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BEHAVIOR OF A PLATE STRIP UNDER SHEAR AND COMPRESSIVE STRESSES BEYOND THE BUCKLING LIMIT*

By A. Kromm and K. Marguerre

The present report is an extension of previous theoretical investigations on the elastic behavior of a plate under compression and shear in the region above the critical. The main object is the clarification of the behavior immediately above the buckling limit since no theoretical expressions for this range have thus far been found and since experimentally, too, any degree of regularity in the behavior of the plate in the range between the critical load and about three to four times the critical, is discernible only with difficulty. The present report thus supplements, for example, the experimental investigations of Lahde and Wagner.

Lahde and Wagner's investigations differ from ours, however, in the following points: Whereas they consider the case of clamped-end condition and rigid lateral stiffening, we shall consider hinged-end conditions and the two limiting cases of rigid and vanishingly small lateral stiffening, respectively. (Through interpolation, the intermediate case of elastic lateral support is thus taken into account.) Lahde and Wagner's chart 4 refers to the particular case of pure shear, while our figures 2 and 3 refer to the more general case of combined loading in compression and shear. There is some deviation in the results - our computations leading to a somewhat smaller supporting strength of the sheet than is obtained on the basis of the test results of Lahde and Wagner.

I. INTRODUCTION

The present paper is a continuation of two previous papers on the behavior of plates beyond the buckling limit.

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*"Verhalten eines von Schub- und Druckkraften beanspruchten Plattenstreifens oberhalb der Beulgrenze." Luftfahrtforschung, vol. 14, no. 12, December 20, 1937, pp. 627-639.

In the first paper (reference 1) there was investigated, with the aid of the energy method, the behavior immediately above the buckling limit under the approximating assumption that for a small excess of load beyond the buckling load, the waves maintained the same shapes they assumed at the critical load. The investigation differed essentially in the method employed from those of other authors, in that first the theory of plates with "large" deflection was based on considerations from differential geometry; and secondly, in the derivation of the equilibrium conditions (expressed in terms of the displacements u, v, w); the principle of virtual displacements and the Ritz expression was applied strictly to the normal displacement w only (hence not to u and v). The essential result obtained was that the apparent stiffness $E_{red} = dp_1/d\epsilon$ was reduced to half its value at the instant of buckling (reference 2).

In the second paper (reference 3) there was investigated (with the aid of an expression by Ritz for w containing several parameters) the behavior of the plate when the critical point was far exceeded. The principal result obtained was the simple approximate formula for the "effective width"

$$b_m = b\sqrt[3]{\frac{p_{cr}}{p_l}} \qquad (p_l > p_{cr})$$

(b = width of sheet, p_l = the stress in the longitudinal reinforcing members (fig. 1)).

The present investigation is a continuation of the previous as regards both subject matter and method. The former is extended by the addition of shear loading to the pressure loading which alone had been considered up to the present ("combined" shear and compressive stress). The method is extended by taking into account a variability in the wave length and, in the presence of shear, the change in the angle of inclination of the waves (angle α, fig. 1) with increasing load beyond the critical. There are thus obtained with a far less expenditure of computation work (and this is a most important factor in the complicated shear problem), results that are only slightly impaired as compared with those previously obtained (reference 3, p. 126) for the case of pure compressive load.

II. STATEMENT OF THE PROBLEM, CHOICE OF INDEPENDENT VARIABLES

We consider (as in the paper cited under reference 1) a strip extending infinitely in the x direction, simply supported by flexurally rigid longitudinal stiffeners. The latter may be supported against each other by cross ribs which, however, are not to make contact with the sheet, so that the buckling waves may be formed undisturbed along x. As shown on figure 1, the cross sections of the stiffeners are denoted by F_l and F_q, respectively, the reference cross sections being taken as $s\,b$ and $s\,a$, respectively, where a is the distance between the transverse stiffeners. Denoting by p_x and p_y the mean externally applied pressures, then there are the following relations between the latter, the pressures p_l and p_q in the stiffeners, and the mean pressures p_1 and p_2 in sheet:

$$p_x = \frac{p_1\ sb + p_l\ F_l}{sb + F_l},\quad p_y = \frac{p_2\ sa + p_q\ F_q}{sa + F_q} \qquad (2.1)$$

The system of longitudinal and transverse stiffener members is assumed not to be stiff at the edges so that the (mean) shear stress τ is taken up only by the sheet. Let the mean displacement be denoted by γ and the wave inclination angle by α, the mean compressive strains in the x and y direction, respectively, by ϵ_1 and ϵ_2, so that the pressures of the longitudinal and transverse members are:

$$p_l = E\,\epsilon_1,\quad p_q = E\,\epsilon_2 \qquad (2.2)$$

The problem in its most general form will consist in determining the elastic condition of the strip as a function of the eight parameters

$$s,\ b,\ a,\ F_l,\ F_q,\ p_x,\ p_y,\ \tau \qquad (2.3)$$

If, in place of these eight independent variables we introduce nondimensional combinations, then s and b drop out as independent parameters since they enter into the expressions for the critical loads only in the form of a quotient $(s/b)^2$. In addition, a and F_q occur only in the combination F_q/sa. Since there still remain five independent parameters, we must restrict somewhat the generality of the investigation.

We choose as the most important particular case $p_y = 0$ (i.e., the absence of an external load in the y direction) and restrict ourselves as regards the geometrical magnitudes a and F_q, to the two limiting cases of very weak and very strong transverse stiffeners, that is:

$$\frac{F_q}{sa} = 0, \quad \frac{F_q}{sa} = \infty$$

in other words, we consider the two limiting cases:

$$p_2 = 0 \quad \text{and} \quad p_q = E\,\epsilon_2 = 0 \qquad (2.4)$$

so that as independent variables there remain only the three magnitudes:

$$\frac{F_l}{sb}, \quad \tau, \quad \text{and} \quad p_l \quad \text{or} \quad p_1$$

The object of our computation is the determination of the two functions:

$$p_1 = p_1\ (p_l,\ \tau), \quad \gamma = \gamma\ (p_l,\ \tau) \qquad (2.5)$$

in which p_1 (or p_l) may be replaced according to (2.1) by the values p_x and F_l/sb.

III. BASIC EQUATIONS

METHOD OF RITZ AND GALERKIN

The basic equations for the determination of the changes in the stress and strain condition of the buckled plate were derived in the three works cited under section I, and we shall briefly set them down here. For the changes in the coefficients of the linear element there is obtained after neglecting the square terms in the strain portion of the tangential displacements u, v and all higher members in the bending portion:

$$\left.\begin{aligned} \gamma_{11} = 2u_x + w_x^2 - 2z\, w_{xx},\ \gamma_{22} = 2v_y + w_y^2 - 2z\, w_{yy} \\ \gamma_{12} = u_y + v_x + w_x\, w_y - 2z\, w_{xy} \end{aligned}\right\} \quad (3.0)$$

The stresses $\bar{\sigma}$, $\bar{\tau}$ in the middle plane of the plate are given by:

$$\left.\begin{aligned} \bar{\sigma}_x - \nu\, \bar{\sigma}_y = E\left(u_x + \frac{w_x^2}{2}\right),\ \bar{\sigma}_y - \nu\, \bar{\sigma}_x = E\left(v_y + \frac{w_y^2}{2}\right) \\ \bar{\tau} = G\,(u_y + v_x + w_x\, w_y) \end{aligned}\right\} \quad (3.1)$$

The expression for the stored-up strain energy in a strip of plate of length l is:

$$A = \frac{Es}{2} \int_{-l/2}^{l/2} \int_{-b/2}^{b/2} \left\{ \frac{1}{E^2} \left[(\bar{\sigma}_x + \bar{\sigma}_y)^2 - 2\,(1+\nu)\,(\bar{\sigma}_x \bar{\sigma}_y - \bar{\tau}^2)\right] \right.$$

$$\left. + \frac{s^2}{12(1-\nu^2)} \left[(\Delta w)^2 - 2(1-\nu)\,(w_{xx}\, w_{yy} - w_{xy}^2)\right] \right\} dx\, dy \quad (3.2)$$

The three equilibrium conditions are obtained according to the principle of virtual displacements from the minimum condition:

$$\delta\,(A + V) = 0$$

If we consider the displacements at the edges as given (so that the edges are kept fixed during the variation), the variation of the potential V of the external forces vanishes (the external forces perform no virtual work on the displacements δu, δv, δw vanishing at the edges), and there remains:

$$\delta A = Es \int_{-l/2}^{l/2} \int_{-b/2}^{b/2} \Big\{ \frac{1}{E^2} [(\bar\sigma_x \, \delta(\bar\sigma_x - \nu \, \bar\sigma_y) + \bar\sigma_y \, \delta(\bar\sigma_y - \nu \, \bar\sigma_x)$$

$$+ 2(1+\nu) \, \bar\tau \delta \bar\tau] + \frac{s^2}{12(1-\nu^2)} \, \delta \Big[\frac{1}{2} \, w_{xx}^2 + \frac{1}{2} \, w_{yy}^2 + w_{xy}^2$$

$$+ \nu \, (w_{xx} \, w_{yy} - w_{xy}^2) \Big] \Big\} \, dx \, dy$$

$$= E \, s \int_{-l/2}^{l/2} \int_{-b/2}^{b/2} \Big\{ \frac{1}{E} [\bar\sigma_x \, \delta \, u_x + \bar\sigma_y \, \delta \, v_y + \bar\tau \, (\delta \, u_y + \delta \, v_x)$$

$$+ \bar\sigma_x \, w_x \, \delta \, w_x + \bar\sigma_y \, w_y \, \delta \, w_y + \bar\tau \, (w_x \, \delta \, w_y + w_y \, \delta \, w_x)]$$

$$+ \frac{s^2}{12(1-\nu^2)} \, w_{xx} \, \delta \, w_{xx} + w_{yy} \, \delta \, w_{yy} + 2w_{xy} \, \delta \, w_{xy}$$

$$+ \nu \, (w_{xx} \, \delta \, w_{yy} + w_{yy} \, \delta \, w_{xx} - 2w_{xy} \, \delta \, w_{xy})] \Big\} \, dx \, dy = 0$$

By integration by parts there is obtained (taking into account the end conditions (δu, δv, $\delta w = 0$ for $y = \pm \, b/2$, δu, δv, δw periodic in x) in the usual manner:

$$\delta A = - \, s \int_{-l/2}^{l/2} \int_{-b/2}^{b/2} \Big\{ \Big(\frac{\partial \bar\sigma_x}{\partial x} + \frac{\partial \bar\tau}{\partial y} \Big) (\delta u + w_x \, \delta w)$$

$$+ \Big(\frac{\partial \bar\tau}{\partial x} + \frac{\partial \bar\sigma_y}{\partial y} \Big) (\delta v + w_y \, \delta w)$$

$$+ \Big(\bar\sigma_x \, w_{xx} + \bar\sigma_y \, w_{yy} + 2\bar\tau \, w_{xy} - \frac{E \, s^2}{12(1-\nu^2)} \, \Delta \, \Delta \, w \Big) \, \delta w \Big\} \, dx \, dy$$

$$+ \Big[\frac{E \, s^3}{12(1-\nu^2)} \int_{-l/2}^{l/2} (w_{yy} - \nu \, w_{xx}) \, \delta \, w_y \, dx \Big]_{y=\pm b/2} = 0 \quad (3.3)$$

The approximate method for the solution of the differential equilibrium equations involved in (3.3) is the following:

