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SUMMARY

As the initial step in the analysis of stress distribution in three dimensionally curved rings (as employed as stiffeners in stressed skin aircraft designs) the ring (fig. 1) formed by the intersection of two circular cylinders is explored for three categories of load: tension in both cylinders (produced by hydrostatic pressure on the cylinder walls), axial force in the large cylinder, and lastly, shear in the large cylinder. The discussion of these three load cases enables general conclusions concerning the behavior of the ring stressed by the shell forces and affords numerical data for the most important load categories (obtainable from the computed by superposition). The quantitative results are illustrated in figures 12, 14, and 15, and condensed in simple approximate formulas through (4.9), (6.2), and (7.8). Qualitatively, it can be stated that, on wings which do not depart excessively from the plane, the moment M_2 about the normal axis (hence that of the three moments which is other than zero even on a perfectly straight wing) remains the paramount stress; and not until there is a very appreciable three-dimensional curvature (when the ratio b/a of the cylinder radii approaches 1) do the two other components of the three-dimensional moment vector, the bending moment M_1 and the torque M_T become perceptible. Since M_2, as the graphs indicate, varies but little with $\epsilon = b/a$ if suitable reference quantities are chosen, rings for "small" openings can be computed as straight rings with very good approximation.

The (closed) ring is statically indeterminate. It effectively evades an excessive stress induced by insufficient torsional stiffness by responding to the load largely with bending moments M_2 – still, on rings fully ineffective in torsion, it is recommended that the existence of the shear stresses within permissible limits be confirmed by approximating with the help of the cited empirical formulas.

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*"Spannungen in Ausschnittversteifungen." Luftfahrtforschung, vol. 18, no. 7, July 19, 1941, pp. 253-61.

The three explored load distributions are three column loads for the unstiffened cylindrical shell; hence they create in the undisturbed shell a pure membrane stress attitude. The calculation is predicated on the assumption that this membrane stress attitude is not materially disturbed by the elastic interference effect between the stiffener opening and the skin. This assumption is met in the "extreme" case of a very stiff ring and a thin-wall shell without frames (or with frames located at some distance from the opening). For the frameless shell would have to attempt to terminate the "interference loads" returned by the ring through cross stresses and bending moments; since these do not become large in the thin shell and are damped quickly besides, any "aid" of the shell for the ring can manifest itself merely in the formation of a small effective border zone which takes nothing essential away from the ring.

In the opposite extreme case (not discussed here) of a ring rigid in strain but flexible in bending and of a shell closed all around by closely placed stiff frames or curved floors – "egg surface" – the state of stress and strain is utterly different. Shell and ring are for the most part subject to diaphragm and axial stresses, and stressed in bending solely by the constrained stresses due to the incompatibility of the form changes. The case is of little practical concern, since structural reasons usually call for rings which are far from ineffective in bending.

The true shell lies between the two extremes. If the ring is distinctly rigid in bending and the shell either is thick-walled or forms an egg surface, a complicated elastic interference effect results which defies calculation and must be ascertained experimentally. The present solution supplies the basis for such experiments by enabling the estimation of the maximum bending stresses to be expected through the determination of their upper limit.

INTRODUCTION

The Flat Ring

Concerning the exact stress distribution of three dimensionally curved rings, such as are used as stiffeners on openings in shells of all kinds, little data are available. The present study treats as a typical example a ring the center line of which is produced by the intersection of two circular cylinders of different diameter.* Three load cases are analyzed:

1. Axial and circumferential stresses in both cylinders, the cylinder stresses themselves to be in the ratio conformal to the cylinders loaded under internal pressure

2. Pure longitudinal tension in the large cylinder

3. Pure shear (torsion) in the large cylinder

To simplify the calculation, it is assumed that the ring, compared to the shell, is very strong, so that its deformations have no perceptible effect on the stress condition in the shell. This provides an upper limit for the ring stresses actually produced in a shell design, for, according to the theory of stressed skin statics the shells, by elastic flexibility of the ring, regroup the forces deposited on it in such a manner that the ring is relieved.

Load case 1.- The solution can be given immediately in the extreme case $a \gg b$ (figs. 1 and 7), that is, if the ring is "practically" flat. Then the forces exerted by the small cylinder are secondary alongside those of the large cylinder: the force pa along the circumferential circles and the force $pa/2$ along the generating axis. Since an equal tension $pa/2$ from all sides stretches the ring without twisting it, the bending stress can be computed as if the ring were loaded in the manner shown in figure 2. (Load cases 1 and 2 become identical except for the exchange of axes.)

The equilibrium conditions on the ring element $ds = b d\varphi$ are expressed in vectorial form from the very start in view of their subsequent application to the three-dimensional problem. Thus $\underline{t}$ denotes the unit vector of the tangent pointing toward increasing arc length s, $\underline{n}$ the unit vector of the normal toward the center of the circle, $\underline{k}$ the unit vector at right angles to the plane of the ring (fig. 3); $\underline{N}$ is to indicate the resultant,

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*In fig. 1 the smaller cylinder is shown outside the large one. But the ring formulas apply exactly, if extending wholly or partly in the large cylinder; the small cylinder can be arbitrarily short; it can be formed by the ring itself, for instance.

$\underline{M}$ the moment of the section stresses applied at a section with the outside normal $\underline{t}$; at the section boundary for which $\underline{t}$ is the inside normal the resultants $-\underline{N}$, $-\underline{M}$ are effective. Since $\underline{N}$ and $\underline{M}$ vary with s, the amounts on "front" and "rear" of a piece of length ds differ by $d\underline{N}$, $d\underline{M}$. In consequence, the force equilibrium specifies according to figure 4:

$$d\underline{N} + \underline{p}\, ds = 0$$

the moment equilibrium (in absence of external moment loading)

$$d\underline{M} + (\underline{t}\, ds) \times \underline{N} = 0$$

(the choice of moment reference point within length ds being immaterial, since the differences of higher order accruing therefrom become small within the limit $d \to 0$). These vectorial equilibrium expressions for the bar element

$$\left.\begin{aligned} \frac{d\underline{N}}{ds} + \underline{p} &= 0 \\ \frac{d\underline{M}}{ds} + \underline{t} \times \underline{N} &= 0 \end{aligned}\right\} \qquad (1.1)$$

can, if load, sectional force, and moment are divided into components along $\underline{t}$, $\underline{n}$, $\underline{k}$, be written in the form

$$\frac{d}{ds}(N\,\underline{t} + Q\,\underline{n}) + p_t\,\underline{t} + p_n\,\underline{n} = 0$$

$$\frac{d}{ds}(M\,\underline{k}) + \underline{t} \times (N\,\underline{t} + Q\,\underline{n}) = 0$$

and because of

$$\frac{d\underline{t}}{ds} = \frac{1}{b}\,\underline{n}, \quad \frac{d\underline{n}}{ds} = -\frac{1}{b}\,\underline{t}, \quad \underline{t} \times \underline{n} = \underline{k} \qquad (1.2)$$

are equivalent to the scalor

$$\frac{dN}{ds} - \frac{Q}{b} = -p_t, \quad \frac{dQ}{ds} + \frac{N}{b} = -p_n, \quad \frac{dM}{ds} + Q = 0 \qquad (1.3)$$

which, in this two-dimensional case could naturally have been read off as well from figure 5.

In this particular load study the components p_t, p_n

(dimensions: force per unit length) should be replaced by

$$p_t = \frac{pa}{2} \sin\varphi \cos\varphi, \quad -p_n = \frac{pa}{2} \sin^2\varphi \tag{1.3'}$$

according to figure 6. The integration of (1.3) presents no difficulty. With $ds = b d\varphi$, the first two equations give:

$$\left.\begin{aligned} N &= A \cos\varphi + B \sin\varphi + \frac{pab}{2} \cos^2\varphi \\ Q &= B \cos\varphi - A \sin\varphi - \frac{pab}{2} \cos\varphi \sin\varphi \end{aligned}\right\} \tag{1.4}$$

The two integration constants A and B follow from the symmetry requirements according to which the cross stress $\varphi = 0$ and $\pi/2$ must disappear. Then

$$A = B = 0 \tag{1.4'}$$

Entering (1.4) and (1.4') in (1.3) and integrating affords

$$M = X - \frac{pab^2}{4} \cos^2\varphi \tag{1.5}$$

Integration constant X remains statically indeterminant; it can be computed by means of Castigliano's principle of least-strain energy

$$\frac{\partial}{\partial X} \int \frac{M^2}{2EJ} ds = 0 \tag{1.6}$$

For the specific case $\overline{EJ}$ = const, equations (1.5) and (1.6) give

$$M = -\frac{pab^2}{8} \cos 2\varphi \tag{1.7}$$

In consequence, the two extreme values of the moment at points $\varphi = 0$, $\varphi = \pi/2$ are inversely equivalent and amount to

$$M_{max} = \frac{pab^2}{8} \tag{1.7'}$$

Equation (1.5) for the moment can equally be derived by another process which is much more simple, to wit: According to figure 6, the first vector equation (1.1), when resolved along the vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ (place independent) characteristizing the space directions x, y, z, instead of along the "natural" variable directions $\underline{t}$, $\underline{n}$, $\underline{k}$ read

$$\frac{d\underline{N}}{b\,d\varphi} + \frac{pa}{2}\,\underline{j}\sin\varphi = 0$$

the integration of this equation is even simpler than that of (1.3); we get

$$\underline{N} = \underline{N}_0 + \frac{pab}{2}\,\underline{j}\cos\varphi = C_1\underline{i} + C_2\underline{j} + \frac{pab}{2}\,\underline{j}\cos\varphi \qquad (1.8)$$

that is,

$$\left.\begin{aligned} Q &= \underline{N} \times \underline{n} = -C_1\cos\varphi - C_2\sin\varphi - \frac{pab}{2}\sin\varphi\cos\varphi \\ N &= \underline{N} \times \underline{t} = -C_1\sin\varphi + C_2\cos\varphi + \frac{pab}{2}\cos^2\varphi \end{aligned}\right\} \qquad (1.8')$$

