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TWO-DIMENSIONAL POTENTIAL FLOW PAST A SMOOTH WALL WITH PARTLY CONSTANT CURVATURE

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TWO-DIMENSIONAL POTENTIAL FLOW PAST A SMOOTH WALL WITH PARTLY CONSTANT CURVATURE*

By Werner v. Koppenfels

SUMMARY

The speed of a two-dimensional potential flow past a smooth wall, which evinces a finite curvature jump at a certain point and approximates to two arcs in the surrounding area, has a vertical tangent of inflection in the critical point as a function of the arc length of the boundary curve. Close to this point the speed is a function of the arc length σ in the form

$$\frac{W}{W_0} = 1 + \frac{1}{\pi}\left(1 - \frac{\rho_1}{\rho_2}\right)\ln\left|\frac{\sigma}{\rho_1}\right| + \underline{P}\ (\sigma)$$

where $\underline{P}\ (\sigma)$ is an expansion in powers starting with the linear term, which defines the further variation of the speed in the individual case. The method discloses that in the vicinity of the curvature jump the assumptions for Pohlhausen's boundary layer equation into which the first two speed derivatives enter, are no longer given. In consequence of the pronounced pressure change at the critical point a breakdown of the boundary layer is anticipated.

I. INTRODUCTION

Local Character of the Conformal Function at The Critical Point

1. General Theorem

Within the framework of the analysis no restriction as to generality is implied by the choice of an arc (with

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*"Ebene Potentialströmung längs einer glatten Wand mit stückweise stetiger Krümmung." Luftfahrtforschung, vol. 17, no. 7, July 20, 1940, pp. 189-95.

nondisappearing curvature) which tangentially terminates in a straight line (fig. 1) as contour of the surface along which the flow is being studied.

We establish the character of the conformal function

$$Z = f(z) \quad z = x + iy, \; Z = X + iY \tag{1}$$

which permits the piece of the x axis on either side of the zero point to correspond to the described contour in the Z place. It is found particularly that $z = 0$ changes to $Z = 0$ and the straight piece of the contour falls in the axis of real positive Z, which is expressed in the requirements:

$$\left.\begin{aligned} f(0) &= 0 \\ f(x) &= X \quad x > 0, \; X > 0 \end{aligned}\right\} \tag{2}$$

In consideration of the smooth transition of the two boundary pieces in the critical point, the conformity must not be disturbed, that is, the first derivation of $f(z)$ must exist at $z = 0$ (which is followed, as is known, by the equivalence of the two boundary values $f'(x)$ from right and left at $x = 0$). Posting

$$\lim_{z \to 0} f'(z) = 1 \tag{3}$$

we conclude that $f(z)$ in the immediate vicinity of $z = 0$ must act as Z. Then the conformal function (1) reads

$$f(z) = \frac{1}{\frac{1}{z} + g(z)} \tag{4}$$

and it is necessary to inquire after the conditions which the function $g(z)$ must fulfill in order that the transformation has the required characteristics. The subsequent investigation uncovers the inner reason the formula reads $\frac{1}{Z} = \frac{1}{z} + g(z)$ rather than $Z = z + g^*(z)$ as might be suspected at first. Formulating

$$f'(z) = \frac{1 - z^2 g'(z)}{[1 + zg(z)]^2} \tag{5}$$

we found that the conformal function has the required characteristics if

$$\left.\begin{array}{l} g(z) \text{ has real value for } z > 0 \\ g(z) \text{ has complex value for } z < 0 \end{array}\right\} \qquad (A)$$

$$\left.\begin{array}{l} \lim\limits_{z \to 0} z\, g(z) = 0 \\ \lim\limits_{z \to 0} z^2 g'(z) = 0 \end{array}\right\} \qquad (B)$$

The conditions (A) express that the reflection of the axis of the real z is composed of the semiaxis of the positive real Z and a curved arc, the conditions (B) insure the smooth merging of the arc.

Take the simple example of

$$g(z) = -\frac{1}{\sqrt{z}} \qquad (6)$$

and analyze the transformation afforded by

$$f(z) = \frac{1}{\frac{1}{z} - \frac{1}{\sqrt{z}}} = \frac{z}{1 - \sqrt{z}} \qquad (7)$$

The conformity is disturbed at $z = \infty$ and $z = 4$ (doubling of angles) (fig. 2). The axis of negative real z is transformed in the curve $X^3 + XY^2 + Y^2 = 0$, which in the zero point merges smoothly but with infinitely great curvature into the axis of the positive real Z.

