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GROUND EFFECT - THEORY AND PRACTICE

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INTRODUCTION

The problem of ground effect has been discussed from time to time in this periodical, the most recent instance being in an article by the author (reference 24) wherein he analyzes the problem on the basis of his developed bi-plane theory. The conclusion of that article is that the increment of lift due to ground effect is largely attributable to the effect of induction of the free vortices, and is practically equivalent to a virtual increase in aspect ratio. The ground clearance was of the order of magnitude comparable to the wing chord.

Recently we have received some new articles and studies on this interesting subject: an informative article by M. Le Sueur (reference 31), a report by G. Dätwyler (reference 22), which treats the case of minimum distance from the ground and is confined to the plane problem only, and lastly, some theoretical studies by Tomotika and others (references 23 and 25), also confined to the plane problem.

We shall briefly review these reports without regard to chronological order.

I. LE SUEUR'S REPORT

Ground effect is a very controversial subject, both as to cause and to its effect. All observations made in free flight are in agreement as regards the existence of certain systematic phenomena such as the greater facility of low-wing airplanes to take off, impossibility of certain

*"Teorie ed esperienze sul problema dell'ala in vicinanza del suolo. From Pubblicazioni della R. scuola d'Ingegneria di Pisa, series VI, no. 261, July 1935, pp. 1-25.

heavily loaded long-distance airplanes to gain altitude, the prolonged glide of low-wing airplanes on landing, etc. Even so, there exist erroneous and misleading opinions as to the true cause of these phenomena.

It is a certainty that ground interference lowers the drag, assuming the lift to be equal, and in quite noticeable proportions. As to the maximum lift, there is no theory to attest to its increase; in fact, divers experiments in accord with certain theories appear to indicate occasionally, a decrease.

On these premises we shall examine the various phases of the problem from the theoretical as well as from the experimental point of view.

A. THEORIES ON INTERFERENCE EFFECT

These theories are able to treat the plane problem (wing of infinite span) or even the three-dimensional problem. In any case, the study on interference reverts to the general study of the biplane making use of the artifice known as "reflection." There are available for this purpose (three-dimensional problem) the noted Prandtl-Betz formulas, which may be applied in our case and which were published by Wieselsberger (reference 5). These formulas were chosen in 1924 by Toussaint for comparison with his experimental values of the coefficient in different cases of monoplanes and biplanes with ground interference. They are generally accepted as sufficiently approximate, though studies made subsequently disclose results more or less at variance with the former.

A report by Rosenhead (reference 18) treats the lift of a flat plate between parallel walls - an analysis based on a method of conformal transformation, and which he compares with Glauert's results (reference 32).

There is also a study by Müller (reference 19) in which he applies to two symmetrical airfoils Ferrari's method of conformal transformation, the results of which are at variance with experiment, as they are indicative of a decrease in lift.

Finally, there is the study by Pistolesi (reference 24) who, in the case of the wing with infinite span, finds

that the circulation increases with the angle of incidence up to a certain value of this incidence, beyond which it decreases again; and, taking into consideration the change in horizontal speed he expresses the change in lift coefficient due to interference by simple formulas (reference 24). He then passes to the case of finite span and finds the percent change of lift in function of the incidence for divers values of h/l ($l = \text{chord}$) - which results are, at least qualitatively, in agreement with experiment. In particular, the increase in lift decreases until it is canceled out as the incidence increases.

Lastly, there are the Japanese reports, the final conclusions of which are identical with those of Pistolesi: At low incidence the lift increases as the distance from the ground decreases; at high incidence, on the other hand, the lift decreases as the distance from the ground decreases.

B. EXPERIMENTAL METHODS

a) Tests with small-scale models.- Not wishing to go back as far as Betz's experiments of 1912 (reference 1) (which, while disclosing negligible interference effects, were quite inaccurate), we record an interesting study by Cowley and Lock (reference 3), based on tests made in England, in July 1920, in the 4-foot No. 1 wind tunnel, at 40 feet per second wind speed, on an R.A.F. 15 biplane of zero stagger, in connection with the "Tarrant" biplane.

The ground was represented, in part (by real ground) by a vertical sheet of tin 4 feet high, 3 feet long; and in part by two identical biplanes symmetrically disposed, in respect to virtual ground, according to the reflection method. Measurements were made of the lift, drag, and pitching moment at angles of attack ranging from -6° to 14° , and for values of h/b ($b = \text{span}$) = 0.167 and 0.306.

At about the same time, the Massachusetts Institute of Technology also made some similar tests in its 4-foot tunnel, at wind speeds of 30 miles per hour, except in two cases where it was raised to 40 and 45 miles per hour. These tests, reported by A. E. Raymond (reference 6), were made on three 3-by 18-inch models: a Martin No. 2, an R.A.F. 15 special, and a U.S.A. 27. Experiments were also made by the flat-plate method (3-ply birch $3/8$ -inch thick, 4 feet high, 3 feet wide, with leading edge chamfered on the side

away from the model) and by the reflection method. In both cases the tests were run for different angles of incidence and different ground distances, varying from $l/4$ to $2l$ ($l = \text{chord}$).

The same experimental method was used in 1921 in Germany to check Wieselsberger's formula (reference 5). The model was a monoplane of 124 cm (48.32 in.) span, aspect ratio 9.

Some years later Toussaint undertook a series of systematic experiments in the $6\frac{1}{2}$ -foot No. 1 wind tunnel at Saint Cyr (reference 9); the ground was represented by a sheet of aluminum 4 mm (0.157 in.) thick, and 1.60 m (5.24 ft.) long. The measurements were made on a wire balance; the wind speed was 32 to 33 m/s (105.0 to 108.3 ft./sec.). The models were a Lioré (S.C. 133 a) wing, a Fokker (S.C. 106 a) wing, a Fokker (S.C. 106 a+b) biplane wing, as well as two Bréguet 14 A 2 airplane models of 1/10 and 1/20 scale. The tests were made very painstakingly to assure accurate results. The incidence was changed in stages of 3° each from -9° to $+15^\circ$, and for three ground distances: 0.530 m (1.74 ft.), 0.438 m (1.44 ft.), and 0.240 m (0.787 ft.). These test data served as check on Betz's formulas, to which we shall refer again later on.

Other tests were made in the Eiffel wind tunnel on models mounted in the presence of a platform representing the ground. The wind speed was 25 m/s (82 ft./sec.); the model was a Caudron R.220; the distance from the ground 0, 100, 200, 300, 400, and 500 mm (3.94, 7.87, 11.81, 15.75, and 19.69 in.). Unfortunately, the use of only two incidences, 0° and 12° , detracts from the value of these experiments.

Next, there are the tests of the Wibault-Penhoet company on an airfoil 172, aspect ratio 5, a low-wing monoplane, type 313, airfoil 209, aspect ratio 7.8; a low-wing monoplane 280, airfoil 125, effective aspect ratio 6.84, real aspect ratio 7.85; a low-wing monoplane 287, airfoil 215, aspect ratio 8.4; a low-wing seaplane 240. The models had spans ranging from 1 meter to 1.30 meters; ground distance and incidence were changed systematically.

Lastly, we mention the tests reported by Dätwyler (reference 22), which refer to very short distances from the ground, and which had been conducted in Göttingen and Zurich.

From among the tests made on aerodynamic carriages, those of Tönnies deserve special mention (reference 21).

b) Full-scale experiments.— Here we have first of all the flights made during 1927 in the United States, as reported by Elliott G. Reid, in Technical Report No. 265 (reference 12). They were made with a Vought VE-7 bi-plane whose aerodynamic characteristics including those of the propeller, had been established in previous tests. The flights were made with the lower wing at 1.50 to 3.75 (5 to 9 ft.) above the ground. An interpolation method was used to eliminate the necessity of maintaining strictly level flight.

Other interesting full-scale tests are those cited in Tönnies' report (reference 21), made on a Klemm 26-2a at Hanover. The records were made with a motion-picture camera at 2, 4, 7, 10, 15, and 20 m (6.56, 13.12, 22.97, 32.8, 49.2, and 65.6 ft.) height. The tests were numerous and followed a set schedule.

