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TWISTING OF THIN WALLED COLUMNS PERFECTLY RESTRAINED AT ONE END

By Lucio Lazzarino

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SUMMARY

Proceeding from the basic assumptions of the Batho-Bredt theory on twisting failure of thin-walled columns, the discrepancies most frequently encountered are analyzed. A generalized approximate method is suggested for the determination of the disturbances in the stress condition of the column, induced by the constrained ~~wrinkling~~ warping in one of the end sections.

INTRODUCTION

The evaluation of the stresses induced by the application of a twisting moment in a structure consisting essentially of thin-walled cylindrical tubing and of arbitrary straight section, is generally carried out with the simple and easily applied formulas given in the well-known Batho-Bredt approximate theory.

The basic assumptions of this theory are as follows:

1. The wall thickness of the tube is thin enough to render the variations in unit stress along the perpendiculars to the walls negligible.

2. The shape of the straight section of the tube is preserved during the strain induced by the application of the twisting moment.

3. The sections of the tube including those at the end are so restrained as to offer no resistance to ~~wrinkling;~~ warping the displacements of points of the tube in direction of the generating axes of the cylindrical surface being freely permitted.

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*"Sulla Torsione di Tubi a Pareti Sottili Perfettamente Incastrati ad una Estremita." L'Aerotecnica, vol. XVII, no. 10, October 1937, pp. 850-861.

While the first two assumptions may be held to be confirmed with sufficient approximation in the majority of practical cases, the third is far from it - at least one of the end sections being joined to the other much more rigid structure or otherwise restrained in such a way that its wrinkling is not permitted. The constrained wrinkling in one of the end sections of the column obviously produces stresses in the direction of the generating axes of the column, the values of which certainly cancel in the free-end section.

On structures whose form would introduce considerable wrinkling, such as those with considerably elongated sections, the previously cited stresses may assume dangerous proportions in certain points, especially if accompanied by stresses due to other concomitant causes (bending, for example).

Hence the importance, in certain cases, attaching to the evaluation - even approximate - of the stresses induced by constrained wrinkling in one of the end sections of the structure. For convenience of treatment, it is admitted that the stress conditions created when applying a torque on a column with one end perfectly restrained, may be considered as the sum of the stress condition as defined by the Batho-Bredt theory and a secondary stress condition, which nullify in the restrained section the wrinkling pertinent to the primary stress condition.

The secondary unit stresses at any section of the column must have zero resultant and zero resultant moment.

According to the Batho-Bredt approximate theory, unitary parallel tangential stresses exist in the plane of a generic section of the column, normal to the generators, in any part of the median contour of the section, and having the value:

$$\tau = \frac{M}{2S\ \delta} \tag{1}$$

where M is the applied torque, S the area enclosed by the median contour of the section, and δ the wall thickness at the point in question.

It should be noted that the section shape has no effect on the value of τ. This, however, does not hold for the displacements of the points of the column induced by that stress condition.

From the Batho-Bredt theory, it is also seen that all straight sections of the column wrinkle equally, and turn, with the rotation of a rigid body, in relation to each other.

The coordinates of a point of the column wall are:

The distance z from the plane of one of the end sections which we suppose to be perpendicular to the generators of the walls;

The length s of the arc of the median contour of the straight section to which the point in question belongs, contained between it and any generator assumed as origin.

To simplify the writing of the formulas, the unit length is chosen so that the perimeter of the median contour of the section of the column is equal to 2π.

Let w and t represent the components of the displacement of a generic point of the wall with respect to z and with respect to the tangent to the median contour of the straight section of the column at the point under consideration, visualized in the position preceding the application of torque.

The relation of general character is verified in each point:

$$\frac{\partial w}{\partial s} + \frac{\partial t}{\partial z} = \frac{\tau}{G} \tag{2}$$

where G is the modulus of tangential elasticity of the material. Let θ be the angle through which a generic section has rotated in relation to its initial position under the effect of the Batho-Bredt stress condition; further, let r represent the length of the segment contained between a generic point of the wall and the point O about which the straight section of the column, to which the considered point belongs, has rotated. Lastly, let α represent the angle formed by this segment with the tangent to the wall at the point in question.

