TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 999

STRESS ANALYSIS OF CIRCULAR FRAMES

By H. Fahlbusch and W. Wegner

Luftfahrtforschung
Vol. 18, No. 4, April 22, 1941
Verlag von R. Oldenbourg, München und Berlin

Washington
December 1941

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 999

STRESS ANALYSIS OF CIRCULAR FRAMES*

By H. Fahlbusch and W. Wegner

SUMMARY

The stresses in circular frames of constant bending stiffness, as encountered in thin-wall shells, are investigated from the point of view of finite depth of sectional area of frame. The solution is carried out for four fundamental load conditions. The method is illustrated on a worked out example.

I. NOTATION

P force

s plate thickness

S static moment

I inertia moment

EI bending stiffness

τs shear flow

φ_o variable angle at center

φ angle at center

r distance of neutral axis from center

R distance of shear flow from center

e distance of shear flow from neutral fiber

x distance of shear center from center

M moment

---

*"Berechnung der Beanspruchung kreisförmiger Ringspante." Luftfahrtforschung, vol. 18, no. 4, April 22, 1941, pp. 122-127.

B bending moment

N normal force

Q transverse force

X statically undetermined quantity

δ load and coefficient, respectively

II. THE FRAME EQUILIBRIUM

For the application of transverse forces in a circular shell with large ratio $\frac{R}{s}$, circular frames are provided. They are in equilibrium with the concentrated loads and the shear forces from the shell. Each loading can be divided into the transverse force passing through the elastic centroid of the shell and the moment (fig. 1).

The transverse force produces, as a result of bending under transverse force, a sinusoidally distributed shear flow in the shell that reaches to the frame (fig. 1a).

$$\tau s = \frac{P}{I} S$$

With

$$I = \pi R^3 s$$

denoting the moment of inertia of the circular shell, the function

$$\tau s = \frac{P}{\pi R} \sin \varphi$$

represents the shear flow variation.

The distance of the shear center of a circular half for sinusoidal variation from the center amounts to

$$x = \frac{R \int_0^\pi \tau s \, du}{\frac{P}{2}} = \frac{4}{\pi} R \approx 1.27 R$$

The shear flow applied at the frame as result of a moment is constant and amounts to (fig. 1b):

$$\tau s = \frac{M}{2\ R^2\ \pi}$$

The distance of the shear center of a circular half for constant variation from the center is:

$$x = \frac{\pi}{2} R \approx 1.57\ R$$

The resulting shear flow (fig. 1c) follows from:

$$\tau s = \frac{P}{R\pi} \sin \varphi + \frac{M}{2\ R^2\ \pi}$$

The distance of the shear center of a circular half from the center amounts to

$$x = \frac{4 + \pi}{\pi + 2} R \quad \text{on one side}$$

and

$$x = \frac{4 - \pi}{\pi - 2} R$$

on the other.

III. STRESS ANALYSIS OF CIRCULAR FRAMES OF CONSTANT BENDING STIFFNESS

Load Case A

Localized Radial Force Acting on the Frame

Ordinarily the circular frame is threefold statically undetermined, but in this instance and in the subsequent load cases the solution can be considerably simplified by cleverly chosen sectionalization. At point O of the load in figure 2, the statically undetermined quantity is $X_3 = 0$. The signs for carrying out the analysis are given in figure 4. The depth of the sectional area of the frame is introduced by means of the ratio $\frac{r}{R}$. The subsequent results are valid for $\frac{r}{R} \lesseqgtr 1$ (fig. 3).

The elasticity equations read

$$\delta_{10} + X_1\delta_{11} + X_2\delta_{12} = 0$$

$$\delta_{20} + X_1\delta_{21} + X_2\delta_{22} = 0$$

The displacement quantities generally follow at

$$EI\ \delta_{ik} = \int B_i\ B_k\ du$$

Determination of the bending moment curve B_o in the statically determined principal system referred to neutral fiber (fig. 4). The tangentially applied shear force element

$$\tau s\ du = \tau s\ R\ d\varphi_o$$

sets up at point φ in the frame the bending moment

$$d\ B_o = -\ \tau s\ du\ e$$

The distance e follows from the geometric relation

$$e = R - r(\sin \varphi_o \sin \varphi + \cos \varphi_o \cos\varphi)$$

Then the bending moment B_o at φ is:

$$B_o = -\frac{PR}{\pi}\int_0^{\varphi} \sin \varphi_o\, d\varphi_o + \frac{Pr}{\pi} \sin \varphi \int_0^{\varphi} \sin^2 \varphi_o\, d\varphi_o + $$

$$+ \frac{Pr}{\pi} \cos \varphi \int_0^{\varphi} \sin \varphi_o \cos \varphi_o\, d\varphi_o$$

hence

$$B_0 = \frac{P}{\pi}\left(R \cos\varphi + r\frac{\varphi}{2}\sin\varphi - R\right)$$

