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CORRELATION OF DATA ON THE STATISTICAL

THEORY OF TURBULENCE

By K. Wieghardt

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SUMMARY

The statistical theory of turbulence affords an ex-
cellent medium for representing the kinematic conditions
in turbulent flow and also serves as a valuable aid to
exact experimental research. But it is still not developed
enough for solving dynamic processes.

Even in the simplest case of isotropic turbulence
the calculation of the correlation curve or of the decre-
ment of turbulence invariably reaches a point where clear-
cut assumptions, such as omission of the inertia terms,
or, earlier, mixing length assumptions or even merely
general dimensional considerations, must be made. Since,
on the other hand, the differential equations are simply
evolved from Navier-Stokes' and the continuity equations
it is safe to assume that in their multitude of solutions
the actually arising relations are contained also; but
there still is no physical principle with the aid of which
the real solution could be dug out from the multitude of
mathematically possible solutions.

I. INTRODUCTION

If a solid body is towed through a fluid two es-
sentially dissimilar types of flow are apparent. For,
if the test is made at a very low Reynolds Number Re,
say, by pulling a thin rod slowly through thick oil in
a container, only the fluid in direct proximity of the
rod is in motion. The fluid particles ahead of the rod
are pushed aside to converge again behind it; when the
rod has just passed a certain place the fluid, if in a
small container, is immediately at rest again. On the
other hand the body experiences, because of the viscosity
forces, a drag even at uniform speed; hence work is per-
formed on the rod. This energy reappears in the forms

of heat and so the work performed on the towed body is changed to heat through the effect of the viscosity forces. An entirely different flow appears at large Reynolds Number, say, when pulling a rod quickly through the water. The viscosity forces, now much lower than in the first case, are unable to change the work performed on the rod into heat quick enough. If the rod is already far away or pulled out of the water altogether, only part of the energy is changed to heat. By virtue of the law of conservation of energy the remaining portion must therefore remain in the fluid in the form of motion energy. On the whole, to be sure, the fluid remains at rest, since the effect of the rod is confined to the neighborhood of the track through the fluid; hence the fluid particles simply move back and forth or form small eddies. This irregular back and forth flow is the turbulent motion state. The macroscopically visible motion of the fluid particles is then gradually changed in motion energy, that is, heat, under the effect of the viscosity.

II. DISSIPATION FUNCTION

The work, which must be performed in order to tow a body at uniform speed through a fluid and which is dissipated in the fluid, is termed the dissipation energy A. It corresponds to the work performed by the fluid stresses per unit time by volume and form change on a fluid element referred to unit volume. In incompressible fluids the term for A is analogous to the strain energy of an elastic body. The strains become the rates of strain; Hooke's law of stresses proportional to the strains is replaced by the linear relation between shear stresses and velocity gradients. Thus it affords for A

"strains"

$$A = \mu \left\{ 2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial v}{\partial y} \right)^2 + 2 \left(\frac{\partial w}{\partial z} \right)^2 \right.$$

"shear"

$$\left. + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 \right\}$$

To make these and the subsequent terms more comprehensive, x, y, z, are rewritten as x_1, x_2, x_3, and the speeds u, v, w as u_1, u_2, u_3, so that the foregoing equation then reads:

$$A = \mu \left\{ \Sigma_h \Sigma_1 \left(\frac{\partial u_i}{\partial x_k} \right)^2 + \Sigma_k \Sigma_1 \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i} \right\} \text{ with } i, k = 1,2,3$$

(1)

A is the dissipation for any motion of an incompressible fluid, where, of course, the continuity equation

$$\Sigma_i \frac{\partial u_i}{\partial x_i} = 0$$

(2)

holds. But the fluid is also to satisfy the Navier-Stokes motion equation:

$$\nu = \frac{\mu}{\rho}$$

$$\frac{\partial u_i}{\partial t} + \Sigma_k u_k \frac{\partial u_i}{\partial x_k} = \frac{\partial X_i}{\partial x_i} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \, \Delta u_i, \text{ where}$$

(3)

$$\Delta \equiv \Sigma_1 \frac{\partial^2}{\partial x_1^2}$$

Forming the divergence of this equation, that is, $\Sigma_i \frac{\partial}{\partial x_i}$ affords

$$\frac{\partial}{\partial t} \Sigma_i \frac{\partial u_i}{\partial x_i} + \Sigma_k u_k \frac{\partial}{\partial x_k} \Sigma_i \frac{\partial u_i}{\partial x_i} + \Sigma_k \Sigma_i \frac{\partial u_k}{\partial x_i} \frac{\partial u_i}{\partial x_k}$$

$$\Sigma \frac{\partial X_i}{\partial x_i}$$

$$= \Delta X_i - \frac{1}{\rho} \Delta p + \nu \Sigma_1 \frac{\partial^2}{\partial x_1^2} \Sigma_i \frac{\partial u_i}{\partial x_i}$$

provided that the external forces possess a potential V for which $\Delta V = 0$, $\Delta X_i = 0$, and because of the continuity equation $\Sigma \frac{\partial X_i}{\partial x_i}$

$$\Sigma k \ \Sigma i \ \frac{\partial u_k}{\partial x_i} \frac{\partial u_i}{\partial x_k} = - \frac{1}{\rho} \Delta p \qquad (3a)$$

and hence

$$A = \mu \left\{ \Sigma k \ \Sigma i \left(\frac{\partial u_i}{\partial x_k} \right)^2 - \frac{1}{\rho} \Delta p \right\} \qquad (4)$$