Integration constants C_1 and C_2 disappear because $Q(0) = Q(\pi/2) = 0$; hence

$$\underline{N} = \frac{pab}{2}\,\underline{j}\cos\varphi \qquad (1.8'')$$

and with it follows, because $\underline{t} \times \underline{j} = -\underline{k}\sin\varphi$ and $\underline{M} \equiv M\,\underline{k}$ from the second equation of (1.1), and equation (1.5)

$$M = \frac{pab}{2}\int \cos\varphi\sin\varphi\,d\varphi = X - \frac{pab^2}{4}\cos^2\varphi \qquad (1.9)$$

The calculation of the three dimensionally curved ring also proceeds in two stages: the determination of the intersection resultants (forces and moments) as far as is possible on the basis of the static statements (1.1), and the solution of the statically indeterminate quantities on the basis of the strain conditions. The problem is most easily solved if the vectors for the integration of the differential equations (1.1) are resolved along the place-independent system of unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$, and then the transfer to a system of axes attached to the space curve (tangent $\underline{t}$ and two normals $\underline{n}_1$, $\underline{n}_2$ carried out the prediction of the integration constants from the strain and symmetry conditions – and these only – necessitates resolution along the natural axes $\underline{t}$, $\underline{n}_1$, $\underline{n}_2$, because the strain law and the symmetry expressions are amenable to simple formulation only for such components of the force and moment vectors.

GEOMETRY OF THE SPACE CURVE

The first step in solving the three-dimensional ring problem is the laying down of the formulas characterizing the geometry of the space curve. – With the notation of figure 7 the space curve is given by the two formulas

$$x^2 + y^2 = b^2, \quad y^2 + z^2 = a^2$$

With the choice of the angle φ projected into the xy plane as place parameter and $b/a = \epsilon$, the triple equation reads

$$\frac{x}{b} = \cos\varphi, \quad \frac{y}{b} = \sin\varphi, \quad \frac{z}{b} = \frac{1}{\epsilon}\sqrt{1 - \epsilon^2 \sin^2\varphi} \tag{2.1}$$

From the geometry of the space curve represented by (2.1) two groups of formulas are applied: 1) the expressions for the arc length and for the three unit vectors of an "accompanying triangle," 2) the relations expressing the position of the curve element with respect to the force directions. – The arc length follows from

$$ds = \sqrt{\left(\frac{ds}{d\varphi}\right)^2} d\varphi = \sqrt{\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2 + \left(\frac{dz}{d\varphi}\right)^2} d\varphi$$

with

$$\left.\begin{aligned} \frac{dx}{d\varphi} &\equiv \dot{x} = -b\sin\varphi, \quad \frac{dy}{d\varphi} \equiv \dot{y} = b\cos\varphi \\ \frac{dz}{d\varphi} &\equiv \dot{z} = -b\frac{\sin\varphi\cos\varphi}{\sqrt{1 - \epsilon^2\sin^2\varphi}} \end{aligned}\right\} \tag{2.2}$$

at

$$ds = b\,d\varphi\sqrt{\frac{1 - \epsilon^2\sin^4\varphi}{1 - \epsilon^2\sin^2\varphi}} \tag{2.3}$$

The introduction of the angle ψ of the space curve tangent with respect to its projection

$$\tan\psi = \frac{dz}{b\,d\varphi} = -\frac{\epsilon\sin\varphi\cos\varphi}{\sqrt{1 - \epsilon^2\sin^2\varphi}} \tag{2.4}$$

$$\underline{t} = \frac{dx}{ds}\underline{i} + \frac{dy}{ds}\underline{j} + \frac{dz}{ds}\underline{k}$$

$$= (\dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k})\frac{d\varphi}{ds} = (\dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k})\frac{\cos\psi}{b}$$

with cos and sin of ψ being given by

$$\cos\psi = \sqrt{\frac{1 - \epsilon^2 \sin^2\varphi}{1 - \epsilon^2 \sin^4\varphi}}, \quad \sin\psi = -\frac{\epsilon \sin\varphi \cos\varphi}{\sqrt{1 - \epsilon^2 \sin^4\varphi}} \tag{2.4'}$$

according to (2.4).

Defining the two mormals $\underline{n}_1$ and $\underline{n}_2$ by the stipulation (of itself arbitrary, but in view of the simplicity of the formulas appropriate), to place $\underline{n}_1$ "horizontal;" ($\underline{n}_1 \times \underline{k}=0$, $\underline{n}_1 \times \underline{t}=0$, $\underline{n}_1^2=1$), the formulas for the three axes read*

$$\left.\begin{aligned} \underline{t} &= (-\underline{i}\sin\varphi + \underline{j}\cos\varphi)\cos\psi + \underline{k}\sin\psi \\ \underline{n}_1 &= -(\underline{i}\cos\varphi + \underline{j}\sin\varphi) \\ \underline{t} \times \underline{n}_1 = \underline{n}_2 &= (\underline{i}\sin\varphi - \underline{j}\cos\varphi)\sin\psi + \underline{k}\cos\psi \end{aligned}\right\} \tag{2.5}$$

The forces that stress the ring are applied on it by the cylinder skins.

Consider figure 9, which represents a ring element ds and (slightly shifted) a skin element at one side of which (with the normal $\overline{\nu}$) a shear force $\underline{K}$ is applied. The load $\underline{p}$ acting on the element of the ring is, because of the equilibrium in the skin element, given by

$$\underline{p} = \underline{K}\cos\alpha = \underline{K}(\underline{n}\,\overline{\nu}) \tag{2.6}$$

There the ring normal $\underline{n}$ is characterized by the fact that

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*The use of the so-called natural axes $\underline{t}$, $\underline{n}$, $\underline{b}$ (tangent, principal normal, binormal) for describing the curve is unnecessary and, in general, inappropriate. For a determination of the natural axes which has not implicit connection with the principal inertia axes of the section requires the knowledge of the third derivations of the system (2.1); whereas (2.5) follows from the first derivations only. The fact that one of the curvature components disappears on the natural axes is an advantage which is of no consequence compared to this drawback.

it with $\bar{v}$ and $\underline{t}$ is located in one plane; hence it falls in the tangential plane of the particular cylinder. It is most simply obtained over the surface normal $\underline{n}_I$ or $\underline{n}_{II}$, respectively, to which it must be at right angles. The outside normal of a circular cylinder falls along the radius vector. Therefore, because of (2.1)

$$\left.\begin{aligned} \underline{n}_I &= \underline{j}\,\epsilon \sin\varphi + \underline{k}\sqrt{1 - \epsilon^2 \sin^2\varphi} \\ \underline{n}_{II} &= \underline{i}\cos\varphi + \underline{j}\sin\varphi (= -\underline{n}_1) \end{aligned}\right\} \qquad (2.7)$$

With the identifying signs of figure 10, the desired curve normals become

$$\left.\begin{aligned} \underline{n}_a = \underline{t} \times \underline{n}_I &= \underline{i}\,\frac{\cos\varphi}{\sqrt{1 - \epsilon^2 \sin^4\varphi}} \\ &\quad + \underline{j}\,\frac{(1 - \epsilon^2 \sin^2\varphi)\sin\varphi}{\sqrt{1 - \epsilon^2 \sin^4\varphi}} - \underline{k}\,\epsilon \sin^2\varphi \cos\psi \end{aligned}\right\} \qquad (2.8)$$

as affecting the large cylinder

$$\begin{aligned} \underline{n}_b (= \underline{n}_2) = \underline{n}_{II} \times \underline{t} &= \underline{i}\sin\varphi \sin\psi \\ &\quad - \underline{j}\cos\varphi \sin\psi + \underline{k}\cos\psi \end{aligned}$$

relative to the small cylinder.