The writer of that report offered some suggestions regarding the photographic method and advocated a method based on the emission of radio signals.

C. TEST DATA - THEIR AGREEMENT WITH ONE ANOTHER

AND WITH THEORY

The results of the various experiments can always be satisfactorily interpreted with Wieselsberger's formula:

$$\Delta\alpha = -\sigma \frac{C_p S}{\pi b^2}$$

$$\Delta C_r = -\sigma \frac{C_p^2 S}{\pi b^2}$$

which can be translated in a change in aspect ratio

$$\Lambda' = \frac{\Lambda}{1 - \sigma}$$

However, there is no accord between the reflection and the flat-plate method, especially for very short distances.

Figure 1 offers an example of this discrepancy.

This was noted by Cowley and Lock in 1921 (reference 3) who - for the plate - imputed the disturbance set up by its leading edge, which causes the air flow to deflect upward, while as regards the reflection method, they raise the element of doubt about the assumption of symmetrical flow which could in reality be replaced by an oscillating flow. The authors conclude that ground interference produces an appreciable effect on the pitching moment, but very little on maximum lift; the maximum L/D is increased from 10 to 13 by the reflection method, and from 10 to 15 by the flat-plate method.

For the plate at 38 mm (1.496 in.) height, equivalent to $\frac{h}{b} = \frac{1}{6}$ (h = twice the ground distance), the increase in lift and in L/D for a given angle of attack, is about twice as high by the flat-plate method as by the reflection method.

These findings are in agreement with those of other experimenters, for instance, Raymond (reference 6).

The Eiffel tests, made exclusively by the flat-plate method disclosed systematically a very much greater influence than Wieselsberger's formula stipulated, and furnished interference factors σ which were about twice the theoretical figure.

Le Sueur's conclusion is that the wind-tunnel tests are in agreement with Wieselsberger's theory.

Tönnies' carriage experiments (reference 21) are in accord with the theory; they record a decrease in lift at high incidence as stipulated by the theory (Pistolesi, Tomotika).

The free-flight tests by Reid (reference 12) are in close accord with the theory. This also holds true, in general, at least, for the flight tests described by Tönnies (reference 21).

Le Sueur comes to the conclusion that as yet we are actually unable to make any definite deductions, although for great ground distances and small angles, the different experiments seem to be in agreement with Wieselsberger's theory, which likens the ground interference to a ficti-

tious increase in aspect ratio. The effect in flight corresponds to what in the United States is called "floating." For great angles and short distances from the ground there seems to be a loss of lift; perhaps it is the phenomenon which is called "pancaking" in the United States.

Lastly, at very high angles of attack and very short distances from the ground, a marked increase in lift may result (air-cushioning effect), according to Datwyler's experiments.

In the next chapter, Le Sueur discusses the effect of the phenomenon on the different phases of airplane motion close to the ground, comparing high-wing to low-wing airplanes from the point of view of ground effect which, however, has no direct bearing on the argument which we shall advance. Suffice it to say that a lift increase may be useful in any case; the decrease in drag, useful at take-off, may become detrimental at landing as it increases the landing run.

II. DATWYLER'S THESIS ON GROUND EFFECT

Datwyler starts with some curious experiences. While Wieselsberger's theory leads one to presume a decrease in maximum lift (effect of induced horizontal velocity, at least in the plane problem), he observes the fact reported by Zimmermann in the aerotechnical debate at the Polytechnic Institute, Zurich, that in 1932 an American low-wing monoplane (Lockheed) landing with retracted landing gear, actually had a lower emergency landing speed than expected. The pilots reported they felt as if at the last moment before touching the ground, an air cushion had formed below the wing. The following year, 1933, an airplane of the same type actually had to make a forced landing with wheels retracted, when the retraction mechanism failed to function.

From these occurrences the author concludes that for small ground distances the omission of the finite profile dimension and the substitution of the profile by one single bound vortex of given circulation, is no longer permissible. For wings of finite span the problem becomes, of course, more complicated.

The theoretical part, detailing the Göttingen and

Zurich experiments, is followed by an appendix on the minimum induced drag of a wing with dihedral.

A. THEORETICAL PART

He begins with the treatment of the two-dimensional flow about a flat plate whose rear edge rests on an infinite plane (fig. 2) at an angle α . The problem is resolved by conformal transformation of plane $z = x + i y$ on the plane $t = r + i s$ (fig. 3), whereby the characteristic function is a particular case of the Christoffel-Schwarz differential equation.

Putting $\frac{t}{c} = \tau$ (c may be arbitrarily assumed) and

$$\frac{b}{c} = \beta = \frac{\alpha}{\pi}$$

with $a = 0$, we have:

$$z - z_0 = \int (-\tau)^{\beta-1} (\beta - \tau) (1 - \tau)^{-\beta} d\tau$$

Integration along the contour of the plate, with $s =$ distance of a point of the contour starting from the trailing edge and $l =$ plate length, we obtain:

$$\frac{s}{l} = \left(\frac{\Gamma}{\beta}\right)^{\beta} \left(\frac{1 - \tau}{1 - \beta}\right)^{1-\beta}$$

in which the author uses the function Γ .

He determines the lift of the plate on the following considerations: The drag is zero in a perfect fluid. According to figure 4, the lift P resulting from the normal pressures on the plate (Σp) and the suction force S at the plate tip, is equal to $S/\sin \alpha$. The suction force S is substantially conditional to the velocity distribution in direct proximity to the plate tip. He establishes the lift coefficient at

$$C_p = \frac{\pi}{\sin(\pi\beta)} \left(\frac{\beta}{1 - \beta}\right)^{1-2\beta} \quad (1)$$

as against

$$C_p = 2\pi \sin(\pi\beta) \quad (2)$$

for the free plate. Equation (1) for small values of β can be approximated:

$$C_p = \beta^{-2\beta} \quad (1')$$

Next the author determines the moment with respect to the leading edge at

$$M = \frac{\rho}{2} U^2 l^2 (J_o - J_u) \quad (3)$$

where J_o and J_u indicate the integrals

$$\int_{s=0}^{s=l} \left(\frac{v_o}{U}\right)^2 \left(1 - \frac{s}{l}\right) d\left(\frac{s}{l}\right)$$

$$\int_{s=0}^{s=l} \left(\frac{v_u}{U}\right)^2 \left(1 - \frac{s}{l}\right) d\left(\frac{s}{l}\right)$$

and whereby v_o and v_u denote the values of the velocity on the top and bottom sides of the flat plate. The values of these integrals are plotted. From (3) we deduce:

$$C_m = J_o - J_u \quad (3')$$

Another case treated by this author is the lift of a flat plate close to the ground at 90° angle of attack. He employs the reflection method and a conformal transformation on plane $X = \phi + i\psi$. The result is expressed in

$$C_p = \pi \frac{2\lambda + 1}{\lambda + 1} = \pi \left(1 + \frac{b}{a}\right)$$

while

$$\lambda = \frac{b}{a - b} \quad (4)$$

and b and a denote the ordinates of the two edges (upper and lower) of the flat plate.

Lastly, the author proposes an approximate evaluation in the general case, based on the effect of the reflected wing. This induces on the reflected wing a horizontal velocity which the author disregards as small compared to V , and a vertical velocity w in upward direction. He computes this velocity in the center of gravity of the circu-

lation; that is, at $l/4$ plate chord on the basis of the plotted velocities induced by the flat plate when moving downward at constant speed w . This flow is, up to a small share of the circulation, equivalent to the downwash on the flat plate. The speed w is that which corresponds to the effective angle of attack, or:

$$\frac{W}{V} = \alpha + \frac{w^*}{V}$$

For $w^* = zw$, it is:

$$\frac{W}{V} = \alpha + \frac{zw}{V}$$

hence

$$\frac{W}{V} = \frac{\alpha}{1 - z}$$

and since

$$C_p = 2\pi \alpha_{\text{eff}} = 2\pi \frac{W}{V}$$

$$C_p = 2\pi \frac{\alpha}{1 - z} \quad (5)$$

Thus, we obtain:

$\frac{s}{l}$	= 0	0.025	0.05	0.10	0.20	∞
$\frac{1}{1-z}$	= ∞	6.67	3.51	2.00	1.389	1

where s = distance above ground, and l = chord. Such result, as the author justly recognizes as valid for small angles of attack and therefore $\frac{2\pi}{1-z}$, represents the angular coefficient of the tangent of the lift coefficient.¹⁾ The author then plots the lift coefficient for various s/l (fig. 5).

