ is the qualifying equation for the eigen-

value $\lambda = \frac{4}{\sigma^* \sqrt{\omega}} \sigma$.

Equation (3.8) is somewhat modified by writing out the sums over r and rearranging them

$$\begin{aligned}
 & (1/k_n + k_n - \lambda)a_n + \zeta_2 \left[(k_n + k_{n-2})a_{n-2} + (k_n + k_{n-2})a_{n+2} \right] + \\
 & \zeta_2^2 \left[k_{n-2}a_{n-4} + k_{n+2}a_{n+4} + (k_{n-2} + k_{n+2})a_n \right] + \zeta_4 \left[(k_n + k_{n-4})a_{n-4} + \right. \\
 & \left. (k_n + k_{n+4})a_{n+4} \right] + \zeta_2 \zeta_4 \left[k_{n-2}(a_{n-6} + a_{n+2}) + k_{n-4}(a_{n-6} + a_{n-2}) + \right. \\
 & \left. k_{n+2}(a_{n-2} + a_{n+6}) + k_{n+4}(a_{n+2} + a_{n+6}) \right] + \zeta_4^2 \left[k_{n-4}a_{n-8} + k_{n+4}a_{n+8} + \right. \\
 & \left. (k_{n-4} + k_{n+4})a_n \right] + \zeta_6 \left[(k_n + k_{n-6})a_{n-6} + (k_n + k_{n+6})a_{n+6} \right] + \\
 & \zeta_2 \zeta_6 \left[k_{n-2}(a_{n-8} + a_{n+4}) + k_{n-6}(a_{n-8} + a_{n-4}) + k_{n+2}(a_{n-4} + a_{n+8}) + \right. \\
 & \left. k_{n+6}(a_{n+4} + a_{n+8}) \right] + \zeta_4 \zeta_6 \left[k_{n-4}(a_{n-10} + a_{n+2}) + k_{n-6}(a_{n-10} + a_{n-2}) + \right. \\
 & \left. k_{n+4}(a_{n-2} + a_{n+10}) + k_{n+6}(a_{n+2} + a_{n+10}) \right] + \zeta_6^2 \left[k_{n-6}a_{n-12} + \right. \\
 & \left. k_{n+6}a_{n+12} + (k_{n-6} + k_{n+6})a_n \right] + \dots = 0 \tag{3.9}
 \end{aligned}$$

The system (3.9) contains only even or odd indices - that is, the buckling forms symmetrical and antimetrical with respect to the center do not effect one another (of the two buckling forms, the one to which the smaller buckling load λ pertains will occur).

If the series (3.1) breaks off, as we assumed, after r^* terms, we are dealing with a finite-term system. If (3.1) contains, for instance, only the two first terms, a_{n-4} , a_{n-2} , a_n , a_{n+2} , a_{n+4} ,

appear in every equation - the system has five terms. For $\zeta_4 \neq 0$ it becomes a 9-term system, for $\zeta_6 \neq 0$ a 13-term system, etc.

One determines λ by first completely setting up (as was done here for $r^* = 3$) the system for the given number r^* of ζ -terms, then arranging it according to the unknowns a_n (for instance $a_{n-2} \left[\zeta_2 (k_n + k_{n-2}) + \zeta_4 \zeta_6 (k_{n+4} + k_{n-6}) \right]$ etc.) and calculating the coefficients of the a_n which thus originate. For prescribed ζ -values these coefficients are, furthermore, functions of k_n , that is of ω , β , and (in the main diagonal) of λ . The curvature value ω is to be regarded as prescribed, β is chosen, and λ is then determined from the condition that the coefficient determinant must disappear. The suitable β would have to be determined - as in the special case $1/R = \text{const.}$ - from the condition $\frac{d\lambda}{d\beta} = 0$ which expresses that the wave length $\lambda = U/\beta$ appears pertaining to the smallest buckling value λ ; however, due to the high degree of the λ -equation, this condition can be satisfied only by plotting $\lambda(\beta)$ for a number of neighboring β -values and reading off an approximate λ_{\min} .

The method requires much detailed calculation - still, it is superior to other methods (for instance to the method of reducing the number of equations by a to-be-guessed relation among the coefficients in the expression (3.1) - which can be interpreted as a sort of Ritz method), because it can be highly systematized and, above all, because its accuracy may - starting from rough approximation values - be increased arbitrarily and at any time.

Nevertheless, the calculation is rather troublesome, due to the large number of parameter values ω , ζ for which it must be performed. In order to make a useful choice among the many possible parameter combinations we must therefore, before numerical evaluation, answer the question as to the appearance of the shell forms corresponding to a certain variation of curvature $1/R(s)$.

4. DETERMINATION OF THE CYLINDRICAL FORM FROM THE VARIATION OF CURVATURE

By the equation (3.1)

$$\frac{1}{R} = \frac{1}{R_0} \left(1 + 2\zeta_2 \cos \frac{2\pi s}{U} + 2\zeta_4 \cos \frac{4\pi s}{U} + 2\zeta_6 \cos \frac{6\pi s}{U} + \dots \right) \quad (4.1)$$

the curvature is given as a function of the arc length. Because of

$$\frac{1}{R} = \frac{d\varphi}{ds}$$

it follows therewith that for the angle $\varphi = \varphi(s)$

$$\varphi = \int \frac{ds}{R} = \frac{s}{R_0} + \zeta_2 \frac{U}{\pi R_0} \sin \frac{2\pi s}{U} + \zeta_4 \frac{U}{2\pi R_0} \sin \frac{4\pi s}{U} + \zeta_6 \frac{U}{3\pi R_0} \sin \frac{6\pi s}{U} + \dots \quad (4.2)$$

From (4.2) one obtains by numerical integration because of the relation

$$\frac{d\bar{y}}{ds} = \sin \varphi, \quad \frac{d\bar{x}}{ds} = \cos \varphi \quad (4.3)$$

the ordinate \bar{y} and abscissa \bar{x} of the profile curve:

$$\bar{y} = \int \sin \varphi ds, \quad \bar{x} = \int \cos \varphi ds \quad (4.3')$$

The ratio U/R_0 in equation (4.1) is at first still open. It is determined from an assumption regarding the opening angle $\varphi_0 = \varphi(U) - \varphi(0)$ of the shell; according to equation (4.2) it is

$$\varphi_0 = \frac{U}{R_0}$$

We shall study here, above all, the half oval. (Compare fig. 1.)
For it

$$\varphi_0 = \pi, \text{ that is, } U = R_0\pi \quad (4.4)$$

and equation (4.2) is transformed into

$$\varphi = \frac{s\pi}{U} + \zeta_2 \sin \frac{2\pi s}{U} + \frac{1}{2} \zeta_4 \sin \frac{4\pi s}{U} + \frac{1}{3} \zeta_6 \sin \frac{6\pi s}{U} + \dots \quad (4.4')$$

In order to reduce calculation expenditure in application of the equation system (3.9), it is desirable to manage with as few terms in the series (4.1) as possible. In figures 3a and 3b curvature and course of curves according to equation (4.1) from (4.4') are plotted for the parameter values $\zeta_4 = \zeta_6 = \dots = 0$ and $\zeta_2 = -0.6$ or $-\underline{1}$, respectively.

Figure 4 contains the curves required for conversion from the parameter $\zeta = -\zeta_2$ to the axis ratio $\epsilon = \frac{b}{a}$ and to the quantity $\epsilon' = \frac{b}{R_0} = \frac{b}{U}$.

It is seen that one can just reach the axis ratio $\frac{b}{a} = \frac{1}{2}$ with the two-term expression: for values $|\zeta_2| > \frac{1}{2}$ one would have a change of sign in the curvature course, that is, one would obtain an oval buckled inward at the ends of the small axis - a form that does not occur in the applications.

In order to obtain "reasonable" ovals with an axis ratio $\epsilon < 1/2$, one must therefore start from a multiterm expression of the type (4.1). Figures 5a and 5b show (as the most obvious oval type) an ellipse of the axis ratio $\epsilon = \frac{1}{2}$, together with its curvature variation. The figure explains why the simple expression

$$\frac{1}{R} = \frac{1}{R_0} \left(1 - \zeta \cos \frac{2\pi s}{U} \right) \quad (4.5)$$

in the region of $\epsilon = \frac{1}{2}$, that is, $s = 1$ may no longer produce ellipse-like curves. Simultaneously, it indicates what type of curvature expression must be chosen: For the limiting curve (the oval with vanishing curvature at the end of the small axis) a higher power of $\sin \frac{\pi s}{U}$ comes into consideration for

$$\zeta_6 = \zeta_8 = \dots = 0, \text{ thus, for instance, } \frac{1}{R} \sim \sin^4 \frac{\pi s}{U}$$

that is

$$\frac{1}{R} = \frac{1}{R_0} \left(1 - \frac{4}{3} \cos \frac{2\pi s}{U} + \frac{1}{3} \cos \frac{4\pi s}{U} \right) \quad (4.6)$$

For $\zeta_8 = \zeta_{10} = \dots = 0$

$$\frac{1}{R} \sim \sin^6 \frac{\pi s}{U}$$

that is

$$\frac{1}{R} = \frac{1}{R_0} \left(1 - \frac{15}{10} \cos \frac{2\pi s}{U} + \frac{6}{10} \cos \frac{4\pi s}{U} - \frac{1}{10} \cos \frac{6\pi s}{U} \right) \quad (4.7)$$

Thus one will use for the pertaining oval-families for instance the expression

$$\frac{1}{R} = \frac{1}{R_0} \left[1 - \zeta \left(1.33 \cos \frac{2\pi s}{U} - 0.33 \cos \frac{4\pi s}{U} \right) \right], \quad (4.8)$$

thus

$$\zeta_2 = -1.33\zeta, \quad \zeta_4 = +0.33\zeta,$$

or, respectively,

$$\frac{1}{R} = \frac{1}{R_0} \left[1 - \zeta \left(1.5 \cos \frac{2\pi s}{U} - 0.6 \cos \frac{4\pi s}{U} + 0.1 \cos \frac{6\pi s}{U} \right) \right], \quad (4.9)$$

thus

$$\zeta_2 = -1.5\zeta, \quad \zeta_4 = +0.6\zeta, \quad \zeta_6 = -0.1\zeta$$

Figure 6 represents the oval $\zeta_2 = -1$ from figure 3b together with the oval determined by equation (4.8) with $\zeta = 0.8$ and the ellipse of the same axis ratio. It can be seen that the oval corresponding to the multiterm expression is more ellipse-like, in particular, that it shows no position of zero curvature. In figure 7, $\epsilon = \frac{b}{a}$, $\epsilon' = \frac{\pi b}{U}$,

and $\rho = \sqrt[3]{\frac{R(U/2)}{R(0)}}$ are plotted for the expression (4.9). For an ellipse ρ would be equal to ϵ - one recognizes from the representation that $\rho(\zeta)$ and $\epsilon(\zeta)$ between $\epsilon = 1$ and $\epsilon = 0.35$ lie close together so that the four-term expression (4.9) is sufficient to describe reasonable ovals down to an axis ratio $\frac{b}{a} \approx \frac{1}{3}$.

In the expressions (4.8) and (4.9) the signs alternate. If one chooses the same expressions all with the same signs, it signifies that the positions $\varphi = 0$ and $\varphi = \frac{\pi}{2}$ exchange their roles: One obtains (with exchanged axes) the same oval; only now the two halves are connected to the flanges at the ends of the large axis, if we still visualize the flanges at $\varphi = 0$ and π . Since this case is of less practical interest, we shall not discuss it more closely - but it should be pointed out at least that it is in this simple manner related to the "main case" equations (4.8) and (4.9), respectively.

For reasons of calculation expenditure we refrained in the present investigation from extending the expression for $1/R$ still further; going still further into an individual case does not present any fundamental difficulties; however, the expenditure will hardly ever be worth while for the applications since the calculations performed with the expressions (4.5) to (4.9) led to the result that the axis ratio ϵ is a suitable parameter for characterization of the stability behavior of an oval, that is, that the form variation in detail is of minor effect on the magnitude of the buckling load.

5. COMPUTATIONAL EVALUATION OF THE BUCKLING
DETERMINANT IN THE CASE OF COMPRESSION

Calculation of the eigen-values λ from the condition that the determinant of the homogeneous equation system (3.9') must vanish is possible only "step by step:" Step by step in the sense that one starts with small ζ and ω values and with the two-term expression (4.5) for the curvature variation.

For $\zeta_4 = \zeta_6 = \zeta_8 = \dots = 0$, each of the equations (3.9) contains only the two first lines, and the system reads, arranged according to the unknowns (with $\zeta_2 = -\zeta$),
for odd n

$$\left. \begin{aligned}
 \left[\frac{1}{k_1} + k_1(1 + \zeta)^2 + \zeta^2 k_3 - \lambda \right] a_1 - \zeta \left[(1 + \zeta)k_1 + k_3 \right] a_3 + \zeta^2 k_3 a_5 &= 0 \\
 -\zeta \left[(1 + \zeta)k_1 + k_3 \right] a_1 + \left[\frac{1}{k_3} + k_3 + \zeta^2 k_3 - \lambda \right] a_3 - \zeta \left[k_3 + k_5 \right] a_5 + \zeta^2 k_5 a_7 &= 0 \\
 \zeta^2 k_3 a_1 - \zeta \left[k_3 + k_5 \right] a_3 + \left[\frac{1}{k_5} + k_5 - \lambda \right] a_5 - \zeta \left[k_5 + k_7 \right] a_7 + \dots &= 0 \\
 \zeta^2 k_5 a_3 - \zeta \left[k_5 + k_7 \right] a_5 + \left[\frac{1}{k_7} + k_7 - \lambda \right] a_7 + \dots &= 0 \\
 \dots &
 \end{aligned} \right\} (5.1)$$

and for even n

$$\left. \begin{aligned} \left[\frac{1}{k_2} + k_2 + \zeta^2 k_4 - \lambda \right] a_2 - \zeta [k_2 + k_4] a_4 + \zeta^2 k_4 a_6 &= 0 \\ -\zeta [k_2 + k_4] a_2 + \left[\frac{1}{k_4} + k_4 - \lambda \right] a_4 - \zeta [k_4 + k_6] a_6 + \zeta^2 k_6 a_8 &= 0 \\ \zeta^2 k_4 a_2 - \zeta [k_4 + k_6] a_4 + \left[\frac{1}{k_6} + k_6 - \lambda \right] a_6 - \zeta [k_6 + k_8] a_8 + \dots &= 0 \\ \zeta^2 k_6 a_4 - \zeta [k_6 + k_8] a_6 + \left[\frac{1}{k_8} + k_8 - \lambda \right] a_8 + \dots &= 0 \end{aligned} \right\} (5.2)$$

The second system shows the usual initial irregularities in the first equation, the first system, those in the two first equations. For the first system these irregularities are such that the sign of ζ considerably affects the behavior of the first terms.

In contrast, the eigen-value resulting from the second system does not depend on the sign of ζ (only a_4 , a_8 , a_{12} change their signs); since the even terms in the expression (2.2) have nodes at $s = 0$ and $U/2$ ($\varphi = 0, \pi/2$), it must indeed be a matter of indifference at which of the axis ends the flanges are connected.

The secondary terms are small if ζ is small and - since they contain only the quantities $k_n \sim \sqrt{\omega}$ - if $\sqrt{\omega}$ is small. For sufficiently small values of ζ and $\sqrt{\omega}$ (for instance, $2\zeta = 0.5$, $\sqrt{\omega} = 50$), therefore, a portion of two equations from one of the two systems (5.1 or 5.2) suffices for the determination of the eigen-value. The quadratic equation for λ is easily solvable, the minimum value as a function of β can be determined immediately. If one retains for instance ζ and increases $\sqrt{\omega}$, one can at first still manage with two or three equations, can determine β and λ , and plot both quantities as a function of $\sqrt{\omega}$. By extrapolation one obtains approximation values of β and λ for the next $\sqrt{\omega}$ value; one improves these approximation values by trial substitutions in the determinant, and continues in this manner up to the upper limit for $\sqrt{\omega}$, at about $\sqrt{\omega} = 4000$, which is of practical interest.