Later on, the angle between the tangent

$$\underline{t}' = (\underline{j}\sqrt{1 - \epsilon^2 \sin^2\varphi} - \underline{k}\,\epsilon \sin\varphi)$$

at a circumferential circle of the large cylinder and the normal $\underline{n}_a$ is particularly needed;

$$\cos(\underline{t}', \underline{n}_a) = \underline{t}\,\underline{n}_a = \sin\varphi \cos\psi \qquad (2.8')$$

THE EQUILIBRIUM EQUATIONS

The equilibrium equations for the ring element (distributed outside moments discounted for the time being) read in vector form as in the two-dimensional case:

$$\frac{d\underline{N}}{ds} + \underline{p} = 0, \quad \frac{d\underline{M}}{ds} + \underline{t} \times \underline{N} = 0 \tag{3.1}$$

with $\underline{p}\,ds$ as vector of the external force applied at the ring element, as results from (2.6); the shear force vector

$$\underline{N} = N\underline{t} + Q_1\underline{n}_1 + Q_2\underline{n}_2 \tag{3.1'}$$

has this time the longitudinal force N and the two cross forces Q_1 and Q_2 as components; the shear moment vector

$$\underline{M} = M_T\underline{t} + M_1\underline{n}_1 + M_2\underline{n}_2 \tag{3.1''}$$

has the torsion moment M_T and bending moments M_1 and M_2.

Equation (3.1) is integrated in two stages:

1. $$\underline{N} = \underline{N}(0) - \int_0^s \underline{p}\,ds = \underline{N}(0) - b\int_0^\varphi \underline{p}\,\frac{d\varphi}{\cos\psi} \tag{3.2}$$

2. $$\underline{M} = \underline{M}(0) - \int_0^s \underline{t} \times \underline{N}ds = \underline{M}(0) - b\int_0^\varphi \underline{t} \times \underline{N}\,\frac{d\varphi}{\cos\psi} \tag{3.3}$$

The integration constant $\underline{N}(0)$ is again found by symmetry considerations, one component of the second constant $\underline{M}(0)$ remains indeterminate. The symmetry of system and load requires the disappearance of the three antisymmetrical quantities Q_1, Q_2, and M_T (the shear resultants) at points $\varphi = 0$ and $\varphi = \pi/2$; from $Q_1(0) = 0$ and $Q_1(\pi/2) = 0$ follow the $\underline{i}$ and $\underline{j}$ components, from $Q_2(0) = Q_2(\pi/2) = 0$, the $\underline{k}$ component of $\underline{N}(0)$; from $M_T(0) = 0$ and $M_T(\frac{\pi}{2}) = 0$ the $\underline{i}$ and $\underline{j}$ components of $\underline{M}(0)$ – the $\underline{k}$ component of $\underline{M}(0)$ – the $\underline{k}$ component of $\underline{M}(0)$ remains to be determined by a strain equation.*

Denoting the intensities of the shear load of the two cylinders along the generating axis and the circumferential circle with

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*Since, for reasons of symmetry, $\underline{N}$ and $\underline{M}$ contain only the even-number harmonics in φ, $Q_2(\pi/2) = 0$ follows from $Q_2(0) = 0$. Hence the cited 6 symmetry conditions yield only 5 independent equations for the determination of the 6 integration constants.

p_1, p_2 for the large cylinder

p_3, p_4 for the small cylinder

equations (2.6), (2.8), and (2.8') afford

$$\underline{p} = p_1 \underline{i} \frac{\cos\varphi}{\sqrt{1-\epsilon^2 \sin^4\varphi}} + p_2(\underline{j}\sqrt{1-\epsilon^2\sin^2\varphi} - \epsilon\,\underline{k}\sin\varphi)$$

$$\times \sin\varphi\cos\psi + p_3\,\underline{k}\cos\psi + p_4(\underline{j}\cos\varphi - \underline{i}\sin\varphi)\sin\psi \tag{3.5}$$

whence

$$\underline{N}_1 = -\int p_1 \underline{i} \frac{\cos\varphi}{\sqrt{1-\epsilon^2\sin^4\varphi}}\,ds = -p_1 b\underline{i}\int \frac{\cos\varphi}{\sqrt{1-\epsilon^2\sin^2\varphi}}\,d\varphi$$

$$= -p_1 b\underline{i}\,\frac{1}{\epsilon}\,\text{arc}\sin(\sin\varphi),$$

$$\underline{N}_{21} = -\int p_2\underline{j}\sqrt{1-\epsilon^2\sin^2\varphi}\,\sin\varphi\cos\psi\,ds =$$

$$+\frac{p_2\underline{j}b}{2}\left[\cos\varphi\sqrt{1-\epsilon^2\sin^2\varphi} + \frac{1}{\epsilon}(1-\epsilon^2)\sinh^{-1}\frac{\epsilon\cos\varphi}{\sqrt{1-\epsilon^2}}\right]$$

$$\underline{N}_{22} = \int p_2\underline{k}\,\epsilon\sin^2\varphi\cos\psi\,ds = p_2\underline{k}\,\epsilon\, b\left(\frac{\varphi}{2} - \frac{\sin\varphi\cos\varphi}{2}\right),$$

$$\underline{N}_3 = -\int p_3\underline{k}\cos\psi\,ds = -p_3\,\underline{k}\,b\,\varphi\quad ,$$

$$\underline{N}_{41} = -\int p_4\,\underline{i}\sin\varphi\sin\psi\,ds = +p_4 b\underline{i}\int\frac{\epsilon\sin^2\varphi\cos\varphi}{\sqrt{1-\epsilon^2\sin^2\varphi}}\,d\varphi$$

$$= -p_4\,\underline{i}\,\frac{b}{2\epsilon}\left[\sin\varphi\sqrt{1-\epsilon^2\sin^2\varphi} - \frac{1}{\epsilon}(\sin^{-1})\ (\epsilon\sin\varphi)\right],$$

$$\underline{N}_{42} = \int p_4\,\underline{j}\cos\varphi\sin\psi\,ds = p_4 b\underline{j}\,\epsilon\int\frac{\sin\varphi\cos^2\varphi}{\sqrt{1-\epsilon^2\sin^2\varphi}}\,d\varphi$$

$$= p_4\,\underline{j}\,\frac{b}{2\epsilon}\left[\cos\varphi\sqrt{1-\epsilon^2\sin^2\varphi} - \frac{1}{\epsilon}(1-\epsilon^2)\sinh^{-1}\frac{\epsilon\cos\varphi}{\sqrt{1-\epsilon^2}}\right] \tag{3.6}$$

It is noted that the load portions p_2 and p_3, which produce $\underline{N}_{22}$ and $\underline{N}_3$, of themselves form no equilibrium groups, because the $\underline{N}$ contain a non-periodic portion, which cancels out when p_3 is equated to $\frac{\epsilon}{2} p_2$. This occurs, for instance, if the four forces p_i originate through the same internal pressure p in the two cylinders.

$$p_1 = \frac{pa}{2}, \quad p_2 = pa, \quad p_3 = \frac{pb}{2}, \quad p_4 = pb \tag{3.7}$$

Those values, written in (3.6) and condensed, give

$$\underline{N} = \frac{pab}{2}\Big[-\underline{i} \sin\varphi\sqrt{1-\epsilon^2\sin^2\varphi} + 2\underline{j}\cos\varphi\sqrt{1-\epsilon^2\sin^2\varphi} - \epsilon\,\underline{k}\sin\varphi\cos\varphi\Big] \tag{3.6}$$

Force $\underline{N}$ is represented by (3.6') with the correct integration constants, because the $\underline{i}$ and $\underline{k}$ portion disappears at $\varphi = 0$, the $\underline{j}$ portion at $\varphi = \pi/2$.

The solution of the moments according to (3.3) is predicated on the three vectors $\underline{t} \times \underline{i}$, $\underline{t} \times \underline{j}$, $\underline{t} \times \underline{k}$. According to (2.5)

$$\left.\begin{aligned}
\underline{t} \times \underline{i} &= \underline{j}\sin\psi - \underline{k}\cos\varphi\cos\psi, \\
\underline{t} \times \underline{j} &= -\underline{i}\sin\psi - \underline{k}\sin\varphi\cos\psi, \\
\underline{t} \times \underline{k} &= \underline{i}\cos\varphi\cos\psi + \underline{i}\sin\varphi\cos\psi.
\end{aligned}\right\} \tag{3.8}$$

which, entered along with (3.6') in (3.3) and integrated, gives with $\underline{M}(0) = X\underline{k}$

$$\underline{M} = X\underline{k} + \frac{pab^2}{2}\left(\underline{i}\,\frac{\epsilon\cos^3\varphi}{3} - \underline{k}\,\frac{(1-\epsilon^2\sin^2\varphi)^{3/2}}{3\epsilon^2}\right) \tag{3.9}$$

By means of the transformation equations (2.5) the natural components M_T, M_1 M_2 of the moment vector then follow at

$$\left.\begin{aligned}
M_T &= \frac{pab^2}{6}\,\frac{1-\epsilon^2}{\epsilon}\sin\varphi\cos\varphi\cos\psi + X\sin\psi, \\
M_1 &= -\frac{pab^2}{6}\,\epsilon\cos^4\varphi, \\
M_2 &= -\frac{pab^2}{6}\,\frac{(1-\epsilon^2\sin^2\varphi)^2 + \epsilon^4\sin^2\varphi\cos^4\varphi}{\epsilon^2\sqrt{1-\epsilon^2\sin^4\varphi}} + X\cos\psi
\end{aligned}\right\} \tag{3.9'}$$

these expressions satisfy, as is seen, the symmetry conditions $M_T(0) = M_T(\pi/2) = 0$.

