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By P. Cicala

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The experimental data that are at present known with regard to the phenomenon of wing flutter permit essentially two classes of vibrations to be distinguished; namely, those for which it may be stated that the aerodynamic phenomenon follows the ordinary theory of potential motion, which theory assumes that the point of separation of the flow during the oscillation constantly coincides with the trailing edge of the wing; and vibrations arising from the irregularity in the aerodynamic phenomenon, whose prediction the present state of the theory does not permit. To the latter class belong those denoted by Studer as "detached" vibrations and perhaps also some types of slow oscillation of the aileron.**

A wing structure, in the normal range of angle of attack, may, up to a certain velocity, present oscillations of the second type. When the wing is brought to rather large positive or negative angles of attack, wind tunnel tests show a notable decrease in the critical velocity. In the wind tunnel the phenomenon presents itself as a serious one that would appear to justify the most pessimistic predictions. For a wing of strong camber free to undergo torsional displacements only and with sufficiently small friction at the supporting suspension, tests conducted at Turin showed rather low critical velocities independent almost of the stiffness of the elastic suspension system when negative angles of attack of only a few

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*"La teoria e l'esperienza nel fenomeno delle vibrazioni alari." L'Aerotecnica, vol. 18, no. 4, April 1938, pp. 412-433.

**There might also be distinguished forced vibrations (for example, oscillations of the plane of the tail due to the disturbed flow from the wing), but no sufficient data on these exist at present.

degrees were attained. The phenomenon, however, should be strongly affected by the Reynolds Number so that from these tests there cannot actually be derived quantitative conclusions applicable to real structures. This aspect of the phenomenon is at the present time being investigated at the laboratory at Zurich.

The greater part of both the theoretical and experimental investigations has been directed to the study of oscillations of the first type. In the case of plane motion (wing of infinite aspect ratio) the problem has been solved in all its aspects both as regards the aerodynamic phenomenon and as regards the conditions of stability of the system. To cite only a recent investigation, Kassner (reference 1) has worked out graphs, by the use of which, for any system of values of the fundamental parameters, the solution of the problem may be obtained with sufficient rapidity. When it is desired to consider the case of the finite wing, the study is considerably complicated. Important investigations have recently been conducted by the Aeronautical Ministry (reference 2). The equations which define the motion of the system are derived in a form analogous to that which is obtained by the use of the principle of virtual work (reference 3), the deformations being expressed by means of a linear combination of suitable functions and the stability conditions discussed with the aid of Routh's determinants.

In all the studies so far developed, more or less arbitrary assumptions are made with regard to the aerodynamic actions. One of the most important of these assumptions is that the forces which are developed at a section of the wing are independent of the state of motion of the adjacent sections. In order to remove such first approximation assumptions, which seriously impair the precision of the methods of computation, there is required a knowledge of the aerodynamic forces and moments at the various sections of the wing corresponding to an assigned state of motion. This problem has therefore been analyzed (reference 4) by means of a procedure which is founded on simplificiations analogous to those which, in the study of the stationary motion of the finite wing (Prandtl vortex filament theory), lead to the familiar integrodifferential equation of the circulation along the span. The present note gives the results of the numerical elaboration of this theory. The coefficients of the aerodynamic forces and moments were determined for a wing of elliptic plan form for the two cases of aspect ratio 6 and 3, respectively, and expressed in a form in which these coefficients figure in the equation of work. There is also indicated an approximate procedure by the use of which the results may be extended to values of the aspect ratio and systems of deformation different from those considered in the given computations.

One result that appears from the investigation is that the values of the coefficients of the aerodynamic actions are only slightly influenced by the form of the oscillation. The computation carried out for one of these coefficients (that which gives the change in lift produced by the flectional (or flapping) displacement) gave almost identical values for the cases of a deformation law represented by a straight line and a second-degree parabola, respectively. This reult permits a considerable reduction in the labor of computation required for the applications. Another result of the analysis developed is that the coefficients, even for wings of normal aspect ratio, have values deviating considerably from those given by the theory of the two-dimensional or plane motion. This could have been predicted by the fact that in the vibrational phenomenon the sections which undergo the greatest displacements are those at the wing tips most susceptible to the tip effect. It is found therefore that the experimental investigations in which there tend to be realized the conditions of plane motion are valuable as confirming the theory, but do not furnish quantitatively valid data for the prediction of the critical velocities of actual wings which are of rather low aspect ratios.

Also, in experimental investigations, the aerodynamic side of the problem has not generally been isolated and clarified. Normally the tests are carried out on a model wing in the wind tunnel by determining the velocity at which the system begins to flutter and the corresponding frequency of the oscillation. Investigations of this kind have been conducted and others are being conducted in almost all principal aerodynamic laboratories. The difficulty which this method of investigation presents is that the results on the model are applicable to the wing if they present the same nondimensional ratios of mass and stiffness and the same relative positions of the center of gravity and the elastic axis along the chord. Thus, in order to realize in the wind tunnel test in a complete manner the conditions under which the effective structure is subjected, it would be necessary that the model and the actual wing present at corresponding sections the same values of the above nondimensional parameters. On account of the extraordinary expansion in the extent of the investigations required in this case and the considerable difficulty encountered in the construction of models which are to have the same mass relations as the actual wing, it seems more convenient to determine the aerodynamic actions by direct tests. The coefficients thus experimentally obtained substituted in the equations of stability then permit the determination of the critical velocity and the other characteristics of the pehnomenon for any system of values of the fundamental parameters. In addition to being able thus to cover by a single series of tests the range of variation of the magnitudes which define the phenomenon, there is also eliminated the necessity of the construction of models which possess the assigned mass relations. Naturally, in order that the results be applicable, it is necessary that the aerodynamic coefficients be independent of the Reynolds Number. If this is not the case, none of the critical velocity determinations thus far conducted on models would give results of practical value.

On the basis of the above considerations there have been undertaken at the Aeronautics Laboratory at Turin direct measurements of the aerodynamic actions on an oscillating wing. The tests so far conducted described in the notes of references 5 and 6 had as their essential object the examination of the operation of apparatus designed for this measurement and these tests will be repeated and extended so as to cover a greater field of investigation. The values experimentally obtained for the aerodynamic coefficients are in good agreement with the theory of oscillatory motion of the wing of finite span and show a clear deviation from the values obtained by the theory of plane motion. These determinations have permitted a check, given at the end of the present note, on the measurements carried out at the laboratory on the free oscillation of a model wing with variable stiffness. The agreement between the computation and experiments is satisfactory. However, various points must be clarified before being able to proceed with safety to the applications. When sufficient data has become available on the aerodynamic actions in the case of a wing with or without aileron and also with regard to the effects of the Reynolds Number, the mean angle of attack of the oscillation, and the compressibility of the medium; then the theory of wing vibration will be able to furnish reliable results for practical application.

Collection of Principal Symbols

The principal symbols are the following:

V, the wind velocity

Ω, angular frequency

x, coordinate parallel to V

L, local chord

$\bar{L}$, mean chord (area/span)

$\bar{x} = -\frac{L}{2} \cos \theta$, distance of a point of a profile section from the center point

b, wing span

$z = -\frac{b}{2} \cos \zeta$, coordinate normal to V

$\lambda = \frac{b}{\bar{L}}$, aspect ratio

$\omega = \frac{\Omega L}{2V}$, reduced local frequency

$\bar{\omega} = \frac{\Omega \bar{L}}{2V}$, reduced mean frequency

$$\omega' = \frac{i\Omega}{V}$$

$$\int_b = \int_{-b/2}^{+b/2}$$

Since the motion considered is harmonic, there is adopted the complex notation. All the magnitudes variable with time (displacements, vertical velocities, circulations, and lift and moment coefficients) are measured as deviations from a mean state, the exact knowledge of which is of no interest to the problem. These deviations are simply indicated by means of their complex amplitudes, suppressing the factor $e^{i\Omega t}$. The conditions relative to the mean state are studied separately by means of the equations of the stationary motion. The superposition of the effects is permitted because of the linearity of the fundamental equations.