A twofold difficulty opposes performance of this calculation: First one must decide on the order of the determinant portion Δ_m to be selected which fixes with sufficient approximation the eigen-value

(which actually should be determined from an infinite determinant Δ_{∞}). Since the order of Δ_m increases with $\sqrt{\omega}$ (the secondary terms become more and more important), the buckling determinant finally becomes of too high an order to be still solvable with respect to λ . Neglecting the convergence problem (regarding the mechanically reasonable result as a sufficient confirmation of convergence), we can surmount both difficulties by the following method. Besides the one for β , we insert an approximation value λ_1 - (estimated by extrapolation) for λ also, and reduce the determinant to its main diagonal with the aid of Gauss' algorithm - without first fixing m . If we plot the new diagonal values (the product of which represents the value of the determinant Δ_m) against the pertaining number of equations n , we obtain a point sequence we can connect by a curve. If we continue sufficiently far with m , this curve will either intersect the axis (generally not precisely at an integral n) or bend up. If the curve intersects the axis, the main diagonal terms were too unimportant, thus $\lambda = \lambda_1$ too large. The calculation is repeated with a slightly smaller $\lambda = \lambda_2$; if the curve this time bends up, the number m was chosen correctly, and the desired λ -value must be between λ_1 and λ_2 because the new diagonal terms would, with increasing n , have to decrease asymptotically toward zero ($\Delta_{\infty} \rightarrow 0$) for this λ -value. Figure 8 shows a number of such curves for $2\zeta = 0.5$ and several sets of values for $\sqrt{\omega}$, β , and λ . From the two curves corresponding to $\sqrt{\omega} = 500$, $\beta = 16$, one recognizes, for instance, that the root of Δ_{∞} must be between $\lambda_1 = 1.18$ and $\lambda_2 = 1.16$. The difference between the values λ_1 and λ_2 is a measure of the accuracy of the calculation. The entire procedure must now be repeated for neighboring β -values in order to make reading off the minimum from a $\lambda(\beta)$ curve possible. Since, however, λ as an extreme value in the neighborhood of the correct value is affected very little by the choice of β , two - at the most three - β proofs will generally be sufficient, provided the investigator has had some "experience" with this calculation method.

One thus obtains a family of curves $\lambda(\sqrt{\omega})$ with ζ as parameter. Since the curves run smoothly, one can conveniently interpolate on the basis of four such curves ($2\zeta = 0, 0.5, 0.8, 1.0$) and with the aid of the curves of figure 4 convert to the directly prescribed quantities, the axes a , b of the oval, and to the wall thickness t contained in σ^* and $\sqrt{\omega}$. According to (2.5)

$$\frac{\sigma}{E} = \lambda \frac{\sigma^* \sqrt{\omega}}{4E};$$

according to (4.4) one obtains for the half oval

$$\sqrt{\omega} = \frac{2\sqrt{3(1-v^2)}}{\pi} \frac{U}{t}, \quad \text{thus } \frac{\sigma}{E} = \frac{\lambda}{\sqrt{\omega}} = \lambda \frac{1}{2\sqrt{3(1-v^2)}} \frac{\pi t}{U} \quad (5.3)$$

With the dimensionless $\epsilon' = \frac{\pi b}{U}$ represented in figure 4 one may write instead

$$\frac{\sigma}{E} = \frac{1}{\sqrt{3(1-v^2)}} \lambda \epsilon' \frac{t}{2b} \quad (5.4)$$

and since λ is itself a function of $\sqrt{\omega}$ (that is, U/t) and ζ (that is, U/b), everything can be converted to the two parameters, the ratio $\frac{2b}{t}$ (spar height/skin thickness), and the axis ratio $\frac{b}{a} = \epsilon$ of the oval.

In the region $200 \leq \sqrt{\omega} \leq 4000$ which is interesting in practice (that is, for the wing nose), for ζ -values > 0 , the two λ -values are not distinguished according to equations (5.1) and (5.2). This becomes physically understandable if one considers figure 9 where for a "mean" pair of parameters $\sqrt{\omega}$, ζ the buckling profile, as it results from equation (5.2), is plotted against the developed width. One can see that a large wave originates in the flat region while many tiny waves develop in the region of large curvature; and it is immediately plausible that it cannot affect the magnitude of buckling load whether these small waves happen to have a bulge or a node at the point $\varphi = \frac{\pi}{2}$. (For ζ -values < 0 , of course, the buckling values differ considerably; for, whether or not a node is enforced at the flat place $\varphi = 0$ is decisive for the buckling form.)

The calculation with the multiparameter curvature expressions (4.8) and (4.9) takes fundamentally exactly the same course, except for the fact that here the computation of the coefficients of the equation system (3.8) as functions of ζ and of the quantities k_n is much more troublesome, and that the complexity of the system is of disadvantage also for application of Gauss' algorithm. With a four-term

expression (4.1) as a basis, the first coefficients of the system read (we denote them here simply by their indices put in parentheses):

$$\begin{aligned}
 (11) &= \frac{1}{k_1} + k_1(1 - \zeta_2)^2 + k_3(\zeta_2 - \zeta_4)^2 + k_5(\zeta_4 - \zeta_6)^2 + k_7\zeta_6^2 \\
 (22) &= \frac{1}{k_2} + k_2(1 - \zeta_4)^2 + k_4(\zeta_2 - \zeta_6)^2 + k_6\zeta_4^2 + k_8\zeta_6^2 \\
 (33) &= \frac{1}{k_3} + k_3(1 - \zeta_6)^2 + k_1(\zeta_2 - \zeta_4)^2 + k_5\zeta_2^2 + k_7\zeta_4^2 + k_9\zeta_6^2 \\
 (44) &= \frac{1}{k_4} + k_4 + k_2(\zeta_2 - \zeta_6)^2 + k_6\zeta_2^2 + k_8\zeta_4^2 + k_{10}\zeta_6^2 \\
 (13) &= k_1(\zeta_2 - \zeta_4)(1 - \zeta_2) + k_3(\zeta_2 - \zeta_4)(1 - \zeta_6) + k_5\zeta_2(\zeta_4 - \zeta_6) + k_7\zeta_4\zeta_6 \\
 (24) &= k_2(\zeta_2 - \zeta_6)(1 - \zeta_4) + k_4(\zeta_2 - \zeta_6) + k_6\zeta_2\zeta_4 + k_8\zeta_4\zeta_6 \\
 (35) &= k_1(\zeta_2 - \zeta_4)(\zeta_4 - \zeta_6) + k_3\zeta_2(1 - \zeta_6) + k_5\zeta_2 + k_7\zeta_2\zeta_4 + k_9\zeta_4\zeta_6 \\
 (15) &= k_1(\zeta_4 - \zeta_6)(1 - \zeta_2) + k_3\zeta_2(\zeta_2 - \zeta_4) + k_5(\zeta_4 - \zeta_6) + k_7\zeta_2\zeta_6
 \end{aligned} \tag{5.5}$$

Fortunately, the calculation with the new coefficients need not be performed for all former $\sqrt{\omega}$ and ζ values; the β -values and the much less sensitive λ -values for the ovals of equal axis ratio and equal ratio $\frac{2b}{t}$ are found to lie very close together. Thereby the calculation with the expressions (4.8) and (4.9) assumes the character of a check calculation for the region $1 \geq \epsilon \geq 1/2$ and of an extrapolation calculation for the values of $1/2 \geq \epsilon \geq 1/3$.

That the buckling load depends only on $2b/t$ and b/a (that is, on the form "on the whole"), but not on the curvature variation in detail, can be physically explained by the effects of curvature variability which oppose one another: The parts of stronger curvature support those of small curvature, and - as the calculation shows - it does not make much difference for the load capacity whether a small region with

strong curvature is present, which for the rest has the character of an almost rigid edge support, or whether, as for the oval (4.5), a larger region of medium curvature expands which relieves the rest of some of its load. However, it is noteworthy that not the mean curvature (one parameter ω) is decisive for the critical compressive stress, but that two parameters, axis ratio and spar height, appear as the critical quantities.

The total result is shown in figure 10 in logarithmic representation. Use of the curve table requires a brief experimental method since for reasons of clarity in the extensive U/t range the expression proportional to the compressive force

$$\frac{\sigma}{E} \frac{t}{2b} \quad (5.6)$$

could not be represented, but only the quantity proportional to the stress $\frac{\sigma}{E}$. Thus one must, for prescribed compression force, first estimate the quantities σ or t and may then determine on the basis of the curve in what direction the estimated value must be changed.

6. THE CLOSED CYLINDER UNDER PURE SHEAR

The shear problem differs considerably from the compression problem in mathematical respect. The main equations (2.1) no longer contain in all terms an even number of derivatives with respect to each of the two variables. Consequently, $\sin \frac{\pi x}{l}$ can no longer be factored so that the equations (2.1'), which served as starting point for the compression calculation, lose their validity; one must refer back directly to the partial differential equations

$$\left. \begin{aligned} \Delta \Delta \phi - \frac{E}{R} w_{xx} &= 0 \\ \frac{Et^2}{1 - \nu^2} \Delta \Delta w + \frac{1}{R} \phi_{xx} - 2\tau w_{xs} &= 0 \end{aligned} \right\} \quad (6.1)$$

In order to understand what is typical in the new way of putting the problem, we shall begin with the simplest case: the closed cylinder of infinite length which shows neither edges $c = \text{const.}$, nor edges

$s = \text{const.}$ Since for such a cylinder oblique waves can go around without interference, the appropriate expression for solution is easily guessed: With

$$w = \sum_{-\infty}^{\infty} a_n \sin \frac{\pi}{U}(ns + \beta x)$$

$$\phi = \sum_{-\infty}^{\infty} A_n \sin \frac{\pi}{U}(ns + \beta x)$$
(6.2)

where $2U$ is the circumference of the complete cylinder, the pair of equations (6.1) is transformed - with consideration of the expression (3.1) and of the trigonometric identity

$$2 \cos \frac{2r\pi s}{U} \sin \frac{\pi}{U}(ns + \beta x) = \sin \frac{\pi}{U}((n+2r)s + \beta x) + \sin \frac{\pi}{U}((n-2r)s + \beta x)$$

into

$$\left. \begin{aligned} & \sum_{n=-\infty}^{\infty} \left(\frac{\pi^2}{U^2} (\beta^2 + n^2)^2 A_n + \frac{E}{R_0} \beta^2 a_n \right) \sin \frac{\pi}{U}(ns + \beta x) + \\ & \sum_{n=-\infty}^{\infty} \sum_{r=1}^{r^*} \zeta_{2r} a_n \sin \frac{\pi}{U}((n+2r)s + \beta x) + \\ & \sum_{n=-\infty}^{\infty} \sum_{r=1}^{r^*} \zeta_{2r} a_n \sin \frac{\pi}{U}((n-2r)s + \beta x) = 0 \\ & \sum_{n=-\infty}^{\infty} \left(\frac{E}{1-v^2} \frac{t^2 \pi^2}{U^2} (\beta^2 + n^2)^2 a_n - \frac{1}{R_0} \beta^2 A_n - 2n\beta \tau a_n \right) \sin \frac{\pi}{U}(ns + \beta x) + \\ & \sum_{n=-\infty}^{\infty} \sum_{r=1}^{r^*} \zeta_{2r} A_n \sin \frac{\pi}{U}((n+2r)s + \beta x) + \\ & \sum_{n=-\infty}^{\infty} \sum_{r=1}^{r^*} \zeta_{2r} A_n \sin \frac{\pi}{U}((n-r)s + \beta x) = 0 \end{aligned} \right\} (6.3)$$

The equation system (6.3) is constructed exactly like the system (3.4) except for the fact that $\sin \frac{n\pi s}{U}$ is now replaced by $\sin \frac{\pi}{U}(ns + \beta x)$, and $\beta^2 \sigma$ by $2\beta n\tau$. Nevertheless the further treatment is very different in the two cases. The summand βx in the argument of the sine function causes the terms with positive and negative n no longer to be distinguished simply by the sign; the expression (6.2) must therefore contain all n values from $-\infty$ to ∞ (whereas in the compression case the positive n values had been sufficient for obtaining a solution that satisfied all conditions), and the equation system for determining the a_n and a_{-n} ($\neq -a_n$) extends, therefore, to infinity "in both directions." That, moreover, the factor of the eigen-value is $2n\beta$ instead of β^2 has the result that the minimum value for τ as a function of β lies in an entirely different β range than in the compression case. In order to perceive this last fact which is of utmost importance for the computational evaluation of the equation system (6.3) we shall here, exactly as in the compression case, briefly treat the cylinder of constant curvature at the outset.

For $\zeta_{2r} = 0$ the equations (6.3) are each reduced to the first line. The requirement that the factor of $\sin \frac{\pi}{U}(ns + \beta x)$ must disappear results in a system of equations which contain in every case only unknowns A_n , a_n with the same index. The buckling condition results precisely as it did in the compression case (compare equations (2.4), etc.) and reads with the abbreviations (2.5), (2.5'), and

$$\mu = 8 \frac{\tau}{\sigma^*} \frac{1}{\sqrt{\omega}} = 4 \sqrt{3(1 - \nu^2)} \frac{\tau}{E} \frac{R}{t}$$

according to (2.6)

$$\frac{1}{k_n} + k_n - \frac{n}{\beta} \mu = 0 \quad (6.4)$$

The condition $\frac{d\mu}{d\beta} = 0$ yields

$$\begin{aligned} 0 &= \frac{1}{k_n} + k_n + \beta(1 - k_n^2) \frac{d(1/k_n)}{d\beta^2} \frac{d\beta^2}{d\beta} \\ &= \frac{1}{k_n} + k_n + \frac{2\beta^2}{\sqrt{\omega}} (1 - k_n^2) \left(1 - \frac{n^4}{\beta^4}\right) \end{aligned} \quad (6.5)$$

Since for the complete cylinder $\sqrt{\omega} = \frac{\sqrt{12(1-v^2)}}{\pi} \frac{U^2}{\pi R_0 t} = \frac{\sqrt{12(1-v^2)}}{\pi} \frac{U}{t}$

is a very great number, β must be either very large or very small in order to cause

$$\mu = \frac{1}{n} \left[\frac{(n^2 + \beta^2)^2}{\beta \sqrt{\omega}} + \frac{\beta^3 \sqrt{\omega}}{(n^2 + \beta^2)^2} \right] \quad (6.4')$$

to assume reasonable amounts. With the assumptions

$$\beta^2 \ll n^2, \quad k_n = \frac{\beta^2 \sqrt{\omega}}{n^4} \quad \left| \quad \beta^2 \gg n^2, \quad k_n = \frac{\sqrt{\omega}}{\beta^2}$$

there follow from (6.5) for β and k_n the simple formulas

$$\left. \begin{array}{l} \frac{n^4}{\beta^2} = \sqrt{3\omega} \\ k_n = \frac{1}{\sqrt{3}} \end{array} \right| \left. \begin{array}{l} \beta^2 = \sqrt{\frac{\omega}{3}} \\ k_n = \sqrt{3} \end{array} \right\} \quad (6.5')$$

and therewith for μ

$$\mu_{(kl)} = \frac{n}{4\sqrt{3\omega}} \left(\sqrt{3} + \frac{1}{\sqrt{3}} \right) = \frac{4n}{4\sqrt{27\omega}} \quad \left| \quad \mu_{(gr)} = \frac{4\sqrt{\omega/3}}{n} \left(\sqrt{3} + \frac{1}{\sqrt{3}} \right) = \frac{4}{n} \sqrt{\frac{\omega}{27}} \quad (6.6)$$

The critical shear load therefore attains its minimum for small β values. According to equation (6.2), β/n is the tangent of the angle formed by the node lines and the x-axis; thus $\beta \ll n$ signifies that long wave crests almost parallel to the axis develop. In case of longitudinal compression, on the other hand, wave fields with an aspect ratio of the order of magnitude unity are formed; that is, the reciprocal action of stretching and bending, which is characteristic for the shell, takes full effect in case of compression, but not in shear; hence the critical shearing stress according to equation (6.6) is incomparably lower than the critical longitudinal-compressive stress according to equation (2.8).¹¹

¹¹For longitudinal compression there is $\sigma_k \sim (t/R)^1$, for shear $\tau_k \sim (t/R)^{3/2}$, for normal pressure (the wave crests run exactly axis-parallel) the critical hoop stress becomes $\sim (t/R)^2$.