The turbulent motion state will be illustrated later by small vortices. To formulate this idea mathematically the general term for the dissipation can be expressed by the rot-vector. The rotation is defined by the three vectors

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}$$

$$\omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}$$

$$\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

Forming

$$\Sigma_i \omega_i^2 = \Sigma k \ \Sigma i \left(\frac{\partial u_i}{\partial x_k} \right)^2 - \Sigma k \ \Sigma i \ \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i} = \Sigma_k \Sigma i \left(\frac{\partial u_i}{\partial x_k} \right)^2 + \frac{1}{\rho} \Delta p,$$

affords

$$A = \mu \left\{ \Sigma_i \omega_i^2 - \frac{2}{\rho} \Delta p \right\} \qquad (4a)$$

Analogous to the theorem of minimum strain energy for the dissipation a law by Helmholtz (reference 1) is applicable. If the inertia terms can be neglected the stationary motion of a fluid arising under the effect of constant forces with unique potential, is distinguished by the property that its dissipation for each region is less than that of any other motion with the values u, v, w at the boundary. This theorem was applied by Vogelpohl to the theory of lubricants (reference 2). Wehrle (reference 3) obtained a similar principle for the turbulence problem. According to it, on the basis of statistical probability considerations the turbulent motion state, for which the dissipation

energy is minimum, is the most probable. But physically it does not seem very clear that the same minimum principle should apply to the turbulent state of motion with inertia and viscosity effects of the same order of magnitude, as to the creeping (slow) motion.

III. ISOTROPIC TURBULENCE

In order to be able to adapt the foregoing formulas for the dissipation to the case of turbulent flow the turbulent motion state must be marked more in detail. The chief characteristic of turbulence is the irregularity of motion of the individual particles. So from the very first it is impossible to define say, the path of one single particle, other than obtain data by means of certain statistical averages theoretical or experimental. However, this limitation is not essential since the interest centers practically only around certain mean values. With U_i denoting the velocity component of the principal flow and u_i the superimposed turbulent fluctuations, the momentary speed in i direction is $U_i + u_i$. As U_i is to represent the mean motion, u_i must disappear in the mean: $\bar{u}_i = 0$, regardless of the intensity of the fluctuations. An indication of the degree of turbulence is obtained from the quadratic mean $u_i' = \sqrt{\bar{u}_i{}^2}$. The averaging is always to be understood as mean over a sufficiently long time interval T; hence $u_i' = \sqrt{\dfrac{1}{T} \int_0^T u_i{}^2 dt}$. To illustrate: for a sinusoidal fluctuation it is:

$$u_i = A \sin 2 \pi n t \qquad u_i' = \frac{1}{2} \sqrt{2}\, A$$

The next logical step is to analyze the simplest field of isotropic turbulent fluctuations, wherein the mean quantity of fluctuations in every direction and every point is identically great, that is, when $u_1' = u_2' = u_3'$ for each point. This case is not without practical significance. For instance, the flow downstream from a screen or the airstream of a wind tunnel, some distance away from the turbulence generator (screens, grids) is isotropic turbulent; that is, in a system of coordinates moving along with the principal flow all directions and positions respecting turbulent velocity fluctuations are equivalent.

For this isotropic turbulence a number of relations

between the different average values can be established. Because of independence of position Δp = const every-where, hence permits averaging over a volume V:

$$\Delta p = \frac{1}{V} \int \Delta p \, dv \quad \text{and transforming according to Gauss'}$$

theorem $\quad \Delta p = \frac{1}{V} \int \underline{n}^0 \; \text{grad } p \, d \, f$, $\underline{n}$ being the unit

vector along the outer normals. On integrating, for example, with respect to a spherical space of radius R, f increases with R^2, V with R^3, and as p is to be = const in the mean, $\overline{\Delta p}$ disappears as $1/R$ (reference 4).

   In virtue of the directional independence the mean values resulting from the cyclic transformation of the coordinates, that is, successive groups, are identically great

$$\omega_1{}' = \omega_2{}' = \omega_3{}', \quad a_1 = \left(\frac{\partial u_i}{\partial x_i}\right)'^2 \quad a_2 = \left(\frac{\partial u_i}{\partial x_k}\right)'^2$$

$$a_3 = \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i}} \quad a_4 = \overline{\frac{\partial u_i}{\partial x_i} \frac{\partial u_k}{\partial x_k}}$$

etc., where $i, k = 1, 2, 3$ but $i \neq k$. Average values of the form $\overline{\dfrac{\partial u_i}{\partial x_k} \dfrac{\partial u_k}{\partial x_i}}$ must disappear by isotropic turbulence, as they exchange the prefix when the coordinate system is turned through 180^0. For kinematic reasons and further-more because of the symmetry the mean values a_1 to a_4 are then not mutually unrelated but subject to the follow-ing relations. After squaring and averaging the contin-uity equation affords

$$a_1 + 2 \, a_4 = 0 \tag{5a}$$

$\overline{\Delta p}$ = 0 as previously deduced; hence affords with

$$\overline{\Delta p} = -\rho \, \Sigma_k \, \Sigma_i \; \frac{\partial u_k}{\partial x_i} \frac{\partial u_i}{\partial x_k}$$

$$a_1 + 2 \, a_3 = 0 \tag{5b}$$

Lastly the rotation of the coordinate system yields a third relation. Turned through 45°, which mathematically is most convenient, the transformation reads

$$\sqrt{2}\ x^* = x + y \qquad\qquad \sqrt{2}\ u^* = u + v$$
$$\sqrt{2}\ y^* = -x + y \qquad and \qquad \sqrt{2}\ v^* = -u + v$$
$$z^* = z \qquad\qquad\qquad w^* = w$$