The course of the three moments, particularly in the important practical case of $\epsilon \ll 1$ (small openings), is of interest. Expansion in powers of ϵ affords

$$
\begin{aligned}
M_T &= \frac{pab^2}{6}\,\frac{1}{\epsilon}\sin\varphi\cos\varphi\left[1 - \epsilon^2\left(1 + \frac{\sin^2\varphi\cos^2\varphi}{2}\right) - \dots\right] \\
&\quad - \epsilon X \sin\varphi\cos\varphi\left(1 + \frac{\epsilon^2}{2}\cos^4\varphi + \dots\right), \\
M_1 &= -\frac{pab^2}{6}\,\epsilon\cos^4\varphi, \qquad (3.9'') \\
M_2 &= \frac{pab^2}{6}\left\{-\left[\frac{}{\epsilon^2} - \left(2\sin^2\varphi - \frac{1}{2}\sin^4\varphi\right) - \dots\right]\right\} \\
&\quad + X\left(1 - \frac{\epsilon^2}{2}\sin^2\varphi\cos^2\varphi - \dots\right)
\end{aligned}
$$

SOLUTION OF THE STATICALLY INDETERMINATE X

Simple Formulas for Maximum Moments

The prediction of the integration constant X is predicated upon a strain equation. If expressed in the form of Castigliano's principle of least strain energy, the geometry of the strain condition is secondary (reference 1). In vectorial form Castigliano's requirement reads

$$\int \underline{M}\,\underline{\tilde{k}}\,ds = \text{Min} \qquad (4.1)$$

with $\underline{k}$ the vector of the curvature change

$$\underline{\tilde{k}} = \vartheta\,\underline{t} + \kappa_1\underline{n}_1 + \kappa_2\,\underline{n}_2$$

with three components: twist ϑ and curvature changes κ_1, κ_2. The thin, slightly curved bar serves as basic strain law, the general case of diagonal bending being analyzed at once. Taking into consideration

$$\sigma_x = E(w''\bar{z} + v''\bar{y}) = E(\kappa_1\bar{z} - \kappa_2\bar{y})$$

($\bar{y}$, $\bar{z}$ distances from the centroidal fiber of the bar).

$$M_1 = \int \sigma_x \bar{z} \, d F, \quad M_2 = - \int \sigma_x \bar{y} \, d F$$

the law reads

$$\left.\begin{aligned} M_T &= G J_T \vartheta \\ M_1 &= G J_1 \kappa_1 - E J_{12} \kappa_2 \\ M_2 &= G J_2 \kappa_2 - E J_{12} \kappa_2 \end{aligned}\right\} \qquad (4.2)$$

with J_1, J_2, J_{12} the inertia and the centrifugal moments referred to axes $\underline{n}_1$ and $\underline{n}_2$ ($\bar{y}$ and $\bar{z}$), J_T the torsional resistance. The solution of (4.2) inserted in (4.1) gives, with the abbreviation

$$\Delta = 1 - \frac{J_{12}{}^2}{J_1 J_2}$$

the equation for X:

$$\frac{d}{d X} \int \underline{M} \, \underline{\tilde{k}} \, ds = 0 = \int \frac{1}{E\Delta} \left(\frac{M_2^{(o)}}{J_2} + \frac{M_1^{(o)} J_{12}}{J_1 J_2} \right) \cos\psi \, d s$$

$$+ \int \frac{M_T^{(o)}}{G J_T} \sin \psi \, ds + X \left[\int \frac{\cos^2\psi \, d s}{E J_2 \Delta} + \int \frac{\sin^2 \psi \, ds}{G J_T} \right]$$

$$= \int \frac{b}{E\Delta} \left(\frac{M_2^{(o)}}{J_2} + \frac{M_1^{(o)} J_{12}}{J_1 J_2} \right) d \varphi + \int b \frac{M_T^{(o)}}{G J_T} \tan \psi \, d \varphi$$

$$+ X \left[\int \frac{b \cos \psi \, d \varphi}{E J_2 \Delta} + \int \frac{\sin^2\psi}{\cos \psi} \frac{b}{G J_T} d \varphi \right] \qquad (4.3)$$

where $M_2^{(o)}$, $M_1^{(o)} = M_1$, $M_T^{(o)}$ are the statically determined portions in (3.9').

The evaluation of (4.3) is predicated on the course of the quantities J with s and φ, respectively. To begin with, $\frac{M_1^{(o)} J_{12}}{J_1}$ will usually be small, compared with $M_2^{(o)}$, because $M_1^{(o)}$ is smaller by three ϵ powers than $M_2^{(o)}$ and factor J_{12}/J_1 itself will be perceptibly smaller than 1. So, the second term of the first parenthesis can be discounted for the rough calculation of X. The same applies to factor Δ in the denominator, which, even by marked departure from 1, does not affect the result for X materially, since it acts in the same sense in the numerator and the denominator. If desired, it can be allowed for in the determination of the still remaining essential parameter

$$\alpha = \frac{GJ_T}{EJ_2} \ (<1) \quad \text{or} \quad = \frac{GJ_T}{EJ_2\Delta} \tag{4.4}$$

which indicates the ratio of torsional to flexural stiffness. Assume average values for J_2 and J_T to be independent of s, the X equation simplifies to

$$X = \frac{A + B\alpha}{C + D\alpha} \tag{4.5}$$

with

$$\left.\begin{aligned} A &= \int_0^{\pi/2} M_T^{(o)} \tan\psi \, d\varphi, \qquad B = \int_0^{\pi/2} M_2^{(o)} \, d\varphi \\ C &= \int_0^{\pi/2} \frac{\sin^2\psi}{\cos\psi} \, d\varphi, \qquad D = \int_0^{\pi/2} \cos\psi \, d\varphi \end{aligned}\right\} \tag{4.5'}$$

For large ϵ values the integrals A...D must be numerically evaluated, for small ϵ integration by series expansion is suggested.

$$\left.\begin{aligned} A &= \frac{pab^2}{6} \int \sin^2\varphi \cos^2\varphi\left(1 - \epsilon^2\left(1 - \frac{\sin^4\varphi}{2}\right) - \ldots\right)d\varphi \\ &= \frac{pab^2}{6\times 8}\left(1 - \epsilon^2\left(1 - \frac{5}{2^5}\right) - \ldots\right) \\ B &= \frac{pab^2}{6}\frac{1}{\epsilon^2}\int\left(1 - \epsilon^2\left(2\sin^2\varphi - \frac{1}{2}\sin^4\varphi\right) + \ldots\right)d\varphi \\ &= \frac{pab^2}{6}\left(\frac{1}{\epsilon^2} - \left(1 - \frac{3}{2^4}\right) - \ldots\right) \\ C &= \epsilon^2 \int \sin^2\varphi \cos^2\varphi\left(1 + \frac{\epsilon^2}{2}(\sin^2\varphi + \sin^4\varphi) + \ldots\right)d\varphi \\ &= \frac{\epsilon^2}{8}\left(1 - \frac{\epsilon^2}{4}(1 + 5/4) - \ldots\right) \\ D &= \int (1 - \epsilon^2\sin^2\varphi\cos^2\varphi - \ldots)d\varphi = \left(1 - \frac{\epsilon^2}{4}(1 - 3/4) - \ldots\right) \end{aligned}\right\} \tag{4.5''}$$

Figure 11 illustrates the result of the calculation of X for a specified axis ratio ($\epsilon = 1/2$), along with the three statically determinate quantities $M_T^{(o)}$, $M_1^{(o)}$, $M_2^{(o)}$ and – for different α values – the curves $X \sin\psi$ and $X \cos\psi$, respectively, from which the final moments M_T and M_2 must be marked off.

At $\epsilon = 1$, the ring degenerates to two flat pieces (semi-ellipsoids), which meet under a right angle. The particular loading (3.7) which is symmetrical to the two ellipsoidal planes, stresses then each half of the ring in its plane only. In other words, the statically determinate portion of the torsion moment must disappear (first equation of (3.9')) and the statically determinate portions of both bending moments must combine to a bending moment about the normal to the plane of the ellipse. In point of fact, the moment vector at $\epsilon = 1$ ($a = b$) for the first and fourth quadrants reads, according to (3.9)

$$\underline{M}^{(o)} = \frac{pab^2}{6} \cos^3\varphi(\underline{i} - \underline{k})$$

for the second and third quadrants

$$\underline{M}^{(o)} = \frac{pab^2}{6} \cos^3\varphi(\underline{i} \pm \underline{k})$$

hence is at right angles to the plane of the ellipse. At $\alpha = 0$, that is, vanishing torsional stiffness, the statically indeterminate (4.5) disappears, because the other half cannot absorb a bending moment as torsion moment at $\varphi = \pi/2$. Hence $M^{(o)}(\pi/2) = 0$; – at $\alpha \neq 0$ a reciprocal restraint occurs which produces torsion and bending moments diverging from $M^{(o)}$, that is, twists the ring half out of its plane. (The result of the calculation for any ϵ and α, illustrated in figure 12, indicates that $M_1^{(o)}$ and $M_2^{(o)}$ actually disagree at $\alpha \neq 0$.)