2. The Case of Finite Curvature Jump

In the sense of the initially posted question the case of a finite curvature jump is of particular importance; hence it involves the specialization of the arbitrary function g(z) to the extent where the image curve has a finite curvature in the zero point. To this end the image curve in the vicinity of the zero point is visualized as being replaced by its curvature radius and the next question is to find the additional condition which the function g(z) must meet in order that the image curve of the axis of the negative real z becomes a circular arc.

It is readily seen that in this case $f(z)$ must have the form

$$f(z) = \frac{1}{h(x) + i\pi c} \tag{8}$$

for negative real z, where $h(x)$ is any real-value function of x. In point of fact

$$X + iY = \frac{1}{h(x) + i\pi c}$$

decomposes into

$$\left.\begin{aligned} h(x)\, X - \pi c Y &= 1 \\ \pi c X + h(x)\, Y &= 0 \end{aligned}\right\} \tag{9a}$$

which form the parameter representation of a circle:

$$\pi c (X^2 + Y^2) + Y = 0 \tag{9}$$

with the center $Z = -\frac{1}{2\pi c}\, i$ and the radius:

$$\rho = \frac{1}{2\pi c} \tag{10}$$

In order that $f(z)$ may show the behavior postulated in (8), it is necessary with regard to (4) for $g(z)$ to have a constant imaginary part for negative real z, owing to which the zero point evinces a logarithmic singularity of the function $g(z)$ and whence follows the representation

$$g(z) = c \ln z + \underline{P}(z) \tag{11}$$

where $\underline{P}(z)$ is the expansion in power of a regular function in $z = 0$, the coefficients of which must be real in order that the required behavior on the axis of the real z is not disturbed.

Hence the general representation of the conformal function in the vicinity of the transition point for the case of straight line changing to a circular arc reads:

$$f(z) = \frac{1}{\frac{1}{z} + c \ln z + \underline{P}(z)} \tag{12}$$

In this manner the local character of the conformal function in the vicinity of the critical area is established for the most general case of a finite curvature jump. The effect of the curvature jump

$$2\pi c = \frac{1}{\rho}$$

is readily apparent. The form of the function represented by $\underline{P}(z)$, which is regular at $z = 0$ and of no significance for the local behavior of $f(z)$, largely defines the further aspect of the boundary curve.

3. Examples

a) Circular Arc Triangle with a Straight Angle

The first example consists of a family of circular arc triangles having one straight angle and the sides of which all pass through one point (fig. 3). If this common intersection point which at the same time is the point of variable curvature change, is transferred by linear transformation to an infinitely remote point, the circular arc triangle is transformed in a triangle with straight sides, two of which are parallel (fig. 4).

According to the Schwarz-Christoffel formula the function

$$Z^* = \int \left(\frac{z^* - 1}{z^*}\right)^{\lambda-1} dz^* \tag{13}$$

conformally transforms the lower z^* half plane onto the part of the Z^* plane situated to the left of the contour. The infinitely remote points of both planes match and by suitable choice of integration constants it is insured that point $Z^* = 0$ corresponds to point $z^* = 1$. On transition to the planes of the reciprocal variables

$$z = \frac{1}{z^*}, \quad Z = \frac{1}{Z^*} \tag{14}$$

the critical point falls in the zero point of the Z plane and is the reflection of the zero point of the z plane, while point $z = 1$ gives a corner point situated at $Z = \infty$. Thus the function

$$Z = f(z) = -\frac{1}{\int (1 - z)^{\lambda-1} \frac{dz}{z^2}} \tag{15}$$

affords a transformation which permits the piece of a circular arc corresponding to the semiaxis of the positive real z and the semiaxis of the positive real Z to correspond to the interval $0 < z < 1$, which touches the circular arc in the neutral point (fig. 5). While the integral cannot be evaluated, as a rule, with elementary functions, the expansion of the integrand in the vicinity of $z = 0$ affords

$$\frac{1}{f(z)} = -\int \frac{1 - \binom{\lambda-1}{1} z + \binom{\lambda-1}{2} z^2 - + \ldots}{z^2} dz$$

$$= \frac{1}{z} + (\lambda - 1) \ln z - \underline{P}(z) \tag{16}$$

where the general behavior ascertained in (12) is confirmed.

A comparison of (12) and (10) with (16) reveals that the image circle of the axis of the negative real z has the radius*

$$\rho = \frac{1}{2\pi(\lambda - 1)} \tag{17}$$

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*The transformation of this profile had been investigated back in 1911 by A. Sonnefeld – "Flows about Compound Cylindrical Shells," Thesis, Jena 1911. Purpose of the work was the calculation of the forces on a bent plate with round head; hence it contains no detailed study of the effect of the curvature jump.