Thus it affords for example:

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left(\frac{\partial u^*}{\partial x^*} - \frac{\partial v^*}{\partial x^*} - \frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial y^*}\right)$$

In virtue of the isotropy, $a_1 = \left(\dfrac{\partial u^*}{\partial x^*}\right)^2$ and so forth, must also apply in the new system of coordinates. Thus squaring the above equation followed by averaging gives at last

$$a_1 - a_2 - a_3 - a_4 = 0 \qquad\qquad (5c)$$

Combining (5a), (5b), and (5c) then affords

$$a_1 = a_2/2 = -2\ a_3 = -2\ a_4 \qquad\qquad (5)$$

of which the most important is $a_1 = \dfrac{1}{2}\ a_2$; hence

$$\left(\frac{\partial u_i}{\partial x_i}\right)'^2 = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k}\right)'^2 \qquad with \quad k \neq i \qquad (5d)$$

Now (with eq. (4) and (4a)) the mean dissipation by isotropic turbulence can be written in the form

$$\overline{A} = \mu\Sigma_k\Sigma_i\left(\frac{\partial u_i}{\partial x_k}\right)'^2 = \mu\Sigma_i\omega'^2_i = 3\ \mu\omega'^2_i = 7.5\ \mu\left(\frac{\partial u_i}{\partial x_k}\right)'^2$$

for the sum $\Sigma_k\Sigma_i\left(\dfrac{\partial u_i}{\partial x_k}\right)'^2$ consists of three terms of the

form $\left(\dfrac{\partial u_i}{\partial x_k}\right)'^2$ and six mutually equal terms $\left(\dfrac{\partial u_i}{\partial x_k}\right)'^2$

with $k \neq i$, so that

$$A = \mu \cdot \left(6 + 3x\frac{1}{2}\right)\left(\frac{\partial u_i}{\partial x_k}\right)'^2 = 7.5 \ \mu\left(\frac{\partial u_i}{\partial x_k}\right)'^2 \tag{6}$$

This relation originated with G. I. Taylor (refer-ence 5).

IV. THE CORRELATION FUNCTIONS

A fluctuating fluid particle affects, because of the continuity due to inertia and viscosity, its entire surround-ing. So with a view to establish a criterion for the scope of this effect in theoretical and experimental studies G. I. Taylor applied the statistical concept of correlation to the theory of turbulence (references 5, 6). The correlation coefficient indicates, whether and to what extent the concurrent fluctuation velocities at two different points, are dependent upon each other. If P_1 and P_2 are two

points (fig. 1a) whose line of connection is at right angles to x and whose distance is y, it is possible, for instance, to relate the simultaneous fluctuations u and

u(y) in x direction through correlation factor g:

$$g\ (y) = \frac{\overline{u\ u\ (y)}}{u'u'(y)} \text{ , or, in isotropic turbulence, with:}$$

$$u' = \text{const} \quad g\ (y) = \frac{\overline{u\ u\ (y)}}{u'^2}.$$

The curve of g as function of distance y is as follows (fig. 2): for y = 0, g has the maximum value g = 1, after which g drops toward both sides ±y monotonically to a small negative value and rises again gradually to zero. From a certain distance on, no practical relationship exists between the speeds, that is, a negative or positive fluctuation value of $\overline{u\ u\ (y)}$ is equally prob-able, so that in the mean $\overline{u\ u\ (y)}$ = 0. To take the older mixing theory literally, which it never was meant to be, the correlation curve would be a rectangle, the width of which would correspond to the size of the individual fluid balls or the mixing path itself. The rougher the turbu-lence L:

$$L_2 = \int_0^\infty g\ (y)\ d\ y = \int_0^Y g\ (y)\ d\ y, \text{ if } Y \text{ is large enough so}$$
that $g\ (Y)$ = 0. $\tag{7}$

Any number of such correlation coefficients can be
defined according to position of the line connecting the
two points and the direction of the explored speeds. But
by isotropic turbulence only two correlation functions
exist which do not dissappear nor can they be reduced by
transformation of the co-ordinates for reasons of symmetry,
which therefore appear at first unrelated. If the previous-
ly defined g is one of these functions the other correl-
ation is $f(x) = \dfrac{\overline{u\,u(x)}}{u'^2}$ (fig. 2) that is, the direction

of the investigated fluctuations and the straight line
joining the two points coincide. With this correlation
also the magnitude of turbulence can be defined.

$$L_1 = \int_0^\infty f(x)\, dx = \int_0^X f(x)\, dx \qquad (8)$$

Since the speed fluctuations themselves must, like
the principal flow, satisfy the continuity equation, both
correlations f and g are tied together by a differ-
ential equation. This equation, originally proposed by
Von Kármán (reference 7) can be deduced, according to
Prandtl, by the following line of reasoning: Consider the
inflowing and outflowing fluid volume through a control
area moving along with the uniform principal flow and
consisting of a hemisphere and its diametrical plane
(Fig. 3). Owing to turbulent fluctuations the flow
volume passing through the diametrical plane per unit
time is $\rho \int_0^{r_1} u'\, g(r)\, 2\pi r\, dr$; $g(r)$ replacing $g(y)$
to indicate that the location of the coordinate system
can, by reason of the assumed isotropy of turbulence,
be arbitrary.