The two equations for the equilibrium of the forces in the plate strip, namely:

$$\frac{\partial \bar{\sigma}_x}{\partial x} + \frac{\partial \bar{\tau}}{\partial y} = 0, \quad \frac{\partial \bar{\tau}}{\partial x} + \frac{\partial \bar{\sigma}_y}{\partial y} = 0 \qquad (3.4)$$

are satisfied exactly by the assumed stress function:

$$\bar{\sigma}_x = \Phi_{yy}, \quad \bar{\sigma}_y = \Phi_{xx}, \quad \bar{\tau} = -\Phi_{xy}$$

There is then obtained from the elasticity equation (3.1), by elimination of the displacements u and v, as a first equation for the relation between the stress function Φ and the normal displacement w, the equation

$$\Delta \Delta \Phi = E\left(w_{xy}^2 - w_{xx} w_{yy}\right) \qquad (3.5)$$

As a second equation, there is obtained from (3.3) the equilibrium condition for the forces at right angles to the plane of the plate:

$$\frac{E s^2}{12(1-\nu^2)} \Delta \Delta w - \Phi_{yy} w_{xx} - \Phi_{xx} w_{yy} + 2\Phi_{xy} w_{xy} = 0 \qquad (3.6)$$

This condition is satisfied only approximately. For the normal displacement w, we set up a plausible expression containing the free parameters η_i, and instead of requiring that δA, that is, that the expression:

$$\int_{-l/2}^{l/2}\int_{-b/2}^{b/2} \left\{ \left(\Phi_{yy} w_{xx} + \Phi_{xx} w_{yy} - 2\Phi_{xy} w_{xy} - \frac{E s^2}{12(1-\nu^2)} \Delta \Delta w \right) \delta w \right\} dx\, dy$$

shall vanish for every variation δw (which would lead to the nonlinear differential equation (3.6)), we require the vanishing of δA only on variation of the free values η_i; that is, in place of the differential equation (3.6), we have the equations:

$$\frac{\partial A}{\partial \eta_i} = 0 \qquad (i = 1, 2, 3 \ldots) \qquad (3.7)$$

(Ritz method).

In the particular case that the parameters η_i in the Ritz expression for w enter linearly:

$$w = \Sigma\ \eta_i\ w_i \tag{3.8}$$

(and that each of the functions w_i satisfies the above given boundary and periodicity conditions!) equations (3.7), on account of $\frac{\partial w}{\partial \eta_i} = w_i$, may be put in the form:

$$\int_{-l/2}^{l/2}\int_{-b/2}^{b/2} \left\{\left(\Phi_{yy}\ w_{yy} + \Phi_{xx}\ w_{yy} - 2\Phi_{xy}\ w_{xy} - \frac{E\ s^2}{12(1-\nu^2)}\ \Delta\ \Delta\ w\right) w_i\right\} dx\ dy$$
$$+ \left[\frac{E\ s^3}{12(1-\nu^2)} \int_{-l/2}^{l/2} (w_{yy} - \nu\ w_{xx})\ (w_i)_y\ dx\right]_{y=\pm b/2} = 0 \qquad (3.9_1)$$

The method of using, in place of the minimum conditions (3.7), a system of equations:

$$\int_{-l/2}^{l/2}\int_{-b/2}^{b/2} \left\{\left(\Phi_{yy}\ w_{xx} + \Phi_{xx}\ w_{yy} - 2\Phi_{xy}\ w_{xy} - \frac{E\ s^2}{12(1-\nu^2)}\ \Delta\ \Delta\ w\right) w_i\right\} dx\ dy = 0 \qquad (3.9_2)$$

is known as the method of Galerkin (reference 4). If the functions w_i are so chosen that the boundary integral in (3.9_1) vanishes, the method is identical with that of Ritz. The two methods then differ only in the order in which the operations of integration with respect to x and y and differentiation with respect to η_i are taken.

It is possible, naturally, in the general case where the parameters η_i in the expression for w do not occur linearly (Rayleigh method), to transform the minimum conditions (3.7) in such a manner that equations of the type (3.9) are obtained. The greater ease in integration work in which this method may possibly result (particularly in the case of a multiparameter expression), is generally offset by the requirement of taking into account all the boundary integrals (that enter into the integration by parts) and carrying along more terms in which a function of the limits of integration in the parameters η_i must be taken into account. In the following investigation therefore the two forms (3.7) and (3.9) for the minimum requirements, will be used side by side, depending on which appears most desirable for purposes of practical computation.

IV. DETERMINATION OF THE STRESS FUNCTION Φ

For the normal displacement w, we assume the expression:

$$w = f \cos \frac{\pi y}{b} \cos \frac{\pi}{l} (x - m\, y) \tag{4.0}$$

and consider the amplitude f, the wave length l, and the value $m = \cot \alpha$ as the free parameters (η_i) in equations (3.7).

Expression (4.0) (the only one that leads to useful results with "finite" amount of computation) satisfies the boundary condition $w = 0$, but not, however, (for $m \neq 0$) the condition of exact hinge support. The fact that, in spite of this, it does enable an approximate determination of the actual relations occurring in hinged support (as shown by the deviations of the critical shear stress computed by the aid of it by 6 percent of the exact value), is explained by the fact that the work of the end moments for the deflections w_y of the plate[1] vanishes, not for each point but on the average, over a period. That such a type of end condition (alternately positive and negative clamping coefficient) cannot physically be realized, naturally impairs the value of the conclusions drawn as to the bending stresses in the neighborhood of the edges. As far as the prediction of the over-all supporting strength of the plate is concerned, however, the effect of this indeterminancy is of subordinate importance.

The computation proper, with the corresponding extensions, proceeds entirely in a similar manner to that pro-

[1] The boundary integral in (3.3) where in place of δw_y there is written w_y.

viously given. (See reference 3, pp. 122-123.)

For the second derivatives of w, there are obtained:

$$\left.\begin{aligned} w_{xx} &= -\frac{\pi^2}{l^2} f \cos\frac{\pi y}{b} \cos\frac{\pi}{l}(x - m y) \\ w_{yy} &= -\left(\frac{\pi^2}{b^2} + \frac{\pi^2 m^2}{l^2}\right) f \cos\frac{\pi y}{b} \cos\frac{\pi}{l}(x - m y) \\ &\quad - 2\frac{\pi^2 m}{b l} f \sin\frac{\pi y}{b} \sin\frac{\pi}{l}(x - m y) \\ w_{xy} &= \frac{m \pi^2}{l^2} f \cos\frac{\pi y}{b} \cos\frac{\pi}{l}(x - m y) \\ &\quad + \frac{\pi^2}{l b} f \sin\frac{\pi y}{b} \sin\frac{\pi}{l}(x - m y) \end{aligned}\right\} \qquad (4.1)$$

Equation (3.5) therefore becomes:

$$\Delta \Delta \Phi = -E \frac{f^2}{2l^2}\frac{\pi^4}{b^2}\left\{\cos\frac{2\pi y}{b} + \cos\frac{2\pi}{l}(x - m y)\right\} \qquad (4.2)$$

A particular integral of this equation is:

$$\Phi^{(p)} = -E\frac{f^2}{32}\left\{\frac{l^2}{b^2(1+m^2)^2}\cos\frac{2\pi}{l}(x - m y) + \frac{b^2}{l^2}\cos\frac{2\pi y}{b}\right\}$$

If, in place of l as parameter, there is introduced the ratio of the strip width b to the "wave separation" $l.\sin\alpha$ (see fig. 1):

$$\beta = \frac{b}{l \sin\alpha}, \quad \text{so that} \quad l^2 = \frac{b^2}{\beta^2 \sin^2\alpha} = \frac{b^2}{\beta^2}(1+m^2) \qquad (4.3)$$

then $\Phi^{(p)}$ assumes a somewhat simpler form:

$$\Phi^{(p)} = -E\frac{f^2}{32}\frac{1}{1 + m^2}\left\{\frac{1}{\beta^2}\cos\frac{2\pi}{l}(x - m y) + \beta^2\cos\frac{2\pi y}{b}\right\} \qquad (4.4)$$

For the stresses there are thus obtained:

$$\left.\begin{aligned}
\bar{\sigma}_x = \Phi_{yy} &= E\,\frac{\pi^2}{8}\,\frac{f^2}{b^2}\,\frac{1}{1+m^2} \\
&\quad \left\{\frac{m^2}{1+m^2}\cos\frac{2\pi}{l}(x-m\,y) + \beta^2\cos\frac{2\pi y}{b}\right\} + \Phi^{(h)}_{yy} \\
\bar{\sigma}_y = \Phi_{xx} &= E\,\frac{\pi^2}{8}\,\frac{f^2}{b^2}\,\frac{1}{1+m^2} \\
&\quad \left\{\frac{1}{1+m^2}\cos\frac{2\pi}{l}(x-m\,y)\right\} + \Phi^{(h)}_{xx} \\
\bar{\tau} = -\Phi_{xy} &= E\,\frac{\pi^2}{8}\,\frac{f^2}{b^2}\,\frac{1}{1+m^2} \\
&\quad \left\{\frac{m}{1+m^2}\cos\frac{2\pi}{l}(x-m\,y)\right\} - \Phi^{(h)}_{xy}
\end{aligned}\right\} \qquad (4.5)$$

The integral $\Phi^{(h)}$ of the homogeneous equation, $\Delta\,\Delta\,\Phi = 0$, corresponding to (4.2), which must be made use of for satisfying the boundary conditions, we put first, setting:

$$\frac{2\pi}{l} = \lambda \qquad (4.6)$$

in the form:

$$\begin{aligned}
\Phi^{(h)} = &\Big\{(A\,\lambda\,y\,\sinh\lambda\,y + B\cosh\lambda\,y)\cos\lambda\,x \\
&+ (C\,\lambda\,y\cosh\lambda\,y + D\sinh\lambda\,y)\sin\lambda\,x\Big\} \\
&- \frac{p_1}{2}\,y^2 - \frac{p_2}{2}\,x^2 - \tau\,x\,y
\end{aligned} \qquad (4.7)$$

The seven constants of integration $A \ldots \tau$ are determined from the requirement that the longitudinal stiffeners connected to the strip remain straight; i.e., in addition to a uniform strain, only motions as a whole should be experienced, or expressed in formulas:

$$\left.\begin{aligned} v\,(x, \pm\, b/2) &= \mp\, \epsilon_2 b/2 \\ u\,(x, \pm\, b/2) &= -\,\epsilon_1\, x \pm \gamma\, b/2 \end{aligned}\right\} \qquad (4.8)^2$$