Aside from the trivial case $\lambda = 1$, which affords the same transformation, are mentioned the specific cases (figs. 6 and 7)*

$$\lambda = \frac{3}{2}, \quad \frac{1}{f(z)} = \frac{\sqrt{1-z}}{z} = \frac{1}{2} \ln \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}} \tag{18}$$

and

$$\lambda = 2, \quad \frac{1}{f(z)} = \frac{1}{z} + \ln z \tag{19}$$

The structure of (15) has plainly indicated on what basis the simple formula is evolved for the dependence of the reciprocal variables $1/z$ and $1/Z$, that is, circular arc and straight line meeting in the Z plane become in the $Z^*(= 1/Z)$ plane parallel straight pieces, the distance of which is equal to half the curvature jump at $Z = 0$. (On transformation of $Z^* = 1/Z$ the circle (9) becomes the straight line $Y^* = \pi c$.)

b. Circular Arc Triangle with Two Straight Angles

As further illustration of the general formula (12) we briefly analyze the transformation of the half plane on a circular-arc triangle with two extended angles (reference 1). According to the general theory, the conformal function is the quotient of two linear unrelated solutions of the hypergeometric differential equation

$$z(1-z)\,u''(z) + \frac{(1-\delta)(3-\delta)}{4}\,u(z) = 0 \tag{21}$$

formed with

$$\alpha = -\frac{3-\delta}{2}, \quad \beta = \frac{1-\delta}{2}, \quad \gamma = 0 \tag{20}$$

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*The conformal functions of the circular-arc triangle with the angles $\delta_1\pi = \pi$, $\delta_2\pi = \lambda\pi$, $\delta_3\pi = (2-\lambda)\pi$ are equally obtainable without the Schwarz-Christoffel formula as quotient of two solutions of the hypergeometric differential equation formed with $\alpha = -1$, $\beta = 1 - \lambda$, $\gamma = 0$: $z(1-z)\,u'' + (\lambda - 1)\,z\,u' - (\lambda - 1)\,u = 0$. The one solution is $u_1 = z$, the other follows from $u_1'\,u_2 - u_2'u_1 = (1-z)^{\lambda-1}$.

where $\delta\pi$ is the third angle of the triangle. The solutions of this differential equation can be achieved in the form of certain integrals which are to be extended along continuous paths in the z plane. Suffice it to prove the characteristic behavior on one of the two "smooth" corners. In the vicinity of $z = 0$, the following linear unrelated solutions (reference 2) are afforded:

$$\left.\begin{aligned} u_1 &= z\, F\left(-\frac{1-\delta}{2}, \frac{3-\delta}{2}; 2; z\right) \\ u_2 &= z\left[F_1\left(-\frac{1-\delta}{2}, \frac{3-\delta}{2}; 2; z\right)\right. \\ &\qquad \left. + F\left(-\frac{1-\delta}{2}, \frac{3-\delta}{2}; 2; z\right) \ln z\right] \end{aligned}\right\} \tag{22}$$

where F denotes the hypergeometric series and F_1 a certain power series which starts with term $\frac{1}{\alpha\beta}\,\frac{1}{z}$ and then progresses in positive powers of z. The quotient of these two solutions

$$f(z) = \frac{u_1}{u_2} = \frac{1}{\frac{F_1}{F}\left(-\frac{1-\delta}{2}, \frac{3-\delta}{2}; 2; z\right) + \ln z} \tag{23}$$

shows the generally established behavior.

II. RELATIONSHIP BETWEEN CONFORMAL FUNCTION, CURVATURE OF STREAMLINE, AND SPEED

1. Decomposition of Acceleration Vector

In order to procure a basis for the discussion of the flow velocity in relation to the streamline curvature, we first define the acceleration vector of a potential flow.