The flectional displacement is denoted by φ and the torsional displacement (rotation of the profile about the aerodynamic center) by ψ. It is assumed that there exists no phase difference between the displacements of the various sections. In every case it is possible to reduce the problem to these conditions by splitting up the displacements into two components in quadrature with each other and examining separately the two simple motions. The functions φ and ψ may therefore be considered as real.

On account of the linearity of the fundamental relations, the variation of lift produced by the flectional displacement is proportional to the displacement itself. Breaking up the constant of proportionality, which in general is complex, into real and imaginary parts, we write as for the plane motion (reference 5)

$$c_p = \pi(a_1{*}\varphi/L + ia_2{*}\Omega\varphi/V) \qquad (1a)$$

The nondimensional real coefficients $a_1{*}$ and $a_2{*}$ depend on the ratio $\Omega L/V$ as for the plane motion, on the plan form of the wing and on the law of variation of the displacements. The latter vary along the span as a consequence of the fact that the actions on one section depend also on the state of motion of the contiguous sections.

Analogously, for the variations in lift produced by the torsional motion we write

$$c_p = \pi(a_3{*}\psi + ia_4{*}\ L\Omega\psi/V) \qquad (1b)$$

and for the moment about the aerodynamic center produced by the flectional and torsional displacements, respectively:

$$c_{m_F} = \pi(b_1{*}\varphi/L + ib_2{*}\Omega\varphi/V) \qquad (1c)$$

$$c_{m_F} = \pi(b_3{*}\psi + ib_4{*}\ L\Omega\psi/V) \qquad (1d)$$

The coefficients which represent the actions in phase (a_1^*, a_3^*, b_1^*, and b_3^*) contain the inertia effects of the air mass surrounding the wing. This inertia action, which remains independent of the velocity, is conveniently separated in the computations of the aerodynamic actions. If the mass characteristics of the wing are determined by a dynamic procedure, the actions are included in the measurement. Making use of the results of the theory of the wing of infinite aspect ratio (reference 5), we separate the inertia terms (in ω^2) and write

$$\left.\begin{aligned} a_1^* &= a_1'^* + \omega^2 \\ a_3^* &= a_3'^* - \omega^2/4 \\ b_1^* &= b_1'^* + \omega^2/4 \\ b_3^* &= b_3'^* - 3\,\omega^2/32 \end{aligned}\right\} \qquad (2)$$

For the practical application, it is of greater interest to know the mean values of the aerodynamic coefficients along the span than the local coefficients and we define the former by the relations

$$\left.\begin{aligned} a_1' \int_b \varphi^2\,dz &= \int_b a_1'^* \varphi^2 dz \\ a_2 \int_b \varphi^2 L dz &= \int_b a_2^* \,\varphi^2 L dz \\ a_3' \int_b \varphi\psi\, L dz &= \int_b a_3'^* \varphi\psi\, L dz \\ a_4 \int_b \varphi\psi L^2 dz &= \int_b a_4^* \varphi\psi\, L^2 dz \\ b_1' \int_b \varphi\,\psi\, L dz &= \int_b b_1'^* \varphi\,\psi\, L dz \\ b_2 \int_b \varphi\,\psi L^2 dz &= \int_b b_2^* \varphi\,\psi\, L^2 dz \\ b_3' \int_b \psi^2 L^2 dz &= \int_b b_3'^* \psi^2 L^2 dz \\ b_4 \int_b \psi^2 L^3 dz &= \int_b b_4^* \,\psi^2 L^3 dz \end{aligned}\right\} \qquad (3)$$

It is easy to show that the coefficients a_1', a_2,... b_4 defined by equation (3) satisfy the following condition, namely, that the virtual work which the aerodynamic actions in phase and in quadrature with the flectional or torsional displacements perform against these displacements may be calculated by supposing that the aerodynamic coefficients are constant for the entire span and have the values a_1',...b_4 given by equation (3). A knowledge of these coefficients is therefore sufficient for the discussion of the stability of the wing structure by the method of virtual work (reference 3). It will be understood that the determination of these coefficients requires in each case the complete solution of the problem for the assigned wing plan form and displacements. Since these characteristics have no great effect on the values of the mean coefficients, the solution of the problem for each particular case is not necessary.

THE THEORY OF THE WING OF FINITE ASPECT RATIO IN NONSTATIONARY MOTION

The problem of the oscillatory motion of a wing of finite span may be attacked without excessive complication when there are extended to it the simplifications assumed in the study of the stationary motion of the finite wing according to the vortex filament theory.

Let c (fig. 1) be the wing contour in plan form. A vortex element a, which for the wing of infinite aspect ratio extends indefinitely in a direction normal to V, deviates in the case of the finite wing at a certain point and bends in the direction of the stream. In order to pass from the first to the second configuration we add to the vortex a the angle vortex h. If, in the computation of the induced velocity at a point O, we substitute for the angle vortex h a parallel one having its vertex in Q at the depth of O, we arrive in the case of steady motion at the ordinary theory of the vortex filament. If the motion is not steady, there exist vortices with axis normal to V also in the wake behind the wing and there thus exist also angle vortices for the finite wing in the wake. The relations given in this note are arrived at if the angle vortices whose vertices fall within the contour of the wing are displaced in the manner described for the steady motion and those of the wake are advanced by the amount d (fig. 2), which represents the displacement ascribed to these at the instant at which they leave the wing.

The sides of the angle vortices perpendicular to V serve to eliminate the effect of the prolongation of the lifting vortices, which are cut off by the section through the point O and which are considered as extended indefinitely on the two sides. The angle vortices having their vertices at the right of the section are therefore indefinitely extended toward the right while those with their vertices at the left extend indefinitely toward the left.

As has been said the effects of the angle vortices are computed after displacements have been impressed upon them. This is equivalent in each case to the addition of a transverse horseshoe vortex of which the inducing effect is neglected. (See reference 4.) The theory is therefore approximate inasmuch as an entire system of vortices is neglected. In the case of steady motion, the theory leads to the integrodifferential equation of Prandtl. Since the latter theory in ordinary cases leads to sufficiently satisfactory results, it may be expected that the extension to nonstationary motion leads to results of some importance. In order to arrive at more accurate results, it is possible, having solved the problem by means of the expressions given below, to determine the induced velocities of the neglected vortex system and treat these by the same procedure as the initial velocities and so on successively, using the method of iteration.

In order to consider the effects of the nonstationary state of motion, we shall adopt, as for the plane problem, the method of Birnbaum, which consists in breaking up the total circulation normal to the direction of V into a part that corresponds to the aerodynamic actions (bound circulation) and a part which is shed from the wing in correspondence to the local variations in intensity and is carried along by the stream (free circulation).

For the case of plane motion, it is known that the vertical (downwash) velocity distribution along the chord of the profile represented by one of the functions

$$W_1 = V\left(\frac{1}{2} + \cos\vartheta\right)$$

$$W_n = V \cos n\vartheta \qquad (n = 2, 3\ldots)$$

corresponds to the distribution of the bound circulation given respectively by the expressions

$$\Gamma_1 = V\left(2 \sin \vartheta - \cot \frac{\vartheta}{2} - i\omega \sin \vartheta \cos \vartheta - i\omega \sin \vartheta\right)$$

$$\Gamma_n = V\left(2 \sin n \vartheta - \frac{i\omega}{n+1} \sin (n+1)\vartheta + \frac{i\omega}{n-1} \sin (n-1)\vartheta\right)$$

It is also known that the total circulation associated with the section corresponding to an arbitrary Γ_n is always zero, so that there are no free vortices in the wake. These asumptions lead to the conclusion that for the finite wing, if we assume as valid the same simplifications that lie at the basis of the vortex filament theory, the results holding for the plane motion may immediately be extended and it may be stated that, if along the chords of the various wing sections, the vertical velocity distribution is expressed by means of W_n or a linear combination of them, the distribution of the bound circulation will be represented by the corresponding linear combination of Γ_n, as if each section of the wing worked with an infinite aspect ratio without being influenced by the adjacent sections.

