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No. 968

ON THE CALCULATION OF FLOW PAST AN INFINITE SCREEN OF THIN AIRFOILS

By E. Pistolesi

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ON THE CALCULATION OF FLOW PAST AN INFINITE SCREEN OF THIN AIRFOILS*

By E. Pistolesi

SUMMARY

The present report deals with the flow past an infinite screen of thin airfoil (two-dimensional problem). The vortex distribution across the profile is established with appropriate expansion in series and the velocity distribution lift, moment, and profile shape deduced. Inversely, the distribution is deduced from the vorticity.

The method is the extension of the Birnbaum-Glauert method for the isolated wing.

INTRODUCTION

The present report describes a method for computing infinite screens of airfoils.

This method is the natural but not immediate extension of the Birnbaum-Glauert method for isolated airfoils. By this is meant that the object of the present study involves screens of thin and slightly curved airfoils.

For convenience the screen whose axis (straight line connecting the median points of the airfoil chord) is at right angle to the chord, is called <u>straight screen</u>; that whose axis contains the profile chords, <u>screen in tandem</u>; and that whose axis is oblique with the chord, <u>oblique screen</u>. By the same definition the <u>obliquity of the screen</u> is the angle of the screen axis with the direction at right angle to the chord.

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*"Sul calcolo di schiere infinite di ali sottili." Publicazioni della R. Scuola d'Ingegneria di Pisa, (seventh series) no.323, September 1937.

The term "chord" is employed in a broad sense and it signifies a straight line a little distance from the profile in reference to which the ordinates of the several points of the profile itself are measured, without imposing that it connect the extreme points or other conditions of the kind.

I. THE STRAIGHT SCREEN

1. An infinite screen of equal and equidistant vortices disposed on axis y, with circulation Γ and spacing h induces in a point P of the coordinate $z = x + iy$ the velocity

$$w = - \sum_{-\infty}^{\infty} \frac{i\,\Gamma}{2\pi(z-nih)} = - \frac{i\Gamma}{2h} \coth \frac{\pi z}{h} \qquad (1)$$

corresponding to a complex potential

$$\Phi = \frac{i\Gamma}{2\pi} \ln \sinh \frac{\pi z}{h}$$

(the axes are orientated as in fig. 1).

On a point of axis x, we have:

$$w = - \frac{i\Gamma}{2h} \coth \frac{\pi x}{h} \qquad (1')$$

Denoting the components of the induced velocities by u and v, we put

$$w = u - iv$$

whence it is readily seen that

$$u = 0 \qquad v = \frac{\Gamma}{2h} \coth \frac{\pi x}{h} \qquad (1'')$$

Assume now that the vortex distribution follows a law $\gamma = f(x)$ along the chord of the airfoils of the straight screen.

Simply suppose that the wing chord is of length $2l$

and contains between $x = -l$ and $x = l$ (fig. 2). Then the induced velocity of the whole screen on axis x is:

$$v = \frac{1}{2h} \int_{-l}^{l} \gamma \, dx' \coth \frac{\pi(x-x')}{h} \tag{2}$$

Then we put

$$\tanh \frac{\pi x}{h} = \xi \tag{3}$$

which readily yields

$$\coth \left(\frac{\pi x}{h} - \frac{\pi x'}{h} \right) = \frac{1-\xi\xi'}{\xi - \xi'}$$

$$dx = \frac{h}{\pi} \frac{1}{1-\xi^2} d\xi$$

and hence

$$v = \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \gamma \frac{1-\xi\xi'}{\xi - \xi'} \frac{1}{1-\xi'^2} d\xi' \tag{4}$$

where

$$\lambda = \tanh \frac{\pi l}{h} \tag{5}$$

We next put

$$\frac{\xi}{\lambda} = -\cos\theta \qquad d\xi = \lambda \sin\theta \, d\theta \tag{6}$$

and find:

$$v = \frac{1}{2\pi}\int_0^\pi \frac{1-\lambda^2\cos\theta\cos\theta'}{\cos\theta' - \cos\theta}\ \frac{\sin\theta'}{1-\lambda^2\cos^2\theta'}\ \gamma\, d\theta'$$

$$= \frac{1}{2\pi}\int_0^\pi \frac{\sin\theta'}{\cos\theta' - \cos\theta}\ \gamma\, d\theta' + \frac{\lambda^2}{2\pi}\int_0^\pi \frac{\sin\theta'\cos\theta'}{1-\lambda^2\cos^2\theta'}\ \gamma\, d\theta'$$

$$= \frac{1}{2\pi}\left[\int_0^\pi \frac{\sin\theta'}{\cos\theta' - \cos\theta}\gamma\, d\theta' + \frac{\lambda}{2}\int_0^\pi \sin\theta'\left(\frac{1}{1-\lambda\cos\theta'} - \frac{1}{1+\lambda\cos\theta'}\right)\gamma\, d\theta'\right] \tag{7}$$

It is interesting to note that the first of the two integrals of equation (7) is a function of θ, and consequently of x, while the second is independent or represents a variation of the constant incidence.

We then adopt for $\gamma = f(\theta')$, in Glauert's example, the series expansion

$$\gamma = V\left(a_o \cot\frac{\theta'}{2} + \sum_1^\infty a_n \sin n\,\theta'\right)$$

where V indicates the asymptotic velocity along the positive direction of axis x, and $\gamma_o, \gamma_1, \ldots \gamma_n \ldots$, the single terms of the preceding series, and $v_o, v_1 \ldots v_n \ldots$ the corresponding values of v, so that $v = \sum_o^\infty v_n$.

In addition, $v_n = v_n' + v_n''$ where v_n' is the part related to θ, v_n'' the part not related to θ.

We have*

$$v_o' = V\frac{a_o}{2\pi}\int_0^\pi \frac{\sin\theta'\cot\frac{\theta'}{2}}{\cos\theta' - \cos\theta}d\theta' = V\frac{a_o}{2\pi}\int_0^\pi \frac{1+\cos\theta'}{\cos\theta' - \cos\theta}d\theta' = \frac{V\,a_o}{2}\ {}^{**}$$

$$v_o'' = \frac{Va_o\lambda}{4\pi}\left\{\int_0^\pi \frac{1+\cos\theta'}{1-\lambda\cos\theta'}d\theta' - \int_0^\pi \frac{1+\cos\theta'}{1+\lambda\cos\theta'}d\theta'\right\} =$$

$$= \frac{V\,a_o\,\lambda}{2}\cosh\frac{\pi l}{h}\tanh\frac{\pi l}{2h} = \frac{V\,a_o}{2}\left(\cosh\frac{\pi l}{h} - 1\right)$$

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*For the calculus of the integrals in these formulas, see the appendix at the end of the report.
**For v_o, the first part v_o' itself is independent of θ.

whence

$$v_o = v_o' + v_o'' = \frac{V\ a_o}{2} \cosh \frac{\pi l}{h} \tag{8}$$

Similarly,

$$v_n' = \frac{V\ a_n}{2\pi} \int_0^{\pi} \frac{\sin \theta' \sin n\theta'}{\cos \theta' - \cos \theta} d\theta' = -\frac{V\ a_n}{2} \cos n\theta$$

$$v_n'' = \frac{V a_n \lambda}{4\pi} \left\{ \int_0^{\pi} \frac{\sin \theta' \sin n\theta'}{1 - \lambda \cos \theta'} d\theta' - \int_0^{\pi} \frac{\sin \theta' \sin n\theta'}{1 + \lambda \cos \theta'} d\theta' \right\}$$

$$= \frac{V\ a_n \lambda}{4\pi} \pi \coth \frac{\pi l}{h} \left\{ \tanh^n \frac{\pi l}{2h} + \left(-\tanh \frac{\pi l}{2h}\right)^n \right\} = \begin{cases} 0 & (n \text{ is odd}) \\ \frac{V\ a_n}{2} \tanh^n \frac{\pi l}{2h} & (n \text{ is even}) \end{cases}$$

and hence

$$v_n = \frac{V\ a_n}{2} \left(-\cos n\theta + \begin{matrix} 0 \\ \tanh^n \frac{\pi l}{2h} \end{matrix} \right) \begin{matrix} n \text{ is odd} \\ n \text{ is even} \end{matrix} \tag{9}$$

And finally, using Σ' to indicate that the sum is extended to odd values of n and Σ'' to indicate the extension to include even values:

$$v' = \Sigma v_n' = \frac{V}{2}(a_o - \sum_1^{\infty} a_n \cos n\theta)$$

$$v'' = \Sigma v_n'' = \frac{V}{2} \left[2\ a_o \frac{\tanh^2 \frac{\pi l}{2h}}{1 - \tanh^2 \frac{\pi l}{2h}} + \sum_2^{\infty}{}'' a_n \tanh^n \frac{\pi l}{2h} \right]$$

or, by expansion in series:

$$v'' = \frac{V}{2} \sum_2^{\infty}{}'' (a_n + 2\ a_o) \tanh^n \frac{\pi l}{2h}$$

2. Now the circulation Γ_n corresponding to the various terms γ_n and hence the complete circulation $\Gamma = \Sigma\, \Gamma_n$ can be computed.