From what has already been said regarding the nature of the strain induced by the stress condition, according to the Batho-Bredt theory, follows (fig. 1):

$$\frac{\partial t}{\partial z} = r \sin \alpha \frac{d\theta}{dz} \tag{3}$$

where

$$\frac{d\theta}{dz} = \frac{M}{4G\ S^2} \int_0^{2\pi} \frac{ds}{\delta} \tag{4}$$

as the cited theory stipulates.

By establishing in the plane of the straight section of the column a system of orthogonal cartesian axes x, y, and z, and assuming as positive direction on the arc s, that which brings axis x on axis y, equation (3) may be written in the form:

$$\frac{\partial t}{\partial z} = \frac{d\theta}{dz} \left[(x - a) \frac{dy}{ds} - (y - b) \frac{dx}{ds} \right] \tag{3a}$$

where a, b are the coordinates of O with respect to the chosen reference.

The coordinates a, b cannot be determined on the basis of the Batho-Bredt theory, inasmuch as this, as has been shown, prescinds from the shape of the straight section of the column, upon which a and b depend.

The position of point O, about which the section in fact rotates, depends in reality on the limiting conditions which must comply with the condition of strain.

Those requirements of the Batho-Bredt approximate theory are insufficient to determine the position of O. For different positions of O, then, we can always obtain the stress condition defined by that theory. From among those, any one may be chosen, imposing on the strain condition wholly arbitrary supplementary conditions which, however, must conform to the assumptions upon which the above-mentioned theory is based. Thereupon also, essentially are based the various attempts made by several authors to determine the center of twist of columns in torsion patterned after the Batho-Bredt method.

Here the supplementary conditions which are to satisfy the principal condition of strain are prescribed by the general consideration of the method adopted for the representation of the total stress condition; a and b, however, assume values which nullify the components of the resultant bending moment of the secondary stresses acting along the column generators which reflect the end section, twisted under the effect of the Batho-Bredt stress condition, in the plane in which it is restrained.

Substituting the expression supplied by equation (3a) in equation (2), we have:

$$\frac{dw}{ds} = \frac{\tau}{G} + \frac{d\theta}{dz}\left[(y-b)\,\frac{dx}{ds} - (x-a)\,\frac{dy}{ds}\right] \qquad (5)$$

where $\frac{dw}{ds}$ is given in function of the known quantity τ supplied by equation (1), equation (4), and by the conformation of the straight section of the column as $\frac{dx}{ds}$ and $\frac{dy}{ds}$, and from the coordinates a, b of the still unknown point O.

Let us consider the rectangular flat plate of thickness δ, now assumed to be constant, obtained by cutting the column wall along a generating axis and developed in plane.

If the mass forces and the external pressures acting on the planes bounding the plate are everywhere zero, the unit stresses set up in the plate by any system of forces, acting in its median plane and however applied along its edges, may be expressed in the following form (reference 1):

$$\left.\begin{aligned} \sigma^*_s\ (s_1 z) &= \frac{\partial^2 F(s,z)}{\partial z^2} \\ \sigma^*_z\ (s_1 z) &= \frac{\partial^2 F(s,z)}{\partial s^2} \\ \tau^*\ (s_1 z) &= \frac{\partial^2 F(s,z)}{\partial s\,\partial z} \end{aligned}\right\} \qquad (6)$$

σ^*_s, σ^*_z are the normal stresses in direction of axes s and z, and τ^* is the tangential stress (axes z, s being disposed parallel to the edges of the plate of length 2π and l, respectively, and corresponding to the development of the median contour of one of the end sections of the column and to the section separated along the generating axis $s = 0$).

If the condition is imposed which must be:

$$\left.\begin{aligned} \sigma^*_s\ (0,\bar{z}) &= \sigma^*_s\ (2\pi,\bar{z}) \\ \sigma^*_z\ (0,\bar{z}) &= \sigma_z\ (2\pi,\bar{z}) \\ \tau^*\ (0,\bar{z}) &= \tau\ (2\pi,\bar{z}) \end{aligned}\right\} \qquad (7)$$

where $\bar{z}$ is any value of z between 0 and l, the functions $\sigma^*_s\ (s,\bar{z})$, $\sigma^*_z (s,\bar{z})$, $\tau^*\ (s,\bar{z})$ can be expanded in Fourier series.