The flow in the z plane is given by the complex potential

$$\omega(z) = \varphi + i\,\psi \tag{24}$$

from which, by differentiation, the velocity vector reflected on the real axis

$$\frac{d\omega}{dz} = u - i\,v = \frac{d\bar{z}}{dt} \tag{25}$$

of absolute amount follows at

$$w = |\,\omega'(z)\,| = \sqrt{u^2 + v^2} \tag{26}$$

With the components of the velocity vector*

$$\left.\begin{aligned} \frac{dx}{dt} &= u(x,y) = \varphi_x \\ \frac{dy}{dt} &= v(x,y) = \varphi_y \end{aligned}\right\} \tag{27}$$

and the components of the acceleration vector

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= u\,u_x + v\,v_x \\ \frac{d^2y}{dt^2} &= u\,v_x - v\,u_x \end{aligned}\right\} \tag{28}$$

we form the curvature of the streamline:

$$k = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}^{\,3}} = \frac{(u^2 - v^2)v_x - 2\,u\,v\,u_x}{\sqrt{u^2 + v^2}^{\,3}} \tag{29}$$

and the time rate of change of the absolute speed

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*With consideration to (25) the functions u and $(-v)$ satisfy the Cauchy-Riemann differential equations.

$$\frac{dw}{dt} = \frac{\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}} = \frac{(u^2 - v^2)u_x + 2\,u\,v\,v_x}{\sqrt{u^2 + v^2}} \tag{30}$$

Equations (29) and (30) may be combined into one complex equation:

$$\frac{\overline{\omega'}^2(z)\,\omega''(z)}{|\omega'(z)|} = \frac{dw}{dt} - i\,k\,w^2 \tag{31}$$

and represent the decomposition of the acceleration vector tangentially and normally to the streamline.

2. Transfer to the Image Plane

In the presentation of the plane of flow by an analytic function

$$Z = f(z) \tag{1}$$

the complex potential must be transformed and from the transformed potential

$$\Omega\,(Z) = \omega(z) \tag{32}$$

follows, after differentiation, the complex velocity

$$\Omega'(Z) = \frac{1}{f'(z)}\,\omega'(z) = \frac{d\overline{Z}}{d\tau} \tag{33}$$

where τ indicates the time measured in the reflected plane. This formula makes it plain that the transformation of the flow on the image plane must also include a time transformation, for a comparison of (25) and (33) gives

$$\frac{d\overline{Z}}{d\tau} = \frac{1}{f'(z)}\,\frac{dz}{dt} \tag{34}$$

and, in addition,

$$\frac{dt}{d\tau} = \frac{1}{|f'(z)|^2} \tag{35}$$

With consideration to this new time τ the absolute velocity W in the image plane and the curvature K of the streamline are associated through the relation corresponding to the representation (31):

$$\frac{\bar{\Omega}'^2(Z)\,\Omega''(Z)}{|\Omega'(Z)|} = \frac{dW}{d\tau} - i\,K\,W^2 \tag{36}$$

Assuming in the z plane a parallel flow past the x axis with constant velocity 1, we post

$$\omega(z) = W_0 z, \qquad \omega'(z) = W_0 = \frac{d\bar{z}}{dt} \tag{37}$$

that is,

$$W_0 t = x \tag{37a}$$

The result in the image plane then is, according to (33):

$$\Omega'(Z) = \frac{W_0}{f'(z)}, \qquad |\Omega'| = W = \frac{W_0}{|f'(z)|} \tag{38}$$

so that (36) is transformed into

$$-\frac{W_0^2\,f''(z)}{|f'(z)|^3 f'(z)} = \frac{dW}{d\tau} - iK\,W^2 \tag{39}$$

or, with respect to

$$\frac{dW}{d\tau} = \frac{dW}{dt}\,\frac{1}{|f'(z)|^2} = \frac{dW}{dx}\,\frac{W_0}{|f'(z)|^2} \tag{40}$$

into

$$-\frac{f''(z)}{|f'(z)|\,f'(z)} = \frac{1}{W_0}\,\frac{dW}{dx} - i\,K \tag{41}$$

and, with the insertion of the arc length,

$$\sigma = \int |f'(z)|_{y=\text{const}}\,dx \tag{42}$$

of the streamline into which the streamline $y = \text{const}$ is transformed, we find:

$$-\frac{1}{[f'(z)]}\frac{f''(z)}{f'(z)}\bigg|_{y=\text{const}} = |f'(z)|_{y=\text{const}}\frac{1}{W_0}\frac{dW}{d\sigma} - i\,K \qquad (43)$$

This formula itself suggests that on an area where the curvature of the streamline changes variably, the derivation of the velocity with respect to the arc length becomes infinite, for, real and imaginary parts of an analytic function usually become concurrently singular.