In order to be able to write down these two equations, we must determine the displacements u, v explicitly with the aid of relation (3.1). We shall write down only the final results of the simple but somewhat tedious computation:

$$\begin{aligned} E\,u\,(x,y) = E\,\frac{\pi f^2}{16\,l}\Bigg\{&\left[\frac{1/\beta^2}{1+m^2}\,(m^2-\nu) + 1 + \cos\frac{2\pi y}{b}\right] \\ &\sin\frac{2\pi}{b}\,(x - m\,y) - \frac{2\pi x}{l} + 2m\,\frac{b}{l}\left(\frac{2\pi y}{b} + \sin\frac{2\pi y}{b}\right)\Bigg\} \\ &- (p_1 - \nu\, p_2)\, x + 2\,(1+\nu)\,\tau\, y \\ &- \frac{2\pi}{l}\,[A\,(1+\nu)\,\lambda\, y \sinh\lambda\, y \\ &\qquad + (2A + B\,(1+\nu))\cosh\lambda\, y]\sin\lambda\, x \\ &+ \frac{2\pi}{l}\,[C\,(1+\nu)\,\lambda\, y\cosh\lambda\, y \\ &\qquad + (2C + D\,(1+\nu))\sinh\lambda\, y]\cos\lambda\, x \end{aligned}$$

$$\begin{aligned} E\,v\,(x,y) = E\,\frac{\pi f^2}{16\,l}\Bigg\{&\left[-\frac{m}{\beta^2}\,\frac{m^2+2+\nu}{1+m^2}\right. \\ &\left.\qquad + m\left(1+\cos\frac{2\pi y}{l}\right)\right]\sin\frac{2\pi}{l}\,(x - m\,y) \\ &- \frac{2\pi y}{l}\left(\left(\frac{l}{b}\right)^2 + m^2\right) \\ &+ \frac{l}{b}\left[\left(1 + \cos\frac{2\pi}{l}\,(x - m\,y) - (m^2+\nu)\right]\sin\frac{2\pi y}{b}\right\} \\ &- (p_2 - \nu\, p_1)\, y \\ &+ \frac{2\pi}{l}\,[A\,(1+\nu)\,\lambda\, y\cosh\lambda\, y \\ &\qquad - (A\,(1-\nu) - B\,(1+\nu))\sinh\lambda\, y]\cos\lambda\, x \\ &+ \frac{2\pi}{l}\,[C\,(1+\nu)\,\lambda\, y\sinh\lambda\, y \\ &\qquad - (C\,(1-\nu) - D\,(1+\nu))\cosh\lambda\, y]\sin\lambda\, x \end{aligned}$$

[2] See footnote, p. 12.

Substituting in (4.8) everywhere the value $y = b/2$ (the corresponding condition for $y = -b/2$ is then, on account of the symmetry of the equations, automatically satisfied), and arranging in powers of x and $\sin \lambda x$, $\cos \lambda y$, we obtain the following system of equations:

$$\left.\begin{aligned} E\,\epsilon_1 &= p_1 - \nu\, p_2 + E\,\frac{\pi^2 f^2}{8\, l^2} \\ E\,\epsilon_2 &= p_2 - \nu\, p_1 + E\,\frac{\pi^2 f^2}{8\, l^2}\left(\left(\frac{l}{b}\right)^2 + m^2\right) \\ G\,\gamma &= \tau + G\,\frac{m\,\pi^2 f^2}{4\, l^2} \end{aligned}\right\} \quad (4.9_1)^{3}$$

$$\left.\begin{aligned} &A\,(1+\nu)\,\frac{\pi b}{l}\sinh\frac{\pi b}{l} + (2A + B\,(1+\nu))\cosh\frac{\pi b}{l} \\ &\qquad = \frac{E\,f^2}{32}\,\beta^2\,\frac{m^2-\nu}{(1+m^2)}\cos\frac{m\pi b}{l} \\ &A\,(1+\nu)\,\frac{\pi b}{l}\cosh\frac{\pi b}{l} - (A\,(1-\nu) - B\,(1+\nu))\sinh\frac{\pi b}{l} \\ &\qquad = \frac{Ef^2}{32}\,m\,\frac{m^2 + 2 + \nu}{\beta^2\,(1+m^2)}\sin\frac{m\pi b}{l} \\ &C\,(1+\nu)\,\frac{\pi b}{l}\cosh\frac{\pi b}{l} + (2C + D\,(1+\nu))\sinh\frac{\pi b}{l} \\ &\qquad = \frac{Ef^2}{32}\,\frac{m^2-\nu}{\beta^2\,(1+m^2)}\sin\frac{m\pi b}{l} \\ &C\,(1+\nu)\,\frac{\pi b}{l}\sinh\frac{\pi b}{l} - (C\,(1-\nu) - D\,(1-\nu))\cosh\frac{\pi b}{l} \\ &\qquad = \frac{Ef^2}{32}\,m\,\frac{m^2 + 2 + \nu}{\beta^2\,(1+m^2)}\cos\frac{m\pi b}{l} \end{aligned}\right\} \quad (4.9_2)$$

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[2]We thus consider for the moment, not the forces but the displacements at the edges as given in advance. The mean values of the stresses (which enter expression (4.7) as integration constants $p_1 p_2, \tau$) are determined in this manner as functions of ϵ_1, ϵ_2 and γ. In the final formulas, however, there is nothing to prevent the inverse interpretation of the functional relation.

[3]This important system of equations may also be obtained in somewhat simpler form (3.1) and (4.8) by considering beforehand the relation between the mean values of the stresses and strains, i.e., by an integration over the complete periods the sin-cos terms drop out.

V. THE MINIMUM CONDITIONS FOR DETERMINING THE PARAMETERS

f, β, m

The further procedure in the computation will now be indicated. After the constants have been computed from (4.9), they are substituted in (3.2), and from the three equations (3.7), the parameters f, β, m are computed. Equations (4.9_1) with

$$E\,\epsilon_1 = p_l, \quad E\,\epsilon_2 = p_q$$

then give the required relations:

$$p_1 = p_1\,(p_l,\ \tau) \qquad \tau = \tau\,(\gamma,\ p_l)$$

(for $p_2 = 0$ or $p_q = 0$). With the aid of the first equation (2.1), the longitudinal stiffener stress p_l and the mean sheet stress p_1 (and hence also the effective contributing width p_1/p_l) and the shear displacement γ are then given for each combination of external loads p_x and τ. It is immediately evident that the computation, which is fundamentally simple, is very tedious in practice. The computation is rendered particularly laborious by the contribution of the "homogeneous members" (4.7) which must be taken into account if the boundary conditions are to be strictly satisfied. It may, however, be observed from equations (4.9_2) that in the case of pure compressive load ($m = \cot\alpha = 0$), these terms become extremely small. (See reference 1, pp. 92 and 93.) To obtain an idea of their order of magnitude also in the presence of shear stresses, it is convenient to investigate the opposite limiting case of pure shear. Making use of the Galerkin formulas (3.9_2) (with f as parameter), this computation may be carried out for the critical point. The results are presented in the table below.

Case	(a) $\epsilon_1 = 0,\ \epsilon_2 = 0$	(b) $p_1 = 0,\ \epsilon_2 = 0$	(c) $\epsilon_1 = 0,\ p_2 = 0$	(d) $p_1 = 0,\ p_2 = 0$
Change from 1 to	0.912	0.867	0.722	0.563
" " 1 "	0.914	0.871	0.741	0.610

The upper row shows the decrease in the apparent shear modulus $d\tau/d\gamma$ at the critical point (see also fig. 9) for the four limiting cases of ideally rigid $(\epsilon_{1.2} = 0)$ and ideally yielding $(p_{1.2} = 0)$ longitudinal and transverse stiffeners, respectively, and the lower row gives the values taking the homogeneous terms into account. The error is in no case, large. It is most noticeable in the (practically uninteresting) limiting case of vanishingly small strength of stiffeners (since only in the absence of a "mean value" of the support can a boundary effect come more into evidence). Even in this case, however, it is smaller than the error which enters through the assumption of a certain edge fixing which lies at the basis of the assumed expression (4.0). These terms therefore may safely be omitted, particularly in the case of combined shear and compressive stress - especially, since such neglect (as cannot otherwise be with a "relaxation" of the edge conditions) acts to oppose the error arising from the assumed expression (4.0).

If we consider this fortunate result to be valid also for $\tau > \tau_{cr}$ then no computation difficulties are offered in obtaining the three equations for the determination of the parameters f, l, m. The Galerkin formula (3.9a) cannot be used, however, since the wave length l is at the same time the interval of integration in the expression (3.2) in the x direction, so that equation (3.9) must be completed by the additional terms mentioned above. It is simpler first to carry out the integration in (3.2), making use of expression (4.0) and the relations (4.5), i.e.:

$$\left.\begin{aligned}
\Phi_{yy} &= E\,\frac{\pi^2}{8}\,\frac{f^2}{b^2}\,\frac{1}{1+m^2}\left\{\frac{m^2}{1+m^2}\cos\frac{2\pi}{l}(x-m\,y) + \beta^2\cos\frac{2\pi y}{b}\right\} - p_1 \\
\Phi_{xx} &= E\,\frac{\pi^2}{8}\,\frac{f^2}{b^2}\,\frac{1}{1+m^2}\left\{\frac{1}{1+m^2}\cos\frac{2\pi}{l}(x-m\,y)\right\} - p_2 \\
-\Phi_{xy} &= E\,\frac{\pi^2}{8}\,\frac{f^2}{b^2}\,\frac{1}{1+m^2}\left\{\frac{m}{1+m^2}\cos\frac{2\pi}{l}(x-m\,y)\right\} + \tau
\end{aligned}\right\}\qquad(5.1)$$

and then the differentiation taking into account the changes of the mean values p_1, p_2, τ with the parameters

$$\left.\begin{aligned} p_1 - \nu\, p_2 &= E\left[\epsilon_1 - \frac{\pi^2 f^2}{8\, b^2}\,\frac{\beta^2}{1 + m^2}\right] \\ p_2 - \nu\, p_1 &= E\left[\epsilon_2 - \frac{\pi^2 f^2}{8\, b^2}\left(1 + \frac{\beta^2 m^2}{1 + m^2}\right)\right] \\ 2\,(1 + \nu)\,\tau &= E\left[\gamma - \frac{\pi^2 f^2}{4\, b^2}\,\frac{m\,\beta^2}{1 + m^2}\right] \end{aligned}\right\} \qquad (5.2)$$

there is obtained:

$$\begin{aligned} A = E\, s\, b\, l \Bigg\{ &\frac{\pi^4 f^4}{256\, b^4}\,\frac{1 + \beta^4}{(1 + m^2)^2} \\ &+ \frac{1}{E^2}\left[\frac{1}{2}\, p_1^{\,2} + \frac{1}{2}\, p_2^{\,2} - \nu\, p_1\, p_2 + (1 + \nu)\,\tau^2\right] \\ &+ \frac{s^2\, f^2\, \pi^4}{96\,(1 - \nu^2)\, b^4}\left[(1 + \beta^2)^2 + 4\,\frac{m^2}{1 + m^2}\,\beta^2\right]\Bigg\} \end{aligned} \qquad (5.3)$$

so that:

$$\begin{aligned} \frac{\partial}{\partial f}\left(\frac{A}{s b l}\right) = E\,\frac{\pi^4\, f^3}{64\, b^4}\,\frac{1 + \beta^4}{(1 + m^2)^2} &+ p_1\,\frac{\partial}{\partial f}\,(p_1 - \nu\, p_2) \\ &+ p_2\,\frac{\partial}{\partial f}\,(p_2 - \nu\, p_1) + \tau\,\frac{\partial}{\partial f}\, 2\,(1 + \nu)\,\tau \\ &+ \frac{E}{1 - \nu^2}\,\frac{s^2 f}{48\, b^4}\left[(1 + \beta^2)^2 + 4\,\frac{m^2\beta^2}{1 + m^2}\right] \end{aligned}$$