We can, for example, express $\sigma^*_z\ (s,\bar{z})$ in the form of

$$\sigma^*_z\ (s,\bar{z}) = \Sigma_n\ A_n\ (\bar{z})\ \sin ns + B_n\ (\bar{z})\ \cos ns \qquad (8)$$

where the values of the coefficients $A_n\ (\bar{z})$ and $B_n\ (\bar{z})$ depend on the value $\bar{z}$. Then the whole plate can be expressed as:

$$\sigma^*_z\ (s,z) = \Sigma_n\ A_n\ (z)\ \sin ns + B_n\ (z)\ \cos ns \qquad (9)$$

with A_n, B_n as functions of z.

The last expression twice integrated with respect to s gives:

$$F(s,z) = s\,\Phi(z) + \psi(z) + \Sigma_n \left(-\frac{A_n}{n^2}\sin ns - \frac{B_n}{n^2}\cos ns\right) \qquad (10)$$

Φ and ψ both being functions of z only.

Substituting this last expression in the first and third of equation (6), we obtain:

$$\left.\begin{aligned} \sigma^*_s &= s\,\Phi'' + \psi'' + \Sigma_n \left(-\frac{A_n''}{n^2}\sin ns - \frac{B_n''}{n}\cos ns\right) \\ \tau^* &= \Phi' + \Sigma_n \left(-\frac{A_n'}{n}\cos ns + \frac{B_n'}{n}\sin ns\right) \end{aligned}\right\} \qquad (11)$$

The function $\sigma^*_s(s,\overline{z})$ must be periodic with respect to s, whatever the value of $\overline{z}$, whence equation (11) affords:

$$\Phi\ (z) = H\ z + K \tag{12}$$

where H and k are constants.

Function F(s,z) obviously satisfies (Maxwell's) differential equation:

$$\frac{\partial^4 F}{\partial s^4} + \frac{\partial^4 F}{\partial z^4} + 2\frac{\partial^4 F}{\partial s^2\ \partial z^2} = 0 \tag{13}$$

obtainable from the consideration of the equilibrium of an element ds dz of the plate.

Substituting the values derived from equation (10) in this last equation affords:

$$\psi^{IV}(z) + \Sigma_n \left[\left(2A_n'' - \frac{A^{IV}{}_n}{n^2} - n^2\ A_n\right) \sin ns + \right.$$

$$\left. + \left(2B_n'' - \frac{B^{IV}{}_n}{n^2} - n^2\ B_n\right) \cos ns\right] = 0 \tag{14}$$

this equation, being satisfied for any value of s, should have at the same time for any value of n (whole) and z:

$$\left.\begin{aligned} &\psi^{IV} = 0 \\ &A^{IV}{}_n - 2n^2\ A_n'' + n^4\ A_n = 0 \\ &B^{IV}{}_n - 2n^2\ B_n'' + n^4\ B_n = 0 \end{aligned}\right\} \tag{15}$$

The general integrals of this differential equation are:

$$\left.\begin{aligned} \psi &= M\ z^3 + N\ z^2 + P\ z + Q \\ A_n &= (\alpha_n + \beta_n z) \cosh nz + (\xi_n + \eta_n\ z) \sinh nz \\ B_n &= (\varphi_n + \chi_n z) \cosh nz + (\Upsilon_n + \omega_n\ z) \sinh nz \end{aligned}\right\} \tag{16}$$

where the quantities M, N, P, Q, α_n, β_n, ξ_n, η_n, φ_n, χ_n, Υ_n, ω_n are constants and depend upon the limiting condition which must satisfy the stress and strain condition of the plate.

Such conditions are:

$$\left.\begin{aligned} \sigma_z^* (s,0) &= 0 \\ \tau^* (s,0) &= 0 \end{aligned}\right\} \quad \text{for} \quad z = 0 \tag{17}$$

$$\left.\begin{aligned} t^* (s,l) &= 0 \\ w^* (s,l) &= - w(s) \end{aligned}\right\} \quad \text{for} \quad z = l \tag{18}$$

$w(s)$ being defined by equation (5), except that one constant which is unimportant need not be determined.