The maximum amounts of the moments in relation to α and ϵ are of particular concern. For $\epsilon < 1/2$, they are readily obtained by means of the series expansions (4.5"). For X there is obtained

$$X = \frac{pb^2a}{6} \frac{1}{\epsilon^2} \frac{\alpha(1-0.8125\epsilon^2+0.2275\epsilon^4)+\frac{\epsilon^2}{8}(1-0.8437\epsilon^2)+\ldots}{\alpha(1-0.0625\epsilon^2-0.0175\epsilon^4)+\frac{\epsilon^2}{8}(1+0.4063\epsilon^2)+\ldots} \qquad (4.6)$$

hence at $\alpha \neq 0$

$$= \frac{pab^2}{6}\left[\frac{1}{\epsilon^2} - \frac{3}{4} + \epsilon^2\left(0.1981 - \frac{1}{16\alpha}\right) + \ldots\right] \qquad (4.6')$$

at $\alpha = 0$ (very low torsional stiffness of ring)

$$= \frac{pab^2}{6}\left[\frac{1}{\epsilon^2} - \frac{5}{4} + 0.2738\ \epsilon^2 + \ldots\right] \qquad (4.6'')$$

(The statically indeterminate destroys, as it should, in both cases the strongest term in the expressions (3.9') for M_T and M_2). For the maximum values of bending moment M_2, which are located at the symmetry points φ = 0, $\pi/2$ ... ($\cos\psi = 1$), there is obtained

at $\alpha \neq 0$

$$\left.\begin{aligned} M_2(\varphi = 0) &= \frac{pab^2}{6}\left[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon^2} - \frac{3}{4} + \epsilon^2\left(0.1981 - \frac{1}{16\alpha}\right) + \ldots\right] \\ &= -\frac{pab^2}{8}\left[1 - \epsilon^2\left(0.264 - \frac{1}{12\alpha}\right) + \ldots\right] \\ M_2(\varphi = \pi/2) &= \frac{pab^2}{6}\left[-\frac{1}{\epsilon^2} + \frac{3}{2} - \frac{3\epsilon^2}{8} + \frac{1}{\epsilon^2} - \frac{3}{4} + \epsilon^2(\ldots) + \ldots\right] \\ &= \frac{pab^2}{8}\left[1 - \epsilon^2\left(0.236 + \frac{1}{12\alpha}\right. + \ldots\right] \end{aligned}\right\} \quad (4.7)$$

at $\alpha = 0$

$$\left.\begin{aligned} M_2(\varphi = 0) &= -\frac{pab^2}{8}\left[\frac{5}{3} - 0.365\ \epsilon^2 + \ldots\right] \\ M_2(\varphi = \pi/2) &= \frac{pab^2}{8}\left[\frac{1}{3} - 0.168\ \epsilon^2 + \ldots\right] \end{aligned}\right\} \quad (4.7')$$

The equations (4.7) confirm first the "two-dimensional" result ($\epsilon \longrightarrow 0$) $|M_{max}| = \frac{pab^2}{8}$; they further indicate that M_2 is reduced at $\varphi = \frac{\pi}{2}$, and likewise at $\varphi = 0$, as long as $\alpha > \frac{1}{12 \times 0.264} = 0.315$. For smaller α values the second bracket of the first expression reverses signs: the maximum amount of the "up" bending moment (< 0) at point 0 is greater in the three-dimensional $(\epsilon \neq 0)$ than in the two-dimensional case $(\epsilon = 0)$. That M_{2max} must become greater at small α values should not be surprising: the ring tries above all to evade a torsional stress; hence it can not devote much attention to the reduction of the maximum bending moment (bending and torsion are interconnected on the three-dimensional curved ring). At the limit $\alpha = 0$ this tendency of the ring even results in a radical departure of the $M_2(\alpha, \epsilon)$ curve from the others: curve $M_2(\epsilon)_{\varphi=0,\alpha=0}$ alone passes at $\epsilon = 0$ through $-\frac{5}{24}\ pab^2$ instead of the point $-\frac{1}{8}\ pab^2$, and drops monotonically to $-\frac{1}{6}\ pab^2$ at $\epsilon = 1$ as a function of ϵ. Figure 12, where $M_2(\epsilon)$ has been plotted for different α values, shows that the $M_2(\epsilon)$ curve clings for some distance to the boundary curve $M_2(\alpha = 0)$ for small torsional stiffness values, while the extreme value $5/24\ pab^2$ is not reached for finite values of α.

The maximum torsion moment (other than for $\alpha = 0$, where M_T is on the whole very small) is approximately located midway between the symmetry point 0 and $\pi/2$. Hence the amount $M_T(\pi/4)$ is determined as approximation for $M_{T_{max}}$. For small values ϵ the expansion in series is again recommended.

$$\left.\begin{aligned} M_T &= \frac{pab^2}{6}\,\epsilon \sin\varphi \cos\varphi\left\{\frac{1}{\epsilon^2} - \left(1 + \frac{\sin^2\varphi\cos^2\varphi}{2}\right) - \epsilon^2(\ldots)\right. \\ &\quad \left. -\left(\frac{1}{\epsilon^2} - \frac{3}{4} + \epsilon^2(\ldots) + \ldots\right)\left(1 + \frac{\epsilon^2}{2}\sin^4\varphi + \frac{3\epsilon^4}{8}\sin^8\varphi\right)\right\} \\ M_T(\pi/4) &= \frac{pb^3}{6}\,\frac{1}{2}\left\{-\frac{1}{2} + \epsilon^2\left(\frac{5}{32} - 0.1984 + \frac{1}{16\alpha}\right) + \ldots\right\} \\ &= -\frac{pb^3}{24}\left[1 - \epsilon^2\left(\frac{1}{8\alpha} - 0.842\right)\right] \end{aligned}\right\} \quad (4.8)$$

At $\alpha = 0$ the next term in the series expansion along ϵ likewise disappears for $\varphi = \pi/4$, leaving:

$$\underset{(\alpha=0)}{M_T(\pi/4)} = \frac{pb^3}{12}\left[\epsilon^2\left(\frac{11}{128} + \frac{5}{32} - \frac{3}{128} - 0.2738\right)\right]$$

$$= -p\,b^3\epsilon^2 \times 0.0046 \approx 0$$

The result of this discussion is that, as regards M_2, the stress analysis with a small safety margin can be carried out according to the simple formula

$$M_{2\,max} = \frac{1}{8}\, pab^2 \qquad (4.9a)$$

For the maximum value of the other bending moment M_1, equation (3.9') affords

$$M_{1\,max} = M_1^{(o)} = \frac{\epsilon}{6}\, pab^2 = \frac{pb^3}{6} \qquad (4.9b)$$

For $M_{T_{max}}$ a simple approximation, which is practical up to $\epsilon = 0.7$ and remains on the safe side near $\epsilon = 1$, is given by the zero point tangent in figure 12

$$M_{T_{max}} = \frac{\epsilon}{24}\, pab^2 = \frac{pb^3}{24} \qquad (4.9c)$$

M_1 and M_T are in our particular load case independent of the radius of the large cylinder.

EFFECT OF ECCENTRIC STRESS APPLICATION

The effect of the moments which stress the ring direct following eccentric load application is secondary compared to the moments (3.9) set up by the cross stress because of the small lever arms. However, an appraisal seems desirable; it can be achieved by means of equation (3.5). To determine the order of magnitude of the additional moments an approximate assumption is that the forces p_1 and p_2 apply at a lever arm h_1 along $\underline{k}$, and forces p and p_4 at a lever arm h along $\underline{n}_1$ (fig. 13). Then the localized load (3.5) produces the following distributed moments $\underline{m}$:

$$\underline{m} = -\, p_1 h_1 \underline{i} \frac{\cos\varphi}{\sqrt{1 - \epsilon^2 \sin^4\varphi}} + p_2 h_1\, \underline{i} \sin\varphi \cos\psi \sqrt{1 - \epsilon^2 \sin^2\varphi}$$

$$+\, p_3\, h_2\, (-\, \underline{i} \sin\varphi + \underline{j} \cos\varphi) + p_4 \underline{k} \sin\psi \qquad (5.1)$$

which give rise to the shear moments

$$\underline{M}_{\underline{m}} = \int \underline{m}\, ds = b \int \underline{m} \frac{d\varphi}{\cos\psi} \qquad (5.2)$$

The integration gives

$$\underline{M}_{\underline{m}} = p_1 b h_1 \underline{j} \frac{1}{\epsilon} (\sin^{-1})(\epsilon \sin \varphi)$$

$$+ p_2 \frac{bh_1}{2} \underline{i} \left[\cos \varphi \sqrt{1 - \epsilon^2 \sin^2 \varphi} + \frac{1}{\epsilon} (1 - \epsilon^2) \sinh^{-1} \frac{\epsilon \cos \varphi}{\sqrt{1 - \epsilon^2}}\right.$$

$$+ p_3 b h_2 \left\{\underline{i} \cos \varphi + \underline{j} \sin \varphi\right\} - p_4 b \underline{k} \frac{h_2}{\epsilon} \sqrt{1 - \epsilon^2 \sin^2 \varphi} \quad (5.3)$$

and, with (3.7) added,

$$\underline{M}_{\underline{m}} = \frac{pab}{2} \left\{\underline{i} \left[h_1 \left(\cos \varphi \sqrt{1 - \epsilon^2 \sin^2 \varphi}\right.\right.\right.$$

$$\left.\left. + \frac{1 - \epsilon^2}{\epsilon} \sinh^{-1} \frac{\epsilon \cos \varphi}{\sqrt{1 - \epsilon^2}}\right) + h_2 \epsilon \cos \varphi\right]$$

$$\left. + \underline{j} h_1 \frac{1}{\epsilon} (\sin^{-1})(\epsilon \sin \varphi) + \epsilon h_2 \sin \varphi\right] + 2 \underline{k} h_2 \sqrt{1 - \epsilon^2 \sin^2 \varphi}\right\} \quad (5.4)$$

Since these moments are small compared to (3.9) in the ratio $\frac{h_1}{b}$ and $\frac{h_2}{b}$, a rough estimate in which only the lowest powers of ϵ are retained, is sufficient.