and

$$
\left.
\begin{aligned}
&(1 + m^2) \frac{4b^2}{\pi^2 f \beta^2} \frac{\partial}{\partial f} \left(\frac{A}{sbl}\right) \equiv E \frac{\pi^2 f^2}{16 b^2} \frac{1}{\beta^2} \frac{1 + \beta^4}{1 + m^2} - p_1 \\
&\quad - \left(\frac{1 + m^2}{\beta^2}\right) p_2 - 2m\,\tau \\
&\quad + \frac{p^*}{4} \left[\frac{(1 + \beta^2)^2}{\beta^2} (1 + m^2) + 4m^2\right] = 0 \\
&(1 + m^2) \frac{4b^2}{\pi^2 f^2 \beta} \frac{\partial}{\partial \beta} \left(\frac{A}{sbl}\right) \equiv E \frac{\pi^2 f^2}{16 b^2} \frac{\beta^2}{1 + m^2} - p_1 \\
&\quad - m^2 p_2 - 2m\,\tau + \frac{p^*}{2} [(1+\beta^2)(1+m^2) + 2m^2] = 0 \\
&(1 + m^2)^2 \frac{4b^2}{m \pi^2 f^2 \beta^2} \frac{\partial}{\partial m} \left(\frac{A}{sbl}\right) \equiv - E \frac{\pi^2 f^2}{16 b^2} \frac{1 + \beta^4}{\beta^2 (1+m^2)} \\
&\quad + p_1 - p_2 - \frac{1 - m^2}{m} \tau + p^* = 0
\end{aligned}
\right\} \tag{5.4}
$$

where

$$
p^* = \frac{E}{1 - \nu^2} \frac{\pi^2 s^2}{3b^2} = E\,\epsilon^* \tag{5.5}
$$

is the buckling load of the strip under pure longitudinal pressure. The system of equations (5.4) may further be somewhat simplified by proper combination of terms:

$$
\left.
\begin{aligned}
p_1 + \tau\,m &= \frac{p^*}{2} [1 + \beta^2] \\
&\quad + E \frac{\pi^2 f^2}{16 b^2} \frac{1}{1 + m^2} \left(\beta^2 + \frac{m^2}{\beta^2 (1 + m^2)}\right) \\
p_2 &= \frac{p^*}{4} [1 - \beta^4] + E \frac{\pi^2 f^2}{16 b^2} \left(\frac{1}{1 + m^2}\right)^2 \\
\frac{\tau}{m} + p_2 &= \frac{p^*}{2} [(1+\beta^2) + 2] - E \frac{\pi^2 f^2}{16 b^2} \frac{1}{\beta^2 (1+m^2)^2}
\end{aligned}
\right\} \tag{5.6}
$$

Through equations (5.6) the parameters f, β, m are given as functions of p_1, p_2, τ and hence by means of (5.2) and (2.1) the required stress-strain relation may be found.

In discussing the system of equations (5.6), a direct solution for β and m is not possible; we shall not consider the transverse stress p_2 as an independent parameter, but compute the two limiting cases only:

$$p_2 = 0$$

(perfectly yielding transverse supports) and

$$p_2 = \nu\, p_1 - E\,\frac{\pi^2\, f^2}{8 b^2}\left(1 + \frac{\beta^2\, m^2}{1 + m^2}\right) \qquad (5.8)$$

($\epsilon_2 = 0$, rigid transverse supports).[4]

VI. THE PARTICULAR CASE $\tau = 0$

We consider first the particular case of pure compressive load: $\tau = 0$. From the third of equations (5.4), we must have $m = 0$ and the system of the first two equations assumes the simple form:

$$\left.\begin{aligned} p_1 + \frac{1}{\beta^2}\, p_2 &= \frac{p^*}{4}\,\frac{(1 + \beta^2)^2}{\beta^2} + E\,\frac{\pi^2}{16}\,\frac{f^2}{b^2}\,\frac{1 + \beta^4}{\beta^2} \\ p_1 &= \frac{p^*}{2}\,(1 + \beta^2) + E\,\frac{\pi^2}{16}\,\frac{f^2}{b^2}\,\beta^2 \end{aligned}\right\} \qquad (6.1)$$

From the above there is obtained for the critical value (with $f = 0$):

$$p_{cr} = \frac{p^*}{4}\,\frac{(1 + \beta^2)^2}{\beta^2} - \frac{1}{\beta^2}\, p_{2cr} = \frac{p^*}{2}\,(1 + \beta^2)$$

for $p_2 = 0$ (no transverse support):

$$\beta = 1, \qquad p_{cr} = p^* \qquad (6.2)$$

For $\epsilon_2 = 0$ (rigid transverse supports), i.e., $p_2 = \nu\, p_1$ (see (5.8)), we have:

---

[4]The second assumption, which unfortunately, leads to a disproportionately large amount of computation, approaches very nearly the relations that actually occur in practice. (See examples, section IX.)

$$\beta^2 = 1 - 2\nu, \quad p_{cr} = (1 - \nu)\, p^* \tag{6.3}$$

in agreement with known results. (See reference 1, p. 94.)

For the relation between p_1 and $p_l = E\,\epsilon_1$ above the buckling point, we obtain in the case $p_2 = 0$ by elimination of f from the two equations (6.1):

$$E \frac{\pi^2 f^2}{16\, b^2} = \frac{p^*}{4} (\beta^4 - 1) \tag{6.-}$$

and from the first of equations (5.2) a very simple parametric representation:

$$\left.\begin{aligned} p_1 &= \frac{p^*}{4} [\beta^2(\beta^4-1) + 2\beta^2 + 2] = \frac{p^*}{4} [\beta^6 + \beta^2 + 2] \\ p_l &= \frac{p^*}{4} [3\beta^2(\beta^4-1) + 2\beta^2 + 2] = \frac{p^*}{4} [3\beta^6 - \beta^2 + 2] \end{aligned}\right\} \tag{6.5}$$

It may be seen that with increasing p_1, p_l there is also an increase in β - i.e., the waves become shorter in the longitudinal direction. Furthermore, the effective width, that is, the ratio

$$\frac{p_1}{p_l} = \frac{\beta^6 + \beta^2 + 2}{3\beta^6 - \beta^2 + 2} \tag{6.6}$$

decreases with increasing β from the value 1 for $\beta = 1$ and approaches the value 1/3 as $\beta \longrightarrow \infty$.[5]

A simple measure for the value which β may assume in the elastic range is given by equation (6.4). If for p^* we put in its value from (5.5), there is obtained:

$$(f/s)^2 = \frac{4}{3\,(1 - \nu^2)} (\beta^4 - 1) \tag{6.7}$$

or (with $\nu = 0.3$):

---

[5]With the assumption of unchanged wave length there is obtained with the assumed expression (4.0), the limiting value 1/2. The improvement is therefore considerable and also surprisingly good when compared with the result of the extended computation.

$$\beta = \sqrt[4]{1 + 0.68\ (f/s)^2} \approx 0.91 \sqrt{f/s}$$

for large values of f/s.

The greatest bending stress occurs in the center of the field in the x direction (direction of the shorter waves) and has approximately the value:

$$\tilde{\sigma}_{max} = \frac{E}{1-\nu^2} \frac{s}{2} \frac{\pi^2}{l^2} f = \frac{E}{1-\nu^2} \frac{\pi^2 s^2}{3b^2} \frac{3}{2} \beta^2 \frac{f}{s} \approx 1.24 \left(\frac{f}{s}\right)^2 p^* \quad (6.8)$$

If σ_p denotes the proportionality limit of the material then f/s, taking into account bending alone, must remain below $0.90\sqrt{\frac{\sigma_p}{p^*}}$ if the deformation is still to be elastic. Replacing in (6.8) $f/s = \frac{2}{\sqrt{3(1-\nu^2)}} \beta^2$ approximately by β, then $\tilde{\sigma}_{max} = 1.81\ \beta^4\ p^*$ and comparison with (6.5) shows that for large loads above the buckling limit $(\beta^2 << \beta^6)$ the maximum bending stress as a function of p_l and p^* may be written in the form:

$$\tilde{\sigma}_{max} \approx \frac{1}{\beta^2}\ 2.42\ p_l$$

or also

$$\tilde{\sigma}_{max} = 2.2\ p^* \left(\frac{p_l}{p^*}\right)^{2/3} = 2.2\ p^{*1/3}\ p_l^{2/3} \quad (6.9)$$

This formula gives an indication of how high the loading may be carried before permanent bending deformations may arise.[6]

For the case $\epsilon_2 = 0$, the formulas become much less simple, and we shall content ourselves with referring to the results shown in figures 3 and 4. It may be seen in particular from figure 4, that the effective width curve (with shear absent) is no longer affected by the behavior of the transverse stiffeners in the range $p_l/p^* \geqslant 2$ and correspondingly, the remaining conclusions drawn for the case $p_2 = 0$ retain their validity.

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[6] The formula is valid under the assumption $\beta^2 >> 1$, that is, large loads above the critical stress; i.e., thin-walled sheet.

VII. THE GENERAL CASE OF COMBINED SHEAR AND COMPRESSIVE STRESSES

We shall now investigate the behavior of the plate under combined shear and compressive stresses $(\tau \neq 0, m \neq 0)$. Here, too, we shall give a complete discussion of the more simple case $p_2 = 0$, but for the other limiting case $(\epsilon_2 = 0)$, we shall write down the results only.

From the second of equations (5.6), writing for briefness:

$$1 + m^2 = 1 + \cot \alpha^2 = \frac{1}{\sin^2 \alpha} = t \tag{7.0}$$

there is obtained for $p_2 = 0$:

$$\frac{E}{p^*} \frac{\pi^2}{4} \frac{f^2}{b^2} = t^2 \ (\beta^4 - 1) \tag{7.1}$$

so that

$$\frac{f}{s} = 1.21\ t \sqrt{\beta^4 - 1} = (\text{for} \quad \beta^4 >> 1)\ 1.21 \frac{\beta^2}{\sin^2 \alpha} \tag{7.2}$$

From (7.1) it is evident that at the critical point $(f = 0)$, quite independently of the shear and longitudinal pressure by which this point was attained, the distance between waves is exactly equal to the plate width $(\beta = 1)$; above the critical point $\beta > 1$.