B. EXPERIMENTS

These are also given in two parts. The first experiments were made at Gottingen in the small wind tunnel on a

¹⁾ The last part of this report contains a more exact and complete application of this concept.

thin, symmetrical airfoil (thickness parameter 0.075) evolved from a Joukowski airfoil, as follows: To assure a flat bottom side and, for reasons of symmetry, a flat upper side, straights were drawn from the trailing edge as tangents to the profile curve, forming a trailing-edge angle of exactly 8° . This gave the trailing edge the desired strength. The radius of curvature of the leading edge was also enlarged. The airfoil had a span of 800 millimeters, a wing chord of 200 millimeters, and an aspect ratio of 4. A flat 20-millimeter thick board with symmetrically tapered leading edge represented the ground. It fitted directly to the nozzle flare and extended with a length of 1,560 millimeters, sufficiently far beyond the trailing edge of the airfoil. Being 1,300 millimeters wide, it extended 50 millimeters at either side of the 1.200-millimeter wide air stream. The wind speed was 30 meters per second; the Reynolds Number

$$Re = \sim 3.7 \times 10^5$$

The tests were made with and without end plates, and in the case of the isolated wing, with proper corrections for autoinduction and scale effect, but disregarded as negligible in the case of a wing in proximity of the ground. The ground distance, measured at the trailing edge, was 5, 10, 20, and 40 millimeters.

Apart from secondary phenomena (negative lift at very small angle of attack due to a venturi effect between airfoil and plane), the test data were qualitatively in accord with the theory: diminution of drag for equal lift; increase of lift at equal angle of attack. However, the lift increase falls below the theoretical expectations. Moreover, there is no change when passing from 10- to 5-millimeter slot width. The reason for this is to be found in the retarding effect of the flat plate used to represent the ground which - rather than being immobile with respect to the wing - is stationary with respect to the fluid. Figures 6 and 7, photographed in the water tank, manifest the unlike behavior in both cases and the unlike formation of the boundary layer. (Fig. 7 corresponds to moving fluid and fixed wing; fig. 8, to moving wing and fluid at rest.) To overcome this drawback, the author had recourse to a second series of experiments in the small wind tunnel of the Federal Technical Institute at Zurich, where he used the method of reflection which, although not reproducing exactly the real phenomenon (no allowance for friction of air entrained by the wing against the ground

$$\text{mm} \times 0.03937 = \text{in.} \quad \text{m/s} \times 3.28083 = \text{ft./sec.}$$

and the cumulative lift which it is necessary to overcome). The tests were limited to the determination of the lift. The results are much more in accord with theory than the previous ones, as seen on figure 8.

If, however, the obtained lift curves are extrapolated to zero, it will be noted that the incidence for zero lift varies as the distance. This, the author ascribes to the effect of finite profile thickness. If the curves are shifted approximately 1.8° to the left, so as to afford $C_p = 0$ for $\alpha = 0$, the result will be the curves of figure 9. The results by the reflection method are much more appropriate than by the flat-plate method. The use of side screens tends to prevent the lateral escape of air dammed up on the bottom side of the wing and consequently, a lateral pressure decrease. The lift distribution becomes more complete and higher lifts may be expected.

Pressure measurements in the slot between the two wings showed positive pressure by adhering flow and, consequently, retarded flow. After separation of the flow the negative pressure on the upper side of the wing continues into the slot itself. This separation occurs sooner as the slot becomes narrower; that is, as the angle of attack becomes smaller.

This interesting report concludes with an appendix, in which the author uses the results obtained in the theoretical study to solve the problem of minimum induced drag of a wing with dihedral angle (fig. 10). He determines the potential, also the potential jump $\Delta\varphi = \varphi_0 - \varphi_u$ between the two sides of the plate by a current having an asymptotic velocity equal to unity. This jump multiplied by $2w$ ($w =$ induced velocity (constant)) on the wing supplies the circulation in the corresponding point of the wing, that is, Γ . There is, according to Kutta-Joukowski, a lift

$$P = 2\rho V \Gamma_0 \cos \epsilon \int_0^l \frac{\Gamma}{\Gamma_0} ds$$

whereby Γ_0 indicates the circulation at wing center; also an induced drag

$$R_i = \frac{w}{V} P$$

$$\Gamma = 2w (\varphi_0 - \varphi_u).$$

With ψ as the ratio between induced drag R_i of the wing with dihedral, and that of the wing with elliptical distribution and equal span (b), we have:

$$\psi = 2\pi \frac{\cos^2 \epsilon}{(\pi - 2\epsilon) \frac{\pi + 2\epsilon}{2\pi} (\pi + 2\epsilon) \frac{\pi - 2\epsilon}{2\pi}} \quad (6)$$

The value ψ is plotted in figure 11 against ϵ .

The report closes with a notation on Tomotika's experiments which pertain, however, to ground distances still amounting to 20 percent of the wing chord.

III. OTHER THEORETICAL STUDIES

A) MÜLLER (reference 19)

Müller proposes to effect the conformal transformation of two circles symmetrically disposed with respect to axis x in two profiles, also symmetrically disposed with respect to the same axis. His transformation is as follows:

$$z = \zeta + \frac{a}{(\zeta - \zeta_0)^2} + \frac{\bar{a}}{(\zeta - \bar{\zeta}_0)^2} \quad (7)$$

The singular points in which $dz/d\zeta$ cancels out must be found within the two circles. Thus,

$$\frac{dz}{d\zeta} = 1 - \frac{a}{(\zeta - \zeta_0)^2} - \frac{\bar{a}}{(\zeta - \bar{\zeta}_0)^2} \quad (8)$$

Equated to zero, we find the four singular points ζ_1 ζ_2 $\bar{\zeta}_1$ $\bar{\zeta}_2$.

If a is real, we obtain a biquadratic equation, and that is the case which Müller elaborates. In this case the singular points ζ_1 ζ_2 have the same ordinate. Taking the centers of the two circles as the points ζ_0 and $\bar{\zeta}_0$ the resulting top of the profile is straight and parallel with the axis of x (zero angle of attack). The

profiles have a certain thickness, maximum at the center, and the tips are not sharp-edged although they still resemble a pair of parallel wings. Quite obviously this case is of no interest in the problem of wings in proximity of the ground. It is necessary to this end that the profiles have an angle of attack so that

$$a = p^2 e^{-2ia}$$

may be obtained. Then it may be maintained that the upper wing has an incidence α and the lower wing $-\alpha$ (wind in positive direction of the axis of x). The study of the form of the profile is, however, quite complicated.

In any case, what interests us is the value of the circulation and the value of the lift; but Müller slighted this point, confining himself to stating that the obtained lift, in contradiction to experience, is less than for the isolated wing. The method evidently consists in passing from the flow around the two circles, studied by Müller (reference 30), by Bonder (reference 10), and Lagally (reference 28), to the flow about the profile, and then computing the lift by Blasius' formula. It is quite curious, indeed, that Müller did not enlarge upon this method, since it is altogether simple.

Bonder's findings were just the opposite (the lift increases to decrease with the ground distance), but the fact that he uses a theory which is an extension of Witoszynski's, instead of that by Joukowski, makes it impossible to compare his results with those of the other two. His results, moreover, expressed in series, are quite complicated and require the use of tables compiled by the author. Besides, Bonder's study has lost much of its importance in comparison with the much more recent work by Tomotika. The same holds for similar studies, such as that by Riabouchinsky (reference 34).