The series of W_n is not complete, however. It is sufficient to think of the case of the rectilinear profile, for which every W_n except W_1 is zero. The bound circulation may therefore be expressed by means of Γ only if the vertical velocity correspond to the rear neutral point ($\cos \vartheta = -1/2$) always remains zero, as results from the expression W_1.

In order to complete the series of Γ_n, we consider the bound vortex distribution

$$\Gamma_o = 2 A (z)\left(\frac{H_1^{(2)}}{H_1^{(2)} + i\, H_o^{(2)}} \cot \frac{\vartheta}{2} + i\omega \sin \vartheta\right)$$

in which A is a function of z and H the Hankel function of parameter ω. Denoting by K the total circulation associated with the section, we have

$$K = \pi\, A\, L/\mu$$

where $$\mu = \frac{\pi\omega}{2}\left(H_0{}^{(2)} - i\,H_1{}^{(2)}\right)e^{i\omega}$$

Since the total circulation associated with the various sections does not remain constant, it will be necessary to consider the wake vortices. With the simplifications previously given, the scheme of figure 2 is arrived at in which is represented the configuration of the system of inducing vortices in the plane of the wing. In computing the velocity at a point O, there are to be considered:

a) The bound and free vortices a and l corresponding to the state of motion of the section through O, which with the expression assumed for Γ_1 (see reference 7) induce along the chord a constant velocity given by

$$-A = -\mu\,K/\pi\,L$$

b) The angle vortices m whose vertices lie on the normal to V through O with density of circulation dK/dz. The velocity induced by all of these vortices is

$$\int_b \frac{dK}{dz}\,\frac{dz}{z - z_1}$$

c) The angle vortices h external to the wing. In the wake the vertices of these elements are distributed with density

$$-\omega'\,e^{-\omega' x_1}\frac{dK}{dz}$$

The velocity induced by one of these is

$$\frac{1}{4\pi}\left(\frac{1}{z - z_1} + \frac{1}{x_1} - \sqrt{\frac{1}{(z - z_1)^2} + \frac{1}{x_1{}^2}}\right)$$

where x_1 and $z - z_1$ give the coordinates of the vertex of h with respect to O. We introduce the function

$$F = \int_0^\infty \left(\frac{1}{z - z_1} + \frac{1}{x_1} - \sqrt{\frac{1}{(z - z_1)^2} + \frac{1}{x_1{}^2}}\right)e^{-\omega' x_1}\,d\,x_1$$

The induced velocity of the vortex system c) is expressed by

$$- \frac{\omega'}{4\pi} \int_b F \frac{d\,K}{d\,z} d\,z$$

Equating the induced velocity of the vortex system to W_o, which the stream should possess on account of its being tangent to the profile, there is obtained the integro-differential equation, which defines the function K:

$$- \pi\, W_o = \frac{\mu\, K}{L} - \frac{1}{4} \int_b \left(\frac{1}{z - z_1} - \omega' F \right) \frac{d\,K}{d\,z} d\,z \qquad (4)$$

If there is therefore given the vertical velocity distribution expressed by means of the series

$$W(x,\ z) = W_o(z) + W_1(z) \left(\frac{1}{2} + \cos\,\vartheta \right) + W_2(z) \cos\,2\,\vartheta + \ldots$$

the total circulation K may be computed by (4) and the bound circulation distribution obtained by the series

$$\Gamma = \frac{2\,\mu\,K}{\pi\,L} \left(\frac{H_1^{(2)}}{H_1^{(2)} + i\,H_o^{(2)}} \cot\,\frac{\vartheta}{2} + i\,\omega \sin\vartheta \right) + W_1\Gamma_1/V + W_2\Gamma_2/V + \ldots \qquad (5)$$

In the case of a nondeformable section, the summation is reduced to the first two terms.

From the bound circulation, there are calculated the aerodynamic actions bearing in mind that the difference in pressure between the two faces of the section is given by the relation

$$p = \rho\, V\, \Gamma$$

Denoting the lift by $c_p\ \rho\, L\, V^2\, d\,z$ and the moment of the aerodynamic actions about the aerodynamic center of an elementary strip by $c_{m_F}\ \rho\, L^2 V^2\, d\,z$ and making use of the expression (5) for Γ, there are found for the coefficients c_p and c_{m_F}, complex functions of z and of the state of motion of the system, the following expressions:

$$c_p\ L\ V = \int_{-L}^{+L} \Gamma d\ x = K\ \mu_1 + i\ \pi\ \omega L (W_2 - W_1)/4 \qquad (6)$$

$$c_{m_F}\ L^2 V = \int_{-L}^{+L} \Gamma \left(\bar{x} + \frac{L}{4} \right) d\ \bar{x} = i\ K\ L\ \omega\ \mu/8 +$$

$$+ \pi\ W_1 L^2 \left(1 - \frac{i\ \omega}{4}\right)/8 - \pi\ W_2\ L^2 \left(1 - \frac{i\ \omega}{2}\right)/8 - \pi\ W_3 L^2/32$$

where

$$\mu_1 = \left(\frac{H_1^{(2)}}{H_1^{(2)} + i\ H_0^{(2)}} + \frac{i\ \omega}{2} \right) \mu$$

For practical application, it is necessary, first of all, to calculate the function F, which, with

$$X = \Omega x/V$$

$$Z = \Omega (z - z_1)/V$$

may be written in the form

$$F(Z) = \int_0^\infty \left(\frac{1}{X} + \frac{1}{Z} - \sqrt{\frac{1}{X^2} + \frac{1}{Z^2}} \right) e^{-i\ X}\ d\ X$$

This function is determined for possible values of the parameter Z. If Z is negative F is put in (4) with sign changed. This result immediately follows by considering the configurations of the vortices h to the right and left of the section through point O. Since this function is fundamental for practical applications, it was determined with particular care. For values of the parameter less than 4, there was made use of the approximate expression

$$F(Z) = -\int_0^Z \frac{e^{-iX}\,dX}{X+Z}\left(1 + \frac{X}{2(X+Z)} - \frac{3\,X^3}{8(X+Z)^3} - \frac{1.03\,X^4}{2(X+Z)^4}\right) +$$

$$+ \int_Z^\infty \frac{e^{-iX}\,dX}{X+Z}\left(1 + \frac{Z}{2(X+Z)} - \frac{3\,Z^3}{8(X+Z)^3} - \frac{1.03\,Z^4}{2(X+Z)^4}\right)$$

For values above 4, there was used the expression obtained by integration by parts

$$F(Z) = -i/Z + 1/2Z^2 - \int_0^\infty \frac{e^{-iX}\,dX}{X^3}\left[2 - Z(3\,X^2 + 2\,Z^2)(X^2 + Z^2)^{3/2}\right]$$

The integral in the second member was calculated numerically by extending the integration up to a limit such that the remainder, which could easily be estimated, remained less than the predetermined limits of approximation. The values obtained are given in table I.

The function μ of the parameter ω, also fundamental for the computation, is given by the graphs of figure 3 in which μ' and μ'' are respectively the real and imaginary parts of the function.

APPLICATION TO AN ELLIPTIC WING

The theory given above was applied to a wing with elliptic plan form. In this case:

$$L = 4\,\bar{L}\,\sin\zeta/\pi$$

We express the total circulation by means of the summation

$$K = \pi\,\bar{L}\,V\,\Sigma\,K_n\,\sin n\,\zeta \qquad (7)$$

where K_n is the numerical complex constant.