For the critical value of τ, there is obtained with the aid of (7.1) from the third of equations (5.6)

$$\tau_{cr} = 2m\ p^* \tag{7.3}$$

independently of the value of the simultaneously acting pressure p_1 $(\leq p^*)$. A relation between τ_{cr} and $p_1 = p_{cr}$ may be obtained through elimination of m from (7.3) and the third of equations (5.4):

$$\tau_{cr} = \frac{m}{1 - m^2} (p_{cr} + p^*)$$

$$m^2 = \frac{1}{2}\left(1 - \frac{p_{cr}}{p^*}\right), \qquad \tau_{cr}^{\ 2} = 2p^{*2}\left(1 - \frac{p_{cr}}{p^*}\right)$$

If we denote by τ^* the critical shear in the absence of p_1:

$$\tau^* = \sqrt{2}p^* \qquad (m^2 = \tfrac{1}{2},\ \alpha \approx 55^\circ)$$

there follows the known relation:[7]

$$\left(\frac{\tau_{cr}}{\tau^*}\right)^2 = 1 - \frac{p_{cr}}{p^*} \tag{7.4}$$

For the relation between p_1 and p_l, τ and γ in the above critical range, there are obtained the following parameter relations:

$$\left.\begin{aligned}
\frac{p_1}{p^*} &= \frac{1}{4}\left(t\ \beta^2(\beta^4-1) + 2\beta^2 - 6t + 8 - \frac{2}{\beta^2}(t-1)\right)\\
\frac{p_l}{p^*} &= \frac{p_1}{p^*} + \frac{t\ \beta^2}{2}(\beta^4 - 1)\\
&= \frac{1}{4}\left(3t\ \beta^2(\beta^4-1) + 2\beta^2 - 6t + 8 - \frac{2}{\beta^2}(t-1)\right)\\
\frac{\tau}{p^*} &= \frac{\sqrt{t-1}}{4}\left[\frac{(1+\beta^2)^2}{\beta^2} + 4\right] = \frac{\sqrt{t-1}}{4}\left[6 + \beta^2 + \frac{1}{\beta^2}\right]\\
\frac{\gamma}{\epsilon^*} &= \frac{2(1+\nu)\tau}{p^*} + \sqrt{t-1}\ t\ \beta^2(\beta^4-1)\\
&= \frac{\sqrt{t-1}}{2}\left[\left(6 + \beta^2 + \frac{1}{\beta^2}\right)(1+\nu) + 2t\ \beta^2(\beta^4 - 1)\right]
\end{aligned}\right\} \tag{7.5}$$

[7]See reference 5. In the general case, $p_2 \neq 0$, the relation reads:

$$\left(\frac{\tau}{p^*}\right)^2 = \left[\frac{1}{2}(1+\beta^2) - \frac{p_1}{p^*}\right]\left[\frac{1}{2}(3+\beta^2) - \frac{p_2}{p^*}\right], \quad \beta^2 = \sqrt{1 - 4\frac{p_2}{p^*}}$$

which is obtained in the simplest way through elimination of n from equations (5.6_1) and (5.6_3). In the corresponding formula of Wagner there is a typographical error which was also passed on in the formula collection of Hock (Luftfahrtforschung, vol. 12 (1935), p. 215); (p_1 instead of p_2 in the second bracket).

In the above system, the directly given external stress p_x does not enter, being connected with the stresses p_1 and p_l according to (2.1) through an additional intermediate parameter F_l/sb. A simple representation of the required magnitudes, namely, the effective width p_1/p_l and mean shear modulus $G_m = \tau/\gamma$ as direct functions of p_x and τ, is therefore not possible. Since the relation (2.1) between p_1 and p_l is linear, however, a very simple procedure may be indicated for the determination of the required relation. On figure 2 is shown a plot of p_1/p^* against p_l/p^* with τ/p^* as parameter. In terms of these coordinates, equation (2.1) is a straight line which is most simply determined by its intercepts on the coordinate axes, the point of intersection with the p_l axis being $p_l^{(o)} = p_x\left(1 + \frac{sb}{F_l}\right)$, and with the p_1 axis $p_1^{(o)} = p_x\left(1 + \frac{F_l}{sb}\right)$. Joining[8] these two points by a straight line, there may be read off at the point of intersection with the $p_1 - p_l$ curve for the given value of τ/p^* the corresponding values of the mean sheet stress p_1 and the stiffener stress p_l. The effective width p_1/p_l is then obtained by simple division.[9] (See also figs. 5 and 6.) From the point of intersection, there is then also found immediately the mean decrease in the shear stiffness with the aid of the τ/γ curves of figure 2. At individual points of the $p_1 - p_l$ curves there have been indicated the corresponding β^2 and t values, in order to obtain a picture of the geometrical deformation conditions. (With the aid of (7.2) there is obtained in a simple manner from β^2 and t also the buckling amplitude f.)

The maximum bending stress may be obtained from the formula:

---

[8]If one of these two points falls outside the limits of the chart there will be found no difficulty in the determination of this line since the slope of the angle of inclination with the negative p_l axis is given by F_l/sb.

[9]For another definition of effective width in the presence of shear, see reference 6.

$$\sigma_{max} = \frac{E}{1 - \nu^2} \frac{s}{2} \left(\frac{\partial^2 w}{\partial \xi^2} + \nu \frac{\partial^2 w}{\partial \eta^2}\right)$$

where ξ is the direction of the maximum, η the direction of the minimum curvature of the surface $w = w(x,y)$. If we neglect (as above) the unimportant second term in the case of large buckling deformations, there is obtained:

$$\sigma_{max} = \frac{E}{1 - \nu^2} \frac{s}{2} \frac{1}{\rho_1}$$

The first "principal curvature" $\frac{1}{\rho_1} = \frac{\partial^2 w}{\partial \xi^2}$ is obtained by the following consideration. The sum and product of the two principal curvatures $1/\rho_1$ and $1/\rho_2$ are, as is shown in differential geometry (see, for example, reference 7) invariants and may be given in terms of the curvatures w_{xx}, w_{yy}, and twist w_{xy} by

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = w_{xx} + w_{yy}, \quad \frac{1}{\rho_1}\frac{1}{\rho_2} = w_{xx} w_{yy} - w_{xy}^2 \qquad (7.6)$$

Eliminating $1/\rho_2$ from these two equations, there is obtained for $1/\rho_1$ the quadratic equation:

$$\left(\frac{1}{\rho_1}\right)^2 - \Delta w \frac{1}{\rho_1} + (w_{xx} w_{yy} - w_{xy}^2) = 0$$

whose solution is:

$$\frac{1}{\rho_1} = \frac{1}{2}\left\{\Delta w - \sqrt{(w_{xx} - w_{yy})^2 + 4w_{xy}^2}\right\}$$

For the maximum curvature occurring at $x = 0$, $y = 0$ of the entire sheet panel, there is therefore found, using (4.1):

$$\frac{1}{\rho_1} = -\frac{\pi^2 f}{2b^2}\left\{(1+\beta^2) + \sqrt{\left(1+\beta^2-2\frac{\beta^2}{1+m^2}\right)^2 + 4\frac{m^2\beta^4}{(1+m^2)^2}}\right\} =$$

$$= -\frac{\pi^2 f}{2b^2}\left\{(1+\beta^2) + \sqrt{(1+\beta^2)^2 - \frac{4\beta^2}{1+m^2}}\right\}$$

For $\beta^2 \gg 1$ this expression may be considerably simplified by expanding the root:

$$-\frac{1}{\rho_1}\frac{b^2}{\pi^2} \approx \frac{f}{2}\left\{(1+\beta^2) + (1+\beta^2)\left(1 - \frac{2\beta^2}{(1+\beta^2)^2\,(1+m^2)}\right)\right\}$$

$$\approx f\ (1+\beta^2)\left(1 - \frac{1}{\beta^2(1+m^2)}\right) \approx f\ \left(\beta^2 + \frac{m^2}{m^2+1}\right)$$

and for the maximum bending stress there is obtained:

$$\frac{\tilde{\sigma}_{max}}{p^*} = \frac{3}{2}\ f/s\ \left(\beta^2 + \frac{m^2}{m^2+1}\right) = \frac{3}{2}\ f/s\ \left(\beta^2 + \frac{t-1}{t}\right) \qquad (7.7)$$

which for $m = 0$, that is, $t = 1$ (and particularly, for $\beta^2 \gg 1$) is in agreement with (6.8).

For large loads above the buckling $(\beta^4 \gg 1)$, f/s may be replaced by $1.21\ t\ \beta^2$ and, according to (7.5) $t\ \beta^6$ by $\frac{4}{3}\frac{p_l}{p^*}$; if we also neglect $\frac{t-1}{t}$, as compared to β^2 (which is justifiable, particularly for predominating pressure stress), there is obtained approximately:

$$\tilde{\sigma}_{max} = 1.81\ \beta^4\ t\ p^* = 2.42\ \frac{1}{\beta^2}\ p_l \qquad (7.8)$$

With the aid of this relation, which we had previously found for the particular case $\tau = 0$, it is possible to obtain the maximum bending stress also from figure 2. Since (7.8) is true for $\beta^2 \gg 1$ (for $\beta = 1$, f, and hence also $\tilde{\sigma}$ becomes zero), then (according to our theory) it is not the bending stress but the stiffener compressive stress that determines the strength of the structural part. It should be observed, however, that the secondary buckling (reference 8) in the neighborhood of the edge that occurs at very large loads above the buckling and which is not taken into account by our theory may, under certain circumstances, lead to higher bending stresses.

The exceptional case $\epsilon_2 = 0$ of particular interest in practice (limiting case of rigid transverse stiffeners) presents much greater difficulties in the computation than the case $p_2 = 0$. Since nothing fundamental, however, is changed in the discussion, we shall content ourselves with merely indicating the system of formulas which leads to the construction of chart 3 similar to chart 2. It is found to be most convenient to allow p_2 to remain as an intermediate parameter, since it is then possible to make direct use of a large part of the computations carried out for the case $p_2 = 0$. (The values given by equation (7.5) for $p_2 = 0$ are denoted by $\hat{p}_1$, $\hat{p}_l$, $\hat{\tau}$, $\hat{\gamma}$):

$$\left.\begin{aligned}
\frac{p_2}{p^*} &= \frac{1}{4}\,\frac{(\beta^4-1)\;2t\beta^2\left[t\left(1+\frac{1}{\beta^2}\right)-1\right]-4\nu\,\hat{p}_1}{-2t\beta^2\left[t\left(1+\frac{1}{\beta^2}\right)-1\right]-1+\nu\left\{t\beta^2+(t-1)\left(1+\frac{2}{\beta^2}\right)\right\}}\\
\frac{p_1}{p^*} &= \hat{p}_1+\frac{p_2}{p^*}\left\{t\beta^2+(t-1)\left(1+\frac{2}{\beta^2}\right)\right\}\\
\frac{p_l}{p^*} &= \frac{p_1}{p^*}+\frac{1}{2}\,t\beta^2\,(\beta^4-1)+(2t\beta^2-\nu)\,\frac{p_2}{p^*}\\
&= \hat{p}_l+\frac{p_2}{p^*}\left(3t\beta^2+(t-1)\left(1+\frac{2}{\beta^2}\right)-\nu\right)\\
\frac{\tau}{p^*} &= \hat{\tau}-\frac{p_2}{p^*}\left(1+\frac{1}{\beta^2}\right)\sqrt{t-1}\\
\frac{\gamma}{\epsilon^*} &= \frac{2\,(1+\nu)\tau}{p^*}+\sqrt{t-1}\left(t\beta^2(\beta^4-1)+\frac{p_2}{p^*}\,4t\beta^2\right)\\
&= \hat{\gamma}+\frac{p_2}{p^*}\left(4t\beta^2-2\,(1+\nu)\left(1+\frac{1}{\beta^2}\right)\right)\sqrt{t-1}\\
\left(\frac{f}{s}\right)^2 &= 1.46\;t^2\;(\beta^4-1)+5.85\;t^2\;\frac{p_2}{p^*}
\end{aligned}\right\}\quad(7.9)$$

The result of the elimination (possible only graphically) of the parameters t and β^2 is given in figure 3. As in figure 2, there are indicated at individual points the corresponding values of t and β^2, so that in each case f/s (and hence by (7.7) also $\tilde{\sigma}$) may be computed. A simple approximate expression of the type of (7.8) for the bending stresses could not be obtained this time.