The outflow amounts to

$$\rho \int_0^{\pi/2} u' \cos\alpha\, f(r_1)\, 2\pi r_1 \sin\alpha\, r_1 d\alpha = \pi u' f(r_1)\, r_1^2$$

the continuity therefore demands

$$\int_0^{r_1} u'\, g(r)\, 2\pi r\, dr = \pi u' f(r_1)\, r_1^2$$

which, after differentiation with respect to r_1, leaves

$$r_1\, g(r_1) = r_1 f(r_1) + \frac{r_1^2}{2} \cdot \left. \frac{f(r)}{dr} \right|_{r=r_1}$$

or, since the radius of hemisphere r is arbitrary:

$$g(r) = f(r) + \frac{1}{2} r \frac{d f(r)}{d r} \tag{9}$$

Integration of (9) with respect to r from 0 to ∞ then affords

$$L_2 = L_1 + \left[\frac{f(r) r}{2} \right]_0^\infty - \frac{1}{2} L_1 = \frac{1}{2} L_1$$

Another equation (generalization of equation (5d)) ties the mean values of the speed gradients $\left(\frac{\partial u}{\partial r} \right)'$ on one point in any direction Θ with respect to the y axis:

$$\left(\frac{\partial u}{\partial r} \right)' = \left(\frac{\partial u}{\partial y} \right)' \sqrt{\cos^2 \Theta + \frac{1}{2} \sin^2 \Theta} \tag{10}$$

The experimental proof of these purely kinematic relations (9) and (10) by hot wire measurements in wind tunnels behind various grids indicate that the flow some distance downstream from a grid is actually turbulent and the correlation test method is correct.

From the aspect of g (y) the value $\left(\frac{\partial u}{\partial y} \right)'$ important for the dissipation can be obtained as follows: expansion of g (y) in Taylor series affords

$$g(y) = \frac{\overline{u\, u\,(y)}}{u'^2} = \frac{1}{u'^2} \left[u'^2 + y\, \overline{u \frac{\partial u}{\partial y}} + \frac{y^2}{2!}\, \overline{u \frac{\partial^2 u}{\partial y^2}} + \dots \right]$$

From $\overline{u^2}$ = const follows

$$\overline{\frac{\partial u^2}{\partial y}} = \overline{u \frac{\partial u}{\partial y}} = 0$$

whence, after another differentiation:

$$\overline{u \frac{\partial^2 u}{\partial y^2}} + \overline{\left(\frac{\partial u}{\partial y} \right)^2} = 0$$

For small y it leaves

$$g\ (y) = 1 - \frac{y^2}{2!\ u'^2}\ \left(\frac{\partial u}{\partial y}\right)^{'2}$$

and

$$\left(\frac{\partial u}{\partial y}\right)^{'2} = 2\ u'^2 \lim_{y \to 0} \frac{1 - g\ (y)}{y^2} = 2\ u'^2\ \frac{1}{\lambda^2} \qquad (11)$$

The thus defined length λ can be geometrically interpreted as the distance from the zero point in which the osculating parabola at $y = 0$ intersects the abscissa $g = 0$ (fig. 2). Expressed with λ the dissipation (6) reads

$$\overline{A} = 15\ \mu\ \frac{u'^2}{\lambda^2} \qquad (12)$$

The smaller λ is, the greater is the dissipation. Taylor, therefore, considers λ as the diameter of the smallest vortices absorbed by the viscosity. The smaller these vortices, the more pointed the aspect of g at $y = 0$, and the quicker such turbulence is decelerated by the viscosity, while vortices of large diameter die away much more gradually.

One principal problem of the statistical theory of turbulence is the prediction of the correlation curve, as it affords a complete characterization of the turbulent state of motion. Von Kármán attempted to solve this problem for the case of isotropic turbulence (references 7, 8, 9) but the results, in the absence of an essential declaration so far, are unsatisfactory.

V.   THE SPECTRUM OF TURBULENCE

This far the description of turbulent velocity field proceeded on the correlation of the speed fluctuations at two different points at the same times. Taylor (reference 10) showed that the concept spectrum of turbulence is also appropriate. He visualized the speed fluctuations on a point harmonically divided in sinusoidal fluctuations, each of a different frequency n, so that $u'^2 F\ (n)\ d\ n$

is the value of the fluctuations with the frequencies

between n and $n + dn$; because $u'^2 = \int_0^\infty u'^2 F(n)\,dn$

we have $\int_0^\infty F(n)\,dn = 1$, by definition.   Between $F(n)$

and $f(x)$ the following relation exists: $f(x) = \dfrac{\overline{u\,u\,(x)}}{u'^2}$,

or written $f(x) = \dfrac{\overline{u\,(t)\,u\,(t + x/U)}}{u'^2}$,   when

$$U \gg u', \qquad t = \frac{x}{U + u} = \frac{x}{U}\left(1 - \frac{u}{U}\cdots\right)$$

In this manner f is represented as correlation of the
velocity fluctuations on a point at different times.

f and $\dfrac{U\,F\,(n)}{2\sqrt{2\pi}}$ are then transformed Fourier series, accord-

ing to Taylor, for example, the equation

$$F(n) = \frac{4}{U}\int_0^\infty f(x)\cos\frac{2\pi n x}{U}\,dx \qquad (13a)$$

and

$$f(x) = \int_0^\infty F(n)\cos\frac{2\pi n x}{U}\,dn \qquad (13b)$$

are applicable.