It is clear (and is confirmed by computational spot checks), that this behavior of the infinitely long shell under shear cannot change fundamentally, due to the variable curvature. In discussion of the system (6.3) we may, therefore, make use of the assumption

$$\beta^2 \ll n^2$$

As a consequence, the quantities

$$k_n \approx \beta^2 \frac{\sqrt{\omega}}{n^4}$$

decrease very rapidly with increasing n , thus the secondary terms in the determinant of buckling for large n become negligibly small so that the discussion of the equation system resulting from equation (6.3) may be limited to a few terms around $n = 0$. Equation (6.6) is maintained as an approximation result; it permits the conclusion that the terms $n = \pm 2$, as the terms with the smallest n , will play the decisive role; for, according to equation (6.4), $n = 0$ has no meaning, and $n = 1$ leads to an entirely different type of buckling (Greenhill buckling) with which we do not want to deal here and for the treatment of which the simplified equations (6.1) would not be sufficient.¹²

Thus we consider the system (6.3) for even n and limit ourselves, for reasons of clarity, to a three-term curvature expression (3.1) and to five equations. One then obtains by eliminating the A_n , arranging and putting equal to zero the factors of

$$\sin \frac{\pi}{U}(ns + \beta x) \quad \text{for } n = -4, -2, 0, 2, 4$$

a system of five equations, the determinant of the coefficients of which

$$\Delta \equiv \begin{vmatrix} (-4, -4) & (-4, -2) & (-4, 0) & (-4, 2) & (-4, 4) \\ (-2, -4) & (-2, -2) & (-2, 0) & (-2, 2) & (-2, 4) \\ (0, -4) & (0, -2) & (0, 0) & (0, 2) & (0, 4) \\ (2, -4) & (2, -2) & (2, 0) & (2, 2) & (2, 4) \\ (4, -4) & (4, -2) & (4, 0) & (4, 2) & (4, 4) \end{vmatrix}$$

¹²Compare Flügge: Statik und Dynamik der Schalen (Statics and dynamics of shells), p. 199 ff., Berlin 1935; furthermore, Kappus: Elastizitätstheorie grosser Verschiebungen, (Elasticity theory of large displacements), ZAMM 19, 1939, p. 351 ff.

must vanish. The coefficients (i,k) of the determinant are

$$(0,0) = \frac{1}{k_0} + k_0 + 2\xi_2^2 k_2 + 2\xi_4^2 k_4$$

$$\left. \begin{matrix} (-2,-2) \\ (2,2) \end{matrix} \right\} = \frac{1}{k_2} + k_2 + \xi_2^2 (k_0 + k_4) + \xi_4^2 (k_2 + k_6) \pm \frac{2\mu}{\beta}$$

$$\left. \begin{matrix} (-4,-4) \\ (4,4) \end{matrix} \right\} = \frac{1}{k_4} + k_4 + \xi_2^2 (k_2 + k_6) + \xi_4^2 (k_0 + k_8) \pm \frac{4\mu}{\beta}$$

$$(-2,2) = (2,-2) = -(\xi_2^2 k_0 + 2k_2 \xi_4)$$

$$(-4,4) = (4,-4) = -\xi_4^2 k_0$$

$$-(-2,0) = -(0,-2) = (2,0) = (0,2) = \xi_2(k_0 + k_2) + \xi_2 \xi_4 (k_2 + k_4)$$

$$-(-4,0) = -(0,-4) = (4,0) = (0,4) = \xi_2^2 k_2 + \xi_4(k_0 + k_4)$$

$$(-4,2) = (2,-4) = (4,-2) = (-2,4) = -\xi_2 \xi_4 (k_0 + k_2)$$

$$(-4,-2) = (-2,-4) = (4,2) = (2,4) = \xi_2(k_2 + k_4) + \xi_2 \xi_4 (k_0 + k_6)$$

Thus the determinant is built in a manner permitting an easy survey; by appropriate addition of the lines or columns it may readily be simplified so that the very large term

$$k_0 = \frac{\sqrt{\omega}}{\beta^2}$$

appears only at the point (0,0). Since k_2, k_4, k_6 are, compared to k_0 , small in the ratio $\frac{\beta^4}{n^4}$, a four-series determinant remains which can be further reduced if one also neglects everywhere k_4, k_6 compared to k_2 and makes sure that the large term

$$\frac{1}{k_4} = \frac{256}{\beta^2 \sqrt{\omega}}$$

occurs only in the elements (-4,-4)(4,4). The condition $\Delta = 0$ is then reduced to the statement that the inner two-series determinant must vanish. In this manner a quadratic equation for μ originates in

which the linear term also vanishes (the sign of μ remains undetermined, as it must be), so that one has as buckling condition:

$$\frac{2\mu}{\beta} = \sqrt{\frac{1}{k_2} + k_2(1 - \zeta_4)^2} \sqrt{\frac{1}{k_2} + k_2(1 + \zeta_4 - 2\zeta_2^2)^2} \quad (6.7)$$

This formula does not vary if a multiterm expression for $1/R$ is used, because the ζ_6 , ζ_8 , in the decisive two-series determinant occur only in combination with k_4 , k_6 . . . , and thus must be eliminated corresponding to the other neglects leading to (6.7).

The sign of ζ_2 is insignificant (as it must be for the closed cylinder); the deviation from circular form causes a reduction in critical load which can easily be calculated for the ovals of the type (4.8) or (4.9) that are of interest. First we must determine from

$$\begin{aligned} \mu &= \frac{\beta}{2} k_2 \sqrt{\left(\frac{1}{k_2}\right)^2 + (1 - \zeta_4)^2} \sqrt{\left(\frac{1}{k_2}\right)^2 + (1 + \zeta_4 - 2\zeta_2^2)^2} \\ &\approx \frac{\beta^3}{32} \sqrt{\omega} \sqrt{\frac{256}{\omega\beta^4} + (1 - \zeta_4)^2} \sqrt{\frac{256}{\omega\beta^4} + (1 + \zeta_4 - 2\zeta_2^2)^2} \end{aligned} \quad (6.7')$$

the critical β -values by differentiation with respect to β or $1/\beta$. The simple calculation yields for $1/\beta^4$ the quadratic equation:

$$\begin{aligned} \frac{256}{\omega\beta^4} - \left[(1 - \zeta_4)^2 + (1 + \zeta_4 - 2\zeta_2^2)^2 \right] - \\ 3 \frac{\omega\beta^4}{256} \left[(1 - \zeta_4)^2 (1 + \zeta_4 - 2\zeta_2^2)^2 \right] = 0 \end{aligned} \quad (6.8)$$

The two expressions

$$(1 - \zeta_4)^2 \quad \text{and} \quad (1 + \zeta_4 - 2\zeta_2^2)^2$$

for the ovals of type (4.8), (4.9) in the ζ -range which is, according to figure 7, of interest, differ numerically only slightly so that it is permissible to replace, for the determination of the extreme, the

square of the geometric mean occurring in the last term of the equation (6.8) by the square of the arithmetic mean; (6.8) then reads

$$\frac{256}{\omega\beta^4} - 2 \frac{(1 - \zeta_4)^2 + (1 + \zeta_4 - 2\zeta_2^2)^2}{2} - 3 \frac{\omega\beta^4 \left[\frac{(1 - \zeta_4)^2 + (1 + \zeta_4 - 2\zeta_2^2)^2}{2} \right]^2}{256} = 0$$

and has the positive root

$$\begin{aligned} \frac{256}{\omega\beta^4} \equiv \frac{1}{k_2^2} &= 3 \frac{(1 - \zeta_4^2)^2 + (1 + \zeta_4 - 2\zeta_2^2)^2}{2} \\ &= 3(1 + \zeta_4^2 - 2\zeta_2^2(1 + \zeta_4) + 2\zeta_2^4) \end{aligned}$$

If we substitute this in equation (6.7), the two radicands may be written in the form

$$4(1 + \zeta_4^2 - 2\zeta_2^2(1 + \zeta_4) + 2\zeta_2^4) \mp 2(\zeta_4 - \zeta_2^2(1 + \zeta_4) + \zeta_2^4)$$

We have therefore

$$\begin{aligned} \mu &= \frac{8}{4\sqrt{27\omega}} \left[1 + \zeta_4^2 - 2\zeta_2^2(1 + \zeta_4) + 2\zeta_2^4 \right]^{1/4} \left[1 - \left(\frac{\zeta_4 - \zeta_2^2(1 + \zeta_4) + \zeta_2^4}{1 + \zeta_4^2 - 2\zeta_2^2(1 + \zeta_4) + 2\zeta_2^4} \right)^2 \right]^{1/2} \end{aligned} \tag{6.9}$$

The last factor in (6.9) may be replaced by 1 for an oval of the type (4.9). Then it can be written in the form

$$\left[1 - \frac{1}{16} \left(1 - \frac{(1 - \zeta_4)^2}{1 + \zeta_4^2 - 2\zeta_2^2(1 + \zeta_4) + 2\zeta_4^2} \right)^2 \right]^{1/2}$$

which makes it clear that even in the extreme case $2\zeta_4 = 0.6 \times 0.9$ (compare fig. 7) the reduction caused by the second summand stays below

1 percent. Thus there results, with consideration of $U = R_0\pi$, for the critical shearing stress

$$\tau = \frac{\sigma^*\sqrt{\omega}}{8} \mu = \frac{E}{(1 - \nu^2)^{3/4}} \frac{\sqrt{2}}{3\sqrt{3}} \left(\frac{t}{R_0}\right)^{3/2} \left[1 + \zeta_4^2 - 2\zeta_2^2(1 + \zeta_4) + 2\zeta_2^4\right]^{1/4} \quad (6.9')$$

Aside from the factor $[1 + \dots]^{1/4}$ the formula (6.9') is built exactly like the formula for the circular cylinder derived by Flügge,¹³ except that the numerical factors deviate slightly. According to Flügge's theory the factor $\frac{1}{3\sqrt{2}} = 0.236$ replaces $\frac{\sqrt{2}}{3\sqrt{3}} = 0.272$, thus a numerical value smaller by 15 percent. This error in the formula (6.9') stems from the simplifications in the initial equations (6.1) which were permitted only under the assumption that "many" waves developed in the hoop direction; however, the deviation is still within technically tolerable limits, in spite of the buckling form of extremely low wave number ($n = 2$).

Thus the mean curvature $1/R_0$ is, according to (6.9'), essentially decisive for the load capacity of the cylinder with oval directrix under shear load. One determines t/R_0 from the spar height $2b$ and the axis ratio $\epsilon = \frac{b}{a}$, with the aid of the ϵ' -curve of figure 4 or 7 (according to the form of the directrix) on the basis of the relation $\frac{t}{R_0} = 2\epsilon' \frac{t}{2b}$; ϵ' is read off as a function of ϵ by interpolation of the variable ζ . Accordingly, (6.9') can be written in the form

$$\frac{\tau}{E} = \left(\frac{t}{2b}\right)^{3/2} \psi(\epsilon)$$

In figure 11 the factors $\psi(\epsilon)$, which - with consideration of Flügge's correction - are given by

$$\psi(\epsilon) = \frac{1}{(1 - 0.09)^{3/4}} \frac{1}{3\sqrt{2}} (2\epsilon')^{3/2} \left[1 + \zeta_4^2 - 2\zeta_2^2(1 + \zeta_4) + 2\zeta_2^4\right]^{1/4}$$

¹³Schalenbuch (Book on shells), p. 206.

are drawn (dot-dashed) for the ovals (4.5) and (4.9). The two curves practically coincide. The expression $\psi(\epsilon) = 1.34\epsilon - 0.37\epsilon^2 - 0.25$ may serve as approximation formula for both ovals (and of course also for (4.8)); thus

$$\frac{\tau}{E} = \left(\frac{t}{2b}\right)^{3/2} (1.34\epsilon - 0.37\epsilon^2 - 0.25) \quad (6.9'')$$

7. THE BUCKLING CONDITIONS FOR THE CYLINDRICAL SEGMENT UNDER SHEAR LOAD; ITS SOLUTION IN FIRST APPROXIMATION

More important for the application than the closed cylinder under shear load is the cylindrical segment of variable curvature supported along two edges $s = \text{constant}$ under shear load (and shear-compression). We think for instance of a half oval with the longitudinal edges supported according to the requirements of the conditions (2.9). The expressions (3.2) and (6.2), which for the previous problems had led to finite-term equation systems, are not usable: One does not satisfy the differential equation, and the other cannot be adapted to the boundary conditions. In contrast, the problem leads to an infinite equation system in which every equation has an infinite number of terms, with the aid of the expressions¹⁴

$$w = \sum_1^{\infty} \left(a_n \sin \frac{\pi x}{l} + b_n \cos \frac{\pi x}{l} \right) \sin \frac{n\pi s}{U} \quad (7.1)$$

$$\phi = \sum_1^{\infty} \left(A_n \sin \frac{\pi x}{l} + B_n \cos \frac{\pi x}{l} \right) \sin \frac{n\pi s}{U}$$

which satisfy term by term the boundary conditions (2.9). The calculation expenditure is nevertheless smaller than in the case of compression because the critical range is again the region of small β -values where the system converges extremely rapidly. For obtaining

¹⁴These expressions are also used by Kromm in the report mentioned in footnote 5, Luftf.-Forschg. 1938, p. 517. Kromm gives a description of the physical importance of the expression (compare the formulation of (7.8) below in the summary) in the yearbook 1940, p. 834.

the equation system itself, one must employ considerations similar to those in section 3; it is particularly useful to introduce again the stipulation (3.6). If we limit ourselves, for reasons of clarity, preliminarily to the simplest curvature expression

$$\frac{1}{R} = \frac{1}{R_0} \left(1 - 2\zeta \cos \frac{2\pi s}{U} \right)$$

there result from the first equation (2.1) (compare (3.7))

by equating to zero the factor of $\sin \beta \frac{\pi x}{U} \sin n \frac{\pi s}{U}$

$$A_n = - \frac{\beta^2}{(\beta^2 + n^2)^2} \frac{EU^2}{\pi^2 R_0} (a_n - \zeta(a_{n+2} + a_{n-2})) \quad (7.2)$$

and by equating to zero the factor of $\cos \beta \frac{\pi x}{U} \sin n \frac{\pi s}{U}$

$$B_n = - \frac{\beta^2}{(\beta^2 + n^2)^2} \frac{EU^2}{\pi^2 R_0} (b_n - \zeta(b_{n+2} + b_{n-2}))$$

The second equation (2.1) where one can at first not factor $\sin \frac{n\pi s}{U}$ yields correspondingly

$$\begin{aligned} & \sum_1^{\infty} \left[\frac{E}{1 - \nu^2} \frac{t^2 \pi^2}{U^2} (\beta^2 + n^2)^2 - \sigma \beta^2 \right] a_n \sin \frac{n\pi s}{U} - \\ & \frac{1}{R_0} \left(1 - 2\zeta \cos 2\pi \frac{s}{U} \right) \beta^2 \sum_1^{\infty} A_n \sin \frac{n\pi s}{U} + \\ & 2\tau \beta \sum_1^{\infty} n b_n \cos \frac{n\pi s}{U} \sum_1^{\infty} \left[\frac{E}{1 - \nu^2} \frac{t^2 \pi^2}{U^2} (\beta^2 + n^2)^2 - \right. \\ & \left. \sigma \beta^2 \right] b_n \sin \frac{n\pi s}{U} - \frac{1}{R_0} \left(1 - 2\zeta \cos 2\pi \frac{s}{U} \right) \beta^2 \sum_1^{\infty} B_n \sin \frac{n\pi s}{U} - \\ & 2\tau \beta \sum_1^{\infty} n a_n \cos \frac{n\pi s}{U} = 0 \quad (7.2') \end{aligned}$$