From the first condition of equation (17) by virtue of (9) and (16) follows:

$$\left.\begin{aligned} \alpha_n &= 0 \\ \varphi_n &= 0 \end{aligned}\right\} \tag{19}$$

and from the second, while keeping in mind the second of equations (11), (12), (16), and (9):

$$\left.\begin{aligned} H &= 0 \\ \xi_n &= - \frac{\beta_n}{n} \\ \Upsilon_n &= - \frac{\chi_n}{n} \end{aligned}\right\} \tag{20}$$

From the general relation:

$$\frac{\partial t}{\partial s} = \frac{1}{E} \left(\sigma_s - \frac{\sigma_z}{m}\right) \tag{21}$$

(E = Young's modulus, $\frac{1}{m}$ = Poisson's ratio) and from the first condition of equation (18) evaluated as before from the expressions σ^*_s and σ^*_z supplied by equation (9) and from the first of equation (11), we obtain, while allowing for (16), (9) and (20):

$$\left.\begin{aligned} N &= -3M\,l \\ \beta_n &= R_n\,\eta_n \\ X_n &= R_n\,\omega_n \end{aligned}\right\} \qquad (22)$$

Hence,

$$R_n = \frac{\frac{2}{n} + \frac{m+1}{m}\,l\,\tanh n\,l}{\frac{1-m}{m\,n}\,\tanh n\,l - \frac{m+1}{m}\,l} \qquad (23)$$

a quantity easily determined.

Deducing equation (2) with respect to s, and then substituting the second complex derivative of t supplied by means of the derivation relative to z of equation (21), the formula assumes the general character of

$$\frac{\partial^2 w}{\partial s^2} = \frac{1}{G}\,\frac{\partial \tau}{\partial s} - \frac{1}{E}\,\frac{\partial \sigma_s}{\partial z} + \frac{1}{Em}\,\frac{\partial \sigma_z}{\partial z} \qquad (24)$$

Substituting in this last relation the values obtained for $z = l$, deriving the ratio of z (equation 9), the first part of equation (11), and the second part of equation (11) with respect to s, and utilizing the conditions imposed by the second of equation (18) the coefficients η_n, ω_n - and consequently, with equations (20) and (22) - the values of the coefficients β_n, X_n, ξ_n, and Y_n can be determined.

Carrying out the operations indicated above and developing equation (5) in Fourior series, putting

$$\frac{dw(s)}{ds} = \Sigma_n \; (\mu_n' + \mu_n'' \, a + \mu_n''' \, b) \sin n\,s +$$
$$(\nu_n' + \nu_n'' \, a + \nu_n''' \, b) \cos n\,s \qquad (25)$$

we find:

$$
\left.
\begin{aligned}
M &= 0 \\
\eta_n &= \frac{+\ (\nu_n{}' + \nu_n{}''\ a + \nu_n{}'''\ b)}{T_n} \\
\chi_n &= \frac{-\ (\mu_n{}' + \mu_n{}''\ a + \mu_n{}'''\ b)}{T_n}
\end{aligned}
\right\} \qquad (26)
$$

where

$$
T_n = \frac{1}{E}\left[\left(R_n\ l\ \frac{3m + 3}{m} + \frac{5m + 3}{m\ n}\right) \sinh n\ l + \right.
$$

$$
\left. + \left(\frac{3m + 3}{m}\ l + \frac{2R_n}{n}\right) \cosh n\ l\right] \qquad (27)
$$

It is readily seen that the stress condition in the flat plate thus defined differs very little from the secondary stress condition in the column.

The limiting conditions manifestly coincide, so that they are assigned to the state of stress of the plate relative to the stress condition of the column.

Now consider a distance (or length) of the cylindrical wall of the column of depth dz and length s_0; $\epsilon\ (s)$ denoting the angle of the tangent to the median contour of the straight section of the column at any point of the arc considered and the normal to it at the origin of this arc, the expression of the equilibrium of displacement according to this last direction of the path of the previously defined cylindrical wall, and assumedly constant thickness δ:

$$
\sigma_s^*\ \delta\ \cos\ \epsilon(s_0)\ dz = \delta \int_0^{s_0} \frac{\partial \tau^*}{\partial z}\ \cos\ \epsilon(s)\ ds\ dz \qquad (28)
$$