It is:

$$\Gamma_n = \int_{-l}^{l} \gamma_n\, dx$$

where

$$dx = \frac{h}{\pi} \frac{\lambda \sin \theta \, d\theta}{1-\lambda^2 \cos^2\theta}$$

and therefore

$$\Gamma_0 = \int_{-l}^{l} \gamma_0 dx = \frac{V a_0 \lambda h}{\pi} \int_0^{\pi} \frac{1 + \cos \theta}{1-\lambda^2\cos^2 \theta} d\theta = V a_0 h \sinh \frac{\pi l}{h} \quad (10)$$

and

$$\left.\begin{aligned} \Gamma_n = \int_{-l}^{l} \gamma_n \, dx &= \frac{V a_n \lambda h}{\pi} \int_0^{\pi} \frac{\sin n\theta \sin\theta}{1-\lambda^2\cos^2 \theta} d\theta \\ &= V a_n h \begin{cases} 0 & (n \text{ is even}) \\ \tanh^n \frac{\pi l}{2h} & (n \text{ is odd}) \end{cases} \end{aligned}\right\} \quad (11)$$

whence

$$\left.\begin{aligned} \Gamma = \Sigma \, \Gamma_n &= V h \left(a_0 \sin h \frac{\pi l}{h} + \sum_1^{\infty}{}' a_n \tanh^n \frac{\pi l}{2h} \right) \\ &= V h \sum_1^{\infty}{}' (a_n + 2 a_0) \tanh^n \frac{\pi l}{2h} \end{aligned}\right\} \quad (12)$$

The velocity at infinity (negative for $x = -\infty$, positive for $x = +\infty$, but equal in both cases in magnitude) is readily obtained from:

$$v_\infty = \frac{\Gamma}{2h} = \frac{V}{2} \sum_1^{\infty}{}'(a_n + 2a) \tanh^n \frac{\pi l}{2h} \quad (13)$$

From the foregoing we can now draw some inferences about the differences between the direction of the tangent to the profile on the trailing edge and the direction of the velocity at ∞, a difference which may be termed <u>angular exaggeration</u> of the blade screen.

In effect, the trigonometric tangents of the two angles are expressed by $v(\pi)/V$ and v_∞/V and their difference δ (which, knowing the assumed smallness of these angles, is dissipated in the difference of the angles) is expressed by

$$\delta = \frac{v(\pi) - v_\infty}{V}$$

where $v(\pi)$ indicates the value of v for $\theta = \pi$.

Likewise, δ can be considered as the sum of δ_n, each δ_n corresponding to Γ_n; whence follows

$$\left.\begin{aligned} \delta_o &= \frac{a_o}{2}\left[\cosh\frac{\pi l}{h} - \sinh\frac{\pi l}{h}\right] = \frac{a_o}{2}e^{-\frac{\pi l}{h}} \\ \delta_n &= \pm\frac{a_n}{2}\left(1 - \tanh^n\frac{\pi l}{2h}\right)\begin{matrix} + & n & \text{odd} \\ - & n & \text{even}\end{matrix} \end{aligned}\right\} \tag{16}$$

3. If the flow velocity has a component $-V \sin \alpha$ on axis y, the shape of the profile remaining the same, a_o may change in such a way that the v_o do not change. In other words, a change in a_o does not change the incidence of the profile.

In the case where there is only a_o and all the other terms are nullified, v would also remain constant and the wing would be flat, with an incidence α expressed by

$$\alpha = \frac{a_o}{2}\cosh\frac{\pi l}{h} \tag{17}$$

or

$$a_o = 2\alpha\frac{1}{\cosh\frac{\pi l}{h}} \tag{17'}$$

and with the known formula (reference 1, p. 96).

$$\Gamma_o = 2\pi V l \alpha \frac{h}{\pi l}\tanh\frac{\pi l}{h} \tag{18}$$

or, in comparison with the isolated wing, the circulation, and hence the lift, would be reduced in the ratio

$\tanh\frac{\pi l}{h}\Big/\frac{\pi l}{h}$.

In the general case equation (18) is still applicable, but the absolute incidence is expressed by

$$\alpha = \frac{a_o}{2}\cosh\frac{\pi l}{h} + \frac{\sum_1^{\infty}{}' a_n \tanh^n\frac{\pi l}{2h}}{2\tanh\frac{\pi l}{h}} \tag{19}$$

We now have for lift P:

$$P = \rho \Gamma V = 2\pi\rho V^2 l \alpha k \tag{20}$$

with k signifying the reduction ratio

$$k = \frac{\tanh \frac{\pi l}{h}}{\frac{\pi l}{h}} \tag{21}$$

4. Proceeding to the determination of the shape of the wing, we have:

$$\frac{dy}{dx} = \frac{v}{V} = \frac{a_o}{2} \cosh \frac{\pi l}{h} + \Sigma'' \frac{a_n}{2} \tanh^n \frac{\pi l}{2h} - \Sigma \frac{a_n}{2} \cos n\theta$$

and so, as in the case of the isolated wing:

$$a_n = \frac{4}{\pi} \int_0^{\pi} \frac{dy}{dx} \cos n\theta . d\theta \tag{22}$$

It is understood that $\frac{dy}{dx}$ should be expressed in relation to θ.

Practically, given y $\left(\text{and hence } \frac{dy}{dx}\right)$ as a function of x, we can pass from x to θ by means of the relation

$$\cos \theta = - \frac{\tanh \frac{\pi x}{h}}{\tanh \frac{\pi l}{h}}$$

<u>Example</u>.- In the majority of cases the evaluation of the first three terms of the series will be sufficient to obtain profiles of single and double camber.

For $n = 0$, it is simply

$$\frac{dy_o}{dx} = \frac{a_o}{2} \cosh \frac{\pi l}{h}$$

which represents a straight line of inclination $\frac{a_o}{2} \cosh \frac{\pi l}{h}$.

For $n = 1$:

$$\frac{dy_1}{dx} = -\frac{a_1}{2}\cos\theta \qquad y_1 = -\int \frac{a_1}{2}\cos\theta \frac{h}{\pi} \frac{\lambda \sin\ \theta\, d\,\theta}{1 - \lambda^2 \cos^2\ \theta}$$

from which, after integration, follows

$$\frac{y_1}{l} = a_1 \frac{1}{2\,\frac{\pi l}{h}\ \tanh\frac{\pi l}{h}} \ln\cosh\frac{\pi x}{h} + K$$

Curve $\ln\cosh\frac{\pi x}{h}$ is shown in figure 3.

For $n = 2$:

$$\frac{dy_2}{dx} = \frac{a_2}{2}\tanh^2\frac{\pi l}{2h} - \frac{a_2}{2}\cos 2\theta$$

which, integrated, gives:

$$y_2 = a_2\left[\frac{\tanh\frac{\pi x}{h}}{\frac{\pi l}{h}\ \tanh^2\frac{\pi l}{h}} - \frac{x}{l}\,\frac{\cosh\frac{\pi l}{h}}{\sinh^2\frac{\pi l}{h}}\right]$$

The second term represents a constant inclination; the first, proportional to $\tanh\frac{\pi x}{h}$, has the shape of the curve in figure 3 and is repsonsible for double camber of the profile.