B) GLAUERT'S METHOD (reference 32)

Glauert analyzes the case of the wing in presence of its image on an infinite plane. His method - which is that used by Prandtl and others - consists of taking the curvature of the induced flow into consideration. The particular feature of Glauert's method is the admission that the induced velocity must be applied with reference

to the midpoint of the wing, while the point vortex representing the circulation around the wing must be placed at the center of pressure, which is the centroid of the bound vortices distributed along the chord of the wing, that is:

$$C_m = -\theta C_p$$

(C_m = moment coefficient about the leading edge with respect to the center of pressure θl . The coordinate of the midpoint of the wing with respect to the point at which the point vortex is assumedly placed, results in:

$$x = (0.5 - \theta) l$$

Then with h = distance between the two wings, a supplementary incidence is the result - the effect of which is additive to that of the curvature.

Finally, the moment coefficient is written in the form

$$C_m = \mu - \frac{1}{4} C_p$$

which gives:

$$C_p - C_{p_0} = \frac{1}{4} \left(\frac{l}{h}\right)^2 (C_p + 2\mu) \quad (9)$$

or else, reverting to the circulation (obtained by multiplying with $\frac{1}{2} v_0 l$)

$$\Gamma - \Gamma_0 = \frac{1}{4} \left(\frac{l}{h}\right)^2 (\Gamma + \mu v_0 l) \quad (9')$$

It is readily seen that this formula is almost identical with that proposed by the writer. Paragraph 3 of reference 24 reveals in fact with some simplifications:

$\left(v = \frac{c_m' l^2}{2\pi h^2} \right)$; note that h represents the distance between the two wings which in this article is expressed by $2h$;
 $c' = \pi$):

$$\Gamma - \Gamma_0 = \frac{l}{2h} \left(\frac{l}{2h} - \alpha \right) \Gamma - \frac{l^2}{4h^2} 2c_m' v_0 l$$

The equivalence follows from $C_m = 2c_m' v_0$ the moment coefficient (constant) with respect to the centroid indicated by $-\mu$ by Glauert; the equivalence, however,

is limited to very small α . In addition, it is of interest to note that in passing from C_p to Γ , the change in the horizontal component of the velocity is completely disregarded; this omission, nevertheless, is not justified and may lead to erroneous results. In the case analyzed by Glauert, it is legitimate.

Rosenhead, in his study (reference 18), analyzed the lift on a flat plate between parallel walls; but the case of the off-center wing, from which he deduces the case of the wing with interference, seems to be afflicted with errors (reference 25a).

C. REPORT BY ITIRO TANI (reference 20)

This study extends Birnbaum's method to include the analysis of a thin wing (two-dimensional problem).

Assuming the wing with a chord equal to 2, the vorticity is supposedly distributed chordwise according to the law:

$$\gamma = a \sqrt{\frac{1-x}{1+x}} + b \sqrt{1-x^2} + c x \sqrt{1-x^2}$$

The self-induced orthogonal velocity becomes:

$$v' = -\frac{1}{2} a - \frac{1}{2} b x + c \left(\frac{1}{4} - \frac{1}{2} x^2 \right)$$

and the induced velocity at a point x, y of the plane

$$v_x' = \frac{1}{2} a \frac{x f (1-q)}{y^2 + (x-q)^2} + \frac{1}{2} b y \left(\frac{1}{f} - 1 \right) + c y \left(\frac{x-q}{f} - x + \frac{q}{2f} \right)$$

$$v_y' = \frac{1}{2} a \left(\frac{f}{1+q} - 1 \right) + \frac{1}{2} b \left(\frac{x-q}{f} - x \right) + c \left(x - \frac{1}{2q} \right) \left(\frac{x-q}{f} - x + \frac{q}{2} \right)$$

where

$$q + \frac{1}{q} = \frac{x^2 + y^2 + 1}{x} \quad f = \sqrt{1 - \frac{q}{x}}$$

The coefficients $a, b,$ and c are determined on the basis of

$$v' + v_y' + v \sin \alpha = 0$$

where α = angle of attack and v_y' is computed as velocity induced by the reflected wing on the objective wing.

Posing:

$$\Gamma = \int_{-1}^1 \gamma dx = \frac{1}{2} a + \frac{1}{4} b = 2\pi v \sin \alpha n$$

we can compute the values of n in function of the ratio $2/h$, where h is the distance of the wing from the plane (in general, we say of t/h , where t = wing chord), which the author represents in two charts: one giving n versus t/h for divers α , the other giving n versus α for various t/h .

To pass from the two- to the three-dimensional problem (wing of finite span), the author computes the self-induced velocity assuming elliptic lift distribution and the velocity induced by the free vortices of the reflected on the real wing on the basis of a uniform distribution (almost exactly like that adopted by the writer, reference 24).)

The first is known to be constant; from the second, median along the span, follows:

$$\frac{1}{b} \int_{-\frac{1}{2}b}^{\frac{1}{2}b} u dz = \frac{\Gamma}{4\pi b} \ln \left(1 + \frac{b^2}{4h^2} \right)$$

Finally, putting:

$$l = \frac{1}{1 + \frac{2t}{b}}$$

$$m = \frac{1}{1 - \frac{t}{4b} n l \ln \left(1 + \frac{b^2}{4h^2} \right)}$$

$$n = \frac{1}{2} \frac{a}{v \sin \alpha} + \frac{1}{4} \frac{b}{v \sin \alpha}$$

we have

$$\Gamma = \pi t v \alpha l m n$$

where l takes account of the autoinduction, m of the mutual induction, and n of the curvature of flow due to the induction.

Lastly, passing to the lift, we write:

$$P = \rho b v \Gamma \left(1 - \frac{\Gamma}{4\pi h v} \right)$$

hence

$$\frac{P}{P_0} = m n \left(1 - \frac{t}{4h} \alpha l m n \right)$$

Tani's method is, as already pointed out, very much like Pistolessi's method; the only difference consists in the method of computing the effect of the flow curvature.

IV. RESEARCHES BY SUSUMU TOMOTIKA AND OTHERS

These researches, also cited in the appendix of Dätwyler's report, are published as Report No. 97, of the Aeronautical Research Institute, of the Tokyo Imperial University (reference 23). They are patterned on the conformal transformation similar to that employed by Rosenhead, in his study (reference 18): Given the plane z , provided the flow is at first assumed to be irrotational, and the plane $f = \varphi + i\psi$, shown in figure 12, and applying in plane f the cut $C G G' C'$, the plane f may be represented at mid-height of t by means of the Schwarz-Christoffel transformation:

$$\frac{df}{dt} = M \frac{t^2 - b^2}{\sqrt{(t^2 - c^2)(t^2 - g^2)}}$$

where M is a constant.

Introducing then the function p of Weierstrass, of periods $2\omega_1, 2\omega_3$, where $\omega_1 > 0$ and $\omega_3 i > 0$, the transformation

$$t^2 = p(s) - e_3$$

transforms the semiplane t in the rectangle of the plane s shown in the figure.

From the preceding formula, follows:

$$\frac{ds}{dt} = \frac{1}{\sqrt{[p(s) - e_1][p(s) - e_2]}}$$

and
$$\frac{df}{ds} = M [p(s) - p(\mu)]$$

which, integrated, gives:

$$f = - M [\zeta(s) + p(\mu) s]$$

Since the circulation is assumed to be zero, f has the period $2\omega_1$, and that leads to the relation:

$$p(\mu) = - \frac{\eta_1}{\omega_1}$$

The constant M is determined so that f differs from $i\psi_0$ in passing from $s = \omega_1$ to $s = \omega_1 + \omega_3$, and then we have:

$$f = \frac{2\psi_0 \omega_1}{\pi} [\zeta(s) + p(\mu) s]$$

Then the inside of the rectangle is transformed in an annular zone in plane z (fig. 13) by means of the transformation:

$$s = \omega_1 + \omega_3 - \frac{\omega_1}{i\pi} \log Z$$

With w as the velocity conjugated to $v_x - i v_y$ in plane z , with w as its modulus, and with Ξ as the angle which it forms with the direction of axis x , we have:

$$\frac{df}{dz} = w = |w| e^{-i\Xi}$$

or, with
$$\Omega = \Xi + i \log |w|$$

$$\frac{df}{dx} = e^{-i\Omega}$$

Now the direction Ξ of the velocity on the boundary line and on the plate is known; thus, the problem in plane Z reduces to the determination of the analytical function $\Omega(Z)$, provided that the real part on the two enveloping circles assume the prescribed values.