If, in this relation for σ_s and τ^* the values given in equation (6) for the flat plate are substituted, we will have after dividing both terms by dz:

$$
\delta\ \frac{\partial^2 F}{\partial z^2}\ \cos\ \epsilon(s_0) = \delta \int_0^{s_0} \frac{\partial^3 F}{\partial s\, \partial z^2}\ \cos\ \epsilon(s)\ ds \qquad (29)
$$

which, in reality, is identical with

$$\delta \frac{\partial^2 F}{\partial z^2} \cos \epsilon(\bar{s}) = \delta \int_0^{\bar{s}} \frac{\partial^3 F}{\partial S \partial z^2} \cos \epsilon \, ds - \delta \int_0^{\bar{s}} \frac{\partial^2 F}{\partial z^2} \frac{d\epsilon}{dS} \sin \epsilon \, ds \tag{30}$$

However, having disregarded in equation (29) the term

$$\delta \int_0^{s_0} \frac{\partial^2 F}{\partial z^2} \frac{d\epsilon}{dS} \sin \epsilon \, ds = \int_0^{s_0} \frac{\sigma^*_z \delta}{\rho(s)} \sin \epsilon \, ds \tag{31}$$

where $\rho(s)$ is the radius of curvature of the cylindrical wall, the approximation of equation (29) is so much greater as the mean value of the ratio δ/ρ is less.

In practice this ratio is generally quite small, hence the error committed by substituting for the secondary stress condition produced in the column wall as the result of restrained wrinkling in one of the end sections, the corresponding stress condition of the flat plate, is practically negligible.

The stress condition of the column is given through equations (9), (11), (16), (19), (20), (22), (23), (26), and (27) in function of the known quantity and of the still unknown coordinates a, b of O.

The last values can be determined by evaluating the condition which stipulates that the resultant bending moment of σ^*_z acting on the restrained section must be zero.

Taking into consideration the relations (9), (16), (19), (20), (22), and (26), this condition can be expressed in the form:

$$\left.\begin{aligned}
&\int_0^{2\pi} x\,\Sigma_n\,V_n\,(l)\,\left[(\mu_n' + \mu_n''\,a + \mu_n'''\,b)\cos ns - \right.\\
&\qquad\qquad \left. - (\nu_n' + \nu_n''\,a + \nu_n'''\,b)\sin ns\right] ds\\
&\int_0^{2\pi} y\,\Sigma_n\,V_n\,(l)\,\left[(\mu_n' + \mu_n''\,a + \mu_n'''\,b)\cos ns - \right.\\
&\qquad\qquad \left. - (\nu_n' + \nu_n''\,a + \nu_n'''\,b)\sin ns\right] ds
\end{aligned}\right\} \quad (32)$$

where

$$V_n\,(l) = \frac{R_n\,l\cosh n\,l + \left(l - \frac{R_n}{n}\right)\sinh n\,l}{T_n} \quad (33)$$

From equation (32) we can now deduce the values a, b of the coordinates of O, and so determine the secondary condition of stress of the column.

Summing up, having obtained the values a, b from equation (32), and posting

$$\left.\begin{aligned}
\mu_n &= \frac{\mu_n' + \mu_n''\,a + \mu_n'''\,b}{T_n}\\
\nu_n &= \frac{\nu_n' + \nu_n''\,a + \nu_n'''\,b}{T_n}
\end{aligned}\right\} \quad (34)$$

the final expressions of the stresses existing in the column can be written as follows:

$$\sigma_z = \sigma_z^* = \Sigma - \mu_n \left[R_n\ z \cosh nz + \left(z - \frac{R_n}{n} \right) \sinh nz \right] \cos ns +$$

$$+ \nu_n \left[R_n\ z \cosh nz + \left(z - \frac{R_n}{n} \right) \sinh nz \right] \sin ns$$

$$\sigma_s = \sigma_s^* = \Sigma_n - \nu_n \left[\left(\frac{2}{n} + R_n\ z \right) \cosh nz + \right.$$

$$+ \left(\frac{R_n}{n} + z \right) \sinh nz \Big] \sin ns + \mu_n \Big[\left(\frac{2}{n} + R_n\ z \right) \cosh nz +$$

$$\left. + \left(\frac{R_n}{n} + z \right) \sinh nz \right] \cos ns$$

$$\tau = \frac{M}{2S\delta} + \tau^* = \frac{M}{2S\delta} - \Sigma_n\ \nu_n \Big[z \cosh nz +$$

$$+ \left(R_n\ z + \frac{1}{n} \right) \sinh nz \Big] \cos ns + \mu_n \Big[z \cosh nz +$$

$$+ \left(R_n\ z + \frac{1}{n} \right) \sinh nz \Big] \sin ns \qquad (35)$$

where the quantities μ_n, ν_n, and R_n are given in equations (34) and (23).