Figures 2 and 3 refer to the limiting cases of very weak and very strong transverse stiffener members, respectively. In order to obtain at least an approximation for any definite intermediate case F_q/sa, the following method is used. On the charts the limiting compressive strains ϵ_s (for $p_s = 0$) and stresses p_s (for $\epsilon_s = 0$) are shown. The stress p_s is the mean stress with which the sheet "adheres" to the longitudinal stiffeners, and the transverse stiffeners must therefore take up a stress $p_q = p_s \frac{sa}{F_q}$. If the strain corresponding to this stress $\epsilon_s = \frac{p_s}{E}\frac{sa}{F_q}$ is now compared with the strain ϵ_s of the longitudinal members according to figure 2 (which was obtained under the assumption of no transverse stiffening), an estimate may be obtained as to which of the two limiting cases is the more nearly approached and mean values obtained for p_1, p_l, $\tau/G\gamma$, etc., computed from the two charts. How such a mean value is to be obtained in any particular case will clearly be indicated by a computed example, given in section IX.

VIII. THE EFFECTIVE WIDTH p_1/p_l AND THE REDUCED SHEAR MODULUS $\frac{d\tau}{d\gamma}$ FOR THE LIMITING CASE OF VERY STRONG LONGITUDINAL STIFFENERS

Although all the required values for some particular application of our theory may be obtained from charts 2 and 3, a few more figures will be given and explained in this section since they are suited for giving a somewhat clearer picture of the general behavior of the characteristic values of the sheet. In all of the figures the stiffener stress p_l which, in the limiting case of very strong longitudinal members ($F_l >> sb$), is equal to the directly given stress p_x, is taken as the reference stress. (If it is also desired to obtain the numerical values for the case $F_l = sb$, then it is naturally possible to use as reference the given load stress p_x with the aid of charts 2 and 3.)

Figure 4 shows the variation of the effective width with p_l/p^* in the absence of shear. The difference between the limiting cases $p_2 = 0$ (continuous curve) and $\epsilon_2 = 0$ (thin dotted curve) is very slight except for the critical point itself. The presence of a definite transverse tension $p_2 = - p^*, -2p^*$, etc., while it increases the critical load, affects the variation of the effective width only in the lower range.

Figure 5 shows the variation of the critical values τ and p_1 for combined stress (for the continuous curves see (7.4); for the dotted curves, see the formulas in reference 5. with $p_2 = \nu p_1$). The abscissa is chosen as the ratio τ/p_l. There may be observed the very considerable effect of the transverse pressure $p_2 = \nu p_1$ in the case of fixed longitudinal stiffeners $(\epsilon_2 = 0)$; p_l is obtained from p_1 by multiplication with $(1-\nu^2) = 0.91$.

Figures 6 and 7 give a plot of the ratio p_1/p_l, for which the term "effective width" has a simple meaning for the case $\tau = 0$. The abscissa is the ratio p_l/p_{cr} and the parameter the ratio τ/p_l of the shear to the stiffener pressure. It may be seen that in this case the concept of "effective width" $\left[b_m = b\,\frac{p_1}{p_l}\right]$ has lost its clear meaning since p_1 very soon becomes less than zero. Under the simultaneous action of shear and pressure, the tension component in the longitudinal direction due to the shear may become greater than the externally applied compressive stress, so that the longitudinal stiffeners must take up not only the entire external pressure but also the additional pressure arising from the condition of equilibrium with the sheet tension stresses (negative support of the skin).

In the case of pure longitudinal pressure the ratio p_1/p_l is, as we have seen from figure 4, in the two limiting cases $F_q \gtrless sb$ only very slightly different. In the presence of shear, however, the stiffness is quite considerably affected by the behavior of the transverse stiffeners. The mechanical explanation is the following: In the case of pure compressive stress the supporting ability of the sheet arises essentially from its prevention of the buckling deformation in the neighborhood of the longitudinal members and a certain "cushioning" effect which the transverse fibers exert as a result of the periodically changing lateral stresses. These lateral stresses remain small in the mean (fig. 8). As a result of the shear, however, there arise, for static reasons, diagonal tension stresses of considerable magnitude ("tension diagonals") that are transmitted to the longitudinal stiffeners. If the latter, due to stiff transverse members, are practically nondisplaceable these diagonal tension stresses result in a remarkable stiffening of the system against additional compressive and particularly shear stresses. If, however, the longitudinal stiffeners are yielding, then a diagonal tension field cannot be set up at all. The angle of wave inclination becomes very small and the sheet resists mainly through its bending stiffness. The apparently paradoxical result that, with constant external compressive load and increasing shear, the ratio $\tau/G\gamma$ for $\epsilon_s = 0$ in general increases (see fig. 3), finds its explanation in the stiffening action of the transverse stresses p_s arising from the shear.

These relations may be brought out somewhat differently with the aid of figures 9 and 10. Both figures show the variation of the reduced "instantaneous" shear modulus $d\tau/d\gamma$ (not the reduced mean shear modulus τ/γ) - figure 9 for pure shear stress, and figure 10 for constant ratio $k = \tau/p_l$. It may be seen from figure 9 that $d\tau/d\gamma$ depends very much on the stiffness of the longitudinal and transverse stiffeners. Curve a (rigid struts) shows in particular the decrease toward the limiting value known from the tension-field theory; curve b is for the case of no longitudinal stiffening; and curve c, for no transverse stiffening. From curve d, there may be obtained the order of magnitude of the resistance which an unstiffened sheet exerts against further deformation.

Figure 11 shows the variation of the angle $\dot{\alpha}$ of the wave inclination and the principal stress angle $\bar{\alpha}$ with τ/p^*, p_l/p^* being taken as parameter, for $\epsilon_s = 0$. (The dot-dash curves separate the regions below and above critical buckling load.) With predominating shear stress (particularly, therefore, for small values of the parameter p_l/p^*) the angles deviate but little from one another; with increasing τ the curves show a tendency to collect in the strip between 40 and 50°, so that with large loads in excess of the buckling load due to shear no great error will be made in assuming the approximate value α = 45°. The figure partially confirms the correctness of the assumptions of the Wagner tension-field theory and at the same time shows in what direction the assumed expressions for the deflections should be corrected if the compressive load predominates.

IX. COMPUTED EXAMPLES

The use of the charts 2 and 3 will be made clear with the aid of two examples.

1. A panel of a plane reinforced plate girder is to take up such a shear stress that τs = 40 kg/cm, and a longitudinal compressive force P_x = 1,500 kg. The distance between the longitudinal stiffener sections (that is, the sheet width) is 130 mm, and the distance between the transverse stiffener frames is 250 mm.

If we consider a mean shear stress in the sheet of τ_{al} = 500 kg/cm² as allowable, then for the wall thickness we must choose

$$s = \frac{400}{500} = 0.8 \text{ mm}$$

With s = 0.8, b = 130 the reference pressure p^* becomes:

$$p^* = 730{,}000 \, \frac{0.64}{16900} \times \frac{\pi^2}{3 \times 0.91} = 100 \text{ kg/cm}^2$$

so that $\frac{\tau}{p^*} = 5$. If we admit a compressive stress in the longitudinal stiffeners p_l = 1,200 kg/cm², then with the aid of figure 3, we may obtain the required section $\overline{F}_l$ of the longitudinal stiffeners. From the latter figure there corresponds to p_l/p^* = 12, and τ/p^* = 5, a sheet stress p_i/p^* = 3.35; the equilibrium of the forces in the longitudinal direction gives:

$$P_x = p_i \; sb + p_l \; \overline{F}_l$$

so that

$$\frac{\bar{F}_l}{sb} = \frac{\frac{P_x}{sb} - p_1}{p_l} = \frac{1440 - 335}{1200} = 0.92 \tag{9.1}$$

i.e.

$$\bar{F}_l = 0.92\ s\ b = 0.967\ \text{cm}^2$$

We assume that among the stiffener sections there are available[10], those of area $F_l = 1\ \text{cm}^2$. We then find on intersecting the curve $\tau/p^* = 5$ in figure 3, with the straight line:

$$p_1\ \frac{sb}{F_l} + p_l = \frac{P_x}{F_l}$$

that is,

$$1.04\ \frac{p_1}{p^*} + \frac{p_l}{p^*} = 15 \tag{9.2}$$

The points

$$\left.\begin{array}{ll} p_1/p^* = 3.22, & p_l/p^* = 11.65 \\ p_2/p^* = -\ 4.25, & \tau/G\gamma = 0.59 \end{array}\right\} \tag{9.3}$$

The values (9.3) were obtained from figure 3 that was computed under the assumption $\epsilon_2 = 0$, that is, rigid transverse stiffeners. In general, this assumption will not be far from the true condition, but it may nevertheless appear desirable at least to estimate the effect of yielding stiffeners. This is possible with the aid of figure 2. Intersecting the curve $\tau/p^* = 5$ in figure 2, with the straight line (9.2), we find:

$$\frac{p_1}{p^*} = -\ 0.5, \quad \frac{p_l}{p^*} = 15.5, \quad \frac{\epsilon_2}{\epsilon^*} = 147, \quad \frac{\tau}{G\gamma} = 0.14 \tag{9.4}$$

---

[10] We shall assume that the computation is on a series of sheet panels so that for each panel there is computed only one stiffener. If the computation is on a single panel, then in all formulas F_l includes the sum of both transverse stiffeners.

Actually the transverse stiffeners are neither ideally rigid nor yielding but we can, nevertheless, obtain the actual stress and deformation condition of the sheet if we allow an additional external load to act in the transverse direction. The condition of equilibrium between the stiffener stress p_q, the mean sheet stress p_s, and the external stress p_y is:

$$p_y = \frac{p_q F_q + p_s sa}{F_q + sa} = \frac{p_q + p_s \frac{sa}{F_q}}{1 + \frac{sa}{F_q}} \qquad (9.5)$$

If the condition $\epsilon_s = 0$, that is, vanishing compressive stress in the transverse stiffeners (as is assumed in fig. 3) is attained, then in order to offset the transverse tensile force of the sheet it is necessary to apply a stress:

$$(\sigma_y)_{III} = (-p_y)_{III} = - p_s \frac{sa/F_q}{1 + \frac{sa}{F_q}}$$

If, however, the sheet remains, on the average, free from stress in the transverse direction $(p_s = 0)$, then an external pressure:

$$(p_y)_{II} = \frac{p_q}{1 + \frac{sa}{F_q}} = E \frac{\epsilon_s}{1 + \frac{sa}{F_q}}$$

must be applied in order to produce the compressive strain ϵ_s, obtained from figure 2, in the transverse stiffeners.

In the example we obtain with $a = 250$ mm, $F_q = 1 \text{ cm}^2$ $\left(\text{i.e., } \frac{sa}{F_q} = 2\right)$

$$(\sigma_y)_{III} = 2.8 \; p^*$$

$$(p_y)_{II} = 49 \; p^*$$

Actually $p_y = 0$, and if we make the approximating assumption that it is permissible to interpolate linearly then by "averaging" we obtain finally:

$$\frac{\tau}{G\gamma} = \frac{0.59 \times 49 + 0.14 \times 2.8}{49 + 2.8} = 0.566$$

$$\frac{p_1}{p^*} = \frac{3.22 \times 49 - 0.5 \times 2.8}{51.8} = 3.02$$

$$\frac{p_l}{p^*} = \frac{11.65 \times 49 + 15.5 \times 2.8}{51.8} = 11.9$$

As was to be expected, the values do not deviate much from those taken from figure 3, so that in most cases the interpolation may be dispensed with.