Accordingly,

$$\underline{M}_{\underline{m}} \approx \frac{pab}{2} \left\{\underline{i}(2 h_1 + \epsilon h_2) \cos \varphi + \underline{j}(h_1 + \epsilon h_2) + 2 \underline{k} h_2\right\}$$

or

$$\left.\begin{aligned} M_T(\underline{m}) &= \frac{pab}{2} [(-h_1 + 2 \epsilon h_2) \sin \varphi \cos \varphi] \\ M_1(\underline{m}) &= -\frac{pab}{2} [h_1 \cos^2 \varphi + (h_1 + \epsilon h_2)] \\ M_2(\underline{m}) &= \frac{pab}{2} [- \epsilon h_1 \sin^2 \varphi \cos^2 \varphi + 2 h_2] \end{aligned}\right\} \quad (5.5)$$

Since at $a > b$ the component M_2 is greater than M_1, according to figure 11, the inertia moment in M_2 direction itself will be enlarged on the ring dimensions; hence it is logical to assume that $h_1 \approx \epsilon h_2$ (fig. 13). With the subsequent simplifications

$$\left.\begin{aligned} M_T(\underline{m}) &= \frac{3pab}{2}\,\epsilon\, h_2 \sin\varphi \cos\varphi \\ M_1(\underline{m}) &= \frac{pab}{2}\,\epsilon\, h_2 \cos^2\varphi \\ M_2(\underline{m}) &= pab\, h_2 \end{aligned}\right\} \qquad (5.6)$$

it is readily seen that these quantities can have no effect on the determination X, since M_1 had acted no part previously and M_T and M_2 both are small in the ratio $\frac{\epsilon^2 h_2}{b}$ compared to the previous values. The maximum amounts themselves change in all three components by quantities that are small in the ratio h_2/b – hence inclusion of the moments due to eccentric load application are not worth while, which leaves for the ring dimensions the extreme values of figure 12, supplemented perhaps by a small safety margin.

LOADING IN PURE TENSION $\sigma_1 \underline{i}$

The calculating process is the same as before. In the axial-force table (3.6) $p_1 = \sigma_1 t$ and $p_2 = p_3 = p_4 = 0$; for the moment, according to (3.8):

$$\underline{M} = \sigma_1 t\, b^2 \int \frac{1}{\epsilon}(\sin^{-1})(\epsilon \sin\varphi)[\underline{j}\sin\psi - \underline{k}\cos\psi\cos\varphi]\frac{d\varphi}{\cos\psi} \qquad (6.1)$$

The integrations again give

$$M_T = \sigma_1 t\, b^2\left[\frac{1}{\epsilon}\frac{\cos\varphi}{\sqrt{1-\epsilon^2\sin^4\varphi}}\left(\frac{1}{\epsilon}(\sin^{-1})(\epsilon\sin\varphi)\right)\right] + X\sin\psi$$

$$M_1 = \sigma_1 t\, b^2\,\frac{1}{\epsilon}\left[\sin^2\varphi - \sin\varphi\sqrt{1-\epsilon^2\sin^2\varphi}\left(\frac{1}{\epsilon}(\sin^{-1})(\epsilon\sin\varphi)\right)\right]$$

$$M_2 = -\sigma_1 t\, b^2\left[\frac{1}{\epsilon^2}\sqrt{1-\epsilon^2\sin^4\varphi} + \sin^3\varphi\cos\psi\left(\frac{1}{\epsilon}(\cos^{-1})(\epsilon\sin\varphi)\right)\right] + X\cos\psi \qquad (6.1')$$

or, developed for small ϵ :

$$M_T = \sigma_1 t b^2 \frac{1}{\epsilon} \left[\sin\varphi \cos\varphi \left(1 + \frac{\epsilon^2}{6}\sin^2\varphi + \frac{\epsilon^2}{2}\sin^4\varphi + \ldots\right)\right]$$

$$- \epsilon X \sin\varphi \cos\varphi \left(1 + \frac{\epsilon^2}{2}\sin^4\varphi + \ldots\right)$$

$$M_1 = \sigma_1 t b^2 \epsilon \left[\frac{\sin^4\varphi}{3} + \frac{2\epsilon^2}{15}\sin^6\varphi + \ldots\right]$$

$$M_2 = -\sigma_1 t b^2 \frac{1}{\epsilon^2}\left[1 + \frac{\epsilon^2}{2}\sin^4\varphi - \frac{\epsilon^4}{3}\sin^6\varphi + \frac{3\epsilon^4}{8}\sin^8\varphi + \ldots\right]$$

$$+ X\left(1 - \frac{\epsilon^2}{2}\sin^2\varphi\cos^2\varphi - \ldots\right) \qquad (6.1'')$$

The maximum moments are obtained according to the previous considerations. They are shown in figure 14. In the vicinity of $\epsilon = 0$, that is, on a ring that does not depart too much from the flat ring, the curves (figs. 12 and 14) are in complete agreement – the loading is, indeed, approximately the same except for the 90° rotation of load direction. In the region $\epsilon \approx 1$, on the other hand, the stress of the ring is typically different in the two load cases: at point $\varphi = 0$ the ring is smooth, but at $\varphi = \pi/2$ it has a distinct precurvature which at $\epsilon = 1$ degenerates into a discontinuity; hence the load direction (and, of course, the subsidiary effect of the small cylinder in the first load case) is essential for the type of stress in point.

Possible approximating formulas, which fail, however, in this instance near $\epsilon = 1$, are:

$$\left.\begin{aligned} M_{2_{max}} &= \frac{1}{4}\sigma_1 t b^2 \\ M_{1_{max}} &= \frac{\epsilon}{3}\sigma_1 t b^2\left(1 + \frac{2}{5}\epsilon^2\right) \\ M_{T_{max}} &= \frac{\epsilon}{12}\sigma_1 t b^2 \end{aligned}\right\} \qquad (6.2)$$

LOADING OF LARGE CYLINDER IN PURE SHEAR τ

With $\tau_{xy} = \tau_{yx} = \tau$ indicating the two conjugate stresses, the first is applied at an area $x = \text{const.}$;

(hence with normal $\underline{i}$) and falls in vector direction

$$(\underline{j}\sqrt{1-\epsilon^2 \sin^2 \varphi} - \epsilon\, \underline{k} \sin \varphi)$$

hence stresses the ring, according to (2.8), with a load per unit length of

$$\underline{p}_1 = \frac{\cos \varphi}{\sqrt{1-\epsilon^2 \sin^4 \varphi}} (\underline{j}\sqrt{1-\epsilon^2 \sin^2 \varphi} - \epsilon\, \underline{k} \sin \varphi) \qquad (7.1)$$

The second stress is applied at an area with the normal $\underline{t}$ and falls in direction $\underline{i}$; hence, according to (2.8'), stresses the ring by

$$\underline{p}_2 = \underline{i} \sin \varphi \cos \psi \qquad (7.2)$$

The total loading of the ring is $\underline{p} = \underline{p}_1 + \underline{p}_2$, whence, according to (3.2) follows the shear load $\underline{N}$ in the form

$$\underline{N} = \tau t b \left(\underline{i} \cos \varphi - \underline{j} \sin \varphi - \frac{k}{\epsilon} \sqrt{1-\epsilon^2 \sin^2 \varphi} \right) + \underline{N}_0 \qquad (7.3)$$