So far, the thickness δ has been assumed constant but by way of approximation the obtained findings may be extended to include the case of variable δ.

Equations (6) and (13), which are the basis of the treatment, can be derived from the relations expressing the equilibrium of displacement according to s and to z of a plate element $\delta\ ds\ dz$, by substitution for the elementary stresses acting on each surface $\delta\ ds$, $\delta\ dz$ of the element in question, the corresponding resultants, and that is legitimate if the latter lie in the median plane of the plate. It is equivalent to considering σ_s^*, σ_z^*, τ^* as stresses applied to any linear element of the surface.

However, if δ varies so that the resultant of the stresses acting on each element $\delta\ ds$, $\delta\ dz$ always lies in the median plane of the plate, and if the effect of the unit stresses acting perpendicularly to the latter can always be kept negligible, the equations for the equilibrium of displacement of plate element ds, dz may still be written in the same form as for the plate of constant thickness, and equations (6) and (13) can be made to retain their validity - it being, of course, understood in that case that σ^*_s, σ^*_z, and τ^* are stresses acting on element $\delta\ ds$ and $\delta\ dz$, respectively.

EXAMPLE

Consider a cylindrical tube having a straight section as in figure 2; the unit length is chosen so that the perimeter of the section is 2π. The wall thickness δ is constant. The length of the tube is $l = 16$.

In the plane of the section a system of orthogonal cartesian axes is established wherein axis x is the tangent to the median contour of the section in one of the points where it intersects the axis of symmetry, axis y being coincident with it (fig. 2).

For evident reasons of symmetry in points symmetrical with respect to y, the displacements and secondary stresses assume opposite values. It suffices therefore to analyze only the behavior of one-half of the section; for example, that for which $x \geq 0$.

Introducing in equations (1) and (4), the values relative to the present example, we have:

$$\tau = 0.19296 \frac{M}{\delta}$$

$$\frac{d\theta}{dz} = 0.23395 \frac{M}{G\ \delta}$$

The section being symmetrical, it must be that $a = 0$; $\frac{dw}{ds}$ supplied by equation (5), may be written:

$$\frac{dw}{ds} = \frac{dw_1}{ds} + b\ \frac{dw_2}{ds}$$

The appended table I lists the values of $\frac{dw_1}{ds}$ $\frac{dw_2}{ds}$ along with those of the coordinates x, y of the median contour and of $\frac{dx}{ds}\frac{dy}{ds}$ relative to the particular half of the section, in function of s.

In order to quickly determine the values of the coefficients μ_n', μ_n'', μ_n''', ν_n', ν_n'', ν_n''', the method of Runge and Emde (Cf. G. Cassinis, Calcoli numerici grafici meccanici. Cap. XIV) is applied.

Denoting with S', S'', D', D'' the sums and the differences of the values of $\frac{dw_1}{ds}$, $\frac{dw_2}{ds}$ in points symmetrical with respect to y, we have:

$$\frac{1}{8}\,\nu_n' = \sum_0^\infty{}_h S_h' \cos n\, s_h \qquad \frac{1}{8}\,\mu_n' = \sum_0^\infty{}_h D_h' \sin n\, s_h$$

$$\frac{1}{8}\,\nu_n''' = \sum_0^\infty{}_h S_h'' \cos n\, s_h \qquad \frac{1}{8}\,\mu_n''' = \sum_0^\infty{}_h D_h' \sin n\, s_h$$

$D_h' = D_h'' = 0$ for any h by reasons of symmetry, and since, as seen, a = 0, we have:

$$\nu_n'' = \mu_n' = \mu_n'' = \mu_n''' = 0$$

for any value of n.