3. Discussion of the Speed Vicinal to the Critical Point

a) Measure for the Curvature Jump

In order to obtain a practical approximate representation of the speed in proximity of the critical point, it is expedient to pass from the case "circular arc - straight line" to the generalized case "circular arc - circular arc," for, in the first case, it is difficult to introduce the concept of "quantity of curvature jump." But it is always possible to quote a similarity transformation which allocates any desired value to the curvature jump. Naturally, this also holds true in the case where two circular arcs of finite radii ρ_1, ρ_2, meet in the zero point, but then the nondimensional quantity

$$\lambda = 1 - \frac{\rho_1}{\rho_2} \qquad (44)$$

affords a suitable indication for the curvature jump. Particularly, $\lambda \ll 1$ characterizes the case where the curvatures of the circles differ very little from each other. Quantity λ is none other than the curvature jump $K_1 - K_2 = \frac{1}{\rho_1} - \frac{1}{\rho_2}$ measured at the length of the circle radius ρ_1.

The linear function

$$Z^* = \frac{2\,\rho_2\,Z}{2\,\rho_2 + iZ} = \frac{1}{\frac{1}{Z} + \frac{i}{2\rho_2}} \qquad (45)$$

transforms the contour of figure 8a into that of figure 8b, without distortion in the vicinity of the zero point, and gives the radii*

$$\rho_1 = \frac{1}{\frac{1}{\rho} + \frac{1}{\rho_2}}, \quad \rho_2 \tag{46}$$

The conformal function (12), wherein $c = \frac{1}{2\pi\rho}$, takes the form after transformation (45):

$$Z^* = \frac{1}{\frac{1}{Z} + \frac{1}{2\pi}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\ln\frac{z}{E} + \underline{P}(z) + \frac{i}{2\rho_2}} \tag{47}$$

where E is the length unit in the z plane. With ρ_1 chosen as this unit and

$$\frac{z}{\rho_1} = z_1, \quad \frac{Z^*}{\rho_1} = Z_1 \tag{48}$$

as a new variable in the original and in the reflected plane, the nondimensional representation

$$\boxed{Z_1 = f_1(z_1) = \frac{1}{\frac{1}{z_1} + \frac{\lambda}{2\pi}\ln z_1 + \frac{1-\lambda}{2}i + \underline{P}_1(z_1)}} \tag{49}$$

is used in the subsequent discussion, with special emphasis on the case where the parameter λ, which measures the curvature jump and usually is situated in the interval $0 < \lambda < 1$, is small with respect to 1.

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*This is apparent when the ρ circle of the Z plane is taken as the image of the real axis of a Z' plane: $Z = \frac{2\rho}{2\rho + iZ'}$, and transformation (45) applied to it:

$$Z^* = \frac{2\rho\,\rho_2}{\rho + \rho_2}\,\frac{Z'}{\frac{2\rho\,\rho_2}{\rho + \rho_2} + i\,Z'}$$

Hence: $2\rho_1 = \frac{2\rho\,\rho_2}{\rho + \rho_2}$

b. Approximate Representation of the Speed

Posting (49) in the form

$$f_1(z_1) = \frac{1}{\frac{1}{z_1} + g_1(z_1)} \tag{49a}$$

with

$$g_1(z_1) = \frac{\lambda}{2\pi} \ln z_1 + \frac{1-\lambda}{2} i + \underline{P}_1(z_1) \tag{49b}$$

followed by forming

$$f_1'(z_1) = \frac{1 - z_1^2 g_1'(z_1)}{[1 + z_1 g_1(z_1)]^2} \tag{50}$$

gives the logarithmic derivative

$$-\frac{f_1''(z_1)}{f_1'(z_1)} = \frac{2 z_1 g_1' + z_1^2 g_1''}{1 - z_1^2 g_1'} + 2 \frac{g_1 + z_1 g_1'}{1 + z_1 g_1} \tag{51}$$

With the interrelations

$$\lim_{z_1 \to 0} z_1 g_1(z_1) = 0$$

$$\lim_{z_1 \to 0} z_1^2 g_1'(z_1) = 0$$

the approximate representation in the vicinity of the critical point follows at

$$-\frac{f_1''(z_1)}{f_1'(z_1)} = 2 g_1(z_1) + 4 z_1 g_1'(z_1) + z_1^2 g_1''(z_1) \tag{52}$$

If z_1 is a point of the real axis $(z_1 = x_1)$, then

$$\underline{R}\left(-\frac{f_1''(x_1)}{f_1'(x_1)}\right) = 2\,\underline{R}\,(g_1(x_1)) + 4\,x_1 \underline{R}(g_1'(x_1)) + x_1^2 \underline{R}(g_1''(x_1)) \tag{52a}$$

which, confined to the constant term of the ensuing expansion gives, together with (49b)

$$\underline{R}\left(-\frac{f_1''(x_1)}{f_1'(x_1)}\right) = 2\,a_0 + \frac{\lambda}{2\pi}\,(3 + 2\,\ln|x_1|) \tag{53}$$