2. As a second example, we choose a case of pure shear stress:

$$\tau s = 60 \text{ kg cm}, \quad \tau_{al} = 750 \text{ kg/cm}^2$$

so that $s = 0.8$ mm, and with $b = 130$ mm, we have:

$$p^* = 100 \text{ kg/cm}^2$$

If we take, as in the first example, $F_l = 1 \text{ cm}^2$, we find at the point of intersection of a straight line of slope $\frac{1.00}{1.04} = 0.96$ through the origin, from figure 3, the values:

$$\left.\begin{aligned} \frac{p_1}{p^*} &= -2.35, & \frac{p_l}{p^*} &= 2.45 \\ \frac{p_2}{p^*} &= -4.65, & \frac{\tau}{G\gamma} &= 0.76 \end{aligned}\right\} \qquad (9.6)$$

and from figure 2

$$\left.\begin{aligned} \frac{p_1}{p^*} &= -11.1, & \frac{p_l}{p^*} &= 11.6 \\ \frac{\epsilon_2}{\epsilon^*} &\approx 850; & \frac{\tau}{G\gamma} &= 0.115 \end{aligned}\right\} \qquad (9.7)$$

Linear interpolation gives:

$$\frac{\tau}{G\gamma} = \frac{0.76 \times 120 + 0.115 \times 3.1}{123} = 0.745$$

$$\frac{p_1}{p^*} = -2.57, \quad \frac{p_l}{p^*} = 2.68$$

The computation thus far was valid for the inner panel of a series of sheet panels for which the longitudinal stiffeners may be considered as remaining straight. In the case of the end panels, it will not be found possible even when the outside stiffeners are made strong, to prevent the edges from bending under the effect of the transverse stress $\sigma_s = - p_s$. In the same manner as the axial elasticity of the transverse stiffeners, the effect of the bending elasticity of the longitudinal stiffeners may be approximately determined. If we consider the stringer between two transverse frames as a beam clamped at the two sides under constant lateral load, there is obtained for the mean value of the deflections v by the known formulas:

$$v_m = \frac{1}{a}\int_0^a v\,dx = p_s\,\frac{s\,a^4}{720\,E\,J}$$

This deflection we shall consider as having been offset by an external force. If the inner longitudinal stiffeners are so weak that the contraction due to v may be taken as uniform in all of the panels, then the stress to be applied is (assuming two equal outer members):

$$p_y = E\,\frac{v_m}{nb/2}\,\frac{F_q}{F_q + sa}$$

In general, however, the inner longitudinal stiffeners will not be ideally flexible in bending and the inner panels can take part only imperfectly in the deformation. It is safe (with respect to the outer panel) to assume that the outer panel must balance the yielding of the stiffeners alone. There will then be a stress:

$$(\sigma_y)_I = - E\,\frac{v_m}{b}\,\frac{F_q}{F_q + sa} = - p_s\,\frac{s\,a^4}{720\,J\,b}\,\frac{F_q}{F_q + sa}$$

which is to be added to the above determined stress $(\sigma_y)_{III}$ in order that the condition $\epsilon_s = 0$ in the outer panel (at least in the mean over the length a) may be set up. The rule according to which the linear interpolation for magnitude $\xi\left(\frac{T}{V},\ p_1,\ \text{etc.}\right)$ between the results ξ_3 and ξ_s from the two charts is to be made, is therefore:

For an inner panel: $$\xi = \frac{\xi_3 (p_y)_{II} + \xi_2 (\sigma_y)_{III}}{(p_y)_{II} + (\sigma_y)_{III}} \qquad (9.7)$$

For an outer panel: $$\xi = \frac{\xi_3 (p_y)_{II} + \xi_2 (\sigma_y)_{III'}}{(p_y)_{II} + (\sigma_y)_{III'}} \qquad (9.8)$$

where $(\sigma_y)_{III'}$, with $\sigma_2 = - p_2$ is obtained from:

$$(\sigma_y)_{III'} = \sigma_2 \frac{s\ a}{F_q + sa} \left\{1 + \frac{a^3\ F_q}{720\ J\ b}\right\} \qquad (9.9)$$

The moment of inertia J of the edge bars is determined from the condition that the bending stress $\tilde{\sigma}_{max} = \frac{M_{max}}{W}$ must not exceed a certain limit $\tilde{\sigma}_{al}$. If $h/2$ denotes the distance of the extreme fibers from the neutral axis, then

$$J \geqslant \frac{h}{2} \frac{M_{max}}{\tilde{\sigma}_{al}} = \frac{\sigma_2}{\tilde{\sigma}_{al}} \frac{s\ a^2\ h}{24}$$

If desired, this value for J may be substituted in (9.9), and there is then obtained:

$$(\sigma_y)_{III'} = (\sigma_y)_{III} + \tilde{\sigma}_{al} \frac{a^2}{30\ h\ b} \frac{F_q}{F_q + s\ a}$$

a relation which, with given dimensions a, h, b is very convenient. In our first numerical example with $\tilde{\sigma}_{al} = 500\ kg/cm^2$ and $h = 5$ cm, the value of the "bending contribution" becomes 53 kg/cm^2, so that it is not negligible to the same extent as $(\sigma_y)_{III}$.

X. SUMMARY

The elastic behavior of a simply supported plate strip under shear and compressive loading above the buckling limit is investigated in the present report. The investigation of this range was carried out with the aid of the energy method. The main results obtained are presented in the form of a chart (fig. 3), the use of which for practical application purposes is explained with the aid of two computed examples. The curves give the relation between the mean skin stress p_1 and the longitudinal stiffener stress p_l (under the assumption that the longitudinal members are not displaceable in the transverse direction) for various values of the shear τ. The chart contains, besides the reduced "mean" shear modulus τ/γ, the mean stress $\sigma_2 = - p_2$, with which the sheet acts laterally on the transverse stiffeners and for individual points the geometrical magnitudes β^2 (b/β = wave separation) and $t = \frac{t}{\sin^2 \alpha}$ (α = wave inclination). The reference pressure p^* is taken as the critical stress for the hinge-supported sheet under pure axial compression $= \frac{E}{1 - \nu^2} \frac{\pi^2 s^2}{3b^2}$. Figure 3 takes account of the practically important range between the critical load and the load about twenty times in excess of the critical. With given external (shear and compressive) load, it is possible by its aid to determine either the stresses when the cross-sectional areas are given, or the required cross sections when the maximum stresses are prescribed. Figure 2 gives the values p_1, τ/γ, and ϵ_2 for various values of the shear τ for the other limiting case of yielding transverse stiffeners ($p_2 = 0$). It serves (in the manner described in section IX) to take into account the compressive and shear elasticity and the bending elasticity of the longitudinal stiffeners. In many cases it will be possible for a first approximation to dispense with this refinement. Figures 4 to 11 show the variation of the effective width p_1/p_l, the reduced "instantaneous" shear modulus $d\tau/d\gamma$ and the wave inclination angle α, for several particular loading cases.

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

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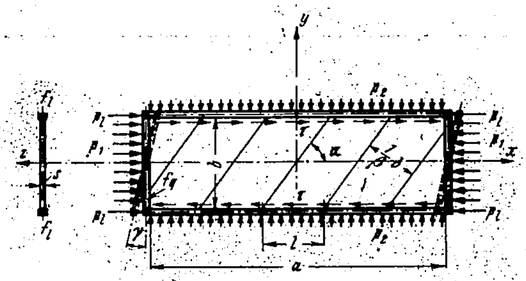

Figure 1.- Plate strip under shear and compressive stress.

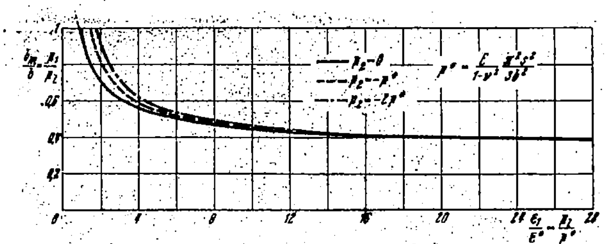

Figure 4.- The effective width p_1/p_l as a function of the excess load beyond the buckling load (pure compressive stress).

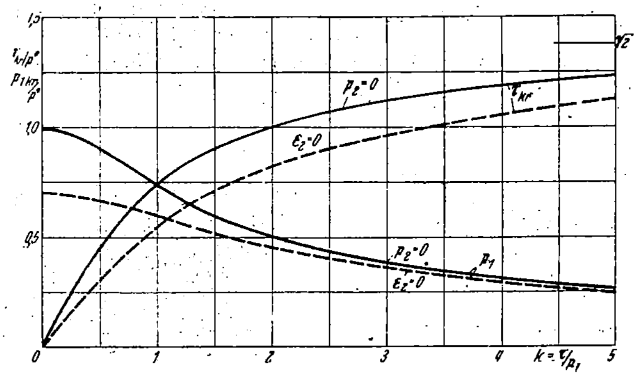

Figure 5.- The critical values τ, p_1 as functions of τ/p_l ($\nu = 0.3$).

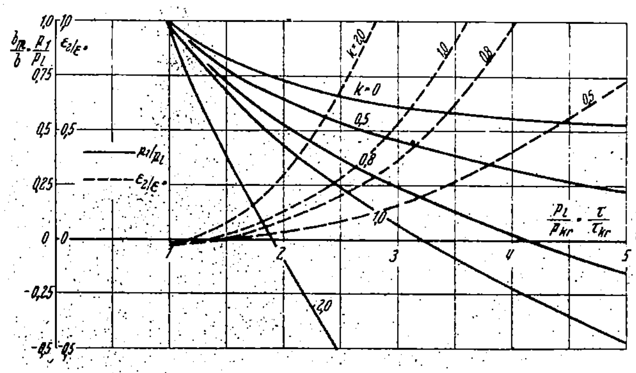

Figure 6.- The effective width p_1/p_l as a function of p_l/p_{kr} ($= \tau/\tau_{kr}$) for various values of the parameter $k = \tau/p_l$, for the case $p_2 = 0$.

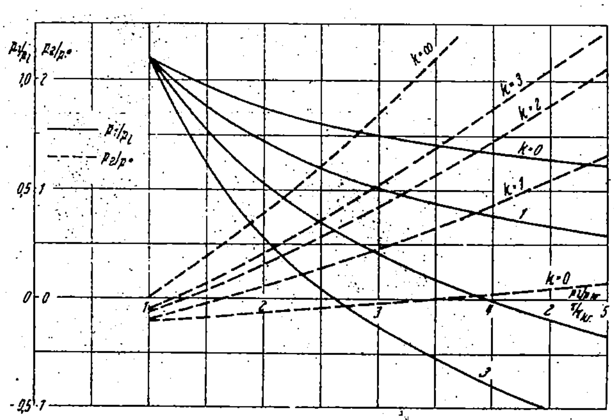

Figure 7.- The effective width p_1/p_l as a function of p_l/p_{kr} ($= \tau/\tau_{kr}$) for various values of the parameter $k = \tau/p_l$, for the case $\epsilon_2 = 0$.