It further is

$B_1 = 1$ due to $X_1 = 1$

$B_2 = r\,(1 - \cos\varphi)$ as a result of $X_2 = 1$

For reasons of symmetry the integration can be limited to a half frame. The following loads and factors are obtained.

$$EI\ \delta_{10} = \int_0^\pi B_1\ B_0\ du = Pr\left(\frac{r}{2} - R\right)$$

$$EI\ \delta_{20} = Pr^2\ R\left(\frac{5r}{8R} - \frac{3}{2}\right)$$

$$EI\ \delta_{11} = r\pi$$

$$EI\ \delta_{22} = \frac{3}{2}r^3\pi$$

$$EI\ \delta_{12} = EI\ \delta_{21} = r^2\pi$$

Solution of the elasticity equations gives the magnitude of the statically undetermined quantities

$$X_1 = -\frac{Pr}{4\pi}, \quad X_2 = \frac{P}{\pi}\left(\frac{R}{r} - \frac{1}{4}\right)$$

whence the ultimate bending moment

$$B = B_0 + X_1B_1 + X_2B_2$$

$$\boxed{B = \frac{Pr}{2\pi}\left(\varphi\ \sin\varphi + \frac{1}{2}\cos\varphi - 1\right)} \qquad (1)$$

The final normal force follows from

$$N = N_0 + X_1 N_1 + X_2 N_2$$

The normal force N_0 in the statically determined principal system at point φ is obtained by splitting the shear force element in the tangential component followed by integration from 0 to φ (fig. 4)

$$dN_0 = - \tau s \; du \cos (\varphi - \varphi_0)$$

$$N_0 = - \frac{P}{2\pi} \varphi \sin \varphi$$

The normal force distribution in the statically undetermined system then is

$$\boxed{N = \frac{P}{2\pi}\left(\frac{2R}{r} \cos \varphi - \frac{1}{2} \cos \varphi - \varphi \sin\varphi\right)} \qquad (2)$$

and the transverse force variation

$$\boxed{Q = \frac{P}{2\pi}\left(\varphi \cos \varphi + \frac{2R}{r} \sin \varphi - \frac{3}{2} \sin \varphi\right)} \qquad (3)$$

Figures 5 to 7 show bending moments, normal force, and transverse force plotted against the frame circumference. The ratio $\frac{r}{R}$ serves as parameter for $\frac{r}{R} = 1$, $\frac{r}{R} = 1.2$, $\frac{r}{R} = 0.8$

Load Case B

Localized Moment Acting Along a

Diameter of the Frame (fig. 8)

For this load the frame is simply statically undetermined at point O. The elasticity equation reads

$$\delta_{30} + X_3\delta_{33} = 0$$

Bending moments, normal force, and transverse force in the statically determined principal system are obtained as for case A. The loads and factors are:

$$EI\ \delta_{30} = Mr^2\left(\frac{r}{4R} - \frac{1}{2}\right)$$

$$EI\ \delta_{33} = \frac{\pi}{2}\ r^3$$

The statically undetermined quantity follows at

$$X_3 = \frac{M}{2r\pi}\left(2 - \frac{r}{R}\right)$$

The final bending moment is

$$\boxed{B = \frac{M}{\pi}\left(\sin\varphi - \frac{\varphi}{2}\right)} \tag{4}$$

the final normal force is

$$\boxed{N = -\frac{M}{r\pi}\sin\varphi} \tag{5}$$

and the final transverse force is

$$\boxed{Q = \frac{M}{r\pi}\left(\cos\varphi - \frac{1r}{2R}\right)} \tag{6}$$

as illustrated in figures 9 to 11.

Load Case C

Localized Tangential Force Acting Along the Neutral Fiber of the Frame at a Distance r from the Center