TABLE I

Z	F	Z	F
0.0	∞-1.571 i	2.2	0.113-0.425 i
.1	2.109-1.375 i	2.4	.097- .395 i
.2	1.490-1.248 i	2.6	.083- .369 i
.3	1.155-1.146 i	2.8	.072- .345 i
.4	.935-1.060 i	3.0	.063- .324 i
.5	.778- .987 i	3.2	.055- .305 i
.6	.658- .922 i	3.4	.048- .289 i
.7	.564- .865 i	3.6	.043- .274 i
.8	.489- .813 i	3.8	.039- .260 i
.9	.427- .763 i	4.0	.035- .248 i
1.0	.376- .726 i	4.2	.031- .237 i
1.1	.333- .689 i	4.4	.028- .226 i
1.2	.297- .654 i	4.6	.026- .217 i
1.3	.265- .624 i	4.8	.024- .208 i
1.4	.238- .593 i	5.0	.022- .200 i
1.5	.214- .567 i	5.2	.020- .192 i
1.6	.194- .540 i	5.4	.018- .185 i
1.7	.176- .517 i	5.6	.017- .179 i
1.8	.160- .496 i	5.8	.016- .172 i
1.9	.146- .475 i	6.0	.015- .167 i
2.0	.134- .458 i	∞	$1/2\, Z^2 - i/Z$

Equation (4) now becomes

$$-W_0 \sin \zeta / V = \frac{\pi \mu}{4} \Sigma K_n \sin n \zeta + \frac{\pi}{2 \lambda} \Sigma n K_n \sin n \zeta +$$

$$+ i \frac{\omega}{2} \sin \zeta \int_b F \, d \zeta \, \Sigma n K_n \cos n \zeta$$

In the computation, only even terms of the series were retained so as to obtain a velocity zero at the center section without having to add a supplementary equation, there being thus assumed equal and opposite displacements for sections equidistant from the center section. Moreover the state of motion of one half-wing has in our case only a slight effect on the field of flow about the other half-wing. This is because in the case of vibration the maximum displacements occur at the tip sections, which are therefore far removed or because the induced velocities of the angle vortices in the case of oscillatory motion decrease with distance more rapidly than in the case of steady motion.

Limiting the summation to the terms in K_2 and K_4, there were applied and the values of W_0 at the sections corresponding to $z = 0.7$ and $z = b/2$, and thus two linear equations were obtained with complex coefficients in the constants K_2 and K_4.

It is convenient to consider the results separately for the various components.

Lift produced by the flectional (flapping) motion.- In this case there are constant velocities along each chord. Of the W_n, only W_0 is different from zero and given by

$$W_0 = i \Omega \varphi$$

where $\varphi(z)$ is the law which defines the flectional displacements along the span.

Having determined the bound circulation in the manner explained above, it is necessary, for the computation of the mean aerodynamic coefficients, to know the quantity

$$L_{\varphi\varphi} = \int_b c_p \, \rho \, L \, V^2 \, \varphi \, d z \tag{8}$$

The real part of $L_{\varphi\varphi}$ represents the virtual work by the aerodynamic actions in phase with the motion; the imaginary part gives the work of the components in quadrature.

By (6) and (7), equation (8) becomes

$$L_{\varphi\varphi} = \rho \, V \int_b K \, \mu_1 \, \varphi \, d \, z = \pi \, \rho \, \bar{L} \, V^2 \int \mu_1 \, \varphi \, d \, z \, \Sigma \, K_n \, \sin n \, \zeta \tag{9}$$

By (1a), (2), and (3), equation (8) may also be written

$$L_{\varphi\varphi} = \pi \, \rho \, V^2 (a_1' \int_b \varphi^2 \, d \, z + \int_b \varphi^2 \, \omega^2 \, d \, z + \frac{i \, a_2 \Omega}{V} \int_b L \, \varphi^2 \, d \, z) \tag{10}$$

Equating expressions (9) and (10) for $L_{\varphi\varphi}$, there is obtained an equation which gives the coefficients a_1' and a_2.

The computation was carried out assuming for the flectional displacements the law

$$\varphi = \frac{V}{\Omega} \cos \, \zeta \, |\cos \, \zeta|$$

In figure 4 is plotted the variation of the real part of the magnitude $K/\pi \, \bar{L} \, V$ along the span for $\lambda = 6$ and for the values of $\bar{\omega}$ indicated on the figure. Figure 5 gives the imaginary part. As may be seen from the curves, the total circulation varies appreciably with the frequency of the oscillation.

Figure 6 shows the distribution of lift produced by the flectional displacement considered. The curves give only the component in quadrature with the displacements (the ordinates of the curves give the values of the real part of c_p/π). The component in phase has not been plotted, since it is of less importance. The reduced frequencies to which the curves refer are indicated on the curves.

The values of the coefficient a_2 obtained for $\lambda = 6$ and $\lambda = 3$ are plotted in figure 7 as a function of $\bar{\omega}$. This coefficient decreases rather slowly with increase in the reduced frequency, while in the case of plane motion it has a steep slope. The values obtained experimentally, indicated on the figure by small circles, have the same general tendency as the theoretical values but lie somewhat above the latter*. The measurements on this coefficient will be repeated in order to ascertain whether the disagreement does not depend on the friction of the oscillating system.

The coefficient a_1' is rather small; the curve in figure 7 corresponds to $\lambda = 6$. For practical applications, this coefficient may be neglected and it may be assumed, if the mass of the system is dynamically determined, that the lift is in quadrature with the displacement which generates it. Also in the test, the coefficient a_1' came out so small that it was not measured.

The computation was also performed assuming a linear law to represent the displacements; that is, putting

$$\varphi = \frac{V}{\Omega} \cos \zeta$$

The values of the coefficients obtained in this case differ only slightly from those of the first case and were not plotted, since the curves would not be distinguishable.

Moments about the aerodynamic center produced by the flectional oscillation.- For the wing of infinite aspect ratio, the moment about the aerodynamic center produced by the flectional oscillation has no component in quadrature with the motion which generates it but is reduced to a pure inertia effect $(b_1 = \omega^2/4;\ b_2 = 0)$. The result may be held valid also for the finite wing. The component of the moment in quadrature with the flectional motion is equivalent to a displacement of the aerodynamic center by less than 5 percent of the chord

$\left(\text{or } \left|\frac{b_2}{a_2}\right| < 0.05\right)$. The component in phase except for the inertia term is also negligible.

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*In regard to the comparison of the results of the tests conducted at Turin, it is necessary to observe that a cause of uncertainty exists in the fact that the aspect ratio of the model under the test conditions employed is not easy to define. The tests were conducted in a free jet tunnel of rather small dimensions and the effect of the jet boundary was rather difficult to estimate. The presence of a plane at the base of the model for the purpose of masking the understructure complicates the phenomenon.

<u>Lift produced by the torsional oscillation.</u>- The focal axis is assumed rectilinear and there are determined the aerodynamic actions corresponding to the torsional oscillations about this axis. In this case

$$-W = V\,\psi\,(1 + i\,\omega/2 - i\,\omega\,\cos\vartheta)$$

and therefore

$$-W_o = V\,\psi\,(1 + i\,\omega)$$

$$W_1 = i\,V\,\psi\,\omega$$

All the other W_n are zero.