The execution of the calculation gives:

$$\nu_0'\,\frac{G\delta}{M} = +\,0.00233\ldots\ldots \qquad \nu_0'''\,\frac{G\delta}{M} = +\,0.00088\ldots\ldots$$

$$\nu_1'\,\frac{G\delta}{M} = +\,0.17111\ldots\ldots \qquad \nu_1'''\,\frac{G\delta}{M} = -\,0.27625\ldots\ldots$$

$$\nu_2'\,\frac{G\delta}{M} = +\,0.07340\ldots\ldots \qquad \nu_2'''\,\frac{G\delta}{M} = -\,0.00878\ldots\ldots$$

$$\nu_3'\,\frac{G\delta}{M} = -\,0.02252\ldots\ldots \qquad \nu_3'''\,\frac{G\delta}{M} = +\,0.05433\ldots\ldots$$

$$\nu_4'\,\frac{G\delta}{M} = -\,0.03663\ldots\ldots \qquad \nu_4'''\,\frac{G\delta}{M} = +\,0.00993\ldots\ldots$$

$$\nu_5'\,\frac{G\delta}{M} = -\,0.01159\ldots\ldots \qquad \nu_5'''\,\frac{G\delta}{M} = -\,0.01453\ldots\ldots$$

$$\nu_6' \frac{G\delta}{M} = +\ 0.00564\ldots\ldots \qquad \nu_6''' \frac{G\delta}{M} = +\ 0.00089\ldots\ldots$$

$$\nu_7' \frac{G\delta}{M} = +\ 0.00803\ldots\ldots \qquad \nu_7''' \frac{G\delta}{M} = +\ 0.00230\ldots\ldots$$

$$\nu_8' \frac{G\delta}{M} = +\ 0.00349\ldots\ldots \qquad \nu_8''' \frac{G\delta}{M} = -\ 0.00292\ldots\ldots$$

The values of the quantities R_n, T_n, and V_n are determined by means of equations (23), (27), and (33), putting $m = 3.3$ which, together with the first part of equation (32), give:

$$b = 0.62031$$

Next, by means of equation (35), we obtain the following values of the stresses σ^*_z, τ^* for the restrained section $(z = 16)$: (See table II.)

We wish to thank Professor E. Pistolesi for the useful advice he has given during the working-up of this report.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

REFERENCE

1. Timoshenko, S.: Theory of Elasticity. McGraw Hill Book Co., New York, 1934.

TABLE I

s	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
x	0	+0.39270	+0.78540	+1.13563	+1.26376	+1.08812	+0.75844	+0.38929	0
y	0	0	0	+.15205	+.51173	+.85038	+1.06376	+1.19522	+1.23980
$\frac{dx}{ds}$	+1.00000	+1.00000	+1.00000	+.69284	-.06743	-.77384	-.89776	-.97411	-1.00000
$\frac{dy}{ds}$	0	0	0	+.73055	+.99783	+.63192	+.44046	+.22608	0
$\frac{dw_1}{ds}\ \frac{G\delta}{M}$	+.19296	+.19296	+.19296	+.02315	-.11013	-.12186	-.10862	-.10001	-.09709
$\frac{dw_2}{ds}\ \frac{G\delta}{M}$	-.23395	-.23395	-.23395	-.15975	+.01578	+.18104	+.21003	+.22789	+.23395

TABLE II

s	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
$\tau_z^*\ \frac{\delta}{M}$	0	-0.01402	-0.03087	-0.02781	+0.00801	+0.02577	+0.02347	+0.01403	0
$\sigma^*\ \frac{\delta}{M}$	-0.00803	-.00719	-.00264	+.00635	+.01254	+.00245	-.00264	-.00625	-.00765

These are the maximum values verified on the column; stresses σ^*_z and τ^* disappear very rapidly from the point of fixation toward the free end.

τ^* & σ^* appear to be [illegible] by 10^{-1}!