In order to make of these two equations - in which we visualize A_n , B_n inserted according to (7.2) - an equation system for determination of the a_n , b_n , we must replace the factors $\cos \frac{n\pi s}{U}$ by $\sin \frac{n\pi s}{U}$, by means of an additional Fourier expression. We first rename the summation index in the last term

$$2\tau\beta \sum m b_m \cos \frac{m\pi s}{U}$$

The Fourier coefficients of the function $\cos \frac{n\pi s}{U}$ are obtained in the known manner from the relation

$$\cos m \frac{\pi s}{U} = \sum_n \frac{\int_0^U \cos \frac{m\pi s}{U} \sin \frac{n\pi s}{U} ds}{\int_0^U \sin^2 \frac{m\pi s}{U} ds} \sin \frac{n\pi s}{U} = \frac{4}{\pi} \sum_n \frac{n}{n^2 - m^2} \sin \frac{n\pi s}{U}$$

$$[m \text{ or } n \text{ odd}]$$

The last term of the first equation (7.2') becomes therefore

$$\frac{8}{\pi} \tau\beta \sum_m \sum_n b_m \frac{mn}{n^2 - m^2} \sin \frac{n\pi s}{U}$$

By exchanging the sequence of the two summations, introducing (7.2), putting the factors of $\sin \frac{n\pi s}{U}$ individually equal to zero and enlarging them by $\frac{4}{\beta^2 \sigma^* \sqrt{\omega}}$, there results from the first equation (7.2')

$$\left(\frac{1}{k_n} - \lambda\right) a_n + a_n (k_n + \zeta^2 (k_{n+2} + k_{n-2})) - \zeta a_{n-2} (k_n + k_{n-2}) -$$

$$\zeta a_{n+2} (k_n + k_{n+2}) + \zeta^2 k_{n-2} a_{n-4} + \zeta^2 k_{n+2} a_{n+4} + \frac{32}{\pi} \frac{1}{\beta} \frac{\tau}{\sigma^* \sqrt{\omega}} \sum_m \frac{m n b_m}{n^2 - m^2} = 0 \quad (7.3)$$

From the second equation (7.2') follows an identically constructed equation in which a_i and b_i appear exchanged compared to (7.3),

and the sign of the last term is reversed. One now readily recognizes that one does not need both equation systems for determination of the critical value of τ .¹⁵ For the second system is transformed into the first if one multiplies the first, third, fifth . . . equation by (-1) and puts

$$b_n = (-1)^n a_n, \quad b_m = (-1)^m a_m;$$

both systems have, therefore, the same eigen-value τ_k^2 .

In order to clarify the further considerations, we introduce besides the abbreviations (2.5') and $\mu = \frac{8\tau}{\sigma\sqrt{\omega}}$ additionally certain quantities $s_{\mu\nu}$ by the equations

$$\left. \begin{aligned} s_{11} &= \frac{\pi}{4} \beta \left[\frac{1}{k_1} + (1 + \zeta)^2 k_1 + \zeta^2 k_3 - \lambda \right] \\ s_{22} &= \frac{\pi}{4} \beta \left[\frac{1}{k_2} + k_2 + \zeta^2 k_4 - \lambda \right] \\ s_{33} &= \frac{\pi}{4} \beta \left[\frac{1}{k_3} + k_3 + \zeta^2 (k_1 + k_5) - \lambda \right] \\ s_{13} &= -\zeta \frac{\pi}{4} \beta \left[(1 + \zeta) k_1 + k_3 \right] \\ s_{15} &= \zeta^2 \frac{\pi}{4} \beta k_3 \\ s_{35} &= -\zeta \frac{\pi}{4} \beta \left[k_3 + k_5 \right], \text{ etc.} \end{aligned} \right\} \quad (7.4)$$

¹⁵The respective consideration also is that of A. Kromm, Luftf.-Forschg. 1938, p. 521.

Then the coefficient scheme of the equation system (7.3) is

$$\begin{array}{ccccccc}
 a_1 & b_2 & a_3 & b_4 & a_5 & b_6 & a_7 \\
 \hline
 s_{11} & \frac{1.2}{3}\mu & s_{13} & \frac{1.4}{15}\mu & s_{15} & \frac{1.6}{35}\mu & 0 \dots \\
 \frac{2.1}{3}\mu & s_{22} & \frac{2.3}{5}\mu & s_{23} & -\frac{2.5}{21}\mu & s_{26} & -\frac{2.7}{45}\mu \dots \\
 s_{13} & -\frac{3.2}{5}\mu & s_{33} & \frac{3.4}{7}\mu & s_{35} & \frac{3.6}{27}\mu & s_{37} \dots \\
 \frac{4.1}{15}\mu & s_{24} & \frac{4.3}{7}\mu & s_{44} & -\frac{4.5}{9}\mu & s_{46} & -\frac{4.7}{33}\mu \dots
 \end{array}
 \quad \left[\mu = \frac{8\tau}{\sigma\sqrt{\omega}} \right]
 \quad (7.5)$$

We investigate first the case of pure shear load ($\lambda = 0$). Precisely as in the case of the complete cylinder, here also the lowest buckling load pertains to the buckling form with the "steep" waves, that is, to small β -values; as there, we need use, therefore, only a small initial portion (from the system (7.5)) and can make appreciable simplifications within the coefficients themselves.

We obtain a first approximation for the critical values of μ from the determinant of buckling of the two first equations

$$\mu_I^2 = \frac{9}{4} s_{11} s_{22} \quad (7.6)$$

The pertaining wave length results from

$$\frac{d(\mu_I^2)}{d(\beta^2)} = \frac{9}{4} \frac{d}{d\beta^2} (s_{11} s_{22}) = 0 \quad (7.7)$$

Because of

$$\beta \ll 1$$

the evaluation of equation (7.7) is very simple. From

$$k_1 = \beta^2\sqrt{\omega}, \quad k_2 = \frac{1}{16} \beta^2\sqrt{\omega}, \quad k_3 = \frac{1}{81} \beta^2\sqrt{\omega}, \quad k_4 = \frac{1}{256} \beta^2\sqrt{\omega}$$

it follows first that the terms $\xi^2 k_3$, $\xi^2 k_4$ cannot be of importance compared to their neighboring terms; if we, moreover, regard k , instead of β^2 , as the unknown to be determined, equation (7.7) is reduced to the simple problem of determining the minimum of the function

$$\frac{1}{16} k_1 \left(\frac{1}{k_1} + (1 + \xi)^2 k_1 \right) \left(\frac{16}{k_1} + \frac{k_1}{16} \right) \equiv \left(\frac{1}{k_1} + (1 + \xi)^2 k_1 \right) \left(1 + \frac{k_1^2}{256} \right)$$

as a function of k_1 . One finds that the second summand of the second parentheses can be omitted, since the differentiation of the first parentheses yields $\frac{1}{k_1^2} - (1 + \xi)^2$, which becomes zero for

$$k_1 \equiv \beta^2 \sqrt{\omega} = \frac{1}{1 + \xi} \quad (7.7')$$

so that $k_1^2/256$ in the neighborhood of the extreme is indeed vanishingly small. With (7.7'), and omission of the small terms, equation (7.6) reads

$$\mu_I^2 = \frac{9}{4} \frac{\pi^2}{16} \beta^2 \left(\frac{1}{k_1} + (1 + \xi)^2 k_1 \right) \frac{16}{\beta^2} \frac{1}{\sqrt{\omega}} = \frac{9\pi^2}{2\sqrt{\omega}} (1 + \xi) \quad (7.8)$$

For the critical shearing stress $\tau = \frac{\sigma^* \sqrt{\omega}}{8} \mu = \frac{E}{\sqrt{12(1 - \nu^2)}} \frac{t\pi}{U} \frac{\mu}{2}$

there results therefore

$$\tau = \frac{3\pi}{2\sqrt{2}} \frac{E}{(1 - \nu^2)^{3/4}} \left(\frac{t\pi}{U} \right)^{3/2} \sqrt{1 + \xi} = 0.55 E \left(\frac{t\pi}{U} \right)^{3/2} \sqrt{1 + \xi} \quad (7.9)$$

wherein, we can, of course, again write $1/R_0$ for π/U .

8. CRITICAL SHEAR LOAD FOR THE HALF CYLINDER
OF ARBITRARY CURVATURE

The formula (7.9) has exactly the same construction as the formula (6.9') for the complete cylinder. However, it is not worth while to refine it or even to discuss it more closely because it represents, after all, among the formulas obtainable from the equation system (7.5) only the first approximation. Calculation of the second approximation does not offer any difficulties since one may now utilize the experiences made in determining the first.

From the requirement that the determinant

$$\begin{vmatrix} s_{11} & \frac{2}{3}\mu & s_{13} \\ \frac{2}{3}\mu & s_{22} & -\frac{5}{6}\mu \\ s_{13} & -\frac{5}{6}\mu & s_{33} \end{vmatrix}$$

has to vanish, the second approximation for the critical value of μ^2 becomes

$$\mu_{II}^2 = \frac{(s_{11}s_{33} - s_{13}^2)s_{22}}{\frac{4}{9}s_{33} + \frac{36}{25}s_{11} + \frac{8}{5}s_{13}} = \frac{9}{4} s_{11}s_{22} \frac{1 - \frac{s_{13}^2}{s_{11}s_{33}}}{1 + \frac{81}{25} \frac{s_{11}}{s_{33}} + \frac{18}{5} \frac{s_{13}}{s_{33}}} = \mu_I^2 \quad (8.1)$$

We can put for the s_{ik} :

$$\begin{aligned} s_{11} &= \frac{\pi}{4} \beta \left(\frac{1}{k_1} + (1 + \xi)k_1 \right), & s_{13} &= -\xi \frac{\pi}{4} \beta (1 + \xi)k_1, \\ s_{22} &= \frac{\pi}{4} \frac{\beta}{k_2} = \frac{4\pi\beta}{k_1}, & s_{33} &= \frac{\pi}{4} \frac{\beta}{k_3} = \frac{81\pi}{4} \frac{\beta}{k_1} \end{aligned} \quad (8.1')$$

The correction terms in (8.1) thus become

$$\frac{s_{13}^2}{s_{11}s_{33}} = \frac{k_1^4}{81} \frac{\zeta^2(1+\zeta)^2}{1+(1+\zeta)^2k_1^2}, \quad \frac{81}{25} \frac{s_{11}}{s_{33}} = \frac{1}{25} \left(1 + (1+\zeta)k_1 \right),$$

$$\frac{18}{5} \frac{s_{13}}{s_{33}} = -\frac{2\zeta}{45} (1+\zeta)k_1$$

We can estimate their magnitude with equation (7.7')

$$\frac{s_{13}}{s_{11}s_{33}} \approx \zeta^2 \frac{1}{162(1+\zeta)^2}, \quad \frac{81}{25} \frac{s_{11}}{s_{33}} + \frac{18}{5} \frac{s_{13}}{s_{33}} \approx \frac{2}{25} - \frac{2\zeta}{45(1+\zeta)} \quad (8.2)$$

One can see that for determining the locus β^2 of the extreme value the numerator correction term may be eliminated; the denominator terms also may be cancelled, since the term $2/25$ is a constant, and the remainder is so small that it is again of no importance. Thus $\frac{dC}{d\beta^2} = 0$ and for determining k_1 and β^2 , respectively, we retain the simple formula (7.7'):

$$k_1 = \frac{1}{1+\zeta}$$

Thus equal signs appear in equation (8.2); by substituting equation (8.2) in equation (8.1), one obtains for the correction C to be made in the first approximation μ_I^2 the expression

$$C = \frac{1 - \frac{\zeta^2}{162(1+\zeta)^2}}{1 + \frac{2}{25} - \frac{2\zeta}{45(1+\zeta)}} = \frac{25}{27} \frac{1 - \frac{\zeta^2}{162(1+\zeta)^2}}{1 - \frac{10}{243} \frac{\zeta}{1+\zeta}} = \frac{25}{27} C_1 \quad (8.3)$$

The quantities $\frac{10}{243} \frac{\zeta}{1+\zeta}$ and $\frac{\zeta^2}{162(1+\zeta)^2}$ affect $\mu_{II} \sim \sqrt{C}$ in the extreme case $\zeta = 0.5$ by less than 1 percent. The only essential difference between the two approximations μ_{II} and μ_I consists, therefore, in the numerical factor $\sqrt{\frac{25}{27}}$ which causes reduction by about

2 percent; thus a further approximation can not bring any further essential improvement.¹⁶ If we now introduce $\tau = \frac{\sigma^* \sqrt{\omega}}{\xi} \mu$ in

$$\mu_{II} = \sqrt{\frac{25}{27}} C_2 \mu_I \quad \text{with}$$

$$C_2 = \sqrt{C_1} = 1 + \frac{5}{243} \frac{\xi}{1 + \xi} - \frac{\xi^2}{324(1 + \xi)^2} \approx 1 + \frac{1}{50} \frac{\xi}{1 + \xi} \quad (8.4)$$

and $\mu_I = \frac{3\pi}{\sqrt{2}} \frac{\sqrt{1 + \xi}}{4\sqrt{\omega}}$, we obtain

$$\tau = \frac{5\pi}{2\sqrt{6}} \frac{E}{(12(1 - \nu^2))^{3/2}} \left(\frac{t\pi}{U}\right)^{3/2} C_2 \sqrt{1 + \xi};$$

for $\nu = 0.3$, thus

$$\frac{\tau}{E} = 0.533 \left(\frac{t\pi}{U}\right)^{3/2} C_2 \sqrt{1 + \xi} \quad (8.4')$$

For the special case $\xi = 0$ this formula is transformed, as it must be, into Kromm's formula (valid for arbitrary U/R_0):

$$\tau = 1.67 E \frac{t}{U} \sqrt{\frac{t}{R_0}} \quad (8.4'')$$

If we introduce instead of U the spar height $2b$, (8.4') reads

$$\tau = 1.50 \left(\frac{t}{2b}\right) \bar{\Psi}(\epsilon) \quad \text{with} \quad \bar{\Psi}(\epsilon) = \varphi(\xi) = (\epsilon'(\xi))^{3/2} C_2 (1 + \xi)^{1/2} \quad (8.5)$$

wherein $\bar{\Psi}$ is a function of the axis ratio which in the special case of the oval (4.5) is obtained from the curves of figure 4. It is drawn in a dashed line in figure 11.

For an arbitrary oval represented by the general expression (4.1), the calculation is hardly different from that for the oval represented

¹⁶For the special case $\xi = 0$ A. Kromm, Luftf.-Forschg. 1938, p. 525, has proved by a check calculation that $\mu_{III} = \mu_{II}$.

by (4.5). The quantities s_{ik} in the more general case $\zeta_4, \zeta_6 \neq 0$ are now, except for the factor $\frac{\pi}{4}\beta$, the quantities denoted by (ik)

(5.5). The considerations made so far show that one may cancel a great deal; one obtains

$$\left. \begin{aligned} s_{11} &= \frac{\pi\beta}{4} \left(\frac{1}{k_1} + (1 - \zeta_2)^2 k_1 \right) \\ s_{22} &= \frac{\pi}{4} \beta \frac{16}{k_1} \\ s_{33} &= \frac{\pi}{4} \beta \frac{81}{k_1} \\ s_{13} &= \frac{\pi}{4} \beta (\zeta_2 - \zeta_4) (1 - \zeta_2) k_1 \end{aligned} \right\} \quad (8.6)$$

and hence

$$\mu_{II}^2 = \frac{9}{4} s_{11} s_{22} \frac{25}{27} \frac{1 - \frac{1}{162} \left(\frac{\zeta_2 - \zeta_4}{1 - \zeta_2} \right)}{1 + \frac{10}{243} \frac{\zeta_2 - \zeta_4}{1 - \zeta_2}} \approx \frac{25}{12} s_{11} s_{22} \left(1 - \frac{1}{25} \frac{\zeta_2 - \zeta_4}{1 - \zeta_2} \right) \quad (8.7)$$

that is,

$$\frac{\tau}{E} = 0.533 \left(\frac{t\pi}{U} \right)^{3/2} (1 - \zeta_2)^{1/2} \left(1 - \frac{1}{50} \frac{\zeta_2 - \zeta_4}{1 - \zeta_2} \right) \quad (8.7')$$

One can see that precisely as for the closed cylinder the sixth harmonic has no significance whatsoever, and that the difference compared to the special oval $\zeta_2 = -\zeta, \zeta_4 = \zeta_2 = 0$ consists only in a modification of the unimportant last correction term. For the ellipse-like ovals we considered especially, the change means a very slight increase of the critical values, since according to equation (4.9) the factor of the correction term assumes in the limiting case $2\zeta = 1$ the

value $\frac{2.1}{1.75} = 1.2$ instead of $\frac{1}{1.5} = 0.67$ according to equation (4.5).