In this instance also the frame is simply statically undetermined at point 0. The procedure is the same as before. The intermediate and final results are:

$$B_o = \frac{Pr}{2\pi}\left(\frac{2R}{r}\sin\varphi - \varphi\cos\varphi - \sin\varphi + \frac{r}{R}\sin\varphi - \varphi\right)$$

$$N_o = \frac{P}{2\pi}\left(\varphi\cos\varphi + \sin\varphi - \frac{r}{R}\sin\varphi\right)$$

$$Q_o = \frac{P}{2\pi}\left(\varphi\sin\varphi + \frac{r}{R}\cos\varphi - \frac{r}{R}\right)$$

furthermore

$$EI\ \delta_{30} = \frac{Pr^3}{2}\left(\frac{R}{r} + \frac{r}{2R} - \frac{5}{4}\right)$$

$$EI\ \delta_{33} = \frac{\pi}{2} r^3$$

and

$$X_3 = -\frac{P}{\pi}\left(\frac{R}{r} + \frac{r}{2R} - \frac{5}{4}\right)$$

$$\boxed{B = \frac{Pr}{2\pi}\left(\frac{3}{2}\sin\varphi - \varphi\cos\varphi - \varphi\right)} \qquad (7)$$

$$\boxed{N = \frac{P}{2\pi}\left(\varphi\cos\varphi + \frac{2R}{r}\sin\varphi - \frac{3}{2}\sin\varphi\right)} \qquad (8)$$

$$\boxed{Q = \frac{P}{2\pi}\left(\varphi\sin\varphi - \frac{2R}{r}\cos\varphi + \frac{5}{2}\cos\varphi - \frac{r}{R}\right)} \qquad (9)$$

B and Q are plotted in figures 13 and 14. N has the same aspect as Q in load case A (fig. 7).

Load Case D

Localized Tangential Force Acting Along the Outer Periphery of the Frame at a Distance R from the Center (fig. 15)

The stresses in this frame loading are obtained when load case C is superposed by a moment of magnitude $M = P\,(R - r)$. Then:

$$\boxed{B = \frac{PR}{2\pi}\left(2\sin\varphi - \frac{r}{R}\varphi\cos\varphi - \frac{r}{2R}\sin\varphi - \varphi\right)} \tag{10}$$

$$\boxed{N = \frac{P}{2\pi}\left(\varphi\cos\varphi + \frac{1}{2}\sin\varphi\right)} \tag{11}$$

$$\boxed{Q = \frac{P}{2\pi}\left(\varphi\sin\varphi + \frac{1}{2}\cos\varphi - 1\right)} \tag{12}$$

Figure 16 illustrates the bending moment distribution; N has the same aspect as N in load case C for $\frac{r}{R} = 1$ and as Q in load case A at $\frac{r}{R} = 1$ (cf. fig. 7). Q in load case D has the same aspect as Q in load case C at $\frac{r}{R} = 1$ (fig. 14).

IV. EXAMPLE

Find the stresses in a circular frame with ratio $\frac{r}{R} = 1.1$ under the following loads. (Fig. 17.)

By division of the force P we get

Load Case A: 0.9 P = 1800 kg acting radially

C: 0.4 P = 800 kg acting tangentially

B: 40 mkg

The resulting bending moment curve is found numerically by superposition of the results from equations (1), (7), and (4), or by graphical superposition of the bending moment curves from the basic loads illustrated in figures 5, 13, and 9. The same method applies to the normal and the transverse force. Of greatest interest is the knowledge of the longitudinal stresses from the bending moments and normal forces.

TABLE I

BENDING MOMENTS (mkg) IN FRAME FROM

THE NUMERICAL SOLUTION

Load case \ φ^o	0	90	180	180	270	360
A	-47.2	53.9	-141.5	-141.5	53.9	-47.2
B	0	-3.0	0	0	3.0	0
C	0	2.7	20.0	20.0	-2.7	0
Result	-47.2	53.6	-161.5	-121.5	54.2	-47.2

TABLE II

NORMAL FORCES (in kg) IN FRAME

FROM NUMERICAL SOLUTIONS

Load case \ φ^o	0	90	180	180	270	360
A	378	-450	-378	-378	-450	378
B	0	41	-400	400	-41	0
C	0	-39	0	0	39	0
Result	378	-448	-778	22	-452	378

Tables I and II give several intermediate values of the mathematical solution, while figure 18 shows the final bending moment and normal force distribution in the circular frame.

For the stress analysis of the rivets or welds between circular frame and shell the shear flow distribution is employed. It is computed by the method indicated in section II and has for the particular frame loading the aspect shown in figure 19. The maximum shear flow amounts to 26 kg/cm.

V. SCOPE OF VALIDITY

1. The solutions hold for circular frames with small sectional depth compared to curvature radius r. In this case the curved member acts similar to a straight member. Hence the stress distribution was assumed linear and the cross sections presumed to remain plane. The effect of the longitudinal and transverse forces on the displacement factors was disregarded.