The load distribution being antisymmetrical, this time with respect to $\varphi = 0$ and $\frac{\pi}{2}$, the axial load N must disappear at points 0 and $\frac{\pi}{2}$ for reasons of symmetry; whence

$$\underline{N}_0 = \frac{1}{b} X\, \underline{k} \qquad (7.4)$$

Quantity X remains statically indeterminate. From (3.8) and (7.3) is obtained

$$-\underline{t} \times \underline{N} = \tau t b \frac{1}{\epsilon} \left[\underline{i} \frac{\cos \varphi}{\sqrt{1-\epsilon^2 \sin^4 \varphi}} \right.$$

$$+ \underline{j} \left(\sin \varphi \cos \psi \sqrt{1-\epsilon^2 \sin^2 \varphi} + \frac{\epsilon^2 \sin \varphi \cos^2 \varphi}{\sqrt{1-\epsilon^2 \sin^2 \varphi}} \cos \psi \right)$$

$$\left. + \underline{k} (\cos^2 \varphi - \sin^2 \varphi) \cos \psi - \frac{1}{b} X \cos \psi (\underline{i} \cos \varphi + \underline{j} \sin \varphi) \right.$$

and then, according to (3.3),

$$\underline{M} = \tau t b^2 \left[\underline{i} \frac{1}{\epsilon^2} (\sin^{-1})(\epsilon \sin \varphi) - \underline{j} \cos \varphi \frac{1}{\epsilon} \sqrt{1-\epsilon^2 \sin^2 \varphi} \right.$$

$$\left. + \underline{k} (\sin \varphi \cos \varphi) - X(\underline{i} \sin \varphi - \underline{j} \cos \varphi) + \underline{M}_0 \right] \qquad (7.5)$$

The integration constant M_0 is zero since the two bending moments $M_1^{(o)}$ and $M_2^{(o)}$, which must be antisymmetrical with respect to $\varphi = 0$ and $\frac{\pi}{2}$ disappear. With (2.5) and (7.5) the three components of the moment are:

$$\left.\begin{aligned}
M_T &= -\tau t b^2 \frac{1}{\epsilon}\left[\frac{1}{\epsilon} \sin\varphi \cos\psi \,(\sin^{-1})(\epsilon \sin\varphi) + \frac{\cos^2\varphi}{\sqrt{1-\epsilon^2 \sin^4\varphi}}\right] + X\cos\psi \\
M_1 &= -\tau t b^2 \frac{}{\epsilon}\left[\frac{1}{\epsilon} \sin\varphi (\sin^{-1})(\epsilon \sin\varphi) - \sin\varphi \cos\varphi \sqrt{1-\epsilon^2 \sin^2\varphi}\right] \\
M_2 &= \left[t b^2 \frac{1}{\epsilon^2} \sin\varphi \sin\psi \,(\sin^{-1})(\epsilon \sin\varphi) + \sin^3\varphi \cos\varphi \cos\psi\right] - X\sin\psi
\end{aligned}\right\} \quad (7.5')$$

or developed again for small ϵ:

$$\left.\begin{aligned}
M_T^{(o)} &= -\tau t b^2 \frac{1}{\epsilon}\left[1 + \frac{\epsilon^2}{6}\sin^4\varphi + \epsilon^4\left(\frac{5}{24}\sin^8\varphi - \frac{2}{15}\sin^6\varphi\right)\right] \\
M_1^{(o)} &= -\tau t b^2 \epsilon \left[\frac{2}{3} + \frac{\epsilon^2}{5}\sin^2\varphi\right]\sin^3\varphi \cos\varphi \\
M_2^{(o)} &= -\tau t b^2 \epsilon^2 \sin^5\varphi \cos\varphi \left[\frac{2}{3} + \frac{\epsilon^2}{5}\sin^2\varphi + \frac{\epsilon^2}{3}\sin^4\varphi + \ldots\right]
\end{aligned}\right\} \quad (7.5'')$$

It is observed that the statically indeterminate (which this time is a cross force rather than a moment) does not reappear in M_1; whereas $M_1^{(o)}$ is no longer small with relation to $M_2^{(o)}$, so that in the execution of the statically indeterminate calculation the effect of M_1 cancels only for the case of non-oblique bending.

When restricted to this particular case ($J_{12} = 0$), the calculating process concerning the determination of X also remains the same as before, with the sole difference that this time the $M_T^{(o)}$ portion contains the lower ϵ powers, so that the extreme case

$$\alpha = \frac{GJ_T}{EJ_2} \rightarrow 0$$

also adjoins the case $\alpha \neq 0$ without discontinuity.

The formula for predicting X can again be written

$$\left. \begin{aligned} X &= \frac{A_1 + B_1\alpha}{C_1 + D_1\alpha} \\ \text{with} \\ A_1 &= \int_0^{\pi/2} M_T^{(o)}\, d\varphi, \quad B_1 = \int_0^{\pi/2} M_2^{\prime(o)} \tan\psi\, d\varphi \\ C_1 &= \int_0^{\pi/2} \cos\psi\, d\varphi, \quad D_1 = \int_0^{\pi/2} \frac{\sin^2\psi}{\cos\psi}\, d\varphi \end{aligned} \right\} \quad (7.6)$$

for $\epsilon > 1/2$ the integral must be again numerically evaluated; for small ϵ

$$X = \frac{1}{\epsilon}\, \frac{1 + \frac{\epsilon^2}{16} + \frac{\epsilon^4}{64} + \frac{10\alpha\epsilon^4}{384}}{1 - \frac{\epsilon^2}{16} - 0.0175\,\epsilon^4 + \frac{\alpha\,\epsilon^2}{8} + 0.4063\,\frac{\alpha\,\epsilon^4}{8}}$$

$$= \frac{1}{\epsilon}\left(1 + \frac{1-\alpha}{8}\,\epsilon^2 + \frac{\epsilon^2}{100}\,(4.09 - 4.81\,\alpha + 1.56\,\alpha^2) + \ldots\right) \quad (7.7)$$

The result is shown in figure 15. This time $M_{T_{max}}$ falls in the symmetry points $\varphi = 0, \pi/2$, $M_{2_{max}}$ near the point $\varphi = \pi/4$. Then, $M_{2_{max}}$ is seen to be little greater than in the extreme case $\epsilon \rightarrow 0$ at $\alpha = 0$; whereas $M_{T_{max}}$ increases considerably near $\epsilon = 1$. The torsion moment has the opposite sign at 0 and $\pi/2$. At $\varphi = \pi/2$ the dependence of the maximum torsion moment on α is as expected; it rises with increasing torsional stiffness. This aspect of $M_T(\alpha)$ is due to the fact that the statically indeterminate portion, which at $\varphi = \pi/2$ is smaller in amount than the statically determinate, decreases with increasing α. — At $\varphi = 0$ the conditions are reversed: Since the statically indeterminate portion governs the sign in this instance, $(M_T)_{extra}$ decreases with increasing α. The curve of the other bending moment M_1 has been omitted in figure 15, since it is almost straight and would intrude, moreover, in the range covered by the M_T curves.

The following simple approximate formulas remain:

$$\left.\begin{aligned} M_{2_{max}} &= \frac{1}{2}\tau t b^2 \\ M_{1_{max}} &= \frac{\epsilon}{6}\tau t b^2(1 + 0.15\,\epsilon^2) \\ M_{T_{max}} &= \frac{\epsilon}{6}\tau t b^2(1 + 0.3\,\epsilon^2) \end{aligned}\right\} \qquad (7.8)$$

the last one fails near $\epsilon = 1$.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

REFERENCE

1. Marguerre, K.: Bestimmung der Verzerrungsgrössen eines räumlich gekrummten Stabes mit Hilfe des Prinzips von Castigliano. Z.f.a.M.M., Aug. 1941.

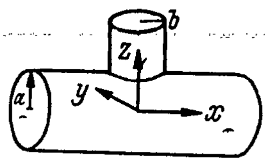

Figure 1.- Section of two cylinders.

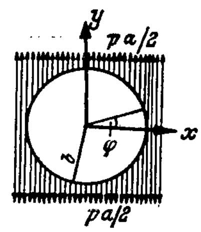

Figure 2.- The first load case within the limit $\frac{b}{a} = \epsilon \longrightarrow 0.$

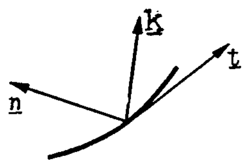

Figure 3.- Identification of axes t, n, k on the flat ring.

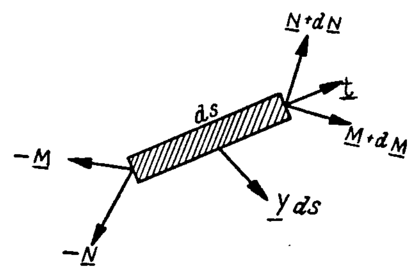

Figure 4.- Equilibrium of ring element d s.

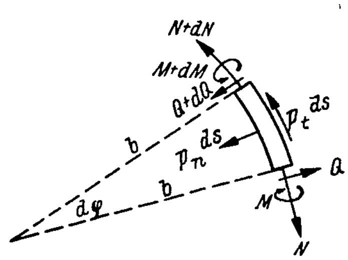

Figure 5.- Equilibrium of flat ring element.

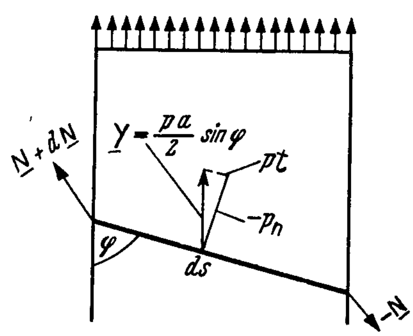

Figure 6.- Resolution of skin stress applied at ring element d s.