More important for the comparison between equations (8.7') and (8.4') is the fact that the factor $1 - \xi_2$ for equal ξ -value is larger this time. For, according to equation (4.9), $\xi_2 = -1.5\xi$, and if one would simulate the oval still more to an ellipse (compare p. 16), one would obtain coefficients still somewhat larger (1.6 for a curvature expression starting from $\sin^8 \frac{\pi s}{U}$, etc.).

The physical significance of this is that among the ovals of equal length U (and therewith of equal mean curvature $1/R_0$) the buckling stiffness increases with the nonuniformity of the curvature variation. Of course this applies only if the spar supports the point of minimum curvature. If one places the support at the ends of the large axis ($\xi_2 > 0$), it becomes more and more ineffective with increasing "ellipticity." We compare for instance the equations (6.9) and (8.7') with one another by substituting, according to equation (4.9), ξ_2 and ξ_4 ; in the extreme case $2\xi = 1$ the numerical factors

$$0.291 \times 0.72 = 0.21 \quad \text{for the unsupported cylinder}$$

$$\text{and} \quad 0.533 \sqrt{0.25 \times 0.99} = 0.26 \quad \text{for the supported cylinder}$$

result which now differ only slightly. If one intensifies the ellipticity by a curvature expression of still more terms (thus with still larger ξ_2 values), the numerical values approach one another still more.

If we again turn from the parameters appearing in equation (8.7') U and ξ to $\frac{2b}{t}$ and ϵ , we may write

$$\tau = 1.5E \left(\frac{t}{2b} \right)^{3/2} \bar{\psi}(\epsilon) \quad (8.8)$$

with $\bar{\psi}(\epsilon) = (\epsilon'(\xi))^{3/2} (1 - \xi_2)^{1/2} \left(1 - \frac{1}{50} \frac{\xi_2 - \xi_4}{1 - \xi_2} \right)$ this time determined

from figure 7. The result of the calculation is also plotted in figure 11 (dashed). Like the curves ψ pertaining to the closed cylinder, the two curves $\bar{\psi}(\xi)$ for the ovals (4.5) and (4.9) also lie very close together; that means that, as before in the compression case, the buckling load of the half oval supported at the "long sides" is almost not at all dependent on the exact form of the oval, but only on the two parameter values $\frac{2b}{t}$ and $\frac{b}{a} = \epsilon$. Since the relations (8.4),

(8.7'), etc., as the comparison of the exact Flügge formula with formula (6.9'), contain (due to the simplifications made in the initial equations) possibilities of error of an order of magnitude of 15 percent, it is sufficient to replace the two $\bar{\Psi}$ curves found here (as well as all others possible) by the curve drawn solidly in figure 11. This approximation curve has the simple equation

$$\bar{\Psi}_0(\epsilon) = 3.1\epsilon - 1.3\epsilon^2 - 0.3 \quad (8.9)$$

To repeat this formula, as the result of the calculations of the two last sections, once more: the simple relation

$$\frac{\tau}{E} = \frac{t}{2b}^{3/2} (3.1\epsilon - 1.3\epsilon^2 - 0.3) \quad (8.9')$$

is valid for the critical shear load of the infinitely long statically supported half oval. It is so simple that it is not worth while to represent it by a diagram. According to its derivation its validity range is limited to the region $1 \geq \epsilon \geq 1/3$; however, most probably not too large errors will arise if it is applied for the region $\epsilon \approx \frac{1}{4}$ to $\epsilon \approx \frac{1}{5}$.

9. THE WING NOSE UNDER COMBINED SHEAR AND COMPRESSIVE LOAD

The extraordinary diversity of the buckling form makes it impossible for compression and shear buckling to "mix"; in case of an appropriate variation in load the one buckling form changes suddenly into the other, that is, the curves for the critical values in a σ -, τ -coordinate system show a break. (Compare fig. 12a.) The calculation which must be performed according to the methods indicated in section 5 shows that the critical compressive load within very wide limits is changed not at all by shear, whereas, as we shall now demonstrate, presence of compression or tension influences the critical shear load.

We start from the equation system (7.5), thus consider first again the special oval $\zeta_4 = \zeta_6 = \dots = 0$. The first approximation has again the form (7.6); however, this time the quantities s_{ik} must be understood to signify the complete expressions (7.4) (that is, with $\lambda \neq 0$). Since the presence of a small compressive force is not able to basically change the critical behavior of the cylinder, we may in the

$\sqrt{\omega}$ region which is of interest for the wing nose again substitute

$k_1 = \frac{\beta^2}{n^4} \sqrt{\omega}$ and obtain (compare also (7.8))

$$\begin{aligned} \mu_I^2 &= \frac{3}{4} \frac{\pi^2}{16} \beta^2 \left(\frac{1}{k_1} + (1 + \zeta)^2 k_1 - \lambda \right) \left(\frac{16}{k_1} - \lambda \right) \\ &= \frac{9\pi^2}{4\sqrt{\omega}} \left(\frac{1}{k_1} + (1 + \zeta)^2 k_1 - \lambda \right) \left(1 - \frac{\lambda k_1}{16} \right) \end{aligned} \quad (9.1)$$

For determining the critical β values from

$$\frac{d(\mu_I^2)}{d\beta^2} = \sqrt{\omega} \frac{d(\mu_I^2)}{dk_1} = 0$$

the small term $\frac{\lambda k_1}{16}$ may be omitted. Therewith there results for the critical value of k_1 the value (7.7') independent of λ :

$$k_1 = \beta^2 \sqrt{\omega} = \frac{1}{1 + \zeta}; \quad (9.2)$$

μ_I^2 thus becomes:

$$\mu_I^2 = \frac{9\pi^2}{2\sqrt{\omega}} \left(1 + \zeta - \frac{9}{16} \lambda + \frac{\lambda^2}{32(1 + \zeta)} \right) \quad (9.3)$$

For the plotting of μ against λ the tangential direction at the point $\lambda = 0$ is of foremost importance; there results from the simplification of (9.3) which is valid for small λ values:

$$\mu_I = \frac{3\pi}{\sqrt{2}} \frac{1}{4\sqrt{\omega}} \sqrt{1 + \zeta} \left(\lambda - \frac{9\lambda}{32(1 + \zeta)} \right) \quad (9.3')$$

Exactly as in the case of pure shear load a corresponding formula results, without considerably larger calculation expenditure, also for

the second approximation pertaining to the system (7.5). One starts again from the formula (8.1):

$$\mu_{II}^2 = \frac{9}{4} s_{11} s_{22} \frac{1 - \frac{s_{13}}{s_{11} s_{33}}}{1 + \frac{81}{25} \frac{s_{11}}{s_{33}} + \frac{18}{5} \frac{s_{13}}{s_{33}}} \quad (9.4)$$

where one has to substitute for s_{ik}

$$\left. \begin{aligned} s_{11} &= \frac{\pi}{4} \beta \left(\frac{1}{k_1} + (1 + \zeta)^2 k_1 - \lambda \right), & s_{13} &= -\zeta \frac{\pi}{4} \beta (1 + \zeta) k_1, \\ s_{22} &= 4\pi \beta \left(\frac{1}{k_1} - \frac{\lambda}{16} \right), & s_{33} &= \frac{81\pi}{4} \beta \left(\frac{1}{k_1} - \frac{\lambda}{81} \right) \end{aligned} \right\} \quad (9.5)$$

The term $s_{13}^2 / s_{11} s_{33}$ in the numerator may be omitted, as we have seen above (p. 38). Then one obtains equation (9.4) in a readily surveyable form if one multiplies throughout in the numerator and the denominator by $1 - \frac{\lambda k_1}{81}$

$$\mu_{II}^2 = \frac{9}{4} \pi^2 \beta^2 \frac{\left(\frac{1}{k_1} + (1 + \zeta)^2 k_1 - \lambda \right) \left(\frac{1}{k_1} - \frac{\lambda}{16} \right) \left(1 - \frac{\lambda k_1}{81} \right)}{\left(1 - \frac{\lambda k_1}{81} \right) + \frac{1}{25} \left(1 + (1 + \zeta)^2 k_1^2 - \lambda k_1 \right) - \frac{2\zeta}{45} (1 + \zeta) k_1^2} \quad (9.4')$$

The position of the minimum of this expression lies again close to

$$k_1 = \frac{1}{1 + \zeta}$$

If one substitutes this value, one obtains

$$\mu_{II}^2 = \frac{9}{2} \frac{\pi^2}{\sqrt{\omega}} \frac{25}{27} (1 + \zeta) \frac{\left(1 - \frac{\lambda}{2(1 + \zeta)} \right) \left(1 - \frac{\lambda}{16(1 + \zeta)} \right) \left(1 - \frac{\lambda}{81(1 + \zeta)} \right)}{1 - \frac{106}{2187} \frac{\lambda}{1 + \zeta} - \frac{10}{243} \frac{\zeta}{1 + \zeta}} \quad (9.6)$$

For the tangents at the point $\lambda = 0$ one obtains by series development (compare also equation (8.4))

$$\mu_I = \frac{5\pi}{\sqrt{6}} \frac{1}{4\sqrt{\omega}} \sqrt{1 + \xi} \left(1 + \frac{1}{50} \frac{\xi}{1 + \xi} \right) \left[1 - 0.263 \frac{\lambda}{1 + \xi} \right] \quad (9.6')$$

or written in τ and σ , with $\tau(\lambda = 0)$ denoted as τ_0

$$\tau(\lambda) = \tau_0 \left[1 - 0.263 \frac{\lambda}{1 + \xi} \right] = \tau_0 \left[1 - \frac{\sigma 2b}{Et} \frac{0.263 \sqrt{3(1 - \nu^2)}}{\epsilon'(1 + \xi)} \right] \quad (9.6'')$$

The formulas (9.6) to (9.6'') are valid for the special oval (4.5). The corresponding formulas for the general oval (4.1) differ from them by the fact that instead of $1 + \xi$ one has to write $1 - \xi_2$ and that the very small correction term $\frac{10}{243} \frac{\xi}{1 + \xi}$ or $\frac{1}{50} \frac{\xi}{1 + \xi}$, respectively, contains, instead of the second fraction, the quantity $\frac{\xi_2 - \xi_4}{1 - \xi_2}$. In

determining τ_0 it had been found that these formal differences are almost completely compensated at transition to the two parameters $2b/t$ and b/a , so that for τ_0 the expression (8.9') which is independent of the oval form resulted. With still better approximation one may replace the factor representing the influence of the longitudinal force

$$\frac{0.263 \sqrt{3(1 - \nu^2)}}{\epsilon'(1 - \xi_2)} = \frac{0.433}{\epsilon'(1 - \xi_2)} = \frac{0.433}{\psi^*(\epsilon)} \quad (9.7)$$

by a "mean" function independent of the oval form. In figure 11 the function $\psi^*(\epsilon)$ is drawn for instance for the two ovals (4.5) and (4.9). The deviations are indeed so slight that - in view of the simplified initial equations - it would be perfectly nonsensical to take them into consideration.

In place of (9.6'') we obtain therefore the formula, valid for arbitrary ovals, for the critical shear in the presence of small compressive or tensile forces (σ is as compression counted positive):

$$\tau = \tau_0 \left[1 - \frac{0.433}{\psi^*(\epsilon)} \left(\frac{\sigma}{E} \frac{2b}{t} \right) \right] \quad (9.8)$$

On the basis of the formulas (9.8) (for small λ) or, respectively, (9.6) (for larger λ values) one may sketch a diagram from which result the admissible critical pairs of values σ , τ . In figure 12 a representation was chosen which saves use of separate diagrams for determination of the critical loads τ_0 , σ_0 (everytime in the absence of the other). This is possible because of two peculiarities of the curves. The dependence of the curve branches starting from the τ -axis (shear with some compression) on the parameter $2b/t$ is such that one may factor this parameter, as equation (8.9') shows, by selection of the easily calculated abscissa

$$\frac{\tau}{E} \left(\frac{t}{2b} \right)^{3/2}$$

The remaining one-parameter curve family is then numbered according to the parameter $\epsilon = \frac{b}{a}$. The curve branches starting from the σ -axis

(compression with some shear) cannot be transformed, in a similarly simple manner, into a one-parameter family dependent only on ϵ , by selection of a suitable ordinate. Instead of this, however, these curves are independent of τ , that is, they are horizontal straight lines which need not be explicitly drawn into the diagram; both parameters on which - if the ordinate $\frac{\sigma}{E} \frac{t}{2b}$ is selected - these "curve"

branches depend can now be indicated by a simple method: one need only mark the pertaining $2b/t$ value at the intersection of the horizontal straight line and the other branch of the σ -, τ -curves numbered according to ϵ . Connecting, in addition, the points of equal $2b/t$ values, one has on the whole a diagram with two one-parameter curve families from which one may read, without additional auxiliary diagrams, the relations between the four quantities σ , τ , $2b/t$, b/a which characterize the critical state.

For use of the diagram (which of course is drawn only for the region 50-1000 of the parameter $2b/t$ that is of practical interest) the following method results: b/a is known, t is selected, $2b/t$ determined. The point of intersection of the curve pertaining to the respective b/a with the abscissa axis represents the pertaining critical τ_0 value in the absence of compression (formula (8.9')).

For smaller τ compression is permissible; it is read off from the curves indexed b/a vertically above τ . If the point comes to lie in the region covered by the dot-dashed curves, one must consider whether or not compression alone produces instability. This is the case when the intersection of the curve indexed b/a with the normal comes to lie above the dot-dashed $\frac{2b}{t}$ -curve, since the piercing points of the dot-dashed with the b/a curves indicate the critical

compression values in the absence of shear. Only if these "critical curves" are not reached, shear is permissible. The method is best made clear by an example

$$\frac{b}{a} = 0.60 \qquad \frac{2b}{t} = 400$$

$$1. \frac{\tau}{E} \left(\frac{2b}{t} \right)^{3/2} = 1.0: \text{ a compression } \frac{\sigma}{E} = 0.16 \frac{t}{2b} \text{ is permissible}$$

$$2. \frac{\tau}{E} \left(\frac{2b}{t} \right)^{3/2} = 0.94: \text{ a compression } \frac{\sigma}{E} = 0.29 \frac{t}{2b} \text{ is permissible}$$

since the dot-dashed curve $\frac{2b}{t} = 400$ lies higher up;

3. $\frac{\tau}{E} \left(\frac{2b}{t} \right)^{3/2} = 0.80$. One would read off $\frac{\sigma}{E} = 0.57 \frac{t}{2b}$, but this value is not admissible, since at $\frac{\sigma}{E} = 0.34 \frac{t}{2b}$ lies the critical value for compression alone.

If the permissible compressive and shear loads prescribed by the problem lie higher (lower) than the critical ones found from the diagram, the reading must be repeated with a larger (smaller) t value.