Length λ is expressed by $F(n)$ in the following
manner:

$$\frac{1}{\lambda^2} = 4\,\pi^2\int_0^\infty \frac{n^2}{U^2}\,U\,F(n)\,\frac{dn}{U} \qquad (14)$$

In figure 4, showing the curve of $U\,F(n)$ plotted
against n/U according to tests by Simons and Salter be-
hind grids (references 10, 11) the spectrum appears inde-
pendent of the speed U of the principal flow. This is
in accord with the fact that the correlation $f(x)$ behind
grids is also almost independent of U. At very small dis-
tances merely the aspect of $f(x)$ and hence length λ depend
on U. But according to (14) the spectrum itself would then
have to be dependent on U. In fact, accurate measurements

revealed that, while the spectrum is independent of U for a large frequency range, a different curve resulted at very high frequencies (from about $n/U = 0.6$ 1/cm) depending upon the mean speed U. On the other hand, the high frequencies are precisely essential for the relationship of the spectrum with λ, according to equation (14).

The concepts of the correlation function or of the spectrum thus affords the possibility to explore the processes in turbulent flows, for the present of course, chiefly by experiment.

VI. DECREMENT OF TURBULENCE

The simplest process in a turbulent flow amenable to theoretical study is the time rate of decrement of isotropic turbulence by viscosity, so important for the turbulence factor of wind tunnels, where screens and honeycomb set up an artificial turbulence which dies out again in the tunnel.

The kinetic energy of turbulence is $E = \dfrac{\rho}{2} \Sigma_i u_i^2 = \dfrac{3}{2} \rho u'^2$ the time rate of decrease $-\dfrac{dE}{dt}$ corresponds to the dissipated energy

$$\overline{A} = 15 \mu \frac{u'^2}{\lambda^2} = -\frac{dE}{dt} = -\frac{3}{2} \rho \frac{d u'^2}{dt}$$

Changing from the moving to a space system of coordinates d/dt can be replaced by $U\, d/dx$ for stationary basic flow, so that

$$U \frac{d u'^2}{dx} = -10 \nu \frac{u'^2}{\lambda^2} \qquad\qquad (15)$$

To integrate eq. (15), λ must be known in relation to u' and x.

1. Mixing Length Assumptions

When a fluid ball having a speed u' relative to the principal flow intermixes the kinetic energy E per unit volume is lowered by the amount of $\Delta E \sim \rho/2\, u'^2$. The time interval t during which this occurs is of the

order of magnitude l/u', where l = mixing length. The

energy loss per unit time $\dfrac{\Delta E}{\Delta t}$ is therefore $\sim \rho\, \dfrac{u'^3}{l}$. A

comparison of this loss with the dissipation discloses

$\dfrac{\rho\, u'^3}{l} \sim \mu\, \dfrac{u'^2}{\lambda^2}$ or $\dfrac{\lambda}{l} \sim \sqrt{\dfrac{\nu}{l\, u'}}$. The mixing length l can be

set down proportional to the turbulence $L_1 = \int_0^\infty f\, d\dot{y}$, so that

$$\frac{\lambda}{L_1} = A \sqrt{\frac{\nu}{L_1 u'}} \tag{16}$$

where A is a numerical constant. L_1 is no longer de-
pendent on u' according to assumption, but solely on
t and x, respectively. The simplest assumption is
L_1 = const (proportional to the size of the mesh). Then

equations (15) and (16) give $\dfrac{U}{u'} \approx \dfrac{5}{A^2} \cdot \dfrac{x}{L_1}$, as confirmed

in many tests (reference 5). According to other tests
by Dryden, $L_1 = L_0 + c\, x$ was recorded, so that

$\dfrac{U}{u'} \sim \log\left(\dfrac{L_0}{M} + \dfrac{cx}{M}\right)$ where M denotes the mesh size of the

screen (reference 12).

2. Dimensional Analysis

While Von Kármán obtains the same equation (15) by
dimensional analysis, he assumes the time rate of decrease
of intensity of turbulence L to be dependent on u',

hence $\dfrac{d\,L}{d\,t} \sim u'$, so that $\lambda^2 = \dfrac{5}{n}\dfrac{\nu x}{U}$ and $\dfrac{U}{u'} \sim \left(\dfrac{x}{M}\right)^n$ results

with $x = U\,t$, $L \sim x^{1-n}$. This result is also confirmed
by experiments; n was defined from the equation for λ^2
and agreement established between the last equation for
U/u' and the measurements (reference 7).