Then, on the basis of a formula established by H. Villat in 1912,* we have:

*Le problème de Dirichlet dans une aire annulaire, Rendiconti del Circolo Matematico di Palermo, 33 (1912), pp. 134-175.

$$\Omega(Z) = \frac{i\omega_1}{\pi^2} \int_0^{2\pi} \Phi(\theta) \zeta_3 \left(\frac{\omega_1}{i\pi} \log Z - \frac{\omega_1}{\pi} \theta \right) d\theta$$

$$- \frac{i\omega_1}{\pi^2} \int_0^{2\pi} \Psi(\theta) \zeta_3 \left(\frac{\omega_1}{i\pi} \log Z - \frac{\omega_1}{\pi} \theta \right) d\theta + iC$$

where $\Phi(\theta)$ and $\Psi(\theta)$ denote the values assumed by Ω on the outside and inside periphery, expressed in terms of angles in the center θ . The two functions Φ and Ψ comply, moreover, to the condition of uniformity:

$$\int_0^{2\pi} \Phi(\theta) d\theta = \int_0^{2\pi} \Psi(\theta) d\theta$$

Applying Villat's formula to the particular case of the problem, we obtain the expression df/dz and that of dz/ds , which for the sake of brevity, is omitted.

The integration of the latter is quite a complicated problem which the author resolves by adding the relation:

$$z = - \frac{2\Psi_0 \omega_1}{\pi U} A(s) + C_0$$

where

$$A(s) = - \frac{\sigma \left[s - \frac{2\omega_1}{\pi} \left(\frac{\pi}{2} + \delta \right) \right]}{\sigma[s] \sigma \left[\frac{2\omega_1}{\pi} \left(\frac{\pi}{2} + \delta \right) \right]} e^{\frac{2\eta_1}{\pi} \left(\frac{\pi}{2} + \delta \right) s}$$

With this function the author supplies another complex series development. Indicating with $2a$ the width of the plate ($2a = l$), we have:

$$2a = \frac{2\Psi_0 \omega_1}{\pi U} e^{i\beta} \left\{ A(s_3) - A(s_4) \right\}$$

$$s_3 = \omega_1 + \omega_3 - \frac{\omega_1}{\pi} \theta_3$$

$$s_4 = \omega_1 + \omega_3 - \frac{\omega_1}{\pi} \theta_4$$

θ_3 and θ_4 being the angles at the center of the two peripheries which correspond to points A and A' .

The author finds other expressions for z and a .

The flow defined by f from which we deduce the conjugated velocity $\frac{df}{dz} = \frac{df}{ds} \frac{ds}{dz}$ has, however, two singular points in A and A' , where the velocity becomes infinite. Superimposing a circulatory flow of proper circulation κ the infinity at A can be removed according to Joukowski. With f' as the complete potential of the circulation flow, and with $\chi = f + f'$ as the complete potential of the built-up flow, the function of the velocity results:

$$\frac{d\chi}{dz} = \frac{d\chi}{dZ} \left(\frac{dz}{ds} \frac{ds}{dZ} \right) = - \frac{i\pi}{\omega_1} Z \frac{d\chi/dz}{dZ/ds}$$

Since dz/ds becomes zero in point A , $d\chi/dZ$ also becomes zero in the corresponding point, or for

$$Z = e^{i\theta_4}$$

Now the circulation in plane Z is simply expressed by

$$f' = - \frac{i\kappa}{2\pi} \log Z$$

hence

$$\chi = f - \frac{i\kappa}{2\pi} \log Z$$

Joukowski's equation gives:

$$\frac{df}{dZ} - \frac{i\kappa}{2\pi} \frac{1}{Z} = 0 \quad \text{for } Z = e^{i\theta_4}$$

which readily gives:

$$\kappa = 2\omega_1 \left[\frac{df}{ds} \right]_{s=s_4}$$

or

$$\kappa = \frac{4\psi_0 \omega_1^2}{\pi} \left[p(s_4) - p(\mu) \right]$$

and in consequence:

$$\frac{d\chi}{ds} = \frac{2\psi_0 \omega_1}{\pi} \left[p(s) - p(s_4) \right]$$

The aerodynamic action, based on Blasius' formula, results in:

$$P_x - i P_y = \frac{1}{2} i \rho \int_C \left(\frac{dx}{dz} \right)^2 dz$$

C being a closed surface containing the flat plate; or, after several transformations:

$$P_x - i P_y = - \frac{1}{2} \rho \frac{\omega_1}{\pi} \int_{\Sigma} \left(\frac{dx}{ds} \right)^2 \frac{ds}{dz} \frac{dz}{dz}$$

where Σ is a surface surrounding the circle inside of plane Z. The calculation, rather tedious, gives the following results:

$$P_x = 0$$

$$P_y = \frac{\psi_0 U \rho}{2 \sin \beta} \frac{\vartheta_1'(0) \left[\vartheta_1 \left(\frac{\beta}{\pi} \right) \right]^2 \vartheta_1 \left(\frac{\theta_3 - \theta_4}{2\pi} \right)}{\vartheta_3 \left(\frac{\theta_3}{2\pi} \right) \left[\vartheta_3 \left(\frac{\theta_4}{2\pi} \right) \right]^3}$$

where β is the angle of attack (fig. 12).

This last is the expression for the lift of the flat plate.

The author demonstrates subsequently that when the distance H between the center of the plate and the wall becomes infinite, the preceding equation tends toward the known value:

$$P_0 = 2\pi a U^2 \rho \sin \beta$$

Taking into consideration an expression previously found for a, we finally have:

$$\frac{P}{P_0} = \frac{1}{2 \sin^2 \beta} \frac{\left[\vartheta_1 \left(\frac{\beta}{\pi} \right) \right]^3 \vartheta_1 \left(\frac{\theta_3 - \theta_4}{2\pi} \right)}{\left[\vartheta_3 \left(\frac{\theta_4}{2\pi} \right) \right]^2 \left\{ \left[\vartheta_3 \left(\frac{\theta_4}{2\pi} \right) \right]^2 - \left[\vartheta_3 \left(\frac{\theta_3}{2\pi} \right) \right]^2 \right\}} \quad (10)$$

The calculation of θ_3 and θ_4 bases upon the two following equations:

$$\theta_3 + \theta_4 = 2\beta \quad (11)$$

$$\frac{\vartheta_3' \left(\frac{\theta_3}{2\pi} \right)}{\vartheta_3 \left(\frac{\theta_3}{2\pi} \right)} + \frac{\vartheta_3' \left(\frac{\theta_4}{2\pi} \right)}{\vartheta_3 \left(\frac{\theta_4}{2\pi} \right)} = 0 \tag{12}$$

The appended tables give the values of θ_3 and θ_4 in function of β and $q = e^{\frac{\omega_3}{\omega_4} \pi i}$. Knowing θ_3 and θ_4 , we may then compute:

$$\frac{2a}{H} = \frac{2}{\sin \beta} \frac{\left[\vartheta_3 \left(\frac{\theta_4}{2\pi} \right) \right]^2 - \vartheta_3 \left(\frac{\theta_3}{2\pi} \right) \left[\vartheta_3 \left(\frac{\theta_4}{2\pi} \right) \right]}{\left[\vartheta_3 \left(\frac{\theta_4}{2\pi} \right) \right]^2 + \vartheta_3 \left(\frac{\theta_3}{2\pi} \right) \left[\vartheta_3 \left(\frac{\theta_4}{2\pi} \right) \right]} \tag{13}$$

Formulas (10), (11), (12), and (13) are fundamental for the calculation.

The author deduces from (12) the following series development:

$$\begin{aligned} \cos \frac{1}{2} (\theta_3 - \theta_4) = \cos \beta [& - 2q + 4q^3 - 4q^5 (3 + 2 \cos^2 \beta) \\ & + 32 q^7 (1 + 3 \cos^2 \beta) \\ & - 2q^9 (37 + 324 \cos^2 \beta + 80 \cos^4 \beta) \\ & + \dots] \end{aligned}$$

This formula combined with (11) is the basis of his appended tabulations.