10. VALIDITY LIMITS - UNSYMMETRICAL OVALS

UNDER BENDING - SUMMARY

The report investigates the stability of the cylindrical shell of infinite length and variable curvature simply supported at the longitudinal edges in case of loading by longitudinal compression and shear. Starting from the cylinder equations (2.1) which are reduced to the "essential" terms, the formulas are developed first in such a general form that the calculation may be performed for a shell of arbitrary curvature (representable by a Fourier polynomial) and of arbitrary opening angle and then be numerically evaluated for the special ovals (4.5), (4.8), and (4.9) with the opening angle π (wing nose). It is found that the shell under compression shows an entirely different behavior from that of the circular-cylindrical shell (the oval shell of equal mean curvature buckles very much earlier), that, however, in case of shear loading - where almost axis-parallel wave crests originate - the variability of the curvature has only little effect. Performance of the calculation is merely more troublesome in case of compression than

in case of shear, because the equation system to be solved in the parameter region of importance for the wing nose converges there very much more slowly. On the other hand, the result in case of compression (fig. 10) has a very much wider range of application - since the waves in longitudinal direction are short, the results are, with good approximation, valid also for the cylinder of finite length. For shear, in contrast, presupposition of great cylinder length is essential - if, due to the transverse reinforcements, the oblique waves can no longer go around without interference, a constraint results which sometimes increases the critical load to a multiple.¹⁷ It is difficult to tell by how much one may, however, assume that it amounts to less than for the circular cylinder because for shortened longitudinal-wave length the variability of curvature (insignificant for the cylinder of infinite length) must have a reducing effect, as in case of compression. Besides, for the breaking load one may count on a considerable reserve also in case of compression; noticeable buckling originates at first only in the slightly curved part, without much affecting the strongly curved part, so that a considerable amount of load may be shifted to the latter before the entire shell collapses. (The shell of strongly variable curvature behaves similarly to a plate stiffened by longitudinal supports above the buckling limit.)

The diagrams and formulas have been calculated under the assumption that the ovals are symmetrical to the large axis, that they show the opening angle $\varphi_0 = \pi$, and that they undergo compression or torsional load. However, they may be used also when the ovals are unsymmetrical, have an opening angle different from π , and undergo bending with or without transverse force. The investigations of section 5 (compare fig. 9) have demonstrated that the two halves of the oval show almost no mutual influence across the strongly curved nose. One calculates, therefore, an unsymmetrical oval under compression by mirroring the flatter one of the two halves (as the more endangered one) at the large axis and taking the dimensions of this substitute oval as a basis for the calculation. If the opening angle is $\varphi_0 \neq \pi$, one provides an equivalent half oval by slightly modifying the more endangered half of the "apexes" so that there the tangent is vertical to the end tangent at the flattest place and mirrors the thus originating quarter oval; this is, as an approximation, permissible, because the region of smallest curvature is of foremost importance for the critical behavior so that a slight change in the region of greatest curvature (which practically remains unbuckled) cannot essentially affect that behavior. If, finally, the wing nose does undergo, not pure compression (bending of the wing about the vertical axis), but bending about the large axis,

¹⁷Compare the circular-cylinder results of Kromm, Jahrbuch 1940 der deutschen Luftfahrtforschung, p. 1832.

one mirrors the compressed half and calculates the thus originating oval as if the maximum bending-compressive stress were uniformly distributed. With this method, one is on the safe side without calculating very unfavorably: for even for the circular cylinder the critical bending stress is not very much higher than the critical compressive stress,¹⁸ and for the oval the difference is a great deal slighter still, due to the width of the pressure zone (which is moreover the least curved).

Transverse forces will, in general, not affect the stability of the wing nose. Vertical forces will not do so because the part transferred by the wing nose is smaller and, moreover, reaches its maximum value in the strongly curved portion, and horizontal ones will not, because the stresses in the nose must change the sign and thus can generally not be large. Therefore only the torsion remains as essential shearing load. In case of combination of torsion and bending about the horizontal axis, the diagram 12 serves for the half under compression. For the half under tension (which sometimes, nevertheless, can be the more endangered one because it usually is the flatter half) we obtain from (8.9) and (9.5) the approximation formula

$$\tau = \tau_0 \left(1 + \frac{0.433}{\psi^*(\epsilon)} \left(\frac{\sigma}{E} \frac{2b}{t} \right) \right)$$

in which σ , as tension, has to be inserted as positive. The quantity $\psi^*(\epsilon)$ is taken from figure 11 - for rough calculations one may put $\psi^* = 1$.

We add a compilation of the abbreviations and of the end formulas important for practical use. (Compare also fig. 2.)

Abbreviations:

$$\beta = \frac{U}{\lambda} \quad (2.3)$$

$$\sqrt{\omega} = \frac{\sqrt{12(1 - \nu^2)}}{\pi^2} \frac{U^2}{R_0 t}, \quad \sigma^* = \frac{E}{1 - \nu^2} \frac{\pi^2}{3} \frac{t^2}{U^2} \quad (2.5)$$

$$\frac{1}{k_n} = \frac{1}{\sqrt{\omega}} \frac{(\beta^2 + \pi^2)^2}{\beta^2}, \quad \lambda = 2\sqrt{3(1 - \nu^2)} \frac{\sigma}{E} \frac{R_0}{t}, \quad \mu = 4\sqrt{3(1 - \nu^2)} \frac{\tau}{E} \frac{R_0}{t} \quad (2.5')$$

¹⁸Flugge, Ing.-Arch. 3, 1932, p. 501 ff.

For the half cylinder $U = R_0\pi$ and thus

$$\sqrt{\omega} = \frac{\sqrt{12(1 - \nu^2)}}{\pi} \frac{U}{t} = 2\sqrt{3(1 - \nu^2)} \frac{R_0}{t} = \sqrt{3(1 - \nu^2)} \epsilon \frac{2b}{t}$$

Formulations:

for the curvature:

$$\frac{1}{R} = \frac{1}{R_0} \left(1 + 2\xi_2' \cos \frac{2\pi s}{U} + 2\xi_4 \cos \frac{4\pi s}{U} + \dots + 2\xi_{2r^*} \cos \frac{2r^*\pi s}{U} \right) \quad (4.1)$$

for the unknowns w and ϕ :

(a) for compression

$$\left. \begin{aligned} w &= \sum_A^{\infty} a_n \sin \frac{\pi x}{l} \sin \frac{n\pi s}{U} \\ \phi &= \sum_A^{\infty} A_n \sin \frac{\pi x}{l} \sin \frac{n\pi s}{U} \end{aligned} \right\} \quad (2.2)$$

(b) for shear (closed shell)

$$\left. \begin{aligned} w &= \sum_{-\infty}^{\infty} a_n \sin \frac{\pi}{U} (\beta x + ns) \\ \phi &= \sum_{-\infty}^{\infty} A_n \sin \frac{\pi}{U} (\beta x + ns) \end{aligned} \right\} \quad (6.2)$$

(c) for shear (shell segment)

$$\left. \begin{aligned} w &= \sum_{n=1,3,5}^{\infty} a_n \sin \frac{\pi x}{l} \sin \frac{n\pi s}{U} + \sum_{n=2,4,6}^{\infty} b_n \cos \frac{\pi x}{l} \sin \frac{n\pi s}{U} \\ \phi &= \sum_{n=1,3,5}^{\infty} A_n \sin \frac{\pi x}{l} \sin \frac{n\pi s}{U} + \sum_{n=2,4,6}^{\infty} B_n \cos \frac{\pi x}{l} \sin \frac{n\pi s}{U} \end{aligned} \right\} \quad (7.1)$$

The critical loads are represented as functions of the two parameters $2b/t$ and b/a (compare fig. 1) in figures 10 (pure compression)

and 12 (shear and compression). For pure shear approximation formulas can be given:

(a) complete cylinder

$$\frac{\tau}{E} = \left(\frac{t}{2b}\right)^{3/2} (1.34\epsilon - 0.37\epsilon^2 - 0.25) \quad (6.9'')$$

(b) half cylinder

$$\frac{\tau}{E} = \left(\frac{t}{2b}\right)^{3/2} (3.1\epsilon - 1.3\epsilon^2 - 0.3) \quad (8.9')$$

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National Advisory Committee
for Aeronautics

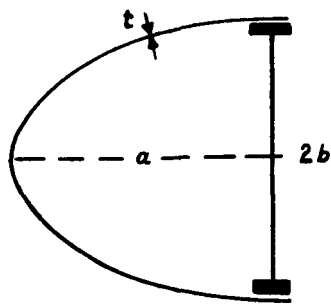


Figure 1.- Shell of variable curvature (wing nose).

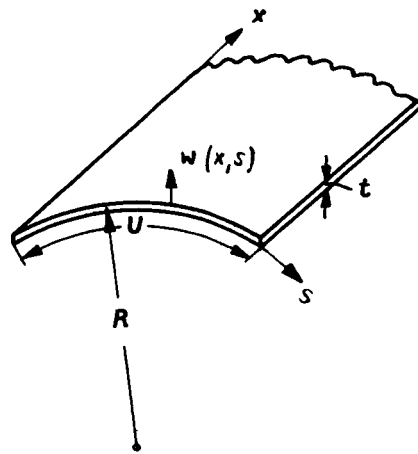


Figure 2.- Shell segment with dimensions and coordinates.

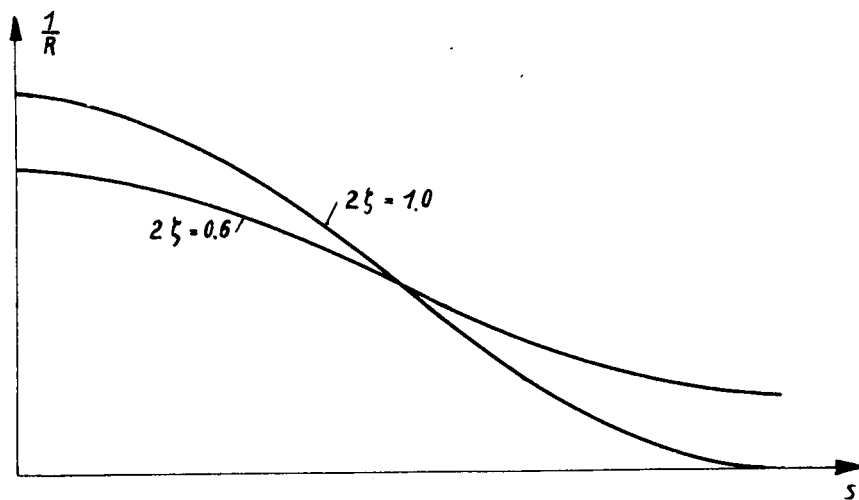


Figure 3(a).- Curvature variation over the developed width for two ovals of the type (4.5).

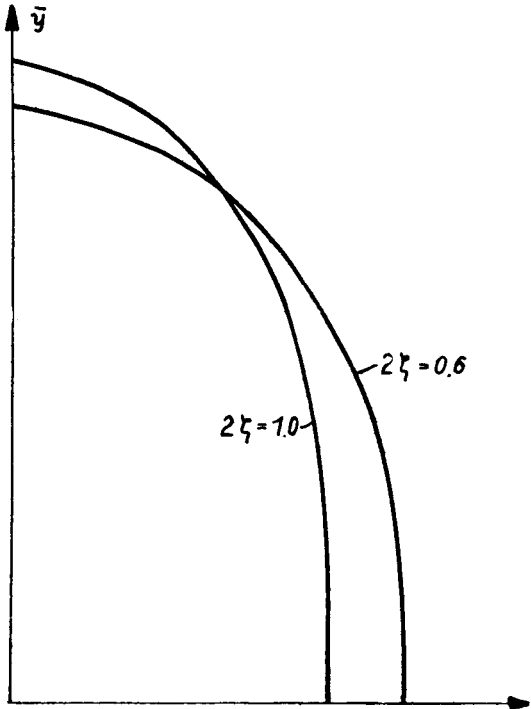


Figure 3(b).- Oval forms for the curvature expression (4.5).

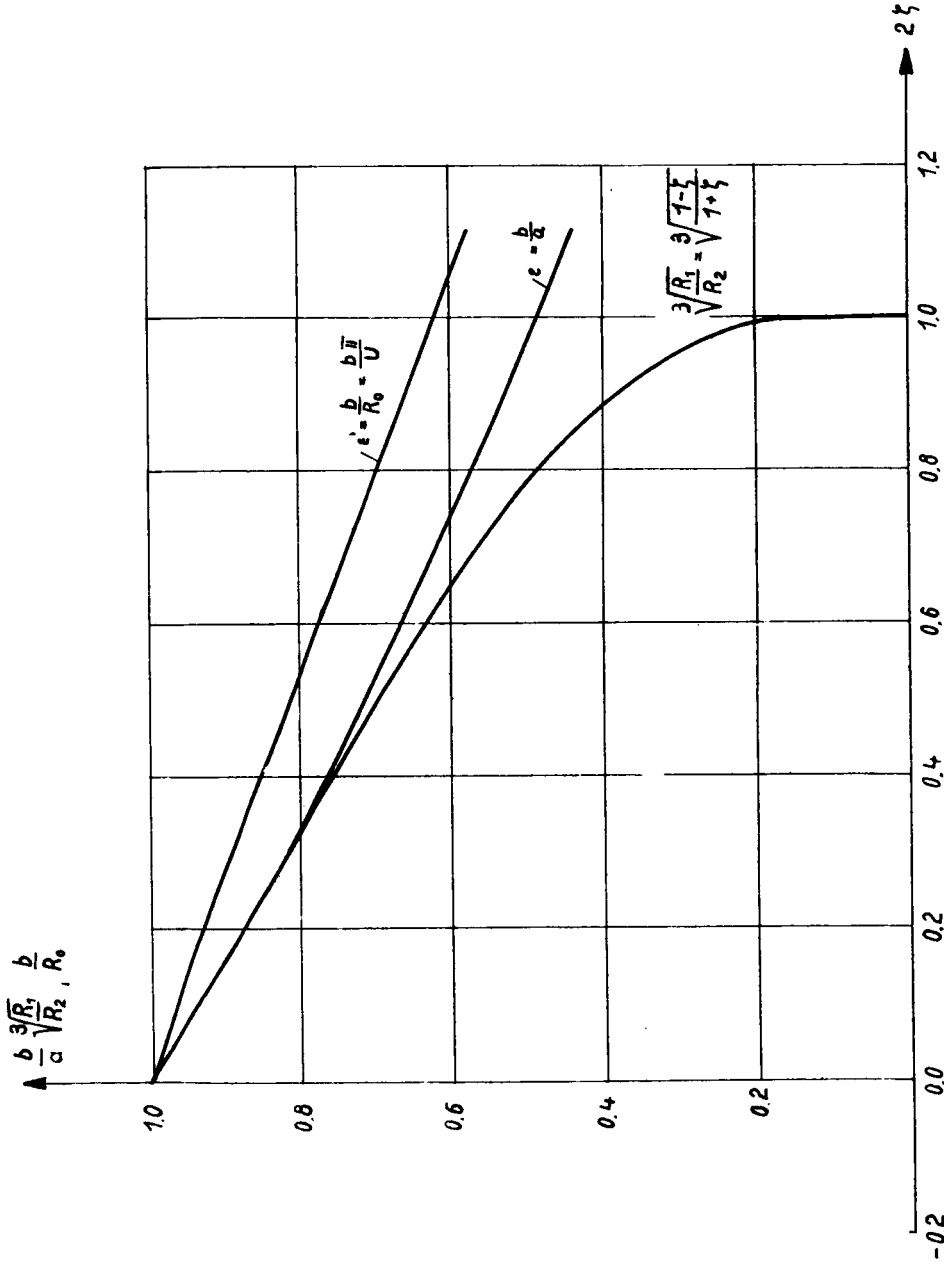


Figure 4.- Expression (4.5) for conversion of ξ to the parameters ϵ and ϵ' . $\sqrt[3]{\frac{R_1}{R_2}}$ is plotted for comparison which for an ellipse coincides with $\epsilon = \frac{b}{a}$ (R_1, R_2 are the apex curvature radii).