Having determined the bound circulation, there was computed the work of the aerodynamic actions, due to the torsional motion, against the flectional displacements. By equations (1b), (2), and (3), this quantity is

$$L_{\psi\varphi} = \int_b c_p\,\rho\,L\,V^2\,\varphi\,d\,z$$

$$= \pi\,\rho\,V^2\,\left(a_3'\int_b \varphi\,\psi\,L\,d\,z - \frac{1}{4}\int_b \omega^2\,\varphi\,\psi\,L\,d\,z + \right.$$

$$\left. + \frac{i\,a_4\,\Omega}{V}\int_b \varphi\,\psi\,L^2\,d\,z\right) \qquad (11)$$

On the other hand, by (6) and (7), this quantity is equal to

$$L_{\psi\varphi} = \pi\,\rho\,\bar{L}\,V^2\int_b \varphi\,\mu_1 d\,z\,\Sigma K_n\,\sin\,n\,\zeta + \frac{\pi\,\rho\,V^2}{4}\int_b \omega^2\varphi\psi L\,dz \qquad (12)$$

Equating the two, there are obtained a_3' and a_4.

In order to represent the angle of torsion, there is assumed the law

$$\psi = \cos\,\zeta$$

The values of the coefficient a_3' for $\lambda = 6$ and $\lambda = 3$ are shown on the curves of figure 8. The coefficient varies little with $\overline{\omega}$, which fact agrees with the results of the test, indicated on the figure by small circles.

Also the coefficient a_4 given for $\lambda = 6$ and $\lambda = 3$ on figure 7 varies slightly, while in the case of plane motion it decreases strongly toward small reduced frequencies. The experimental values indicated on the figure by small circles are somewhat below the theoretical.

Moments about the aerodynamic center due to the torsional oscillation.- The work done by the aerodynamic actions produced by the torsional motion against the torsional displacements from (1d), (2), and (3) is given by the expression

$$L\psi\psi = \pi \rho V^2 \left(b_3{}' \int_b \psi^2 L^2 d z - \frac{3}{32} \int_b \omega^2 \psi^2 L^2 d z + \frac{i b_4 \Omega}{V} \int \psi^2 L^3 d z \right)$$

and from (6) and (7) by

$$L\psi\psi = \frac{i}{8} \pi \rho V^2 \left(\bar{L} \int_b \psi \omega \mu L d z \Sigma K_n \sin n\zeta + \int_b \omega (1 - i\omega/4) \psi^2 L^2 d z \right)$$

Equating these two expressions, there are computed the two coefficients $b_3{}'$ and b_4. The coefficient $b_3{}'$, which for the infinite wing is zero, comes out as a small value for the finite wing. The coefficient b_4, which for $\lambda = \infty$ is equal to 1.8, has for $\lambda = 6$ and $\lambda = 3$ the values given by the curves on figures 9 and 10. The experimental results are in good agreement with the theory.

APPROXIMATE CALCULATION OF THE COEFFICIENTS

For systems of displacement different from those to which the previous curves refer, it is necessary to consider the solution of equation (4). The computation, even if only approximate, is in every case very laborious: that of the velocities induced by an assigned system of vortices sufficiently simple for the case of steady motion must, for the nonstationary motion, be carried out by the method of graphical integration, and is complicated by the presence of points of infinity in the function integrated. We shall therefore indicate an approximate procedure for the determination of the coefficients. The justification of the method lies essentially in the agreement of the results with those obtained by using the more exact method indicated above. In the cases considered no practically appreciable difference was found between the results obtained by the two methods of computation.

From what has been said, what is to be determined in each case is the work which the aerodynamic forces due to an assigned system of displacements S perform when certain displacements S' are impressed on the system. The displacements under consideration may be flectional, torsional, or those of the aileron, the condition required being that the motion of the various sections be in phase. If this condition is not satisfied, it is possible to reduce to this condition by dividing the motion of the system into two components in quadrature. The displacements S and S' may be represented respectively by the real functions s(z) and s'(z). With these functions known, there is first solved the equation

$$s\ L = K_o - \frac{L}{4} \int_b \frac{d\ K_o}{d\ z_1} \frac{d\ z_1}{z_1 - z} \tag{13}$$

The value of K_o obtained by this equation represents the distribution of total circulation, which under the conditions of steady motion would correspond to a displacement law given by s(z). The solution of the equation may be found, for example, by following the procedure given in the paper by Gebelein (Über die Integralgleichung, etc., reference 8).

If for each section there is computed the effect of the tip vortices, concentrating the circulation increase at the distance L/2 k on the two sides of the section, (13) becomes simply

$$s\ L = K_o'(1 + k) \tag{14}$$

We define the numerical factor k (independent of z) by the condition that the virtual work against the displacements s' of the lifts corresponding to s, calculated by the use of (14), have the exact value given by the solution of (13). That is, we write

$$\int_b K_o\ s'\ d\ z = \int_b K_o'\ s'\ d\ z$$

from which

$$k(s,\ s') = \frac{\int_b s\ s'\ L\ d\ z}{\int_b K_o\ s'\ d\ z} - 1$$

Making use of this coefficient k also for the case of oscillatory motion, we concentrate the increase in total circulation at the same distance from the section as for the case of steady motion and equation (14) then becomes

$$-\pi\ W_o\ L = K\ [\mu\ (\omega) + k - i\ \omega\ F(\omega/k)]$$

In order to compute the mean coefficients of the aerodynamic actions, we refer to a mean profile of chord $\overline{L}$. The total circulation will be given by

$$K = \frac{-\pi\ W_o\ \overline{L}}{\mu\ (\overline{\omega}) + k - i\ \overline{\omega}\ F(\overline{\omega}/k)}$$

and therefore setting

$$\left.\begin{aligned} \Lambda &= \frac{\mu}{\mu + k - i\ \overline{\omega}\ F} \\ 1 - \lambda' - i\ \lambda'' &= \frac{H_1^{(2)}}{H_1^{(2)} + i\ H_o^{(2)}} \end{aligned}\right\}$$

there will be obtained from equations (6)

$$\left.\begin{aligned} \frac{1}{\pi}\ c_p\ V &= -W_o\ \Lambda\left(1 - \lambda' - i\ \lambda'' + \frac{i\ \overline{\omega}}{2}\right) + i\overline{\omega}\ (W_2 - W_1)/4 \\ \frac{8}{\pi}\ c_{m_F} V &= i\ \overline{\omega}\ \mu\ \Lambda W_o + W_1\left(1 - \frac{i\ \overline{\omega}}{4}\right) - W_2\left(1 - \frac{i\ \overline{\omega}}{2}\right) - W_3/4 \end{aligned}\right\} \qquad (15)$$

The functions λ' and λ'' are given in table I of reference 5. The values of μ are given in figure 3 and those of F in table I of the present note. The factor Λ, a function of $\overline{\omega}$, and through k, of the plan form of the wing and of the displacement laws, tends in the case of infinite aspect ratio $(k = 0)$, to the value unity and in this case equations (15) furnish equations (1) of the above reference.

We observe that the value of k does not vary much on changing the system of displacements. For an elliptic wing of $\lambda = 6$, there is obtained:

For $s = s' = \cos\zeta$ $k = 0.67$

and for $s = s' = \cos\zeta \, |\cos\zeta|$ $k = 0.70$

The application of the method indicated to the case of flectional torsional oscillations is immediate. If only the first harmonic is considered, the laws of variation of the flectional and torsional displacements are not very different. For the parameters $k(\varphi, \varphi)$, $k(\psi, \varphi)$, $k(\varphi, \psi)$, and $k(\psi, \psi)$, there will be sufficiently close values. Using a mean value of these, it will then be possible to determine the coefficients by means of the relations

$$\left.\begin{aligned} a + 2\,i\,\bar{\omega}\,a_2 &= -2\,i\,\bar{\omega}\,\Lambda\,(1 - \lambda' - i\,\lambda'' + i\,\bar{\omega}/2) \\ a_3 + 2\,i\,\bar{\omega}\,a_4 &= \Lambda(1 + i\,\bar{\omega})(1 - \lambda' - i\,\lambda'' + i\,\bar{\omega}/2) \\ b_1 + 2\,i\,\bar{\omega}\,b_2 &= \Lambda\,\bar{\omega}^2/4 \\ b_3 + 2\,i\,\bar{\omega}\,b_4 &= i\,\bar{\omega}(1 + i\,\bar{\omega})\,\Lambda/8 + i\,\bar{\omega}(1 + i\,\omega/4)/8 \end{aligned}\right\}$$

which are readily deduced from equations (15). The extension to systems with ailerons also is simple.