If the three terms are present, we have

$$\delta = \frac{a_0}{2}e^{-\frac{\pi l}{h}} + \frac{a_1}{2}\left(1 - \tanh\frac{\pi l}{2h}\right) - \frac{a_2}{2}\left(1 - \tanh^2\frac{\pi l}{2h}\right)$$

or else

$$\delta = \frac{a_0}{2}e^{-\frac{\pi l}{h}} + a_1\frac{e^{-\frac{\pi l}{2h}}}{\cosh\frac{\pi l}{2h}} - \frac{a_2}{2\cosh^2\frac{\pi l}{2h}}$$

for $l = h$, for instance, it is:

$$\frac{\pi l}{h} = \pi \quad \text{and} \quad e^{-\frac{\pi l}{h}} = \sim\frac{1}{23}$$

Restricted to the first term, it is:

$$\delta = \alpha \frac{e^{-\frac{\pi l}{h}}}{\cosh \frac{\pi l}{h}} = \sim \frac{1}{529} \alpha$$

The second reaches 0.041 a_1 and the third, -0.08 a_2.

5. We now make the comparison with the theory of substitution vortices (reference 2).

In the case of the flat wing (excluding self-induced velocity, disposing the vortices in the forward neutral point* and computing the induced velocity in the rear neutral point) it simply gives:

$$v' = \frac{\Gamma}{2h} \coth \frac{\pi l}{h} - \frac{\Gamma}{2\pi l} - V \alpha$$

and on the other hand:

$$\Gamma = - 2\pi l v'$$

Therefore:

$$- \frac{\Gamma}{2\pi l} = \frac{\Gamma}{2h} \coth \frac{\pi l}{h} - \frac{\Gamma}{2\pi l} - V \alpha$$

whence

$$\Gamma = 2 \pi V l \alpha \frac{h}{\pi l} \tanh \frac{\pi l}{h}$$

which is the same formula previously worked out. From this it is concluded that the method of substitution vortices gives, for the flat wing, not merely approximate but exact results.

In the case of the cambered wing (single camber) it is observed that the inclination of both the leading and trailing edges $\pm a_1/2$. Now, in the case of the isolated wing, an equal inclination is obtained with an equal value of a_1 to which corresponds a moment coefficient with respect to the forward neutral point, of $\frac{\pi}{8} a_1$ to which

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*The term "forward neutral point" and "rear neutral point" for points situated at quarter-chord length from the tip was introduced by Küssner.

corresponds a doublet $\mu = \frac{\pi}{4} V l^2 a_1$. According to the general procedure of substitution vortices, it happens that we can compute the induced velocity in the rear neutral point of a screen of doublets (exclusive of that on the wing). But it is readily seen that the symmetry of the vortices with respect to the origin is such that the center of gravity of the circulation remains in the same origin for such a wing. In consequence, it is sufficient to consider the induced velocity of a screen of vortices disposed on axis y. We will have:

$$v'_1 = \frac{\Gamma_1}{2h} \coth \frac{\pi l}{2h} - \frac{\Gamma_1}{\pi l}$$

Other than this, the form of the profile produces a change in the incidence, or, what amounts to the same thing, a change in zero lift direction, given (for isolated wing) at $\frac{a_1}{4}$. As a result:

$$\Gamma_1 = -2\pi l \, \Gamma_1 \left\{ \frac{\coth \frac{\pi l}{2h}}{2h} - \frac{1}{\pi l} \right\} + 2\pi l \, V \frac{a_1}{4}$$

from which

$$\Gamma_1 = V a_1 h \frac{1}{2} \frac{\tanh \frac{\pi l}{2h}}{1 - \frac{h}{\pi l} \tanh \frac{\pi l}{2h}}$$

A comparison with the exact formula

$$\Gamma_1 = V a_1 h \tanh \frac{\pi l}{2h}$$

shows the approximation to be no longer satisfactory despite the fact that the ratio $\pi l/h$ becomes of the order of unity or higher.

It is interesting, on the other hand, to notice that the rigorous result can be obtained by assuming the induction of screen vortices to be reduced to half. In effect then we have:

$$\Gamma_1 = -\pi l \, \Gamma_1 \left\{ \frac{\coth \frac{\pi l}{2h}}{2h} - \frac{1}{\pi l} \right\} + 2\pi l \, V \frac{a_1}{4}$$

from which the exact value for Γ_1 given above can be readily obtained.

6. A convenient and sufficiently approximate method for the practical prediction of the characteristics of a wing of given shape is that of stopping with the first three terms of the series, and deriving the coefficients of inclination of the tangents to the profile in three points - 1, 0, 1, by the Birnbaum method.

Denoting the inclination of the tangent in a generic point x, with α_x and the inclinations in points A B C (leading edge, trailing edge, and center of profile (fig. 4) with α_A, α_B, α_C, we find:

$$\alpha_A = \frac{a_o}{2} \cosh \frac{\pi l}{h} - \frac{a_1}{2} + \frac{a_2}{2}\left(- 1 + \tanh^2 \frac{\pi l}{2h}\right)$$

$$\alpha_B = \frac{a_o}{2} \cosh \frac{\pi l}{h} + \frac{a_1}{2} + \frac{a_2}{2}\left(- 1 + \tanh^2 \frac{\pi l}{2h}\right)$$

$$\alpha_C = \frac{a_o}{2} \cosh \frac{\pi l}{h} \qquad \frac{a_2}{2}\left(1 + \tanh^2 \frac{\pi l}{2h}\right)$$

from which we obtain

$$a_o = \frac{\alpha_C}{\cosh \frac{\pi l}{h} \times \cosh^2 \frac{\pi l}{2h}} + \frac{\alpha_A + \alpha_B}{2} \frac{1}{\cosh^2 \frac{\pi l}{2h}}$$

$$a_1 = \alpha_B - \alpha_A$$

$$a_2 = \alpha_C - \frac{\alpha_A + \alpha_B}{2}$$

and from equation (12):

$$\Gamma = V h \left\{ \alpha_C \frac{\tanh \frac{\pi l}{h}}{\cosh^2 \frac{\pi l}{2h}} + 2 \alpha_B \tanh \frac{\pi l}{2h} \right\} \tag{23}$$

or, better:

$$\Gamma = 2 \pi l V \frac{\tanh \frac{\pi l}{h}}{\frac{\pi l}{h}} \left\{ \frac{\alpha_B + \alpha_C}{2} + \frac{\alpha_B - \alpha_C}{2} \tanh^2 \frac{\pi l}{2h} \right\} \tag{23'}$$

In this very simple formula $\frac{\alpha_B + \alpha_C}{2}$ is the angle of lift for the isolated wing. The second term in parentheses disappears when $h/l \longrightarrow \infty$, while the reduction factor $\tanh \frac{\pi l}{h} \Big/ \frac{\pi l}{h}$ tends toward 1.

Equation (23) or (23') can be used for all practical cases of the straight screen.

It is noted that in place of equation (23) the following procedure may be adopted: To compute Γ as the sum of Γ_0 and Γ_1 each computed by the method of substitution vortices of which at n. 5, allocating to the profile incidence α_C in the calculation of Γ_0 and incidence $\frac{\alpha_B - \alpha_C}{2}$ in the calculation of Γ_1. It is easy to see that the two methods lead to the same results.

7. The prediction of the moment is more complicated. The moment M about origin O (center of profile) is expressed by

$$M = \rho V \int_{-l} \gamma x \, dx.$$

but the expression of x obtained from equation (3) and equation (6) is too complicated for direct integration. It is more convenient to expand x in series of cosines.

Posted, that is to say:

$$x = \sum_{1}^{\infty} b_n \cos n\theta$$

we have:

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \cos n\theta \, d\theta = -\frac{2}{\pi n} \int_0^{\pi} \sin n\theta \, \frac{h\lambda}{\pi} \frac{\sin\theta}{1-\lambda^2\cos^2\theta} \, d\theta$$

which is easily obtained by integrating by parts; and, finally,

$$b_n = \begin{cases} -\dfrac{2h}{\pi n} \tanh^n \dfrac{\pi l}{2h} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

Then:

$$x = -\frac{2h}{\pi} \sum_{1}^{\infty}{}' \frac{1}{n} \tanh^n \frac{\pi l}{2h} \cos n\theta$$

and, consequently, with the usual notation:

$$M_o = -\frac{2h^2}{\pi^2} \rho V^2 \lambda \int_0^{\pi} a_o \cot \frac{\theta}{2} \left[\sum_{1}^{\infty}{}' \frac{1}{m} \tanh^m \frac{\pi l}{2h} \cos m\theta \right] \frac{\sin \theta d\theta}{1 - \lambda^2 \cos^2\theta}$$

$$M_n = -\frac{2h^2}{\pi^2} \rho V^2 \lambda \int_0^{\pi} a_n \sin n\theta \left[\sum_{1}^{\infty}{}' \frac{1}{m} \tanh^m \frac{\pi l}{2h} \cos m\theta \right] \frac{\sin \theta d\theta}{1 - \lambda^2 \cos^2\theta}$$

In particular,

$$M_o = -\frac{h^2}{\pi} \rho V^2 a_o \cosh \frac{\pi l}{h} \ln \cosh \frac{\pi l}{h} \qquad (24)$$

The point of application of the lift is obtained by dividing M_o by $P_o = \rho V^2 a_o h \sinh \frac{\pi l}{h}$, whence:

$$x_o = -\frac{h}{\pi} \coth \frac{\pi l}{h} \cdot \ln \cosh \frac{\pi l}{h} \qquad (25)$$

or else:

$$x_o = -\frac{l}{2} \cdot \frac{\tanh \frac{\pi l}{h}}{\frac{\pi l}{h}} \left(1 + \frac{1}{2} \tanh^3 \frac{\pi l}{h} + \frac{1}{3} \tanh^5 \frac{\pi l}{h} + \ldots\right) \qquad (25')$$

It is seen that $x_o \rightarrow -\frac{l}{2}$ for $\frac{l}{h} \rightarrow 0$, whereas it tends toward $-l$ for $\frac{l}{h} \rightarrow \infty$.