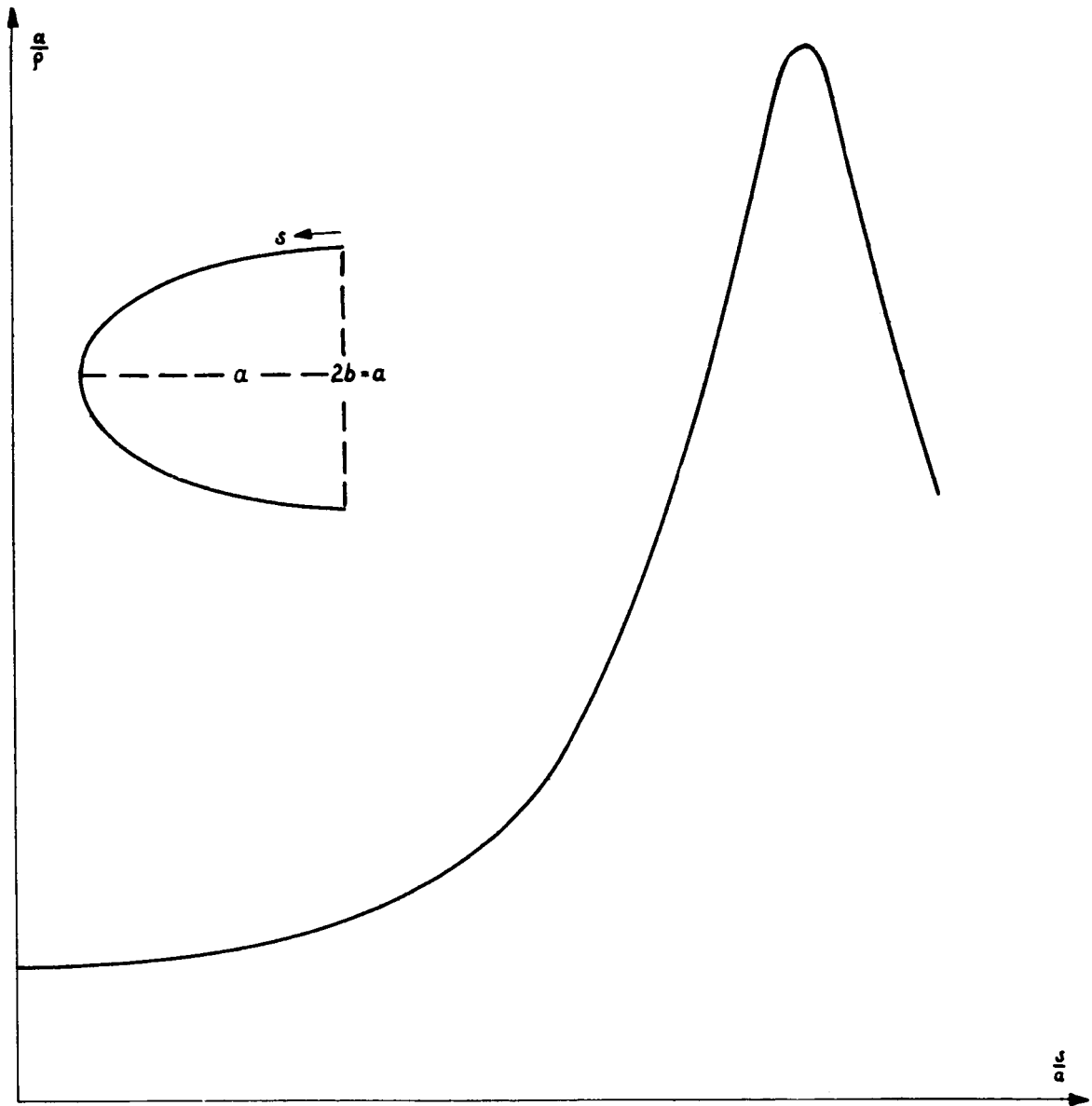


Figure 5.- Curvature variation $\frac{a}{\rho(s)}$ of the ellipse $\frac{b}{a} = \frac{1}{2}$ over the arc length s .

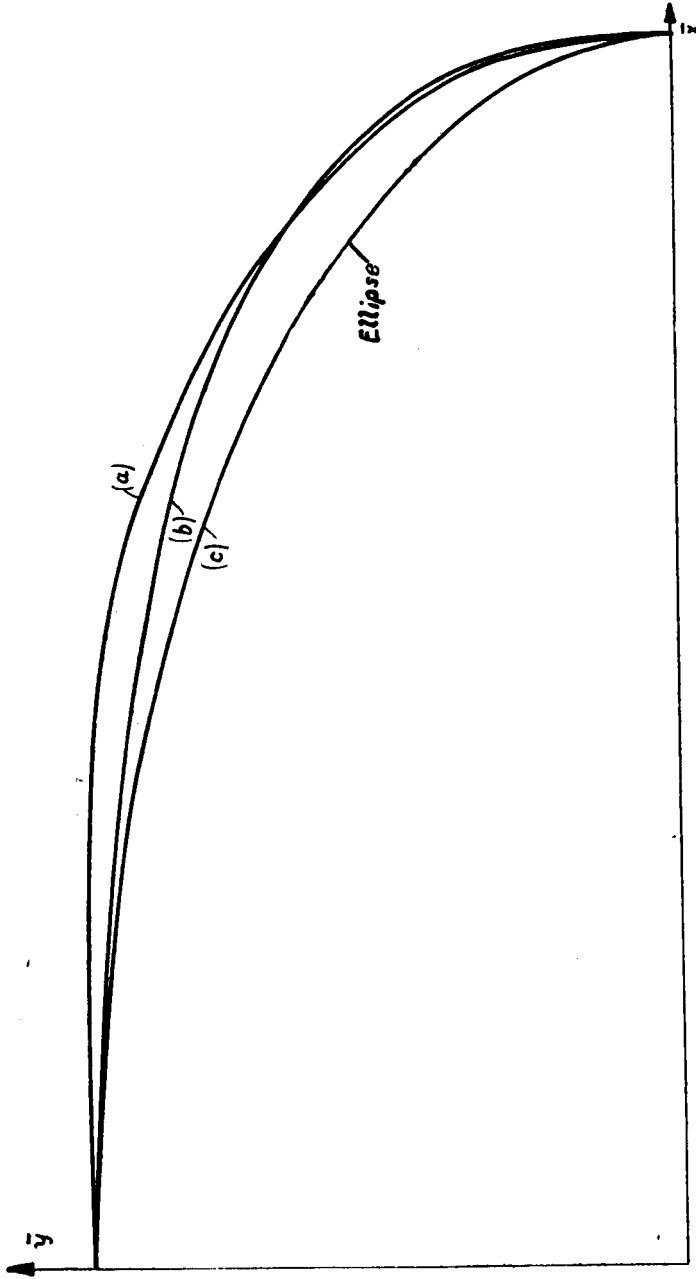


Figure 6.- Three ovals of the axis ratio 1/2.

$$(a) \frac{1}{R} = \frac{1}{R_0} \left[1 - \cos \frac{2\pi s}{U} \right]$$

$$(b) \frac{1}{R} = \frac{1}{R_0} \left[1 - 0.8 \left(\frac{4}{3} \cos \frac{2\pi s}{U} - \frac{1}{3} \cos \frac{4\pi s}{U} \right) \right]$$

(c) Ellipse.

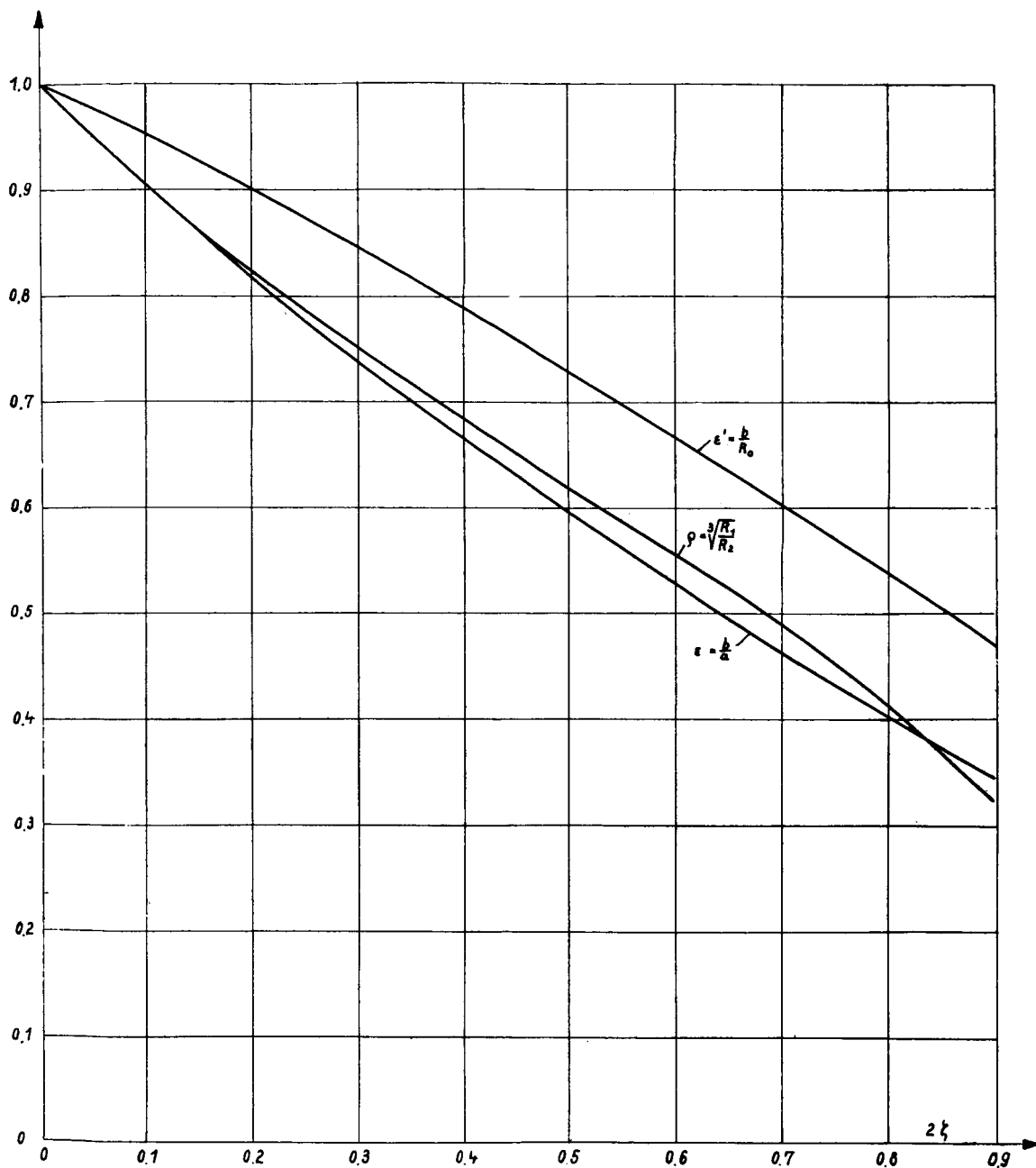


Figure 7.- Expression (4.9). For conversion of the parameter ξ to $\epsilon = \frac{b}{a}$ (axis ratio) and $\epsilon' = \frac{b}{R_0} = \frac{b\pi}{U}$ (ratio between small axis and mean curvature radius), R_1, R_2 are the apex curvature radii.

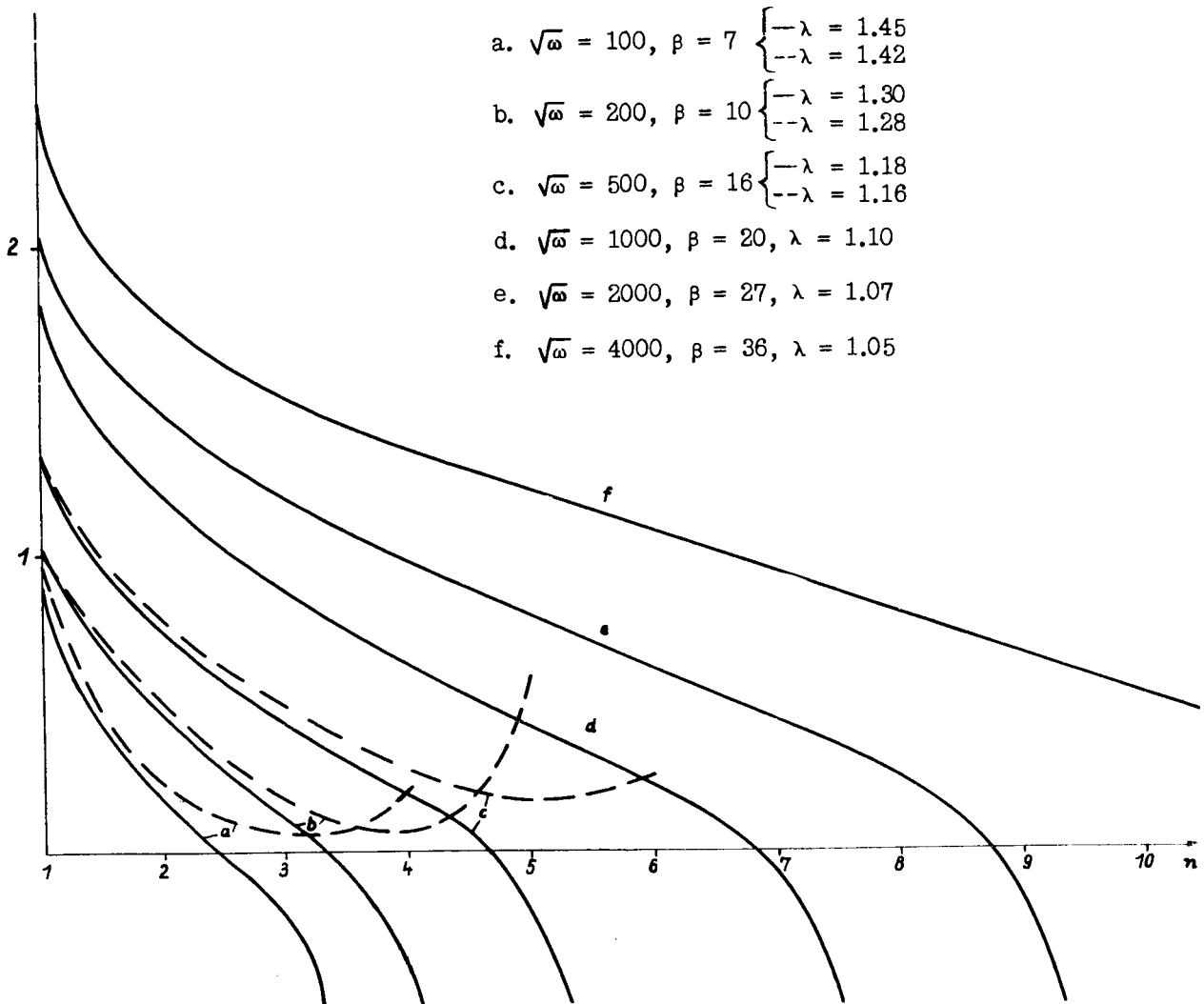


Figure 8.- For determination of the zeros of the determinant (5.2) with the aid of Gauss' algorithm. Curvature expression (4.5); $2\zeta = 0.5$.