CHECK OF THE FREE OSCILLATION TESTS

From the measurements carried out, the mean values of the aerodynamic coefficients were derived and with these a check was obtained on the free oscillation tests conducted at the Aeronautical Laboratory at Turin (reference 9). For the computation, there has been assumed

$$-a_2 = a_3 = 0.55$$

$$a_4 = 0.38$$

$$b_4 = 0.1$$

$$a_1 = b_1 = b_2 = b_3 = 0$$

The values of the other parameters of the structure are given in reference 3. The critical velocities are plotted in figure 11 as a function of the ratio of flexural stiffness R_f to torsional stiffness R_t. The experimental values are indicated by the black circles. On the same figure are also plotted the values of the ratio of the frequency of oscillation corresponding to the critical velocity to that of the pure torsion. The experimental values are in good agreement with the theoretical.

We denote by A the complex amplitude of the flectional displacement at the tip section of the model, by B the amplitude of the torsional oscillation and put $A/BL = f + iq$; f thus representing the component in phase and q the component in quadrature of the flectional motion with respect to the torsional. The curves obtained for these magnitudes are given in figure 12 as functions of the same parameter $R_f/L^2 R_t$. If account is taken of the difficulty of such measurements, complicated by the fact that actually the condition of steady oscillations cannot be realized, it may be stated that the agreement of these values with the theory may be considered as satisfactory. It is necessary also to observe that the two series of tests, namely, those of the free vibration and those of the aerodynamic actions were carried out on entirely different profile sections.

On the basis of the results obtained, it may be stated that it is permissible, for an approximate calculation of the critical velocity, to apply the coefficients thus determined.

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

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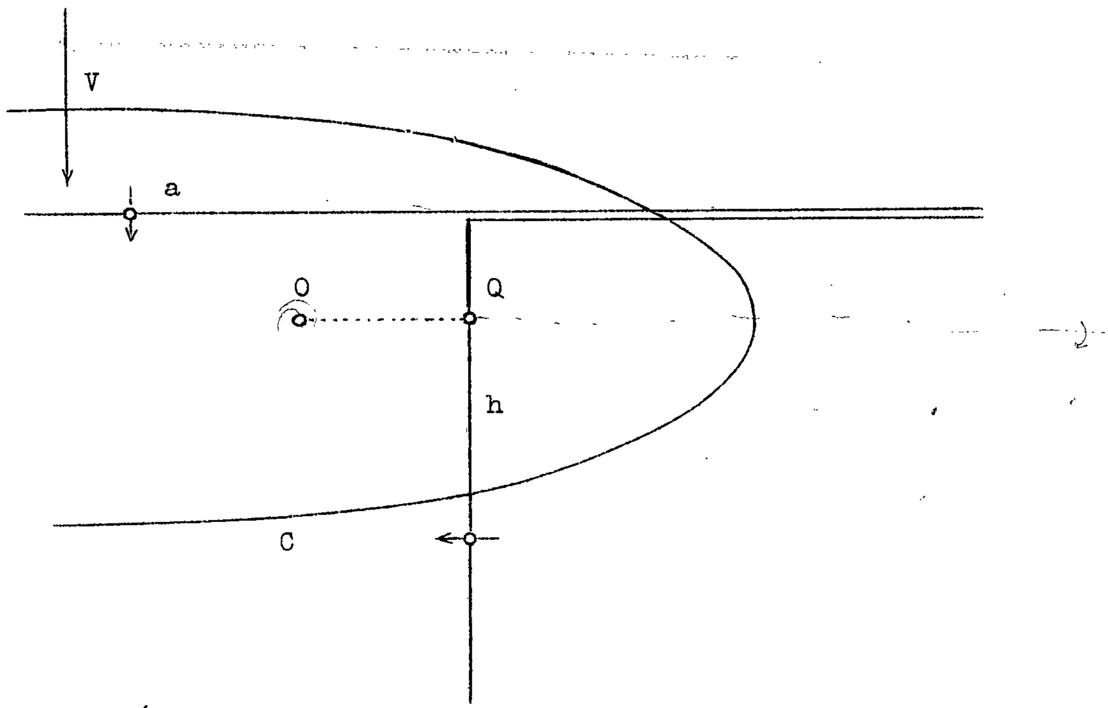

Figure 1

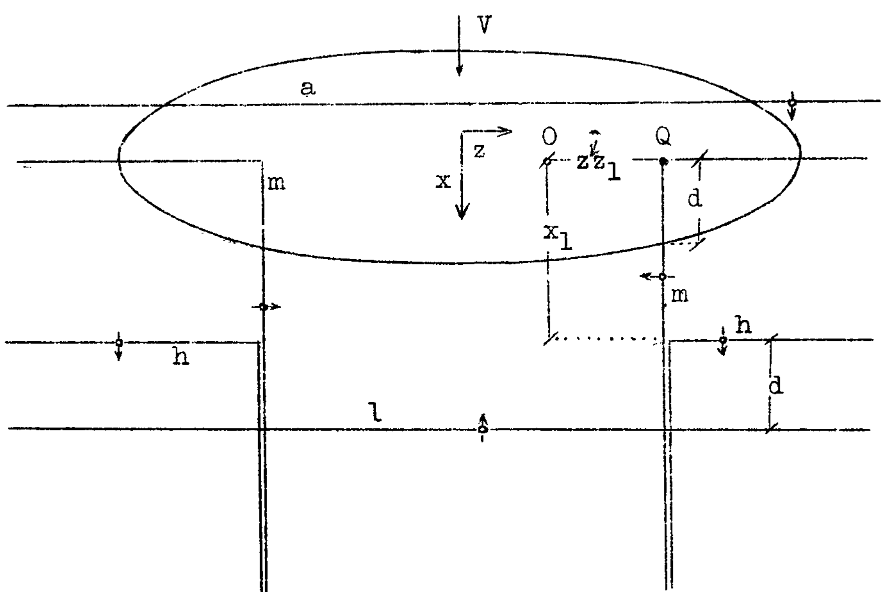

Figure 2

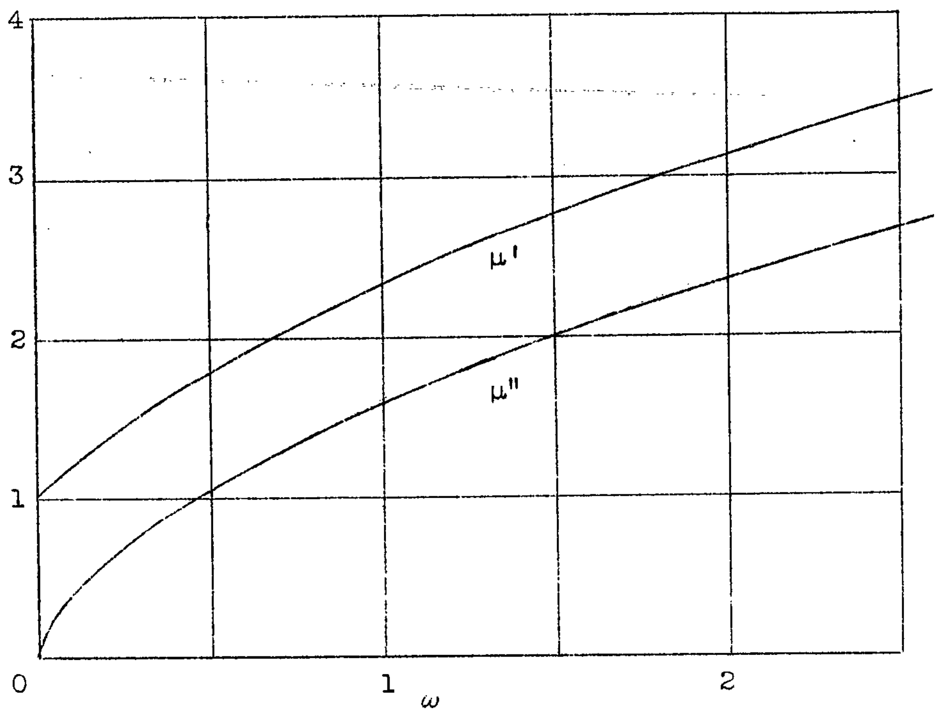

Figure 3.