The values of P/P_0 and of $2a/H$ (or l/H) deduced from (10) and (13) are included in the tables and plotted in figure 14.

Thus it is readily seen that for small $2a/H$, the ratio P/P_0 is always less than 1, regardless of angle of attack; but for small β , the values of P/P_0 become greater than 1 when $2a/H$ exceeds a certain critical value which is an incremental function of β . For medium angles of attack the lift is manifestly increasing in accord with experience, by small distances.

Tomotika's calculations do not include very small distances H . In particular, they do not come to an end at the cancellation of the slot s between wing trailing edge and the plane of infinite length. Obviously, this condi-

tion prevails when $H = a \sin \beta$, or for $\frac{2a}{H} = \frac{2}{\sin \beta}$.

In particular, $\frac{2a}{H} = 6.47$ for $\beta = 18^\circ$, and $\frac{2a}{H} = 3.4$ for $\beta = 36^\circ$. Compared with Dätwyler's data, the results seem to fairly agree, but not enough to remove all doubts. (Fig. 16 indicates the value of P/P_0 obtained by Dätwyler by a small circle; that is, 0.630.)

Lastly, Tomotika gives an approximate development of P/P_0 in powers of $2a/H$, and compares the results with that of the rigorous calculation (applicable to small values of $2a/H$). The expansion for P/P_0 in powers of $2a/H$ finally gives:

$$\begin{aligned} \frac{P}{P_0} = & 1 - \frac{\sin \beta}{2} \left(\frac{2a}{H} \right) + \frac{1}{16} (4 - 3 \cos^2 \beta) \left(\frac{2a}{H} \right)^2 - \\ & - \frac{\sin \beta}{32} (4 - 3 \cos^2 \beta) \left(\frac{2a}{H} \right)^3 \\ & + \frac{1}{512} (32 - 57 \cos^2 \beta + 22 \cos^4 \beta) \left(\frac{2a}{H} \right)^4 \end{aligned}$$

The lift calculation, evolved on Cisotti's ideas, is given in appendix I.

In another report (reference 25) Tomotika attacks the problem of lift on a flat plate in a stream bounded by an infinite plane wall.

Although these problems do not seem to have any practical significance, they are, nevertheless, of interest from the theoretical point of view. The result is shown in figure 15, from which it is readily seen that there is an increase in lift, increasing with the angle of attack and ratio $2a/H$ (decrease with distance from the plate).

The case in which the interval s between leading edge and plate is zero, in Dätwyler's report, leads to a negative lift, as is readily understood when observing that in his treatment there is always a force toward the wing. Dätwyler's problem may be considered the limit of Tomotika's problem.

V. APPROXIMATE FORMULAS DERIVED BY PISTOLESI'S METHOD

It might be of interest at the conclusion of this review to check whether my own method suggested for the calculation of the infinite biplane lends itself to convenient application to assure a high degree of approximation in the solution of the problem of ground interference.

The concept of this method may be briefly summed up as follows:

a) Compute the circulation about the wing on the basis of the velocity (further induced asymptotically) possessed by the wing at a point $1/4$ -chord length from the trailing edge (point K); the lift is obtained by multiplying by $c' l$ the component of this orthogonal velocity of the axis of the profile ($c' = \frac{1}{2} \frac{dC_p}{d\alpha}$; for airfoils with small camber, $c' = \pi$).

b) Substitute, for the effects of induction and lift calculation, the wing with a vortex located in the center of gravity of the circulation; or else in a point situated $\frac{1}{4} l$ length from the leading edge (point I), with a proper doublet added in the same point. In the case of a flat wing, the doublet is zero and the wing may be replaced by a simple vortex located at the said point.

Now let us see how this concept can be applied to the wing in the presence of its reflection; we limit ourselves to the case of a flat wing. The more simple application is that disregarded in the induction calculation: the angle of attack of the wing (fig. 16).

We have then at point K:

$$v_n = v_0 \sin \alpha + \frac{\Gamma}{2\pi} \frac{\frac{l}{2}}{4h^2 + \frac{l^2}{4}}$$

and consequently:

$$\Gamma = \pi l v_n = \Gamma_0 + \Gamma \frac{\frac{l^2}{4}}{4h^2 + \frac{l^2}{4}}$$

which gives:

$$\Gamma = \Gamma_0 \left(1 + \frac{l^2}{16h^2} \right) \quad (14)$$

The velocity in correspondence with point I is

$$v_0 - \frac{\Gamma}{4\pi h}$$

and finally, the lift is:

$$P = \rho \Gamma \left(v_0 - \frac{\Gamma}{4\pi h} \right)$$

or

$$P = \rho \pi l v^2 \sin \alpha \left[1 - \frac{l}{4h} \left(1 + \frac{l^2}{16h^2} \right) \sin \alpha \right] \left[1 + \frac{l^2}{16h^2} \right]$$

$$\frac{P}{P_0} = \left[1 - \frac{l}{4h} \left(1 + \frac{l^2}{16h^2} \right) \sin \alpha \right] \left[1 + \frac{l^2}{16h^2} \right] \quad (15)$$

Applying this formula to the case of $\alpha = 4\frac{1}{2}^\circ$, we find:

$\frac{l}{4h} = 0$	0.1	0.2	0.3	0.4	0.5
$\frac{P}{P_0} = 1$	1.002	1.023	1.062	1.118	1.188

Plotting these results (fig. 14, dotted line), they are seen to remain consistently above the rigorous values of Tomotika, and not widely at variance at high incidence. As regards the inclination of the wing in the calculation of the induced velocity, the approximate formulas (3) and (8) of my previous report could be applied, but other formulas more rigorous and no more complicated than they, are preferable. We have (fig. 17):

$$v_n = v_0 \sin \alpha + \frac{\Gamma}{2\pi d} \sin \gamma$$

$$\Gamma = \pi l \left(v_0 \sin \alpha + \frac{\Gamma}{2\pi d} \sin \gamma \right)$$

$$= \pi l v_0 \sin \alpha + \frac{\Gamma l}{2\pi d} (\sin \gamma_0 \cos \alpha - \cos \gamma \sin \alpha)$$

Then $\sin \gamma_0 = \frac{l \cos \alpha}{2d}$ $\cos \gamma_0 = \frac{2h}{d}$

so that

$$\Gamma = \Gamma_0 + \frac{\Gamma l}{2d^2} \left(\frac{l}{2} \cos^2 \alpha - 2h \sin \alpha \right)$$

or with
$$d^2 = 4h^2 + \frac{l^2}{4} \cos^2 \alpha$$

after simple changes:

$$\Gamma = \Gamma_0 \frac{1 + \left(\frac{l}{4h}\right)^2 \cos^2 \alpha}{1 + \frac{l}{4h} \sin \alpha} = \Gamma_0 k \quad (16)$$

where k is the multiplication factor of Γ_0 . We have also:

$$P = \rho \Gamma \left[v_0 - \frac{\Gamma}{2\pi \left(2h + \frac{l}{2} \sin \alpha \right)} \right]$$

which, after various changes* gives:

$$P = P_0 k' \quad (17)$$

with
$$k' = k (1 - kr) \quad (18)$$

whereby, again:

$$k = \frac{1 + \left(\frac{l}{4h}\right)^2 \cos^2 \alpha}{1 + \frac{l}{4h} \sin \alpha} \quad (19)$$

*In taking account of the wing curvature ($c_m' \neq 0$), we arrive by similar treatment at a result which is always expressible by (18), with

$$k = \frac{\sin \alpha (1 + \lambda^2 \cos \alpha) - \frac{c_m'}{2\pi} \lambda^2}{\sin \alpha (1 + \lambda \sin \alpha)} \quad (19')$$

$$r = \frac{\frac{c'}{\pi} \lambda \sin \alpha}{1 + \lambda \sin \alpha} \quad (20')$$

whereby $\lambda = \frac{l}{4h}$.

$$r = \frac{\frac{l}{4h} \sin \alpha}{1 + \frac{l}{4h} \sin \alpha} \quad (20)$$

Applying this formula to the case of $\alpha = 4.5^\circ$, we obtain the following tabulation:

$\frac{l}{4h} = 0$	0.1	0.2	0.3	0.4	0.5
$\frac{P}{P_0} = 1$	0.9943	1.0085	1.0404	1.086	1.146

These figures are equally superior to those of Tomotika, but much more approximate than the former, especially for small l/h (up to 0.8). They are shown in figure 14 as dot-dashed line.