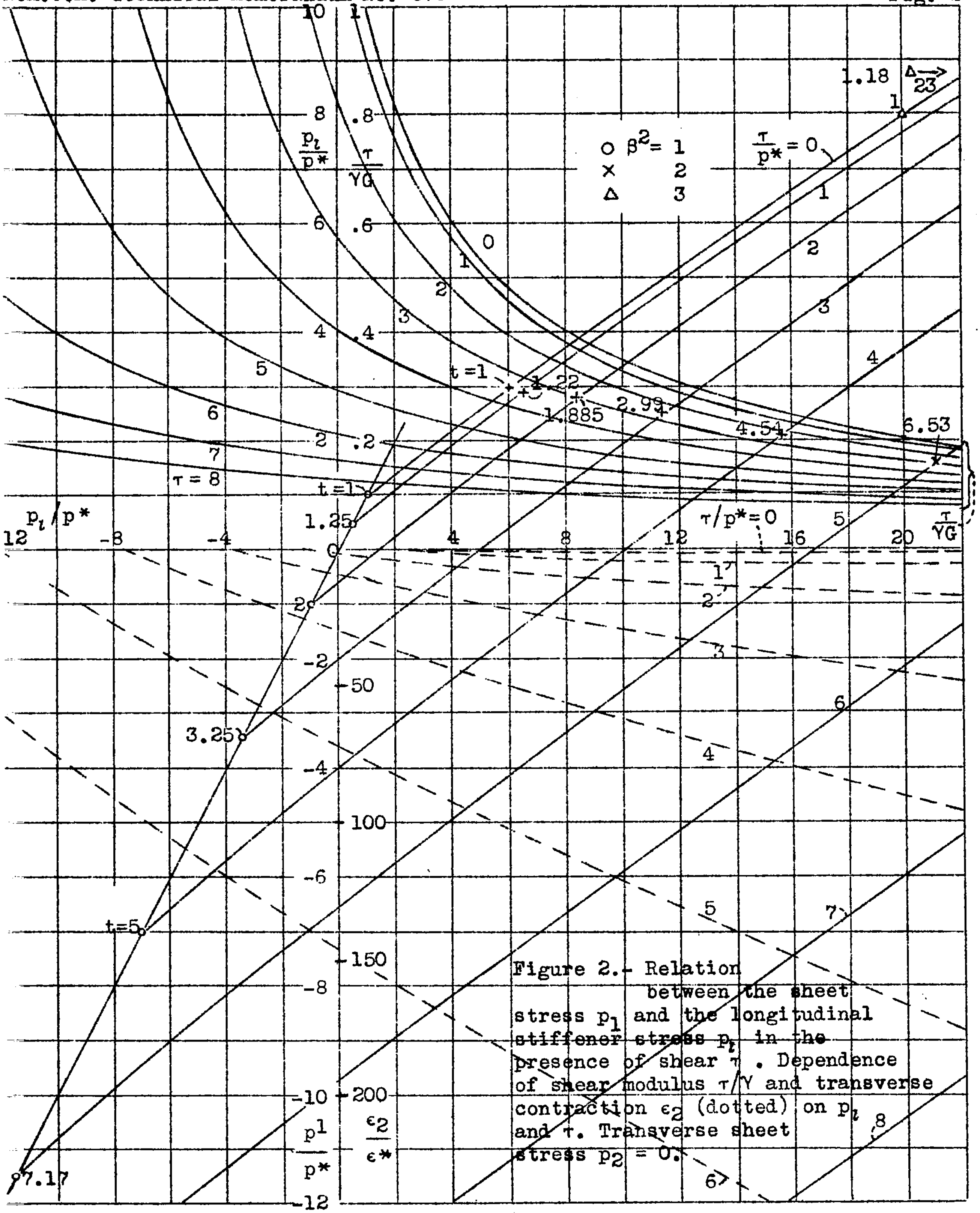

Figure 2.- Relation between the sheet stress p_1 and the longitudinal stiffener stress p_l in the presence of shear τ. Dependence of shear modulus τ/γ and transverse contraction ϵ_2 (dotted) on p_l and τ. Transverse sheet stress $p_2 = 0$.

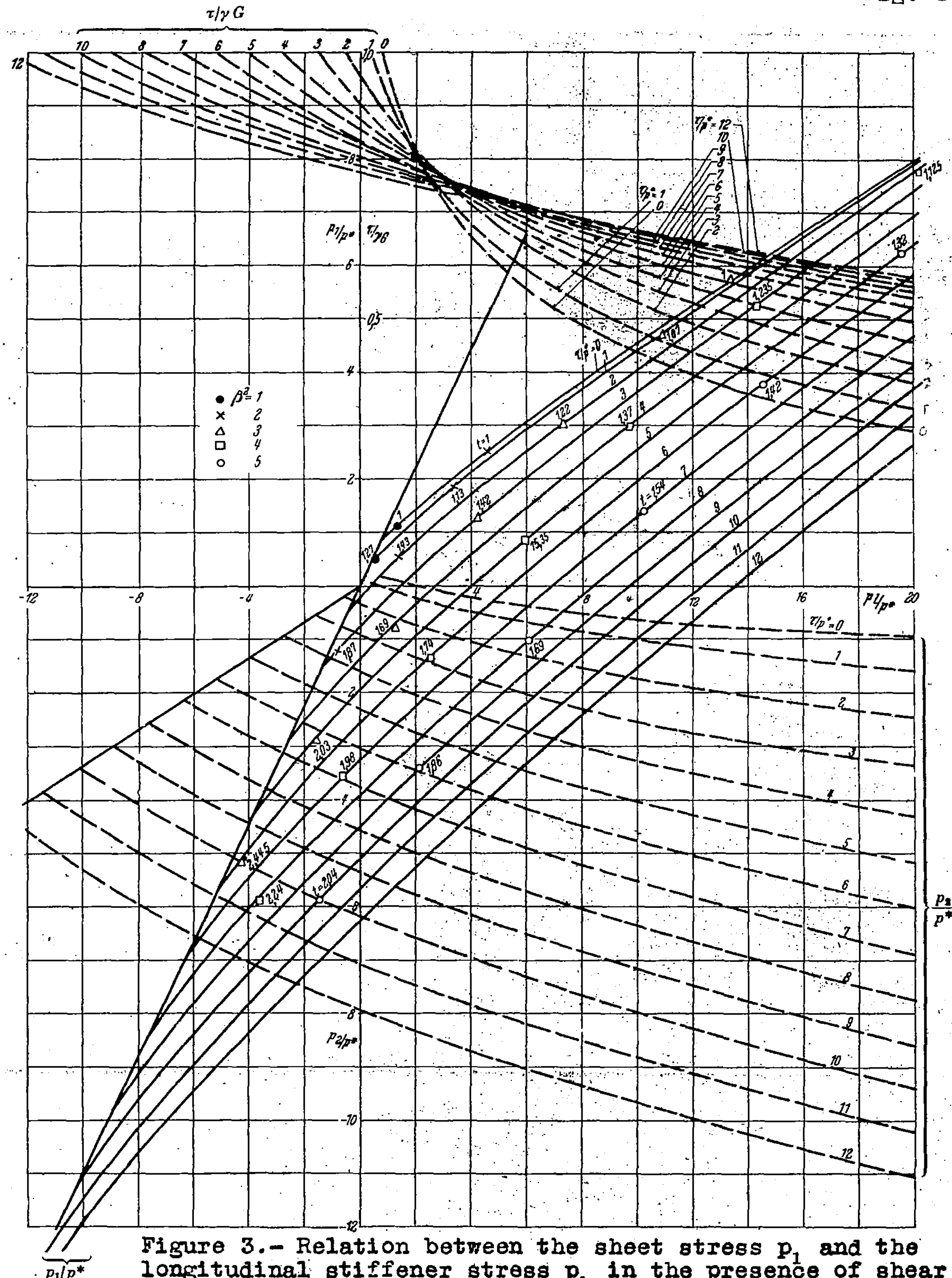

Figure 3.- Relation between the sheet stress p_1 and the longitudinal stiffener stress p_l in the presence of shear τ. Dotted curves give shear modulus τ/γ and transverse sheet stress p_2 as functions of p_l and τ. Transverse contraction $\epsilon_2 = 0$.

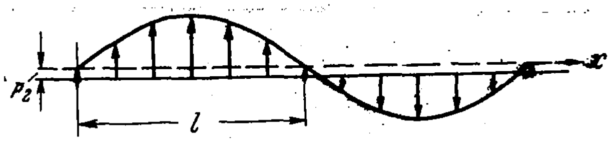

Figure 8

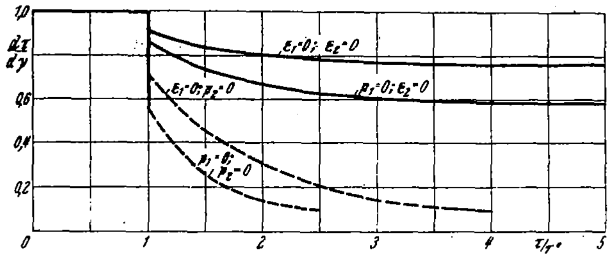

Figure 9.- Reduced "instantaneous" shear modulus $d\tau/d\gamma$ for pure shear stress for the four limiting cases

a) $\varepsilon_1=\varepsilon_2=0$, b) $p_1=\varepsilon_2=0$, c) $\varepsilon_1=p_2=0$, d) $p_1=p_2=0$.

$$\tau^*=\sqrt{2}\,p^*=E\,\frac{\sqrt{2}}{1-\nu^2}\,\frac{\pi^2 s^2}{3b^2}.$$

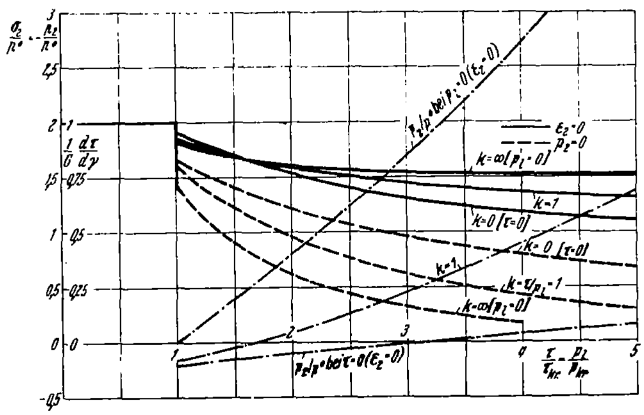

Figure 10.- $d\tau/d\gamma$ as a function of the excess stress beyond the critical for various values of $k = \tau/p_l$.

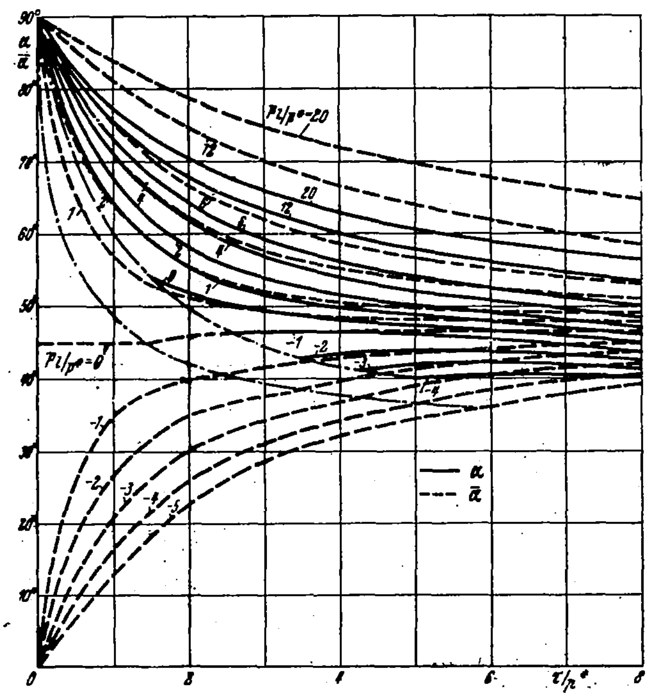

Figure 11.- Wave inclination angle α and principal stress direction $\bar{\alpha}$ as functions of τ and p_l ($\epsilon_2 = 0$).