2. The bending stiffness of the shell plate compared with that of the circular frame was presumed to be small.

3. The departure of the frame contour from the circular shape due to elastic strain was discounted.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

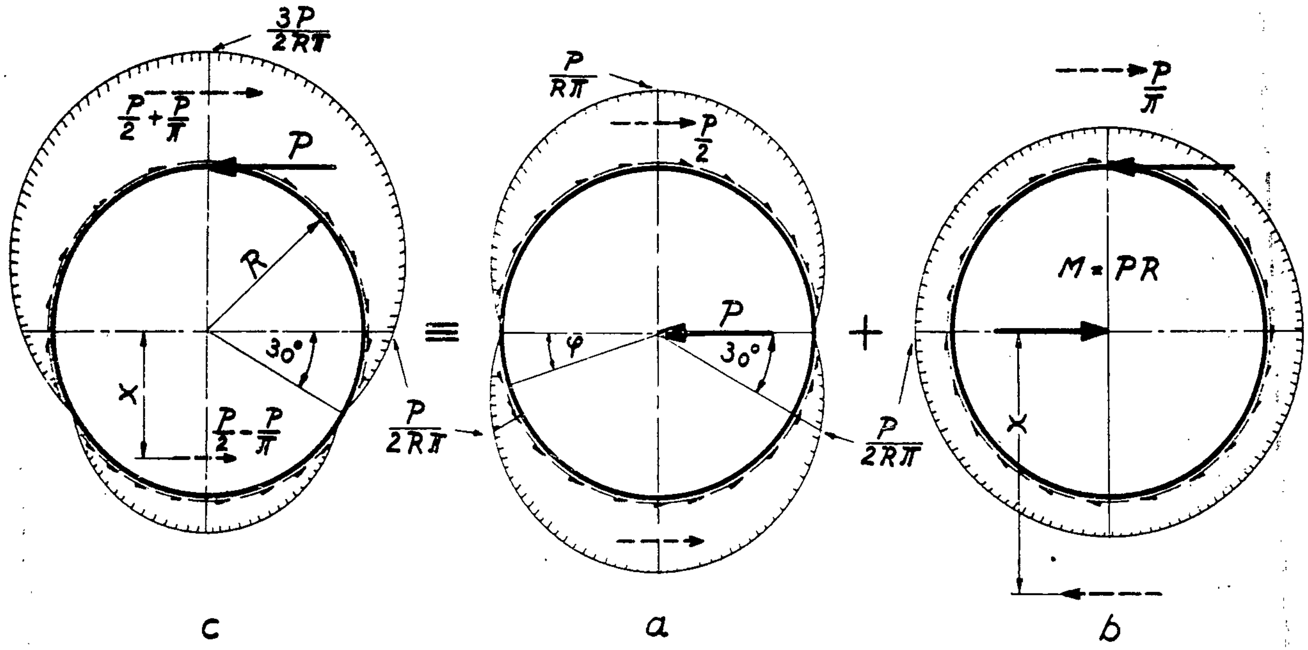


Figure 1.- Equilibrium of frame and division into basic load cases.

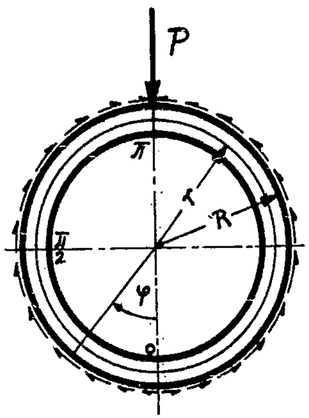


Figure 2.- Load case A; radial loading.

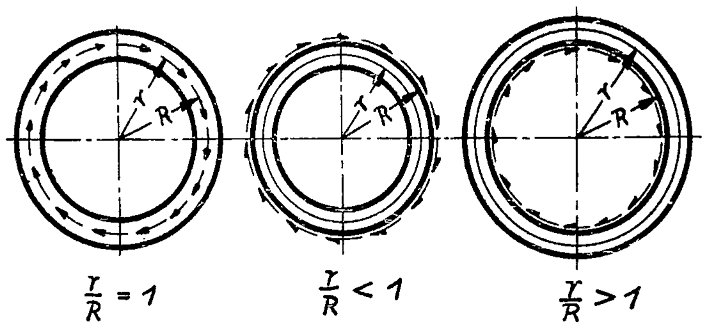


Figure 3.- Representation of ratio r/R by equal shell diameter.

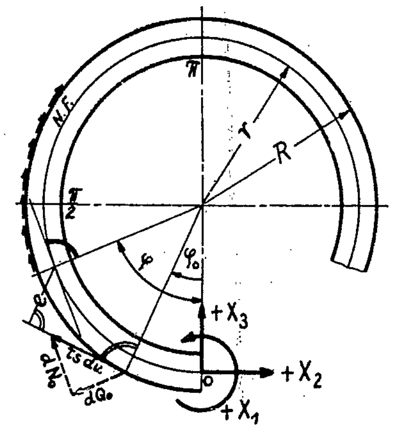


Figure 4.- Identification of B_0, N_0, Q_0 and signs.