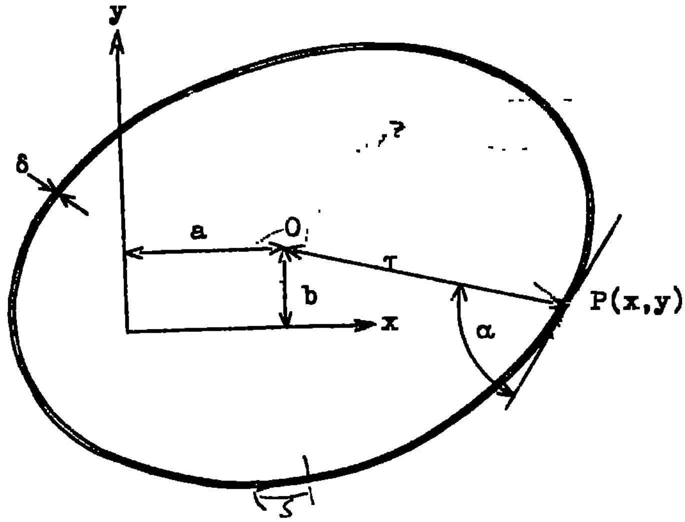
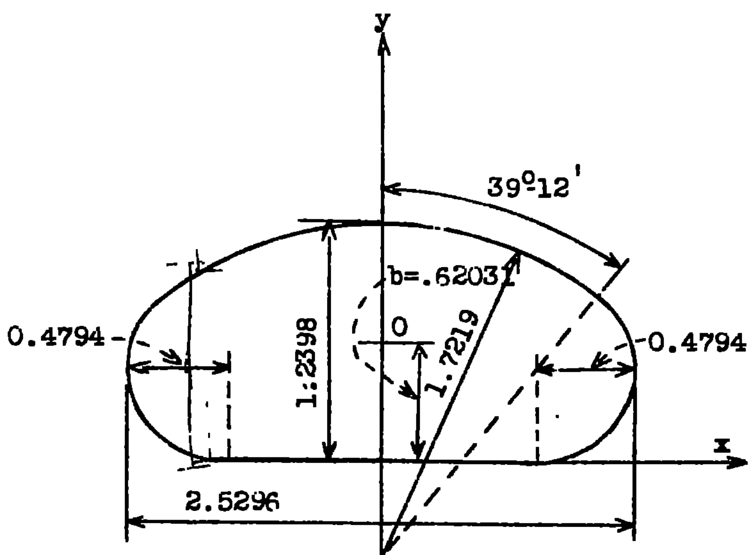


Figure 1.



Area = 2.5912

Figure 2.

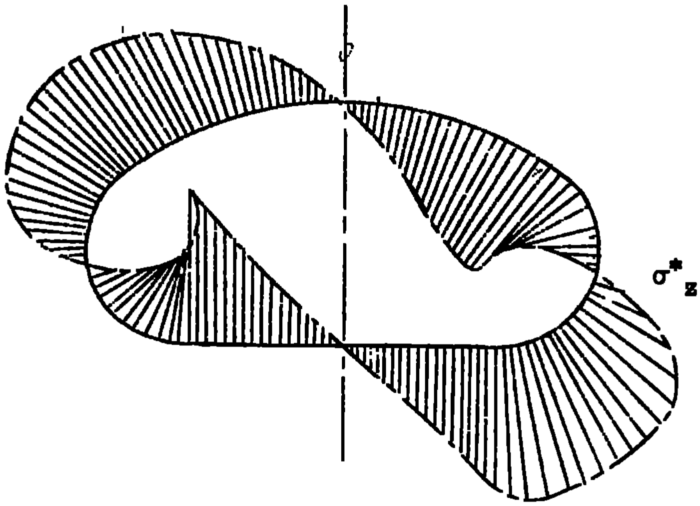


Figure 3.

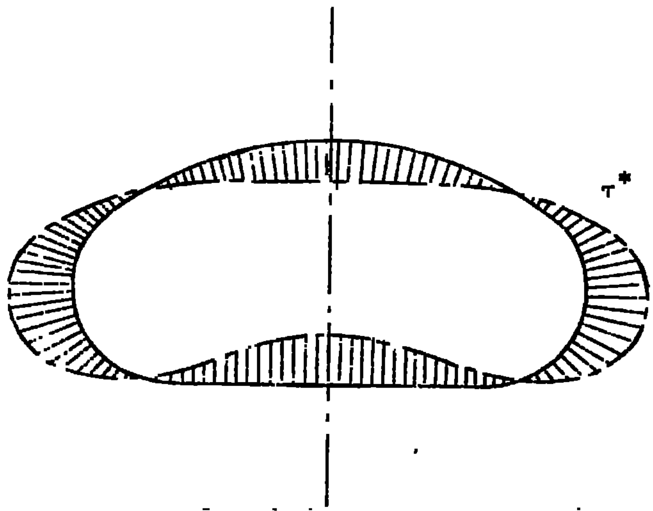


Figure 4.