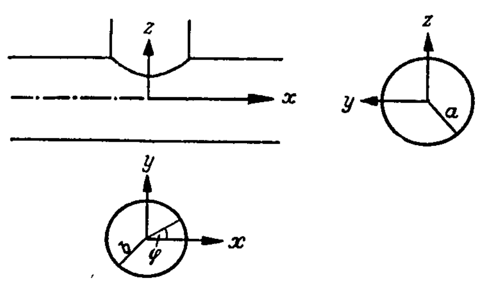

Figure 7.- The three projections of the space curve.

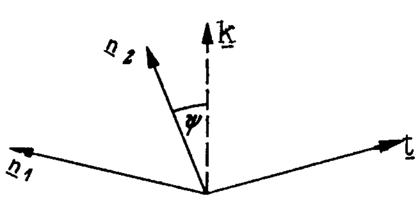

Figure 8.- Definінition of angle ψ.

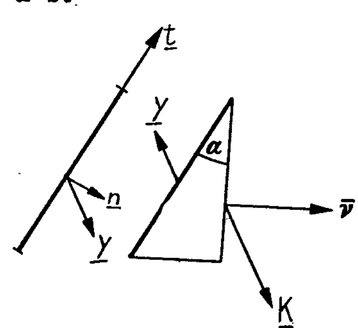

Figure 9.- Solution of linear load p (the plot lies in the tangential plane of the cylinder.

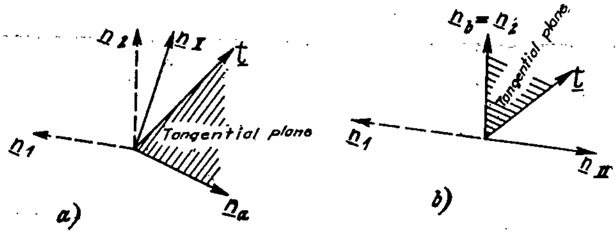

Figure 10.- Relative position of the different normals $\underline{n}$.

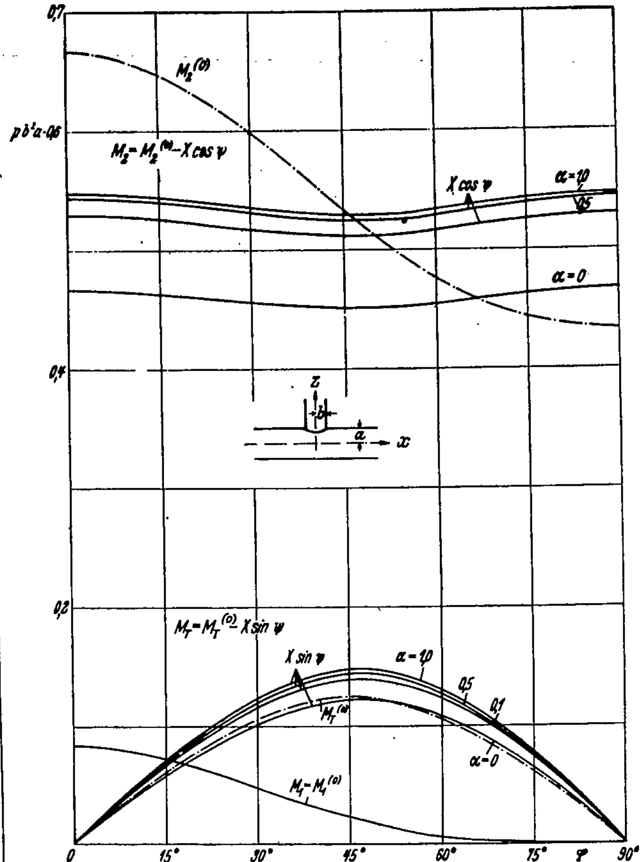

Figure 11.- Moment distribution along φ. The statically indeterminate portion X plotted against $\alpha = \frac{GJ_T}{EJ_2}$. (axes ratio $\epsilon = \frac{b}{a} = \frac{1}{2}$). (internal pressure p.)

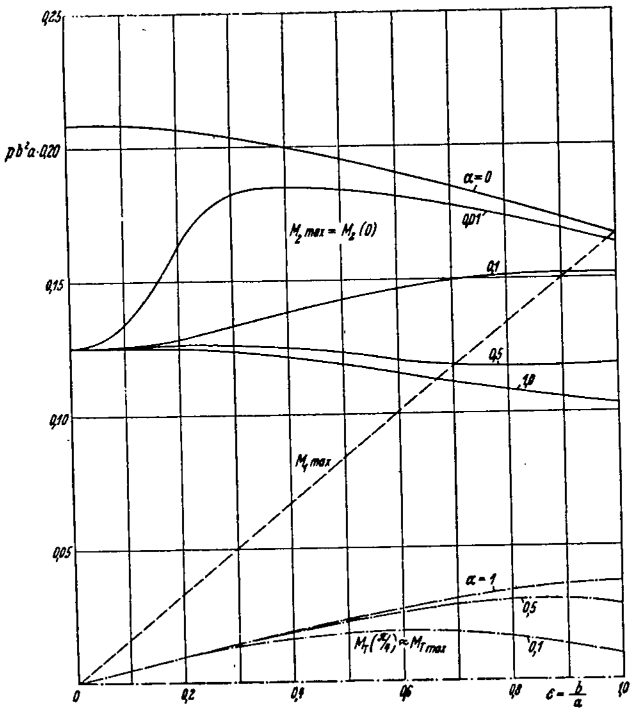

Figure 12.- Maximum moments against $\epsilon = \frac{b}{a}$, parameter $\alpha = \frac{GJ_T}{EJ_2}$.

Load: internal pressure p.

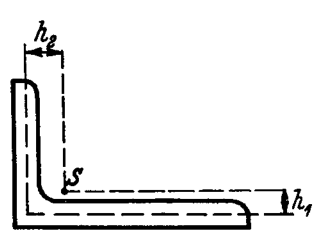

Figure 13.- Ring section, moment lever arms h_1, h_2.

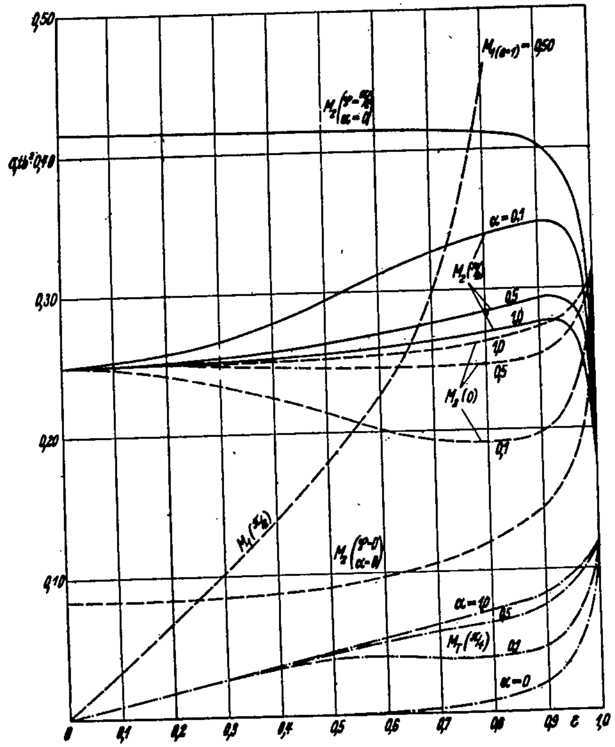

Figure 14.- Maximum moments against $\epsilon = \frac{b}{a}$, parameter $\alpha = \frac{GJ_T}{EJ_2}$

Load: tension $\sigma_1 t$. On comparison with Figure 12.-note that in the extreme case $\epsilon \rightarrow 0$ the loads $\frac{p\ a}{2}$ and $\sigma_1 t$ mutually agree -

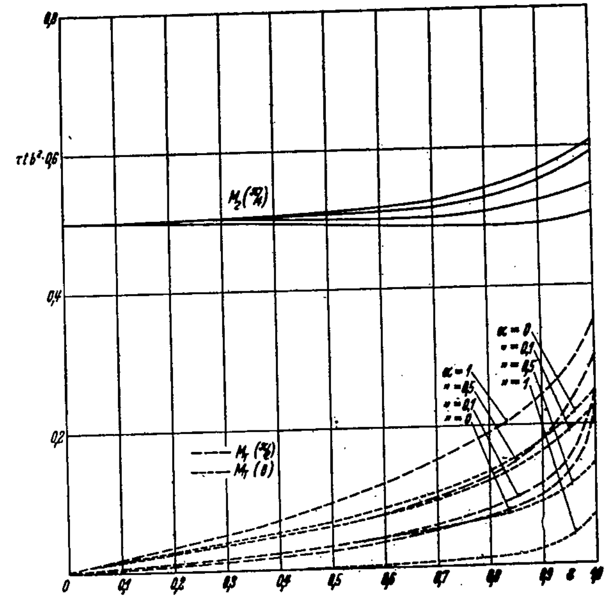

Figure 15.- Maximum moments against $\epsilon = \frac{b}{a}$, parameter $\alpha = \frac{GJ_T}{EJ_2}$.

Load: shear τ t in large cylinder. $M_2\ (\pi/4) \approx M_{2\ max}$ increases by decreasing α.