The foregoing discussion is but a rough summary of
the decremental process. For exploring the physical de-
tails the turbulence can be explained by eddies, as the

equation $\bar{A} = \mu\, \overline{\Sigma_i\, \omega_i^2}$ suggests. Eddies of the order of

magnitude of mesh size are produced on the screen, after

which these eddies split up into smaller ones to be
absorbed by the viscosity effect. Considering two
particles of a vortex filament of angular velocity
ω_0, with mean distance d_0, it is, according to Taylor
(reference 13), more likely owing to the diffusive effect
of the turbulence, that distance d_0 has increased after
some time. If the viscosity is negligible at small ω_0

and great diameter, ω increases: $\omega = \dfrac{\omega_0}{d_0} d$ according

to Helmholtz's vortex theorem. After ω has increased to
a certain amount the viscosity effect can no longer be
neglected, in which instance the foregoing consideration
fails. But it seems problematical whether a statistical
process, namely, diffusion can be assumed for such close
distances as d_0, because, non-statistically argued a
reaction force results on a vortex filament when d in-
creases, which pulls the particles together again. Taylor
and Green (reference 13) tried to follow up the process
mathematically by insertion of a formula for u, v, w,
that roughly corresponds to an isotropically turbulent
flow, in the equation of motion. They obtained a slight
rise in dissipation for small time intervals, which,

because $A = \mu \sum_i \omega_i^2$ also indicates a rise of the mean

rotation. But owing to the inferior convergence of the
very difficult calculation no definite conclusions can be
drawn respecting the behavior for longer time intervals.

VII. LAW OF SIMILITUDE FOR THE TRANSITION POINT

According to Reynolds' law of similitude two flows
are equivalent when the ratio of inertia-viscosity forces
is the same. In most cases this simple law holds even for
turbulent flow. However, in many instances, the turbulence
plays such a prominent part that the Reynolds number alone
no longer suffices for characterizing the flow attitude.
Then the type of turbulence must be closer characterized
by additional phase quantities. For isotropic turbulence
there are afforded two mutually independent characteristic
quantities, namely, the degree of turbulence u'/U and the
size of turbulence L_1 (equation (8)).

A particularly profound effect of the turbulence is
that on the position of the transition point, that is, the
location of the area where the friction layer becomes tur-
bulent. This relation of transition point and turbulence

was explored by Taylor with his statistical turbulence
theory (reference 5), on the basic assumption of the
turbulent fluctuations of the pressure gradients affect-
ing the position of the transition point through the
turbulence.   This is indicated to the extent that the
separation of the laminar friction layer itself is large-
ly defined by the outside pressure gradient.   The cal-
culation pressumes that the body is in an isotropic tur-
bulent flow.   For the pressure fluctuations in direction
of the basic flow (speed U), which are equivalent to the

turbulent velocity fluctuations   u, it is $p \sim \frac{\rho}{2} u^2$;

whence the fluctuations of the pressure gradients are

$\frac{\partial p}{\partial x} \sim \rho\, u\, \frac{\partial u}{\partial x}$, or, because of the directional independence

$\rho\, u\, \frac{\partial u}{\partial y}$ and the mean square

$$\left(\frac{\partial p}{\partial x}\right)' = \sqrt{\overline{\left(\frac{\partial p}{\partial x}\right)^2}} \sim \sqrt{\overline{\rho^2\left(u\frac{\partial u}{\partial y}\right)^2}} \sim \rho\, u'\left(\frac{\partial u}{\partial y}\right)' \qquad (17)$$

Then the mean value $\left(\frac{\partial u}{\partial y}\right)'$ can be put in relation
with u' and L_1.   Together with (11) and (16), equation (17)
gives:

$$\left(\frac{\partial p}{\partial x}\right)' \sim \frac{\rho\, u'^2}{L_1} \sqrt{\frac{u'\, L_1}{\nu}} \qquad (18)$$

Then the previous assumption can be formulated by
dimensional analysis as follows: it is presumed that the
location of the transition point (coordinate X, on the
sphere, for instance, the arc length from forward stag-
nation point to transition point) depends, besides U

and ν, on $\left(\frac{\partial p}{\partial x}\right)'$.   Accordingly the Reynolds Number Re

(X) must be supplemented by a second dimensionless number
consisting of quantity X, dynamic pressure $q = \frac{\rho}{2} U^2$ and

$\left(\frac{\partial p}{\partial x}\right)'$ with which the following nondimensional is formed:

$$\Lambda' = \left(\frac{\partial p}{\partial x}\right)' \frac{X}{q} \sim \left(\frac{\partial p}{\partial x}\right)' \frac{X}{\rho\, U^2} \qquad (19)$$

With the computed expression for $\left(\dfrac{\partial\;p}{\partial\;x}\right)'$ it affords

$$\Lambda' = \left(\frac{u'}{U}\right)^2 \frac{X}{L_1} \sqrt{\frac{u'\;L_1}{v}} = \sqrt{\frac{U\;X}{v}} \sqrt{\frac{X}{L_1}} \left(\frac{u'}{U}\right)^{5/2} \qquad (19a)$$

Now, according to the previous assumption the Reynolds number of the transition point $Re\;(X)$ must be a function of this characteristic factor Λ'.

$$Re\;(X) = \frac{U\;X}{v} = f\;(\Lambda') = f_1\left[\sqrt{Re(X)}\sqrt{\frac{X}{L_1}\left(\frac{u'}{U}\right)^5}\right]$$

or

$$Re\;(X) = F_2\left[\left(\frac{u'}{U}\right)\left(\frac{X}{L_1}\right)^{1/5}\right] \qquad (20)$$

If the turbulence is produced by a screen of mesh size M, one may simply put $L_1 \sim M$ for larger Reynolds numbers of turbulence from about $\dfrac{M\;u'}{v} = 60$ according to Taylor. Then (20) finally reads:

$$Re\;(X) = F\left[\left(\frac{u'}{U}\right)\left(\frac{X}{M}\right)^{1/5}\right] \qquad (20a)$$

This relation has been well confirmed on flat plates (reference 6) and on an elliptic cylinder (reference 14).