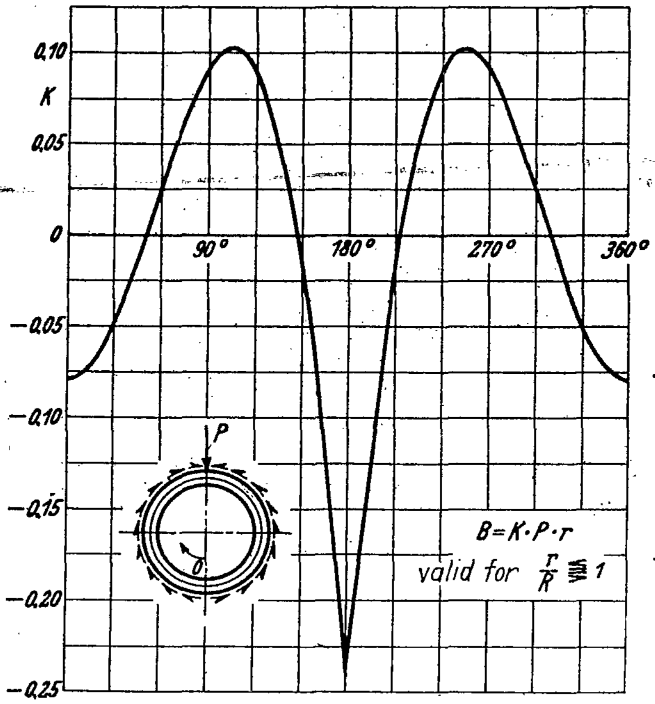


Figure 5.- Bending moments under radial load.

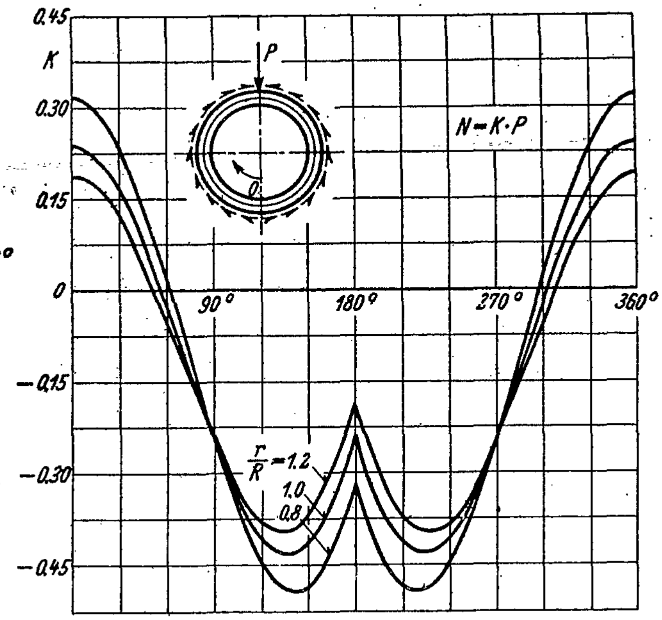


Figure 6.- Normal forces under radial load.

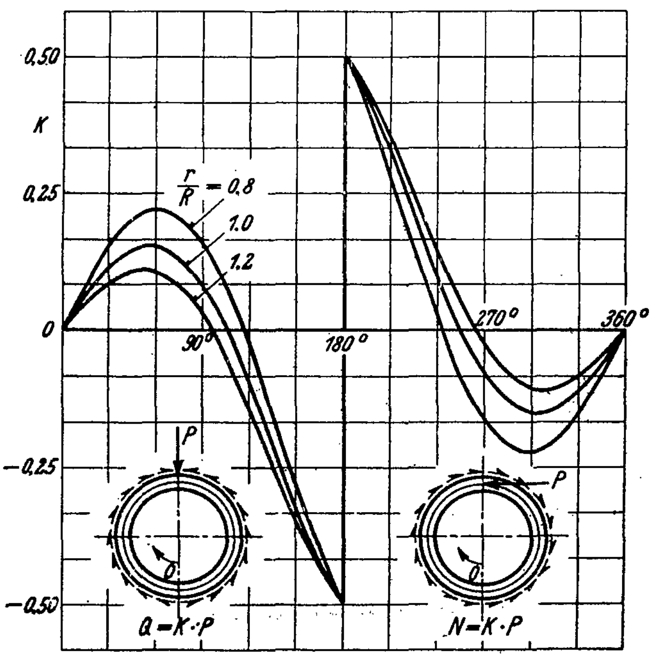
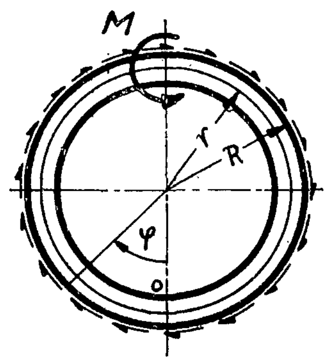