In other words, the aerodynamic center of the profile for the flat plate is shifted toward the leading edge so as to reduce the spacing (h) of the screen.

As for the moments M_n, we immediately have $M_n = 0$ for odd n, as is shown by the symmetry with respect to 0 of the flow distribution, while for even n the moment is other than zero, only the complete circulation and the lift being zero.

M_n'' is easily determined, when bearing in mind that

$\int_0^\pi \frac{\cos n\theta}{1-\lambda^2\cos^2\theta} d\theta$ for $n < 0$ is identical to that obtained for the same value of n with changed sign, as distinguished from that obtained with the material application of the formula that gives the value of the said integral $\left(\pi \cosh \frac{\pi l}{h} \tanh^n \frac{\pi l}{2h} \text{ for even } n\right)$

Therefore:

$$M_n = -\frac{h^2}{2\pi}\rho V^2 a_n \lambda \sum_1^\infty{}' \frac{1}{m}\tanh^m\frac{\pi l}{2h} \left(J_{m+n-1} + J_{m-n+1} - J_{m+n+1} - J_{m-n-1}\right)$$

after posting (see appendix)

$$J_n = \int_0^\pi \frac{\cos n\theta}{1-\lambda^2\cos^2\theta} d\theta$$

Thus we obtain:

$$M_n = \frac{h^2}{\pi}\rho V^2 a_n \Sigma'_m \tanh^m \frac{\pi l}{2h} \cdot \frac{1}{m}\left[\tanh^{|n+m|}\frac{\pi l}{2h} \pm \tanh^{n-m}\frac{\pi l}{2h}\right] \quad (26)$$

where the + sign is valid for $m \leq n - 1$ and the − sign for $m \geq n + 1$.

For $n = 2$, we have:

$$M_2 = \frac{h^2}{2\pi}\rho V^2 a_2 \left[\left(\tanh^2\frac{\pi l}{2h} - \frac{1}{\tanh^2\frac{\pi l}{2h}}\right)\ln\cosh\frac{\pi l}{h} + 2\left(1 + \tanh^2\frac{\pi l}{2h}\right)\right]$$

$$M_2 = \frac{h^2}{\pi}\rho V^2 a_2 \cosh\frac{\pi l}{h}\left[-\frac{2\ln\cosh\frac{\pi l}{h}}{\sinh^2\frac{\pi l}{h}} + \frac{1}{\cosh^2\frac{\pi l}{2h}}\right]$$

or, in more expressive form:

$$M_2 = \frac{h^2}{\pi}\rho V^2 a_2 \tanh^2\frac{\pi l}{2h}\left[1 + 2\tanh^2\frac{\pi l}{2h}\left(\frac{1}{3} + \frac{\tanh^4\frac{\pi l}{2h}}{3\cdot 5} + \frac{\tanh^8\frac{\pi l}{2h}}{5\cdot 7} + \ldots\right)\right]$$

For $\frac{\pi l}{2h} \to 0$, $M_2 \to \frac{\pi l^2}{4}\rho V^2 a_2$, the value characteristics of the isolated wing, while for $\frac{\pi l}{2} \to \infty$, $M_2 \to 0$.

Since a variation in incidence changes a_o only, the point x_o previously obtained is such that with respect to it the moment does not change with the incidence. This is, moreover, the aerodynamic center of the profile of the screen.

II. THE SCREEN IN TANDEM

8. The treatment is wholly similar to that of the straight screen.

The induced velocity of a screen of vortices with equidistant h arranged on axis x (one of the vortices in the origin) is

$$w = -\sum_{-\infty}^{\infty} \frac{i\Gamma}{2\pi(z - n h)} = -\frac{i\Gamma}{2h} \cot \frac{\pi z}{h} \tag{27}$$

and hence in a point on axis x:

$$v = \frac{\Gamma}{2h} \cot \frac{\pi x}{h} \tag{27'}$$

With the same notation as before, we post:

$$\tan \frac{\pi x}{h} = \xi \; ; \; \cot\left(\frac{\pi x}{h} - \frac{\pi x'}{h}\right) = \frac{1 + \xi\xi'}{\xi - \xi'}$$

$$dx = \frac{h}{\pi} \cdot \frac{1}{1+\xi^2} d\xi; \quad \lambda = \tan \frac{\pi l}{h}; \; \frac{\xi}{\lambda} = -\cos\theta$$

hence:

$$v = \frac{1}{2\pi} \int_0^{\pi} \frac{1+\lambda^2 \cos\theta \cos\theta'}{\cos\theta' - \cos\theta} \cdot \frac{\sin\theta'}{1+\lambda^2\cos^2\theta'} \gamma \, d\theta' \tag{28}$$

and, after simple transformations:

$$v = \frac{1}{2\pi} \int_0^{\pi} \frac{\sin\theta'}{\cos\theta' - \cos\theta} \gamma \, d\theta' - \frac{\lambda^2}{2\pi} \int_0^{\pi} \frac{\sin\theta' \cos\theta'}{1+\lambda^2\cos^2\theta'} \gamma \, d\theta'$$

wherefrom, after posting

$$\gamma = V\left(a_o \cot\frac{\theta}{2} + \sum_1^\infty a_n \sin n\theta\right)$$

we obtain:

$$v_o' = V\frac{a_2}{2} \qquad v_o'' = V\frac{a_o}{2}\left(\cos\frac{\pi l}{h} - 1\right) \qquad v_o = V\frac{a_o}{2}\cos\frac{\pi}{h}$$

$$v_n' = -V\frac{a_n}{2}\cos n\theta \qquad v_n'' = \begin{cases} -V\dfrac{a_n}{2}\tan^n\dfrac{\pi l}{2h} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

and finally:

$$v' = \frac{V}{2}\left(a_o - \sum_1^\infty a_n \cos n\theta\right) \tag{29}$$

as in the case of the straight screen

$$v'' = \frac{V}{2}\left\{a_o\left(\cos\frac{\pi l}{h} - 1\right) + \sum_2^{\infty}{}'' \; i^n \tan^n\frac{\pi l}{2h}\right\}$$

or else

$$v'' = \frac{V}{2}\sum_2^{\infty}{}'' \; i^n(a_n + 2a_o)\tan^n\frac{\pi l}{2h}$$

As regards the circulation, we have, correspondingly:

$$\Gamma_o = V a_o h \sin\frac{\pi l}{h} \tag{30}$$

$$\Gamma_n = \begin{cases} V a_n h\, i^{n-1}\tan^n\dfrac{\pi l}{2h} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} \tag{31}$$

and finally, for the flat wing:

$$\Gamma_o = 2\pi V l \alpha \frac{\tan\dfrac{\pi l}{h}}{\dfrac{\pi l}{h}} \tag{32}$$

a formula similar to equation (18).

The tandem arrangement amplifies, as is seen, the circulation considerably.

Similarly, we can study the shape of profiles corresponding to the various terms of the circulation, which is omitted here for the sake of brevity.