A markedly improved approximation is obtained if the calculation of the induced velocity on the basis of a single vortex concentrated at I' is abandoned in favor of the field of the velocity around the reflected wing, for which we proceed as follows:

The velocity about a flat plate of length 2, assumed with a velocity of the components v_1 and v_2 and with a circulation Γ is:

$$w = v_1 - i v_2 \frac{z}{\sqrt{z^2 - 1}} - i \frac{\Gamma}{2\pi} \frac{z}{\sqrt{z^2 - 1}}$$

The application of the above formula to the field about the reflected wing, gives:

$$v_1 = v_0 \cos \alpha \quad v_2 = -v_0 \sin \alpha$$

and we have:

$$w = v_0 \cos \alpha + i v_0 \sin \alpha \frac{z}{\sqrt{z^2 - 1}} - i \frac{\Gamma}{2\pi} \frac{1}{\sqrt{z^2 - 1}} \quad (21)$$

For the determination of the circulation, it is necessary to determine w at point K on the real wing which, relative to the reflected wing, has the ordinates:

$$x_K = 2h \sin \alpha + \frac{1}{2} \cos 2\alpha$$

$$y_K = 2h \cos \alpha - \frac{1}{2} \sin 2\alpha$$

In general, denoting with χ the ratio of distance h and wing chord l , $\chi = h/l$, the previous formulas can be written as:

$$\left. \begin{aligned} x_K &= 4\chi \sin \alpha + \frac{1}{2} \cos 2\alpha \\ y_K &= 4\chi \cos \alpha - \frac{1}{2} \cos 2\alpha \end{aligned} \right\} \quad (22)$$

To determine the lift - whatever the circulation - we must compute w at point I of the real wing which, relative to the reflected wing, has the ordinates:

$$\left. \begin{aligned} x_I &= 4\chi \sin \alpha - \frac{1}{2} \cos 2\alpha \\ y_I &= 4\chi \cos \alpha + \frac{1}{2} \sin 2\alpha \end{aligned} \right\} \quad (23)$$

From (21) follows:

$$\left. \begin{aligned} v_x &= v_0 \cos \alpha - v_0 \sin \alpha (Bx + Ay) + \frac{\Gamma}{2\pi} B \\ v_y &= -v_0 \sin \alpha (Ax - By) + \frac{\Gamma}{2\pi} A \end{aligned} \right\} \quad (24)$$

where

$$\left. \begin{aligned} A &= \pm \frac{\sqrt{\frac{1}{2}(x^2 - y^2 - 1) + \frac{1}{2}\sqrt{(x^2 - y^2 - 1)^2 + 4x^2 y^2}}}{\sqrt{(x^2 - y^2 - 1)^2 + 4x^2 y^2}} \\ B &= \pm \frac{\sqrt{-\frac{1}{2}(x^2 - y^2 - 1) + \frac{1}{2}\sqrt{(x^2 - y^2 - 1)^2 + 4x^2 y^2}}}{\sqrt{(x^2 - y^2 - 1)^2 + 4x^2 y^2}} \end{aligned} \right\} \quad (25)$$

With regard to the signs of the radicals, it is readily seen that A will have the sign of x ; B , the sign of $-y$.

The normal component of the real wing at K will be $v_x \sin 2\alpha + v_y \cos 2\alpha$, where x and y are the coordinates of K :

$$\begin{aligned} v_n = v_0 & \left\{ [\cos \alpha - \sin \alpha (Bx + Ay)] \sin 2\alpha - \sin \alpha (Ax - By) \cos 2\alpha \right\} + \\ & + \frac{\Gamma}{2\pi} (B \sin 2\alpha + A \cos 2\alpha) \end{aligned}$$

which, since $\Gamma = 2\pi v_n$ gives:

$$\Gamma = 2\pi v_0 \frac{[\cos \alpha - \sin \alpha (Bx + Ay)] \sin 2\alpha - \sin \alpha (Ax - By) \cos 2\alpha}{1 - B \sin 2\alpha - A \cos 2\alpha}$$

or

$$\frac{\Gamma}{\Gamma_0} = \frac{1 - (B_K x_K + A_K y_K) \sin 2\alpha - (A_K x_K - B_K y_K) \cos 2\alpha}{1 - B_K \sin 2\alpha - A_K \cos 2\alpha} \quad (26)$$

To pass to the value of lift P , it is necessary to know the horizontal velocity corresponding to point I , which will be $v_x \cos \alpha - v_y \sin \alpha$:

$$v_0' = v_0 \left\{ [\cos \alpha - \sin \alpha (Bx + Ay)] \cos \alpha + \sin \alpha (Ax - By) \sin \alpha \right\} + \frac{\Gamma}{2\pi} (B \cos \alpha - A \sin \alpha)$$

or, writing $\Gamma = 2\pi v_0 k \sin \alpha$ (where $k = \Gamma/\Gamma_0$) and making several changes:

$$\frac{v_0'}{v_0} = 1 - \sin \alpha \left\{ [B_I (x_I - k) + A_I y_I] \cos \alpha - [A_I (x_I - k) + B_I y_I - 1] \sin \alpha \right\} \quad (27)$$

which, for $\frac{v_0'}{v_0} = r$, finally gives:

$$P = P_0 kr \quad (28)$$

The calculation is quite rapid since the quantities A and B are easily computed, after the expressions $x^2 - y^2 = 1$ and $4x^2 y^2$ have been calculated.

Applying the method to the case of $\alpha = 18^\circ$ (i.e., very high incidence), and for the whole range of values χ (or of $\lambda = 1/\chi$), analyzed by Tomotika ($\lambda = 0$ to 3)), the agreement with his test results is practically perfect, so that the two curves coincide. This is valid for the plane problem - that is, the problem of the infinite wing.

To analyze the problem of the wing with finite span necessitates taking into account the velocity induced by the free vortices and the calculation is notably compli-

cated. In view of this complication, I felt justified in my study (reference 24) to confine myself to considering this effect by the method of horseshoe (or trailing) vortices (lift distribution by uniform law); it is probably possible to improve the research method by adopting Ferrari's method (reference 36) for the calculation of the biplane of finite span, which is based upon elliptical lift distribution. Still, even supposing it is possible to substitute the wing with a distribution of vortices on the aerodynamic center of the wing and with a distribution of doublets on the central axis of the profile, its applicability with good approximation is limited to moderate ratios of l/h .