Figure 7.- Transverse forces under radial load, concurrent normal forces under tangential load (distance r).



$$B_0 = \frac{M}{2\pi}\left(\frac{r}{R}\sin\varphi - \varphi\right),$$

$$N_0 = -\frac{M}{2R\pi}\sin\varphi,$$

$$Q_0 = \frac{M}{2R\pi}(\cos\varphi - 1).$$

Figure 8.- Load case B; moment loading.

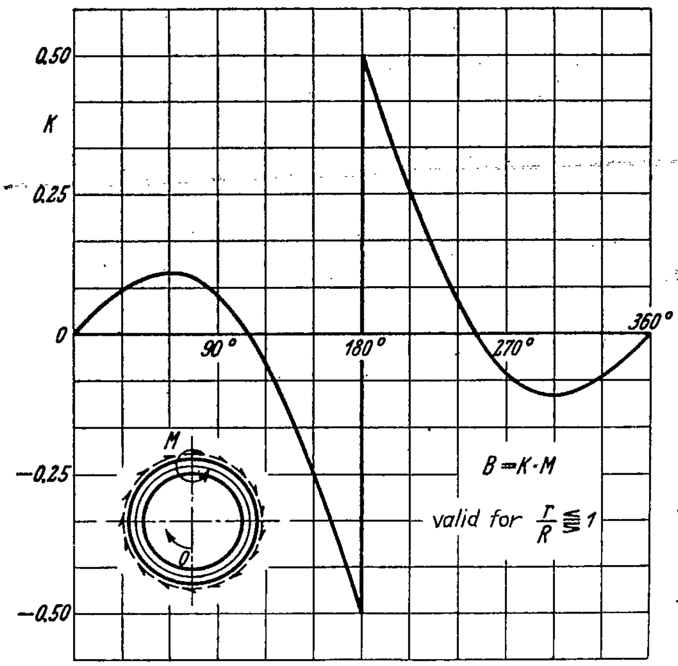


Figure 9.- Bending moments under moment loading.

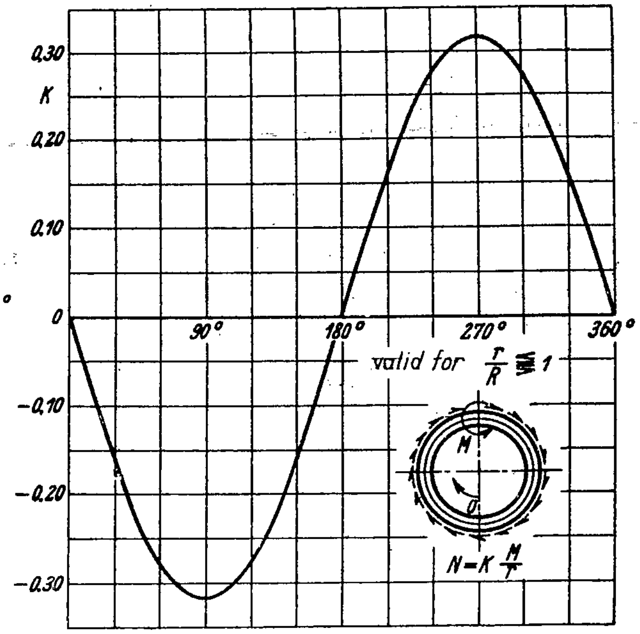


Figure 10.- Normal forces under moment loading.

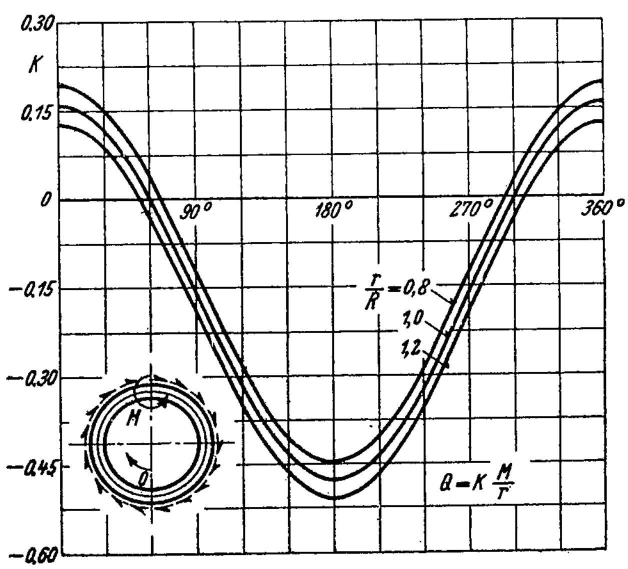


Figure 11.- Transverse forces under moment loading.