9. The comparison with the theory of substitution vortices remains to be made. For the straight wing, the vortices being disposed in the forward neutral point, the induced velocity in the rear neutral point is:

$$v' = \frac{\Gamma}{2h} \cot \frac{\pi l}{h} - \frac{\Gamma}{2\pi l} - V\alpha$$

and

$$\Gamma = -2\pi l v'$$

whence:

$$-\frac{\Gamma}{2\pi l} = \frac{\Gamma}{2h} \cot \frac{\pi l}{h} - \frac{\Gamma}{2\pi l} - V\alpha$$

from which we obtain equation (32).

In this case also, the results obtained are not merely approximate but exact. And as for the straight screen, exact results are obtained for cambered wing by disposing a screen of vortices in points in the center of the profile and reducing the induction by one-half.

For the sake of brevity, other similar considerations for the straight screen are omitted inasmuch as regards the practical calculation of the lift in relation to inclination α_A, α_B α_C, shape of profiles, and calculation of the moment.

III. THE OBLIQUE SCREEN

10. Let β signify the obliquity of the screen, defined as the angle through which axis y' containing the vortices must rotate in order to be superimposed on axis y (fig. 6).

To pass from w referred to axes $x' y'$, to w referred to axes $x y$, it is sufficient to post in the complex potential

$$z' = z e^{i\beta}$$

Thus:

$$w = -\frac{i\Gamma}{2h} \coth \frac{\pi z e^{i\beta}}{h} e^{i\beta} \tag{33}$$

In a point on axis x, the value of w is simply obtained by posing x at the position of z. The value of w is complex, hence there is a real component, that is to say, along axis x, which is neglected, and an imaginary one, that is, along axis y, which alone is of interest.

We write:

$$\tanh \frac{\pi x e^{i\beta}}{h} = \xi; \quad \lambda = \tanh \frac{\pi l e^{i\beta}}{h};$$

$$\frac{\xi}{\lambda} = -\cos\theta \ (\theta \text{ complex}) \tag{34}$$

and merely follow the procedure step by step, as for the straight screen. Thus

$$\gamma_o = V a_o \cot \frac{\theta}{2} = V a_o \frac{1 + \cos\theta}{\sin\theta}$$

$$\gamma_n = V a_n \sin n\theta$$

with complex a_o and a_n. Besides:

$$w = -\frac{ie^{i\beta}}{2h} \int_{-l}^{l} \gamma \, dx' \times \coth \frac{\pi(x-x')e^{i\beta}}{h}$$

whence

$$w = -\frac{i}{2\pi} \int_0^{\pi} \frac{1-\lambda^2 \cos\theta \cos\theta'}{\cos\theta' - \cos\theta} \times \frac{\sin\theta'}{1-\lambda^2\cos^2\theta'} \gamma \, d\theta'$$

It follows that:

$$w_o' = -\frac{i V a_o}{2}; \; w_o'' = -\frac{i V a_o}{2}\left(\cosh \frac{\pi l e^{i\beta}}{h} - 1\right) \qquad (35)$$

and, on the whole:

$$w_o = \frac{i V a_o}{2} \cosh \frac{\pi l e^{i\beta}}{h} \qquad (36)$$

Similarly:

$$w_n = -\frac{i V a_n}{2}\left(-\cos n\theta + \begin{matrix} 0 \\ \tanh^n \frac{\pi l e^{i\beta}}{2h} \end{matrix}\right) \begin{matrix} n \ \text{odd} \\ n \ \text{even} \end{matrix} \qquad (37)$$

Proceeding with the calculation of Γ (generally complex), we find:

$$\Gamma_o = V h e^{-i\beta} a_o \sinh \frac{\pi l e^{i\beta}}{h} \qquad (38)$$

$$\Gamma_n = V h e^{-i\beta} a_n \begin{cases} 0 & n \ \text{even} \\ \tanh^n \frac{\pi l e^{i\beta}}{2h} & n \ \text{odd} \end{cases} \qquad (39)$$

The total Γ must, of necessity, be real; in effect it contains an imaginary part which may represent a total source other than zero, which is naturally not admissible, forcing the wing profile to be closed. Then we would have

$$\underline{I}\left\{e^{-i\beta}\left(a_o \sinh \frac{\pi l e^{i\beta}}{h} + \Sigma' a_n \tanh^n \frac{\pi l e^{i\beta}}{2h}\right)\right\} = 0 \qquad (40)$$

where $\underline{I}$ is the imaginary coefficient.

11. The presence of a source distribution across the chord prompts us to attribute a certain thickness to the profile (E. Pistolesi - "Theory of Thin Airfoils" - in course of publication). We can, however, obtain the shape of the "dorsal spine" of the profile by the conventional method, namely, by ascertaining the vertical component of the induced velocity

$$v = \frac{V}{2} R\left\{a_o \cosh \frac{\pi l e^{i\beta}}{h} - \sum_1^\infty a_n \cos n\theta + \Sigma'' a_n \tanh^n \frac{\pi l e^{i\beta}}{2h}\right\} \qquad (41)$$

where R indicates the real part; and we put $\frac{dy}{dx} = \frac{v}{V}$.

For y_o, resulting from v_o, it yields the rectilinear profile with incidence $\frac{v_o}{V}$, $\alpha = \frac{1}{2} R \left[a_o \coth \frac{\pi l e^{i\beta}}{h} \right]$; the y_n have curved profiles, from which the curve can also be computed; but for simplicity this very complex development is foregone.

If it is the only term in a_o, we should have

$$a_o = A_o \frac{e^{i\beta}}{\sinh \frac{\pi l e^{i\beta}}{h}} \tag{42}$$

with A_o real, and then it follows that:

$$\alpha = \frac{1}{2} A_o R \left[e^{i\beta} \coth \frac{\pi l e^{i\beta}}{h} \right] \tag{43}$$

In general, the inclination of the profile for $\theta = \pi$ (trailing edge) can be calculated. It is:

$$\frac{v(\pi)}{V} = \frac{1}{2} R \left(a_o \cosh \frac{\pi l e^{i\beta}}{h} + \Sigma' a_n - \Sigma'' a_n + \Sigma'' a_n \tanh^n \frac{\pi l e^{i\beta}}{2 h} \right)$$

For ∞, we find: $v_\infty = \frac{\Gamma \cos \beta}{2 h}$, or:

$$\frac{v_\infty}{V} = \frac{1}{2} \cos \beta \left\{ e^{-i\beta} \left(a_o \sinh \frac{\pi l e^{i\beta}}{h} + \Sigma' a_n \tanh^n \frac{\pi l e^{i\beta}}{2 h} \right) \right\}$$

or else, since the part in brackets is itself real:

$$\frac{v_\infty}{V} = \frac{1}{2} R \left(a_o \sinh \frac{\pi l e^{i\beta}}{h} + \Sigma' a_n \tanh^n \frac{\pi l e^{i\beta}}{2 h} \right)$$

whence the "angular exaggeration" δ follows at:

$$\delta = \frac{1}{2} R \left\{ a_o e^{\frac{i\pi l e^{i\beta}}{h}} \pm \Sigma a_n \left(1 - \tanh^n \frac{\pi l e^{i\beta}}{2 h} \right) \right\} \quad \begin{matrix} + \text{ for } n \text{ odd} \\ - \text{ for } n \text{ even} \end{matrix} \tag{44}$$

12. A method for verifying that equation (40) is satisfactory is to put, as previously done in the case where a_o only is other than zero (equation (42)):

$$a_o = A_o \frac{e^{i\beta}}{\sinh \frac{\pi l e^{i\beta}}{h}}$$

and in addition

$$a_n = A_n \frac{e^{i\beta}}{\tanh^n \frac{\pi l e^{i\beta}}{2\,h}} \quad (A_n \text{ real}) \qquad (45)$$

In this case:

$$\Gamma = V\,h\,(A_o + \Sigma'\,A_n) \qquad (46)$$

In the case of the flat wing, the lift reduction factor can be computed much more simply. In fact, posting

$$\Gamma_o = 2\pi l V \alpha k \qquad (47)$$

where k is the reduction factor, leaves

$$k = \frac{A_o}{\alpha}\,\frac{h}{2\pi l} = \frac{\frac{h}{\pi l}}{R\left(e^{i\beta} \cdot \coth \frac{\pi l e^{i\beta}}{h}\right)} \qquad (48)$$

and by simple calculation

$$k = \frac{h}{\pi l}\,\frac{\cosh^2\left(\frac{\pi l}{h}\cos\beta\right) - \cos^2\left(\frac{\pi l}{h}\sin\beta\right)}{\cos\beta \sinh\left(\frac{\pi l}{h}\cos\beta\right)\cosh\left(\frac{\pi l}{h}\cos\beta\right) + \sin\beta \sin\left(\frac{\pi l}{h}\sin\beta\right)\cos\left(\frac{\pi l}{h}\sin\beta\right)} \qquad (49)$$

or else:

$$k = \frac{h}{\pi l}\,\frac{\cosh\left(\frac{2\pi l}{h}\cos\beta\right) - \cos\left(\frac{2\pi l}{h}\sin\beta\right)}{\cos\beta \sinh\left(\frac{2\pi l}{h}\cos\beta\right) + \sin\beta \sin\left(\frac{2\pi l}{h}\sin\beta\right)} \qquad (49')$$

Formula (49) agrees with that by Numachi (reference 3), according to a method previously indicated by Grammel (reference 1), with the difference that the Numachi-Grammel procedure does not exactly fit the flat wing, but rather one with slight camber.*

We can finally verify, for the case in point, the method of substitution vortices.