For the imaginary part, it affords with (49b):

$$\underline{F}\left(-\frac{f_1''(x_1)}{f_1'(x_1)}\right) = 2\,\underline{F}(g_1(x_1)) - \begin{cases} 1 & \text{for } x_1 < 0 \\ 1 - \lambda & \text{for } x_1 > 0 \end{cases} \tag{52b}$$

With regard to

$$f'(0) = f_1'(0) = 1 \tag{54}$$

the arc length in the vicinity of the critical point is, according to (42):

$$\sigma = x = \rho_1 x_1 \tag{55}$$

and hence, according to (43):

$$\frac{dW/W_0}{d\sigma/\rho_1} = \rho_1 \underline{R}\left(-\frac{f''(x)}{f'(x)}\right) = \underline{R}\left(-\frac{f_1''(x_1)}{f_1'(x_1)}\right)$$

$$= 2\,a_0 + \frac{\lambda}{2\pi}\left(3 + 2\,\ln\left|\frac{\sigma}{\rho_1}\right|\right) \tag{55a}$$

$$K = \underline{F}\left(\frac{f''(x)}{f'(x)}\right)$$

$$= \frac{\rho_1}{1}\,\underline{F}\left(\frac{f_1''(x_1)}{f_1,(x_1)}\right) = \begin{cases} -\dfrac{1}{\rho_1} & (x_1 < 0) \\ \dfrac{1}{\rho_1}\,(\lambda - 1) = -\dfrac{1}{\rho_2} & (x_1 > 0) \end{cases} \tag{55b}$$

The negative signs of the curvatures are due to the fact that the region in which the flow is explored is the outside region of the circles.

From (55a) the integration for the speed in the vicinity of the critical point affords the approximate representation

$$\frac{W}{W_0} = 1 + 2\,a_0\,\frac{\sigma}{\rho_1} + \frac{\lambda}{2\pi}\,\frac{\sigma}{\rho_1}\left(1 + 2\,\ln\left|\frac{\sigma}{\rho_1}\right|\right) \tag{56}$$

c. Discussion

The representation

$$\frac{dW/W_0}{d\sigma/\rho_1} = 2\ a_0 + \frac{\lambda}{2\pi}\left(3 + 2\ \ln\left|\frac{\sigma}{\rho_1}\right|\right) \tag{55a}$$

first enables a precise statement in the critical point. It discloses that the flow velocity in the critical point ($\sigma = 0$) where the curvature undergoes a finite jump has a vertical tangent. Owing to the symmetrical behavior of the derivation for positive and negative σ this point is an inflection point (no peak).

As approximate formulas (55a) and (56) are to be construed as follows: On the contour of a given profile, that is, for every specifed function g_1 of the form (49b) an interval can be marked off about the critical point wherein (55a) and (56) are applicable. They apply in this range within the scope of accuracy in which $z_1 g_1(z_1)$ and $z_1^2 g_1'(z_1)$ may be disregarded with respect to 1, that is, within the scope of accuracy within which $f_1'(z_1)$ may be put equal to 1, according to (50). It is apparent that the range of validity is likewise defined by the regular function $\underline{P}_1(z_1)$ additive in (49b) and and this, in turn, depends largely upon the further aspect of the profile. Hence, no generally valid predictions as to the range of validity of these representations can be made without knowledge of the further aspect. Even the constant term a_0 of the power expansion $\underline{P}_1(z)$ in (49) exerts a profound effect. Under these conditions no predictions independent of the length scale (chosen previously equivalent to radius ρ_1), for any other choice of length unit (E' instead of E) modifies in respect to $\ln\frac{z}{E'} = \ln\frac{z}{E} - \ln\frac{E'}{E}$ the constant term a_0 of the complementary expansion. This fact is not surprising from the mathematical point of view, for it is quite natural that on the basis of the given curvature jump alone, that is, on the basis of a statement concerning the unsteady behavior of the conformal function at one particular point, any binding conclusions concern only the conditions at that particular point. The precise conclusion here is the fact that the development of the velocity in the vicinity of the critical point contains the characteristic logarithmic term (cf. (56)), wherefrom follows particularly the existence of the vertical turning tangent. More and farther reaching generally valid predictions are mathematically impossible.