Taylor's calculation is of special importance for the critical Re number of spheres, since it serves a criterion of turbulence of wind tunnels. $Re_{critical}$ is defined as that Reynolds number at which the sphere has the drag coefficient $c_w = 0.3$. This drag value defines on different spheres and degrees of turbulence a certain geometrically similar state of flow; hence the ratio X/D (D = sphere diameter) has a certain value C at $c_w = 0.3$, so that $Re_{critical}\;(D) = \dfrac{U\;D}{v} = \dfrac{1}{C}\;Re\;(X)$. Herewith equation (20) gives

$$Re_{crit}\;(D) = G\left[\left(\frac{u'}{U}\right)\left(\frac{D}{M}\right)^{1/5}\right] \qquad (21)$$

Figure 5 gives the experimental proof of this calculation according to measurements by Dryden (reference 12); all test points on different spheres, screens, and degrees of turbulence are located on one curve, if the critical Reynolds number is plotted against $\left(\dfrac{u'}{U}\right)\left(\dfrac{D}{L_2}\right)^{1/5}$ $\left(L_2 = \dfrac{1}{2} L_1 \sim M\right)$.

Translation by J. Vanier,
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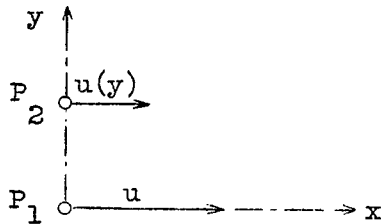   Mechanics, Cambridge, 1938, p. 321.

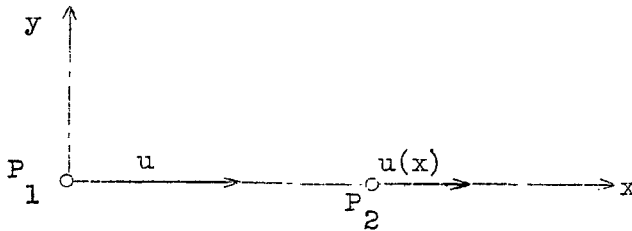Figure 1a.- Definition of correlation coefficient $g(y)$.
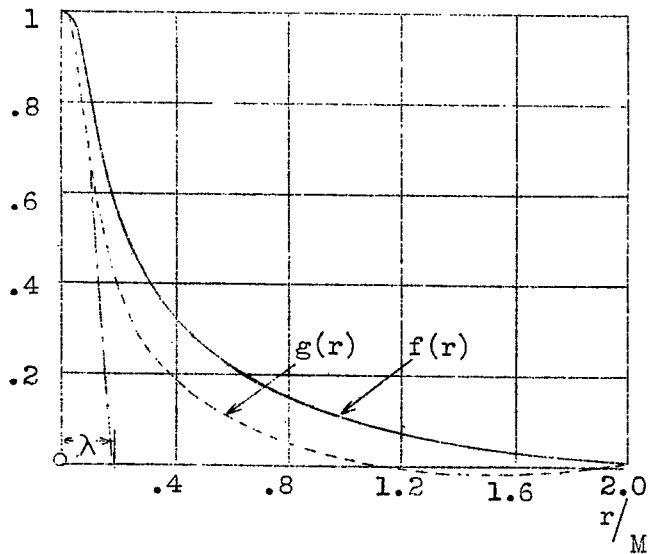
Figure 1b.- Definition of correlation coefficient $f(x)$.

Figure 2.- Aspect of correlation functions downstream
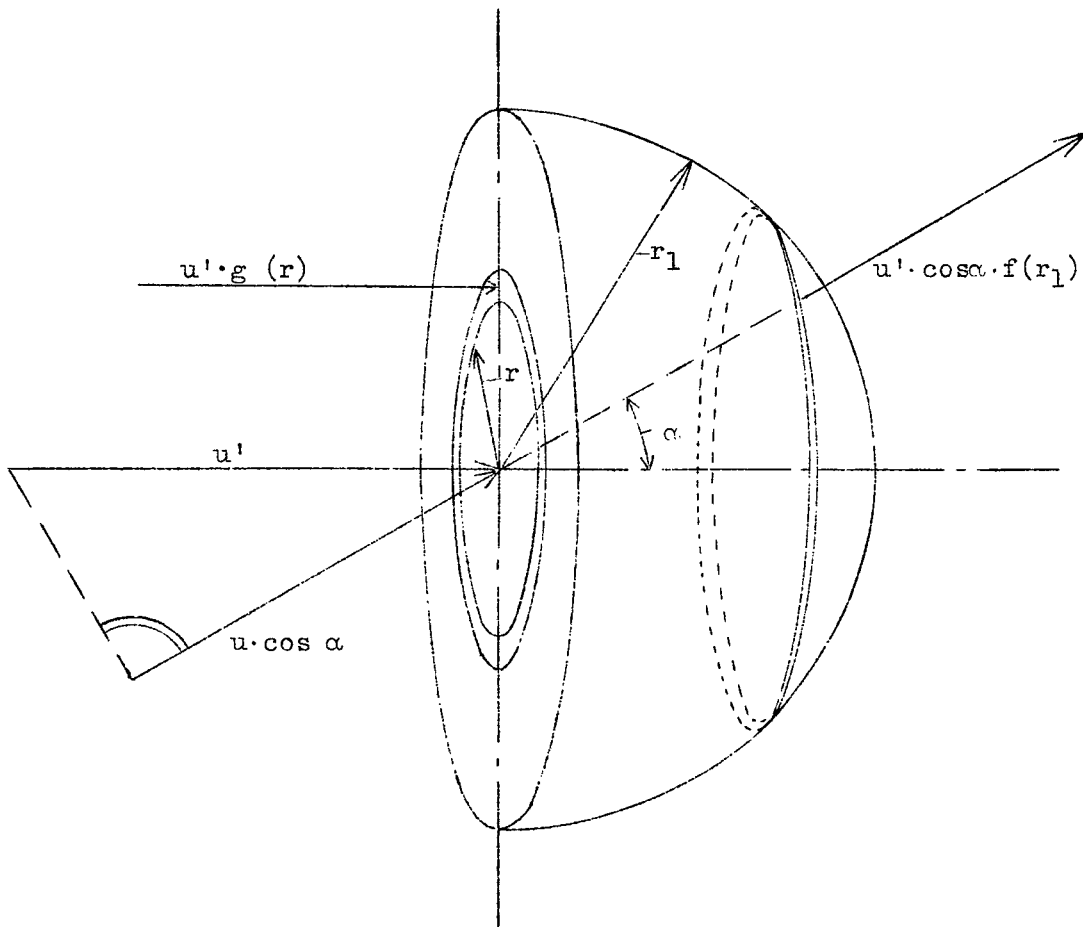from the screen (mesh size M).