The vertical induced velocity in the rear neutral point of the screen of vortices concentrated in the forward neutral point, the self-induced velocity, is

$$v = \frac{\Gamma}{2h} R\left(e^{i\beta} \coth \frac{\pi l e^{i\beta}}{h}\right) - \frac{\Gamma}{2\pi l}$$

On the other hand, we have:

$$\Gamma = 2\pi l \left\{ V\alpha - \frac{\Gamma}{2h} R\left(e^{i\beta} \coth \frac{\pi l e^{i\beta}}{h}\right) + \frac{\Gamma}{2\pi l} \right\}$$

from which equation (48) follows. Again the method of substitute vortices insures exact results.

It will be noted that every change in incidence, without change in profile form, results in a change of a_0; even the Δa_0 corresponding to a change $\Delta\alpha$ must have the form (42), which must correspond to a zero lift source. However, in general, equation (47) holds true for Γ, or:

$$\Gamma = 2\pi l V \alpha k \qquad (50)$$

the terms a_0, a_1, a_2 ...(and hence the shape of the profile) affect only the position of the zero lift curve whose angle of lift α must be computed.

The method of substitute vortices can also be applied for obtaining Γ_1, if equation (45) is taken for a_1,

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*It should be remembered also that Grammel's curved line is rather one-half the profile, the other half being formed by the symmetry of the first with respect to the chord; that it involves a flat rather than a curved profile of a certain thickness, which is more clearly shown in the treatment of the present study.

with the single condition that the screen of vortices Γ_1 are placed in the centers of profiles and the induced velocity is halved.

We have, in fact:

$$\Gamma_1 = -\pi l \Gamma_1 \left\{ \frac{1}{2h} R\left(e^{i\beta} \coth \frac{\pi l e^{i\beta}}{2h} - \frac{1}{\pi l}\right)\right\} + 2\pi l V \alpha_1$$

where α_1, the natural incidence of the curved profile, is given by

$$\alpha_1 = \frac{1}{2} \frac{v_1(0) + v_1(\pi)}{V} = \frac{1}{4} R(a_1)$$

from which follows:

$$\Gamma_1 = \frac{R(a_1)}{R\left(e^{i\beta} \coth \frac{\pi l e^{i\beta}}{2h}\right)} V h$$

or, with equation (45) for a_1,

$$\Gamma_1 = V h A_1$$

conformable to equation (46).

In general, the method of substitute vortices is applicable by computing Γ as the sum of Γ_0 and Γ_1, giving the profile an incidence α_0 (fig. 4) for computing Γ_0 and incidence $\frac{\alpha_B - \alpha_C}{2}$ for computing Γ_1. It affords

$$\Gamma = 2\pi l V \left\{ \alpha_C \frac{\frac{h}{\pi l}}{R\left(e^{i\beta} \cosh \frac{\pi l e^{i\beta}}{h}\right)} + \frac{\alpha_B - \alpha_C}{2} \cdot \frac{\frac{2h}{\pi l}}{R\left(e^{i\beta} \cosh \frac{\pi l e^{i\beta}}{2h}\right)} \right\} \qquad (51)$$

13. The practical calculation of the aerodynamic characteristics of a given airfoil can be carried out by stopping at the first three terms of the expansion of γ.

Equation (40) supplies, however:

$$\underline{I}\left(a_o e^{-i\beta} \sinh \frac{\pi l e^{i\beta}}{h} + a_1 e^{-i\beta} \tanh \frac{\pi l e^{i\beta}}{2\ h} \right) = 0$$

Moreover, with the notation used for the straight screen, we find:

$$\alpha_A = R\left\{ \frac{a_o}{2} \cosh \frac{\pi l e^{i\beta}}{h} - \frac{a_1}{2} + \frac{a_2}{2}\left(-1 + \tanh^2 \frac{\pi l e^{i\beta}}{2\ h} \right)\right\}$$

$$\alpha_B = R\left\{ \frac{a_o}{2} \cosh \frac{\pi l e^{i\beta}}{h} + \frac{a_1}{2} + \frac{a_2}{2}\left(-1 + \tanh^2 \frac{\pi l e^{i\beta}}{2\ h} \right)\right\}$$

$$\alpha_C = R\left\{ \frac{a_o}{2} \cosh \frac{\pi l e^{i\beta}}{h} + \frac{a_2}{2}\left(1 + \tanh^2 \frac{\pi l e^{i\beta}}{2\ h} \right)\right\}$$

from which:

$$\left.\begin{aligned}
&\alpha_B - \alpha_A = R(a_1) \\
&\alpha_C - \frac{\alpha_B + \alpha_A}{2} = R(a_2) \\
&\alpha_C \left(1 - R \tanh^2 \frac{\pi l e^{i\beta}}{2\ h}\right) + \frac{\alpha_B + \alpha_A}{2}\left(1 + R \tanh^2 \frac{\pi l e^{i\beta}}{2\ h}\right) = R\left(a_o \cosh \frac{\pi l e^{i\beta}}{h}\right)
\end{aligned}\right\} \quad (52)$$

The problem generally remains indeterminate; to make it determinate two other conditions prevail (one, represented by equation (40)), which naturally must depend upon the profile thickness.

Reserving the treatment of this point for the next paragraph, let us see what the results are from proceeding on the basis of equation (45). We have:

$$A_o = \frac{\alpha_C\left(1 - R \tanh^2 \frac{\pi l e^{i\beta}}{2\ h}\right) + \frac{\alpha_B + \alpha_A}{2}\left(1 + R \tanh^2 \frac{\pi l e^{i\beta}}{2\ h}\right)}{R\left(e^{i\beta} \coth \frac{\pi l e^{i\beta}}{h}\right)}$$

$$A_1 = \frac{\alpha_B - \alpha_A}{R\left(e^{i\beta} \coth \frac{\pi l e^{i\beta}}{2\,h}\right)}$$

and, substituting in equation (46):

$$\Gamma = 2\pi l V \left\{k\ \alpha_C + k_1 \frac{\alpha_B - \alpha_A}{2} - \frac{\alpha_B + \alpha_A - 2\alpha_C}{4} R\left(\tanh^2 \frac{\pi l e^{i\beta}}{2\,h}\right)\right\} \quad (53)$$

where k is expressed by equation (48), and

$$k_1 = \frac{\frac{2h}{\pi l}}{R\left(e^{i\beta} \cosh \frac{\pi l e^{i\beta}}{2\,h}\right)} \quad (54)$$

the same k in which $\frac{1}{2} l$ replaces length l.

Equation (53) coincides with (51) when $a_2 = 0$ and hence $(\alpha_B - \alpha_C) + (\alpha_A - \alpha_C) = 0$. Other forms into which equation (53) can be transformed are omitted for brevity.

14. It will be noted that, with the effected position, we have as a value of the source for $x = 0$, or for $\theta = \frac{\pi}{2}$, the imaginary coefficient of $V\left(a_0 \cot \frac{\pi}{4} + a_1 \sin \frac{\pi}{2}\right)$, or:

$$\epsilon = V\ I\left(A_0 \frac{e^{i\beta}}{\sinh \frac{\pi l e^{i\beta}}{h}} + A_1 e^{i\beta} \coth \frac{\pi l e^{i\beta}}{2\,h}\right)$$

This source intensity defines an inclination of the tangents to the profile on the top and bottom camber with respect to the tangents to the center line given by $\frac{1}{2}\frac{\epsilon}{V}$, and, in consequence, an angle τ between the two quoted tangents (fig. 7) given by *

$$\tau = A_0\ I\left(e^{i\beta} \operatorname{cosech} \frac{\pi l e^{i\beta}}{h}\right) + A_1\ I\left(e^{i\beta} \coth \frac{\pi l e^{i\beta}}{2\,h}\right) \quad (55)$$

Such values of τ are termed τ_0. They vary, as is seen, with the incidence, since A_0 itself varies as the incidence varies.