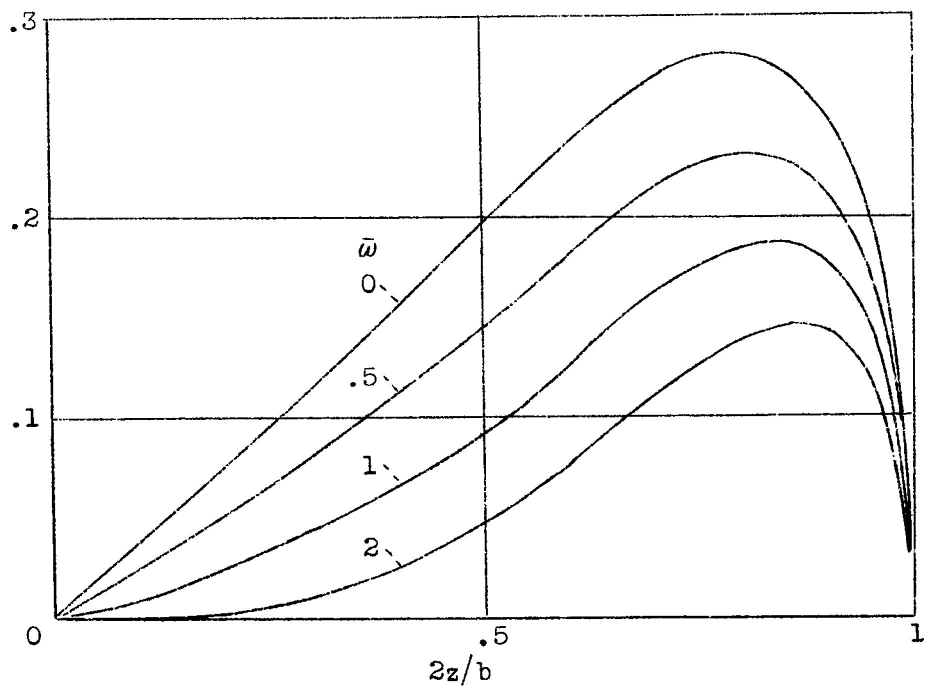

Figure 4.

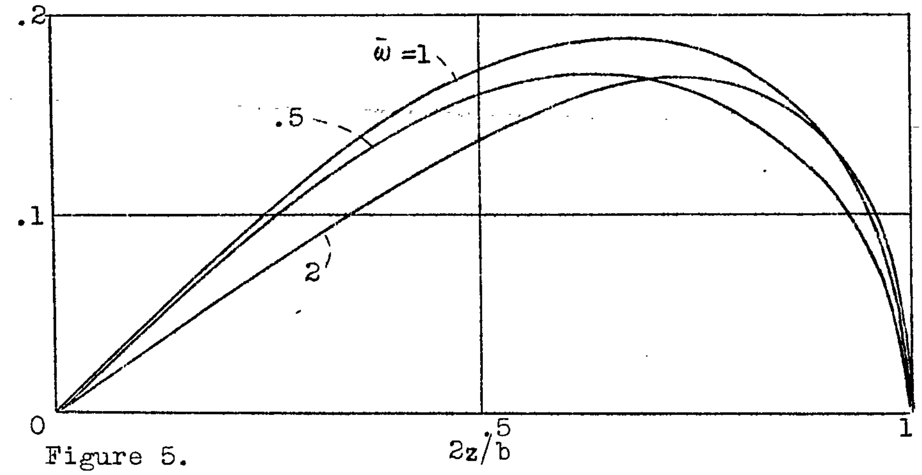

Figure 5.

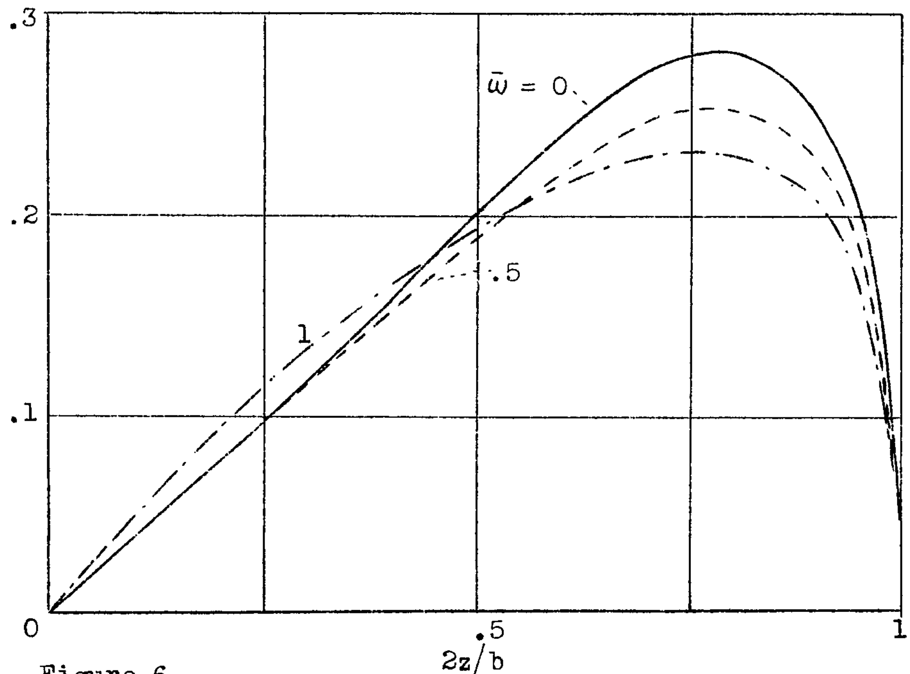

Figure 6.

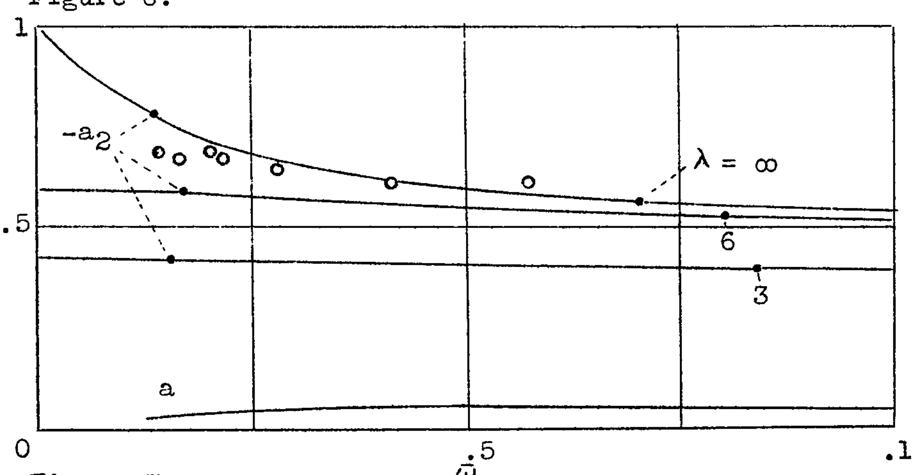

Figure 7.