Figure 3.- Definition of continuity equation expressed in correlation
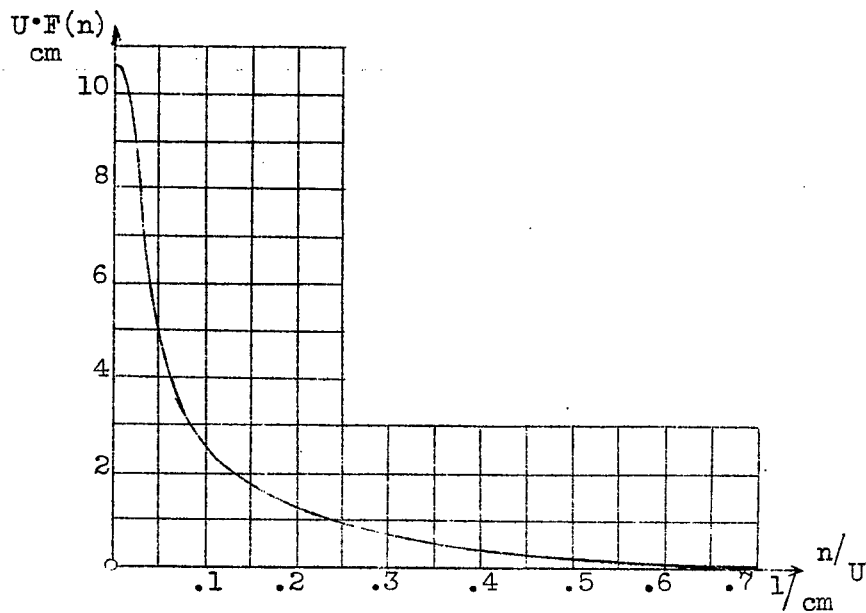functions in isotropic turbulence.

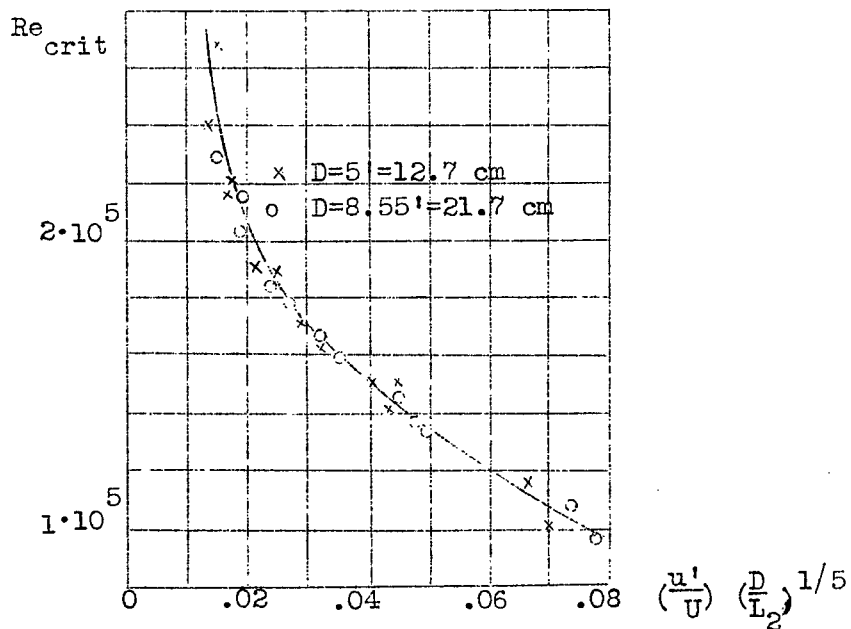Figure 4.- Spectrum of isotropic turbulence.

Figure 5.- Critical Reynolds number of sphere against turbulence. (12)