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*Angle τ is positive when the nose is to the left.

The ensuing solution is a special case. In general,

$$a_o = (A_o + i\,B_o)e^{i\beta}\ \text{cosech}\ \frac{\pi l e^{i\beta}}{h}$$

$$a_n = (A_n + i\,B_n)e^{i\beta}\ \coth^n \frac{\pi l e^{i\beta}}{2\ h}$$

should replace equation (45).

With these positions, equation (40) becomes:

$$B_o + \Sigma'\ B_n = 0 \tag{56}$$

while continuing the evaluation of equation (46), or:

$$\Gamma = V\ h(A_o + \Sigma'\ A_n) \tag{46}$$

Stopping, as usual, with the first three terms, and posting, for simplicity of notation,

$$\left.\begin{aligned} e^{i\beta} \coth \frac{\pi l e^{i\beta}}{h} &= p + iq \\ e^{i\beta} \coth \frac{\pi l e^{i\beta}}{2\ h} &= p_1 + iq_1 \\ e^{i\beta} \coth^n \frac{\pi l e^{i\beta}}{2\ h} &= p_n + iq_n \end{aligned}\right\} \tag{57}$$

$\left(\text{in which } k = \dfrac{h}{\pi l p}\ \text{e}\ k_1 \dfrac{2h}{\pi l p_1}\right)$, we obtain in place of equation (52) the following:

$$\alpha_B - \alpha_A = A_1 p_1 - B_1 q_1 \tag{58}$$

$$\alpha_C - \frac{\alpha_B + \alpha_A}{2} = A_2 p_2 - B_2\ q_2 \tag{59}$$

$$a_C\left(1 - R\ \tanh^2 \frac{\pi l e^{i\beta}}{2\ h}\right) + \frac{\alpha_B + \alpha_A}{2}\left(1 + R\ \tanh^2 \frac{\pi l e^{i\beta}}{2\ h}\right) = A_o p - B_o q \tag{60}$$

These formulas are then supplemented by that obtained by angle τ, which, if it is considered that

$\operatorname{cosech} \frac{\pi l e^{i\beta}}{h} = \coth \frac{\pi l e^{i\beta}}{2\,h} - \coth \frac{\pi l e^{i\beta}}{h}$, is written as:

$$\tau = A_0(q_1 - q) + B_0(p_1 - p) + A_1 q_1 + B_1 p_1 \tag{61}$$

From (56) finally follows

$$B_0 + B_1 = 0$$

After $-B_1$ is substituted for B_0, equations (58), (60), and (61) give three equations in A_1, B_1, and A_0 which, resolved, give:

$$\left.\begin{aligned} A_0 + A_1 &= (A_0 + A_1)_0 + B_1\, M \\ \tau &= \tau_0 + B_1\, N \end{aligned}\right\} \tag{62}$$

where $(A_0 + A_1)_0$ and τ_0 indicate the values of $A_0 + A_1$ and of τ for $B_1 = B_0 = 0$, and, besides:

$$\left.\begin{aligned} M &= \frac{q_1}{p_1} - \frac{q}{p} \\ N &= q_1\, M + p - \frac{q^2}{p} \end{aligned}\right\} \tag{63}$$

As to τ_0, its value given in (55) can be written as:

$$\tau_0 = \left\{ \alpha_C \left(1 - R \tanh^2 \frac{\pi l e^{i\beta}}{2\,h}\right) + \frac{\alpha_B + \alpha_A}{2}\left(1 + R \tanh^2 \frac{\pi l}{2h}\right)\right\} \frac{q_1 - q}{p} + (\alpha_B - \alpha_A)\frac{q_1}{p_1} \tag{64}$$

For the profile of single camber, so that $a_2 = 0$ and hence $\alpha_C = \frac{\alpha_B + \alpha_A}{2}$, it simply gives

$$\tau_0 = 2\,\alpha_C \frac{q_1 - q}{p} + (\alpha_B - \alpha_A)\frac{q_1}{p_1} \tag{64'}$$

and finally:

$$\Gamma = (\Gamma)_o + V h \frac{M}{N} \cdot \Delta \tau \qquad (65)$$

where we post

$$\Delta \tau = \tau - \tau_o$$

and $(\Gamma)_o$ indicates the value of Γ for $B_1 = 0$, given in equation (53).

This formula shows that Γ depends not only on the form of the "mean line" of the profile, as for the straight screen and the screen in tandem, but also on the airfoil thickness or, if preferred, to the curve of the top and bottom camber.

The problem is, as seen, rather complicated and the practical calculation of the oblique screen wich is the one that involves the propeller and the blades of hydraulic machines, is necessarily laborious.

15. A few words on the moment calculation. The elementary moment with respect to origin O is $\rho V R(\gamma x dx)$, and hence

$$M = \rho V R \int \gamma x \, dx$$

To effect this integration, x is expanded in series by posing

$$x = \sum_1^{\infty} b_n \cos n\theta$$

with

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \cos n\theta \, d\theta = -\frac{2}{\pi n} \int_0^{\pi} \sin n\theta \frac{h\lambda}{\pi} e^{-i\beta} \frac{\sin\theta}{1-\lambda^2\cos^2\theta} d\theta$$

whence

$$b_n = \begin{cases} -\dfrac{2h e^{-i\beta}}{\pi n} \tanh^n \dfrac{\pi l e^{i\beta}}{2h} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

Proceeding as for the straight screen, we find

$$M_o = -\rho \frac{h^2 V^2}{\pi} R\left(a_o e^{-2i\beta} \cosh \frac{\pi l e^{i\beta}}{h} \ln \cosh \frac{\pi l e^{i\beta}}{h}\right)$$

If it is assumed that

$$\alpha_o = A_o e^{i\beta} \operatorname{cosech} \frac{\pi l e^{i\beta}}{h}$$

the result is

$$M_o = -\rho \frac{h^2 V^2}{\pi} A_o \, R\left(e^{-i\beta} \coth \frac{\pi l e^{i\beta}}{h} \ln \cosh \frac{\pi l e^{i\beta}}{h}\right)$$

and now the point of application of the lift is characterized by:

$$x_o = \frac{M_o}{P_o} = -\frac{h}{\pi} R\left(e^{-i\beta} \coth \frac{\pi l e^{i\beta}}{h} \ln \cosh \frac{\pi l e^{i\beta}}{h}\right)$$

Similarly for M_n:

$$M_n = \rho \frac{h^2 V^2}{\pi} R\left\{a_n e^{-i\beta} \Sigma_m{}' \tanh^m \frac{\pi l e^{i\beta}}{2h} \cdot \frac{1}{m}\left[\tanh^{n+m} \frac{\pi l e^{i\beta}}{2h} \pm \tanh^{|n-m|} \frac{\pi l e^{i\beta}}{2h}\right]\right\}$$

where the + sign applies to $m \leq n - 1$, and the - sign to $m \geq n + 1$.

For

$$a_n = A_n e^{i\beta} \coth^n \frac{\pi l e^{i\beta}}{2h}$$

the result is:

$$M_n = \rho \frac{h^2 V^2}{\pi} A_n \, R\left\{e^{-i\beta} \coth^n \frac{\pi l e^{i\beta}}{2h} \Sigma_m{}' \tanh^m \frac{\pi l e^{i\beta}}{2h} \times \frac{1}{m}\left[\tanh^{n+m} \frac{\pi l e^{i\beta}}{2h} \pm \tanh^{|n-m|} \frac{\pi l e^{i\beta}}{2h}\right]\right\}$$

APPENDIX

The integrals used in the foregoing study are as follows:

1) $$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_1} d\theta = \frac{\pi \sin n\theta_1}{\sin\theta_1}$$

The value of the integral is that expressed by the second term of the preceding formula even if θ and θ_1 are complete because the line along which the integration is made, starting from 0 and terminating at π, passes through the point θ_1.