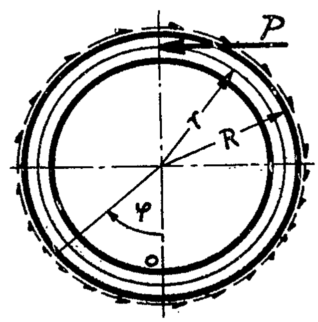


Figure 12.- Load case C; tangential loading (distance r).

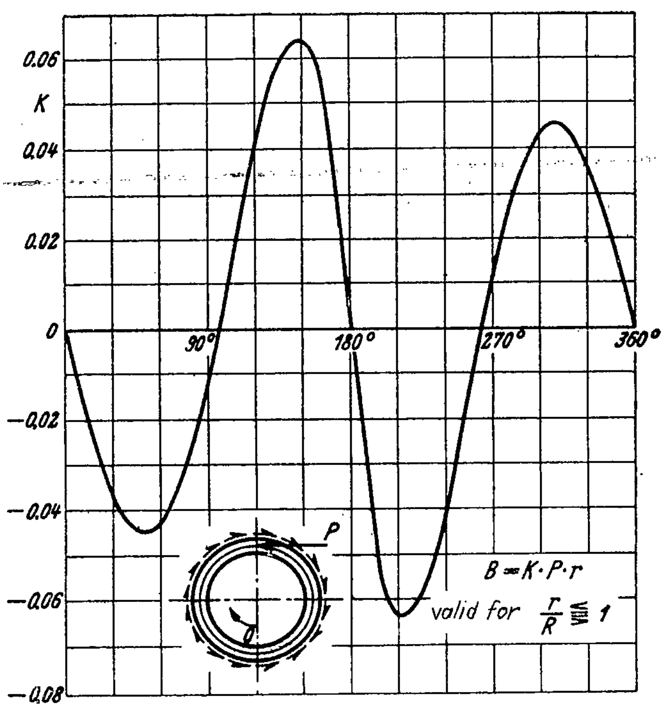


Figure 13.- Bending moments under tangential loading (distance r).

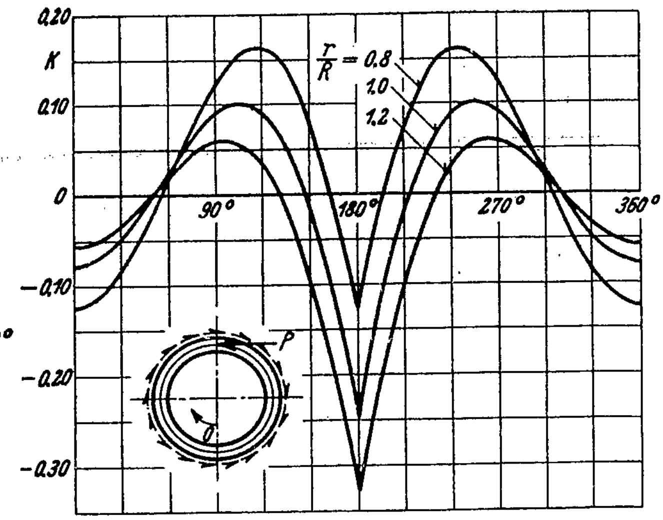


Figure 14.- Transverse forces under tangential loading (distance r).

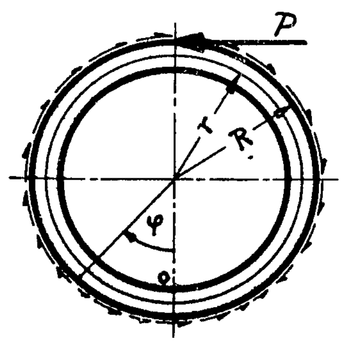


Figure 15.- Load case D; tangential loading (distance r).