In spite of this fundamental reservation regarding the scope of the approximate representations (55a), (56) it is interesting to follow the course of the approximation curve. In the case $\lambda = 0$, which corresponds to the constant curvature change at $\sigma = 0$, the approximate curve is rectilinear with the pitch

$$m = 2\, a_0 = \left(\frac{dW/W_0}{d\sigma/\rho_1}\right)_{\lambda=0} \tag{57}$$

so that (56) can be written in the form

$$\frac{W}{W_0} = 1 + \left(\frac{dW/W_0}{d\sigma/\rho_1}\right)_{\lambda=0} \frac{\sigma}{\rho_1}$$

$$+ \frac{1}{2\pi}\left(1 - \frac{\rho_1}{\rho_2}\right)\left(1 + 2\ln\left|\frac{\sigma}{\rho_1}\right|\right)\frac{\sigma}{\rho_1} \tag{56a}$$

If, on the other hand, a finite curvature jump prevails, it evinces a typical antisymmetric discontinuity with a vertical tangent in the critical point (fig. 9). The place on the edge where the ordinate of the approximate curve shows the greatest divergence from the straight line is that where $\frac{dW/W_0}{d\sigma/\rho_1} = m = 2\, a_0$, that is, according to (55a) at

$$\sigma^* = \pm e^{-3/2}\, \rho_1 = \pm 0.233\, \rho_1 \tag{58}$$

For the approximate curve $\left[\lambda = 1 - \frac{\rho_1}{\rho_2} \ll 1\right]$, the amount of the deviation is proportional to the curvature jump and, measured at right angles to the straight line, amounts to

$$\frac{\Delta W}{W_0} = \frac{1 - \rho_1/\rho_2}{\pi\sqrt{1 + m^2}}\, \frac{\sigma^*}{\rho_1} = 0.075\, \frac{1 - \rho_1/\rho_2}{\sqrt{1 + m^2}} \tag{59}$$

The lengths are referred to ρ_1, and it is assumed that the ratio of the curvature radii is small with respect to 1.

So, while the bulging of the approximate curve is small for small curvature jumps, it is of comparatively wide extent. The maximum departure from the straight line is, according to (58), about one-fourth curvature radius from the transition point. The bent curve gradually adjusts itself to the normal.

It is again emphasized that no general prediction of the extent to which the actual velocity curve follows this approximate curve is fundamentally possible, because of its dependence upon the further aspect of the profile, as exemplified in our equations by the regular function in the critical point (49), the coefficients of which must be defined from one case to the next. To illustrate;

With the profile shown in figure 7, apply the the conformal function, first in the form

$$Z_1 = \frac{1}{\frac{1}{z} + c \ln (c z)} \tag{I}$$

then in the form

$$Z_2 = \frac{1}{\frac{1}{z} + c \ln z} \tag{II}$$

In (I) the lengths are referred to $\frac{1}{c}$, in the other, (II), to 1. This affords a supplementary additive constant $c \ln c$ in the denominator. In both cases the critical point is situated in the neutral point and the image circle of the axis of the positive real z has the radius $\rho = \frac{1}{2\pi c}$. The two differ only in their length of trailing edge, since the reflection of the branching point $z = \frac{1}{c}$ gives first: $Z_1\left(\frac{1}{c}\right) = \frac{1}{c}$; then: $Z_2\left(\frac{1}{c}\right) = \frac{1}{c(1 - \ln c)}$. In the first transformation the ratio of radius to length of trailing edge is constant $= \frac{1}{2\pi}$; in the other it changes with c and amounts to $\frac{1}{2\pi}(1 - \ln c)$.

When $c \longrightarrow 0$ it grows beyond bounds, although both transformations then become identical. On determination of the velocity it affords a maximum only to the right of the critical area, but not to the left, in the first case where the ratio of radius to trailing edge is equal to $\frac{1}{2\pi}$. In the second case, however, there is a maximum to the right and to the left of the critical point for small c, as is seen when the exact value of the velocity is expanded in powers of c and stopped with the linear term, which is permissible for suitably small c. The real course of the velocity curve in case (II) is plotted for $c = 0.25$ and $c = 0.1$ in figure 10. Each of the two curves comes from $+\infty$ (for $x = -\infty$) and has an inflection point for $x < 0$. If $c = 0.1$, the curve is found to follow largely the course of the approximate curve, obtained from small c in the form $W/W = 1 + c\sigma(1 + 2 \ln \sigma)$. It can be obtained from (56a) by posting $\rho_2 = \infty$, $\rho_1 = \frac{1}{2\pi c}$, and $m = -2 c \ln 2 c\pi$. It is to be noted (fig. 7) that in this example it is useless to make the curvature jump equal zero and to speak of the velocity curve of normal variation. This example makes it plain that without knowledge of the further course of the profile no valid conclusions can be made of the extent to which the approximate curve (fig. 9) approahces the true velocity curve.