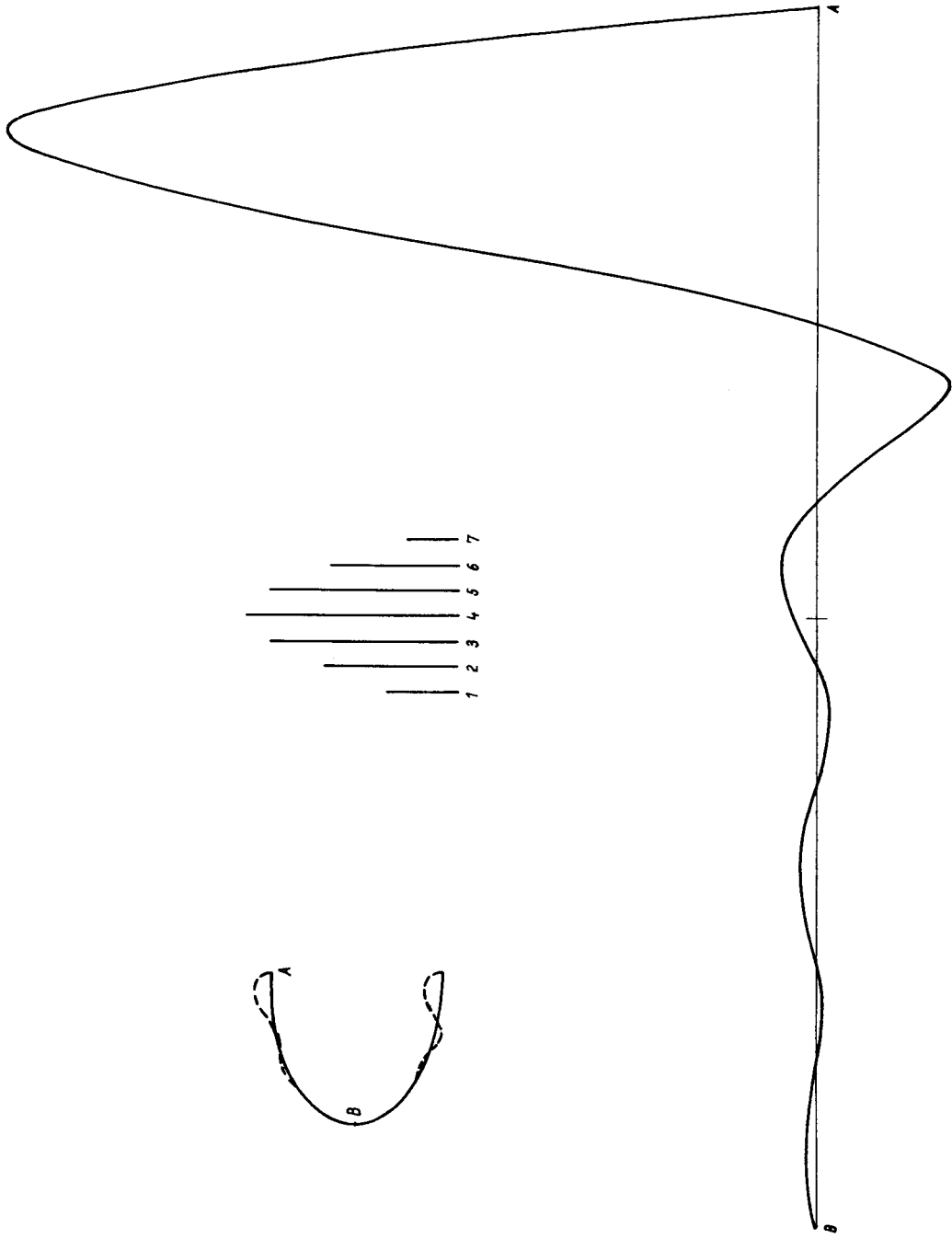


Figure 9.- Compression buckling profile plotted over half the developed width $U/2$ of a half shell; shown in the secondary figure are the amplitudes of the seven harmonics of which the buckling pattern is composed (curvature expression (4.5)).

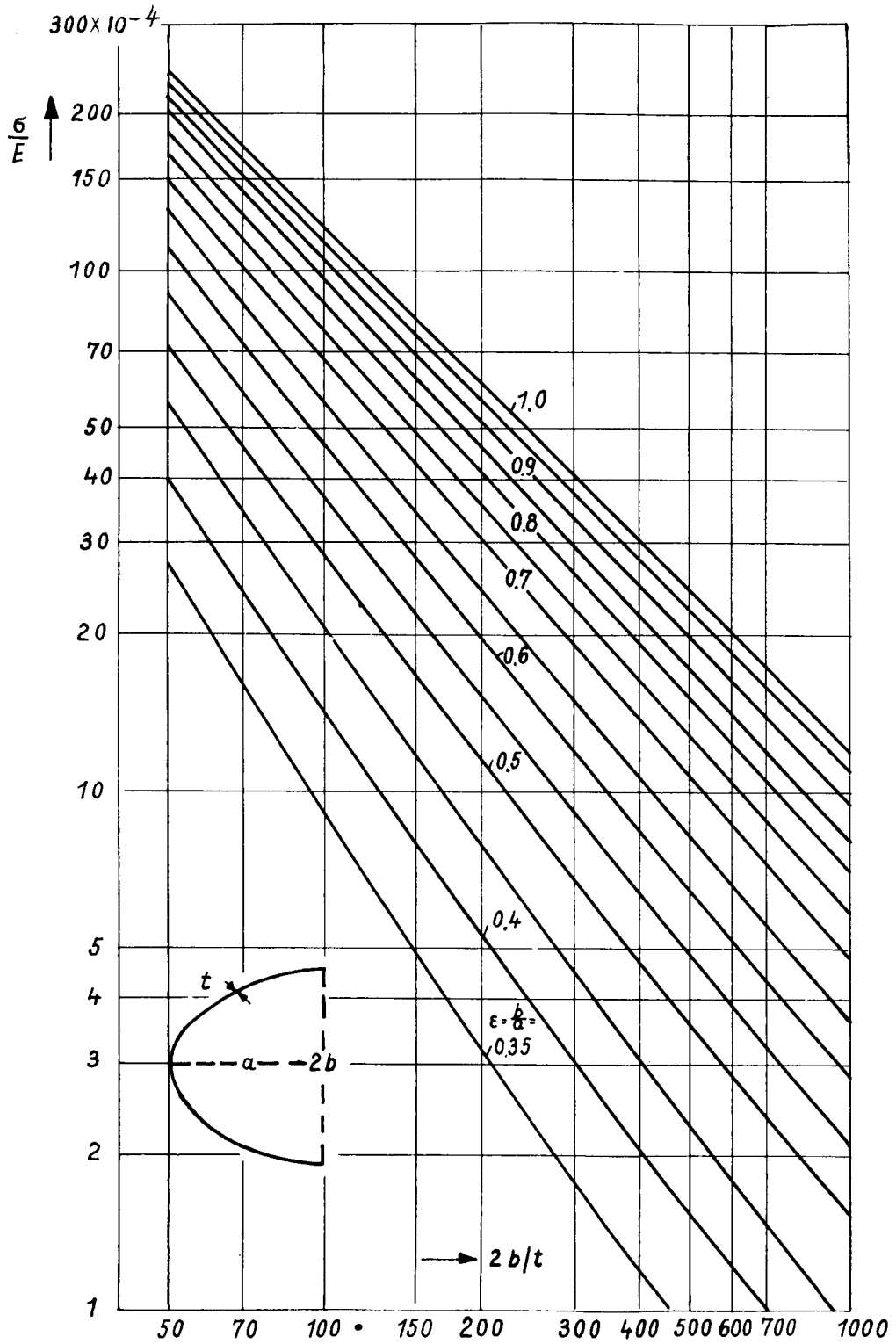


Figure 10.- Critical compressive stress σ as a function of the parameters $2b/t$ and b/a .

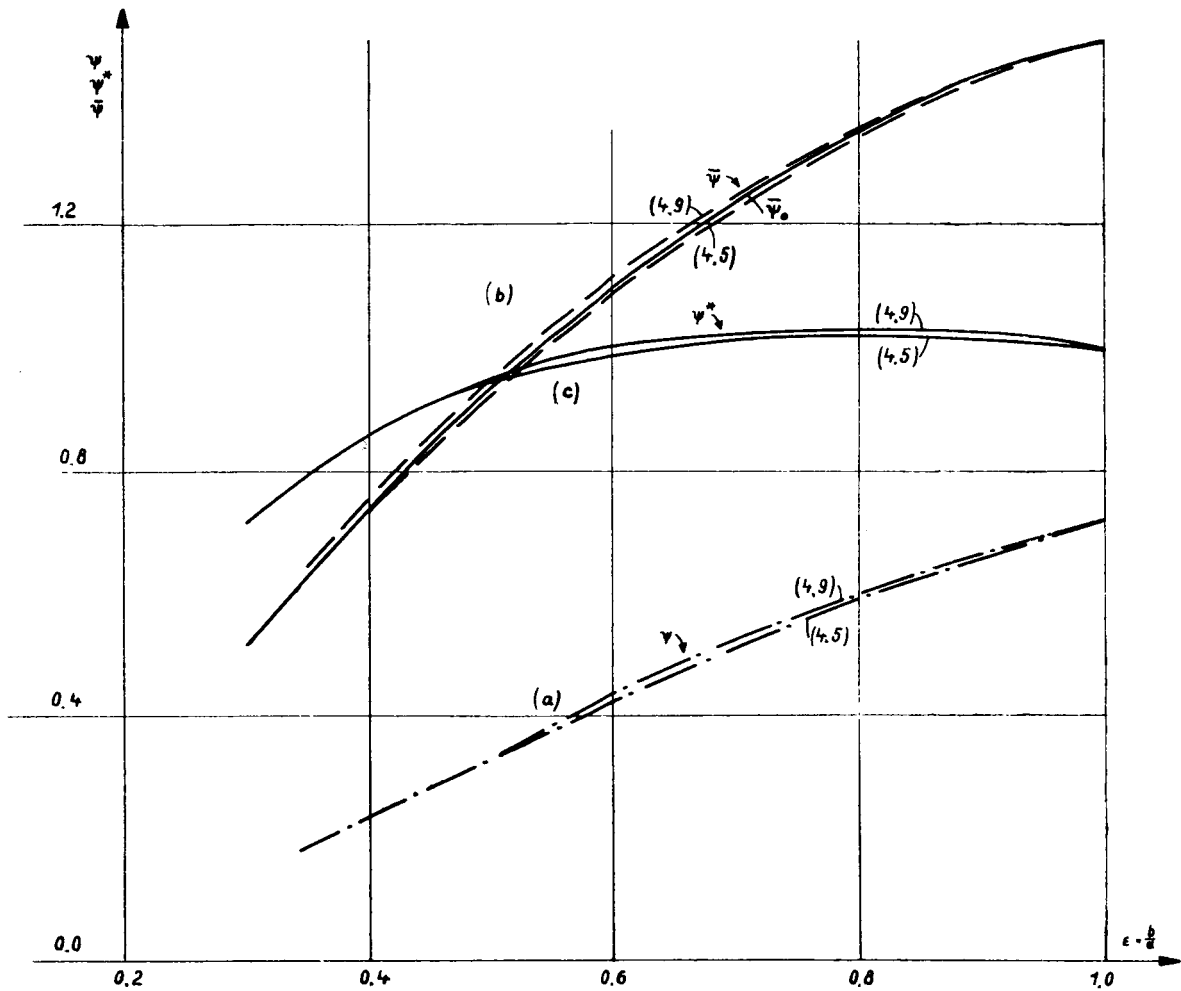


Figure 11.- Shear buckling, $\frac{\tau}{E} = \psi \left(\frac{t}{2b} \right)^{3/2}$

- (a) ψ for the closed cylinder (oval forms, equations (4.5) and (4.9)).
Approximation $\psi = \psi_0 = 1.34\epsilon - 0.437\epsilon^2 - 0.25$.
- (b) $\psi = \bar{\psi}$ for the half cylinder (oval forms, equations (4.5) and (4.9)).
Approximation $\psi = \bar{\psi}_0 = (3.1\epsilon - 1.3\epsilon^2 - 0.3)$.
- (c) The factor ψ^* , decisive for the decrease by compression (oval forms, equations (4.5) and (4.9)). For rough calculations, $\psi^* = 1$.

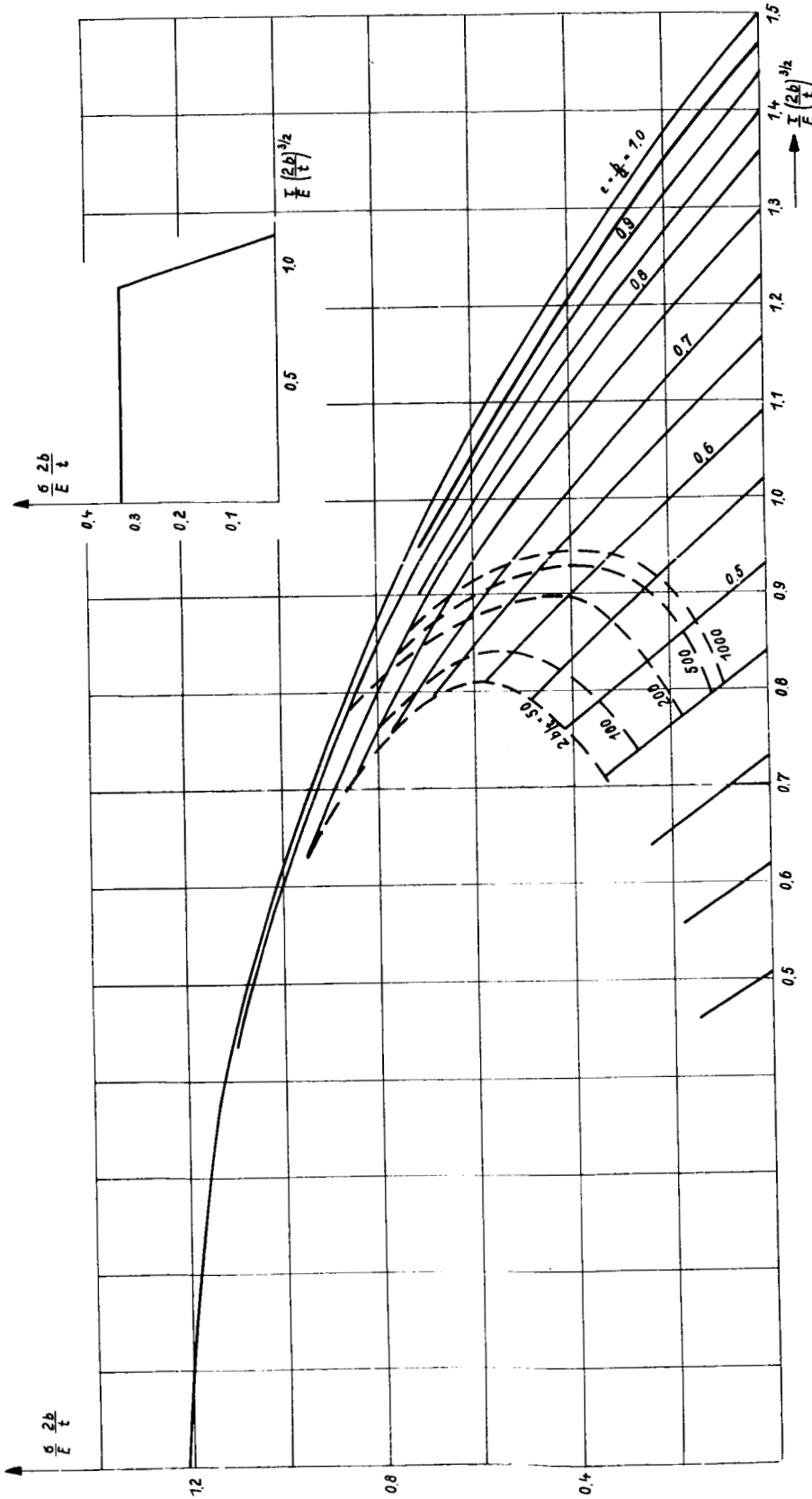


Figure 12.- Oval cylinder of infinite length under compression σ and shear τ . (The σ -, τ -curve pertaining to a certain parameter pair ϵ , $2b/t$ consists of the branch starting from the τ -axis and a horizontal, the height of which is fixed by the indication of the value $2b/t$ on this branch. Compare top, right, where the critical shearing-compression curve for $\frac{b}{a} = 0.6$, $\frac{2b}{t} = 650$ is drawn.)