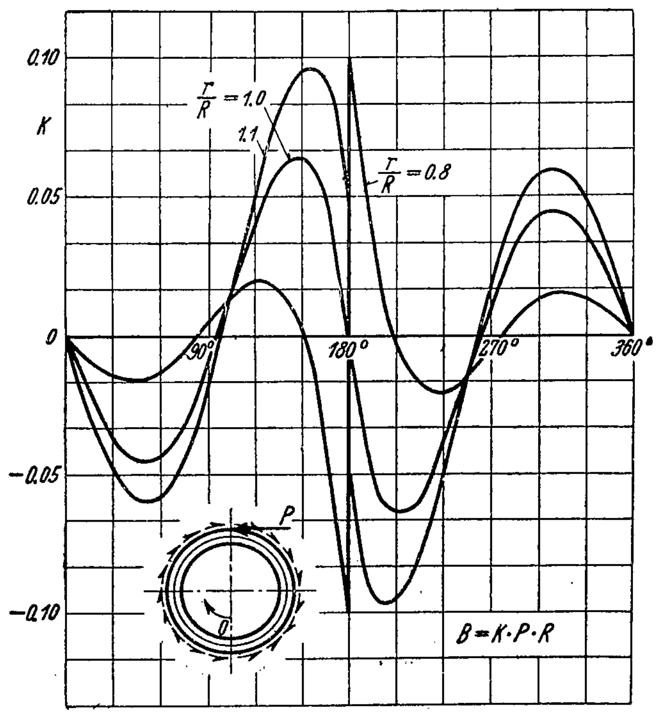


Figure 16.- Bending moments under tangential loading (distance r).

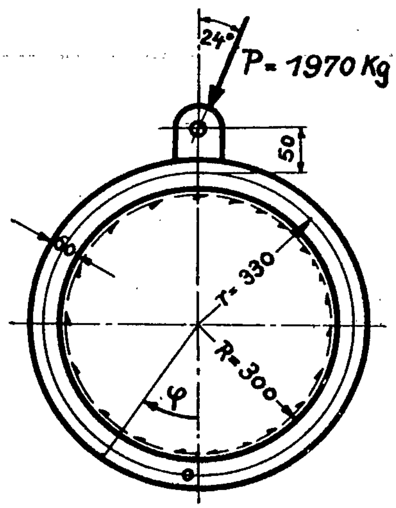


Figure 17.- Sample frame loading.

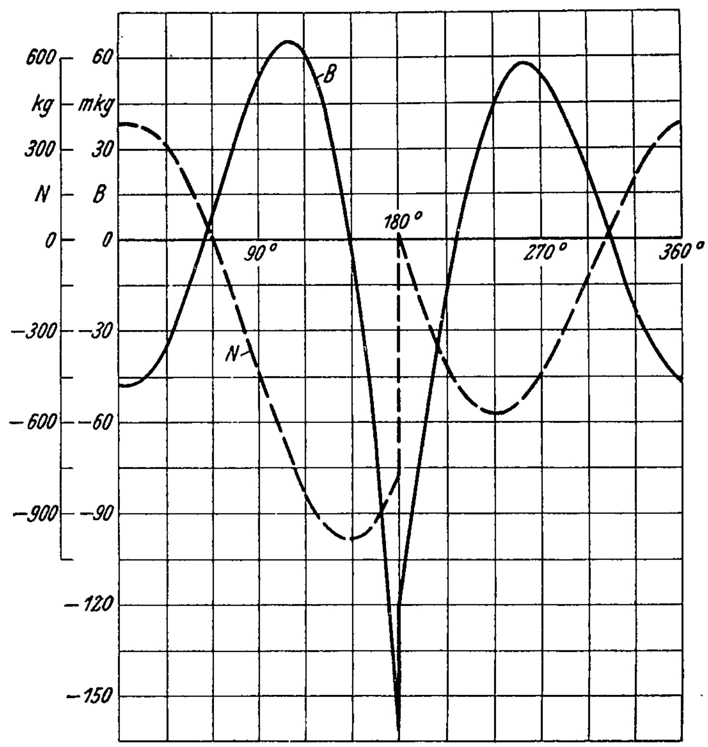


Figure 18.- Bending moments and normal forces.

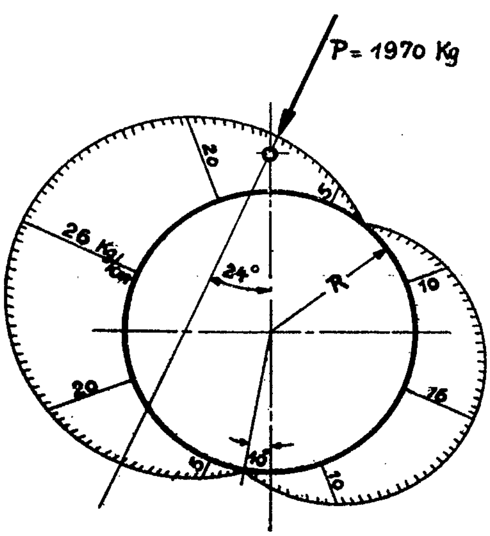


Figure 19.- Magnitude and variation of shear flow at frame.