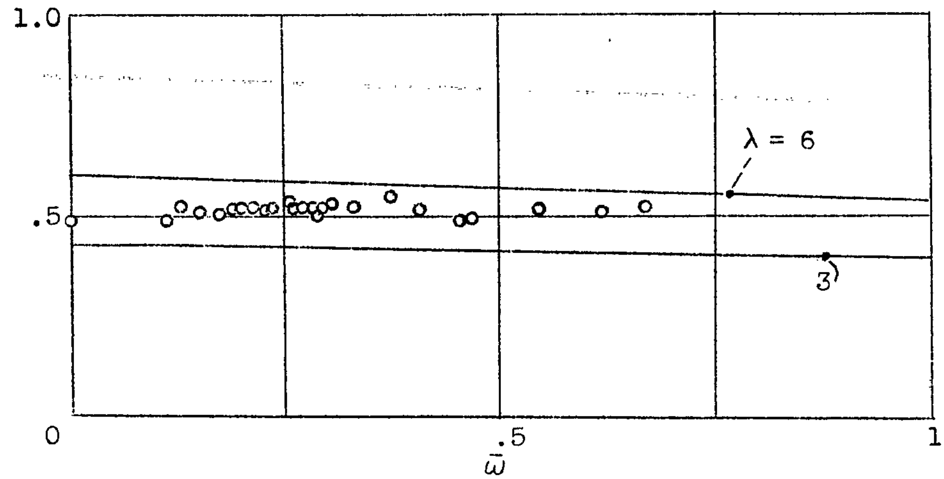

Figure 8

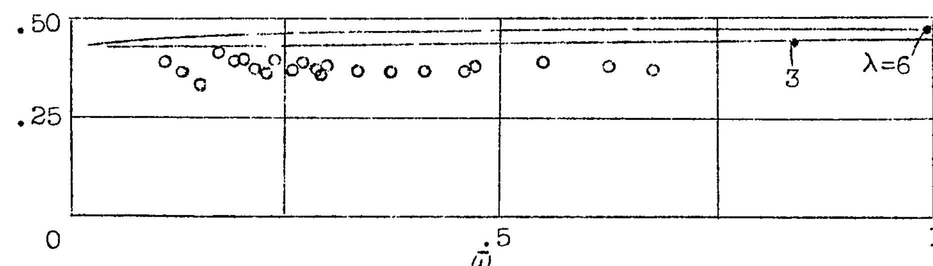

Figure 9

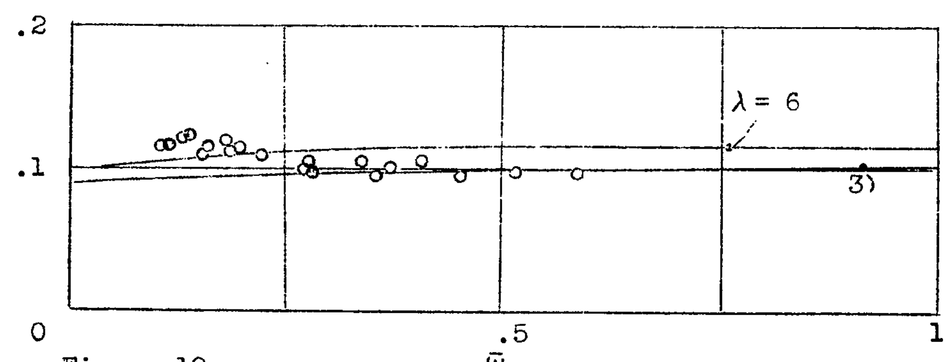

Figure 10

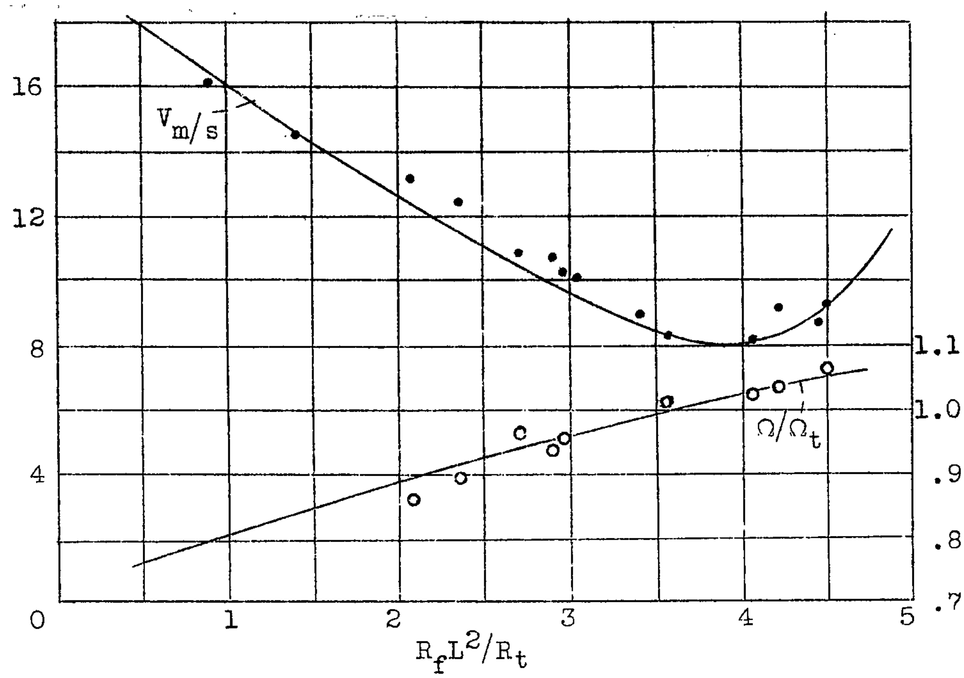

Figure 11.

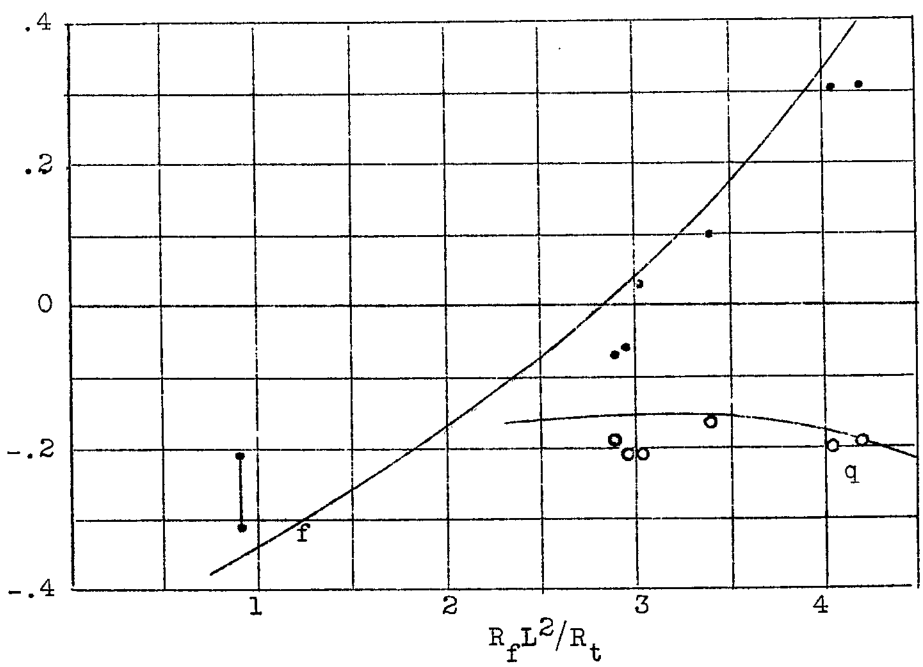

Figure 12