Translation by J. Vanier,
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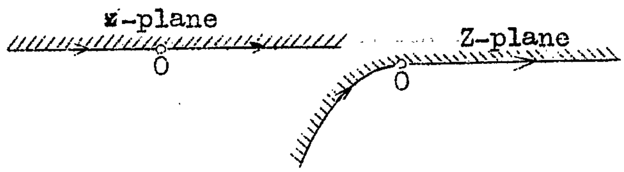
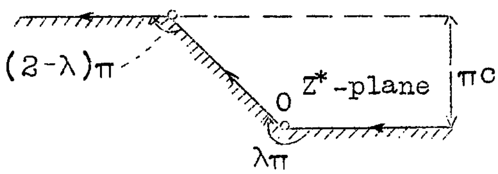


Figure 1.



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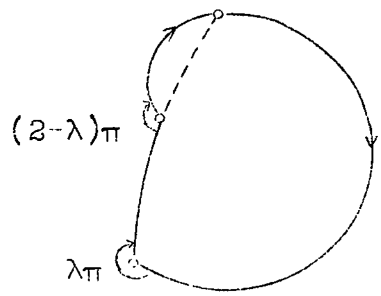

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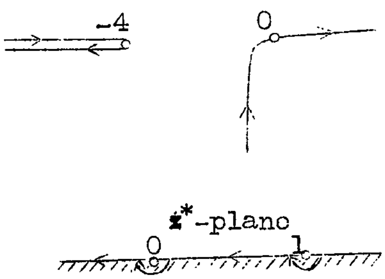

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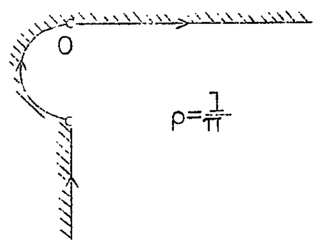

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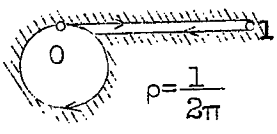

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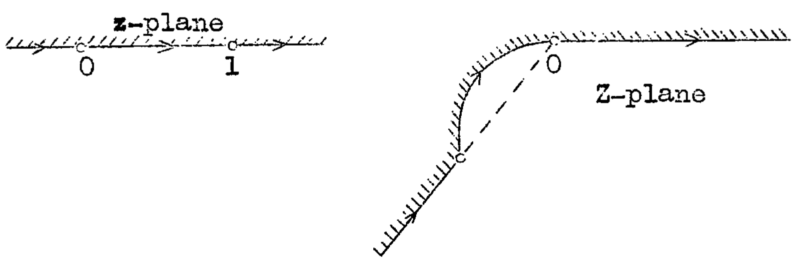

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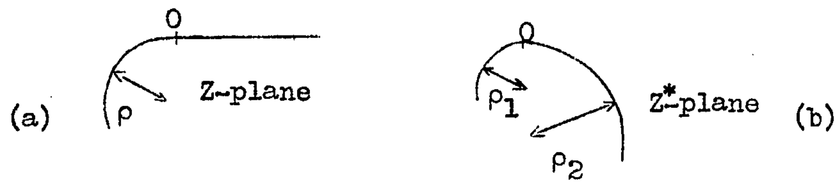

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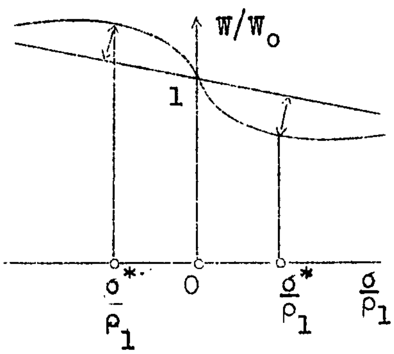

Figure 9.

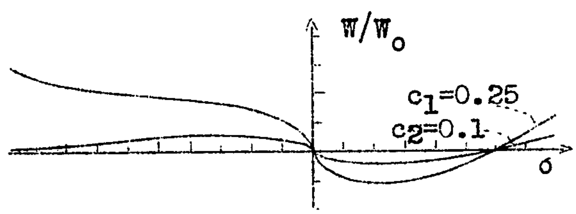

Figure 10.

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