2) Posting $\lambda = \tanh\alpha$

affords

$$I_n = \int_0^\pi \frac{\cos n\theta \, d\theta}{1 - \lambda\cos\theta} = \pi \cosh\alpha \tanh^n \frac{\alpha}{2}$$

$$I_n' = \int_0^\pi \frac{\cos n\theta \, d\theta}{1 + \lambda\cos\theta} = \pi \cosh\alpha \left(- \tanh\frac{\alpha}{2}\right)^n$$

The above integrals reduced to the form

$$\int_0^\pi \frac{\cos n\theta \, d\theta}{1 + p^2 - 2p\cos\theta} = \frac{\pi p^n}{1 - p^2}$$

(See Laska, Collection of Formulas, p. 266), by posing

$$p = \frac{1 - \sqrt{1 - \lambda^2}}{\lambda} \quad (p < 1) = \tanh\frac{\alpha}{2}$$

3)

$$J_n = \int_0^\pi \frac{\cos n\theta}{1 - \lambda^2\cos^2\theta} d\theta = \begin{cases} 0 & \text{for } n \text{ odd} \\ \pi\cosh\alpha\tanh^n\frac{\alpha}{2} & \text{for } n \text{ even} \end{cases}$$

Division of the integral by two shows how easy it is to verify

$$J_n = \frac{1}{2}(I_n + I_n')$$

4)

$$\int_0^{\pi} \frac{\sin\theta \ \sin n\theta}{1 - \lambda^2 \cos^2\theta} d\theta = \frac{1}{2}(J_{n-1} - J_{n+1}) =$$

$$= \begin{cases} 0 & \text{for } n \text{ odd} \\ \pi \coth\alpha \tanh^n \frac{\alpha}{2} & \text{for } n \text{ odd}^{*} \end{cases}$$

5)

$$\int_0^{\pi} \frac{\cos\theta \ \cos n\theta}{1 - \lambda^2 \cos^2\theta} d\theta = \frac{1}{2}(J_{n-1} + J_{n+1})$$

$$= \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{\pi \cosh^2\alpha}{\sinh\alpha} \tanh^n \frac{\alpha}{2} & \text{for } n \text{ odd} \end{cases}$$

The above formulas for the integrals I_n, I'_n, J_n, and their derivatives, are equally applicable in the complex field.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

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*Probably an error.

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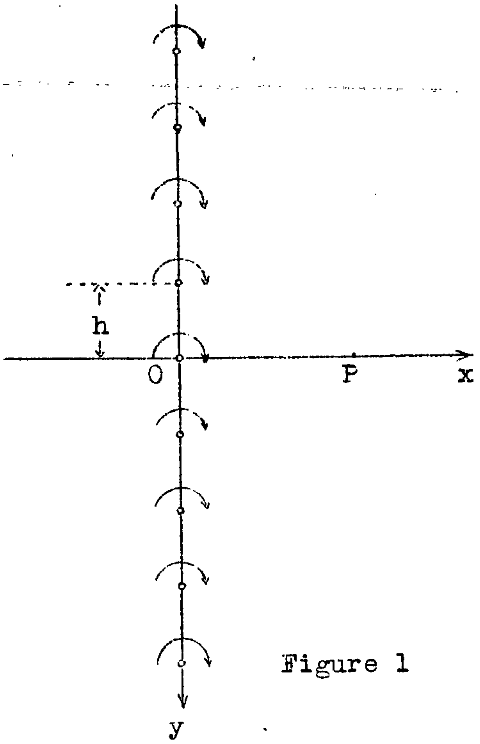
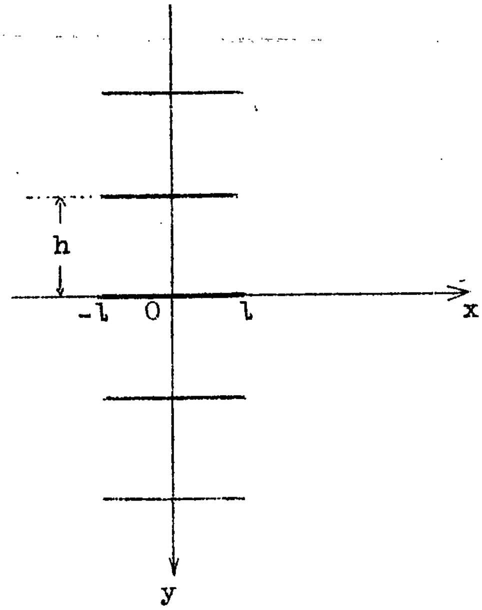


Figure 1



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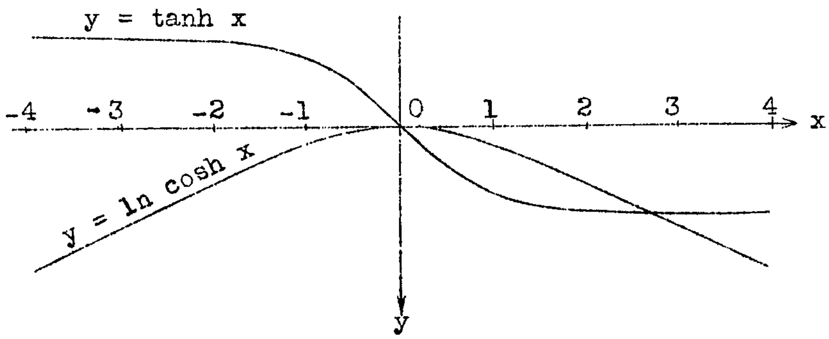

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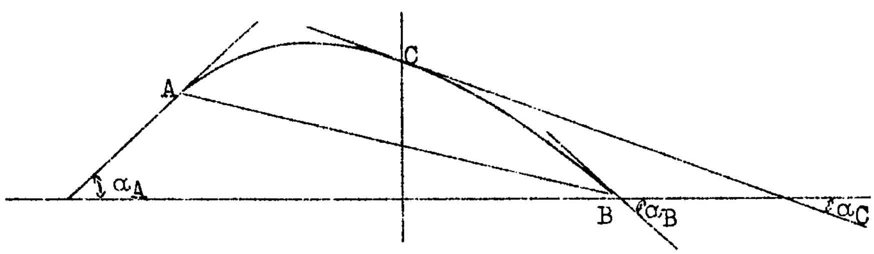

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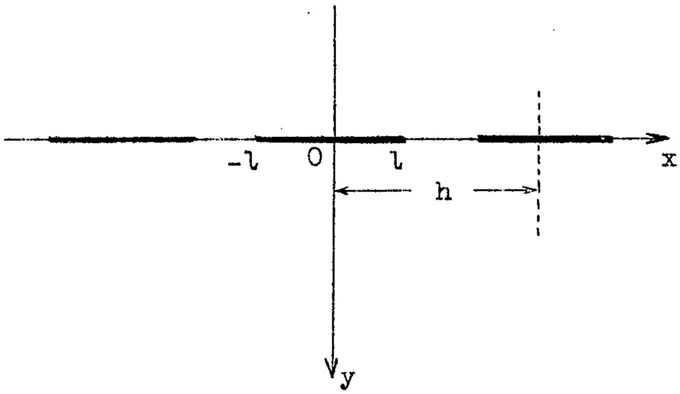

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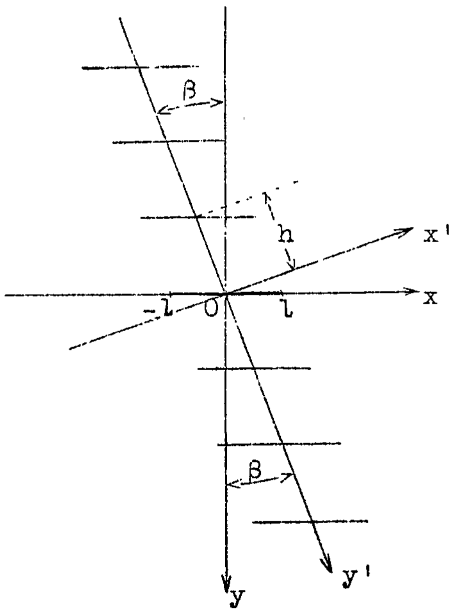

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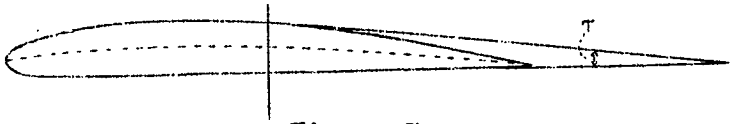

Figure 7

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