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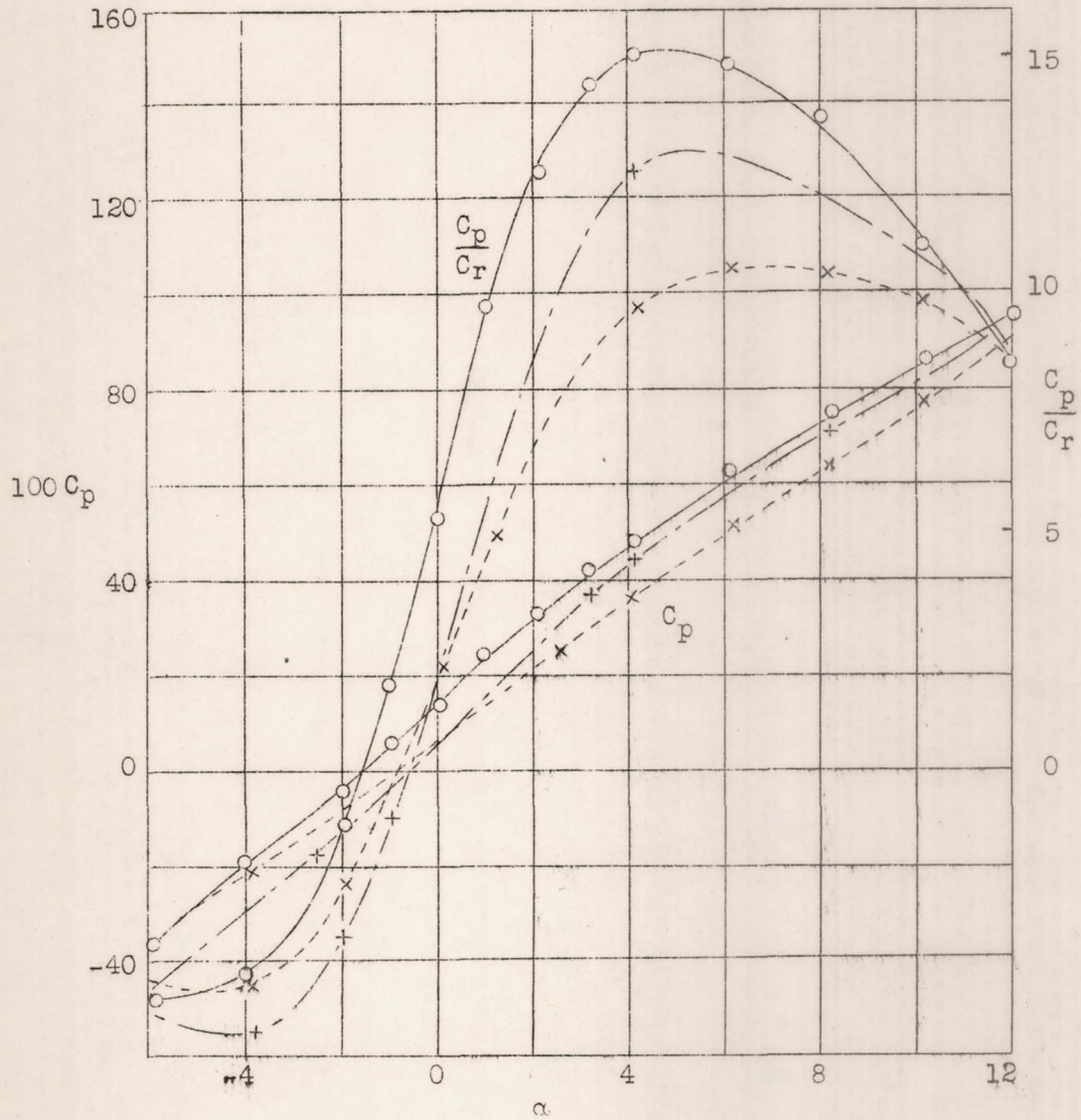
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- x----- Biplane alone
 - o----- Plate 38mm below lower wing
 - +----- Reflection 76mm between lower wings
- Airspeed 12.19 m/sec
 Wings of biplane $76 \times 457 \text{ mm}^2$

Figure 1.

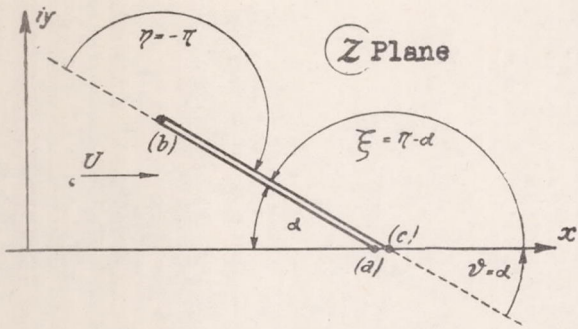


Figure 2.- Z plane (reflected plane).

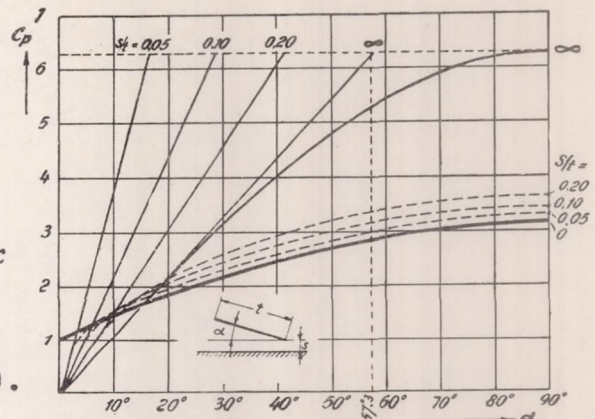
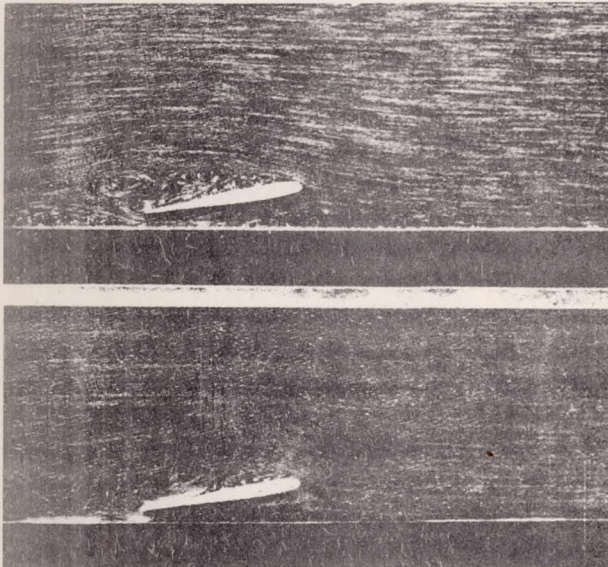


Figure 5.- Lift of flat plate in proximity of the ground (partly approximated).



Figures 6,7.- Flow patterns.

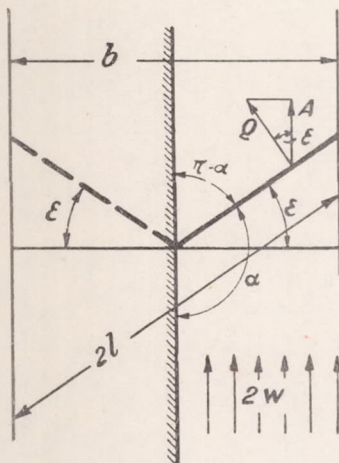


Figure 10.

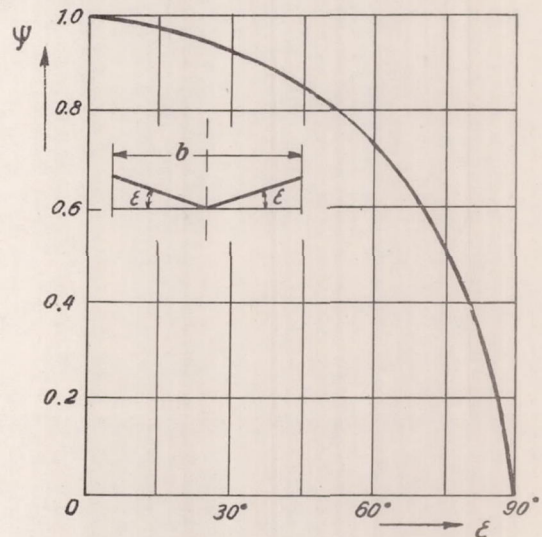


Figure 11.

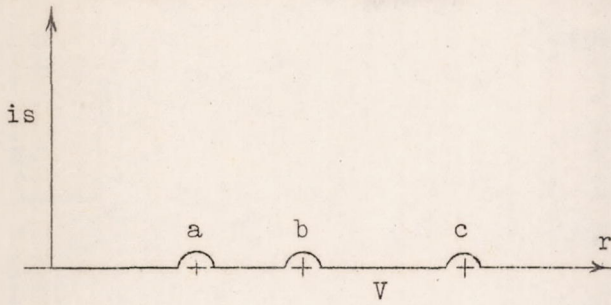


Figure 3.- t plane
(original plane).

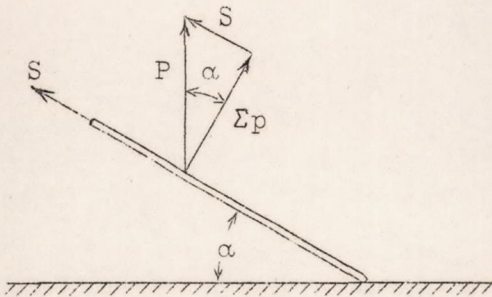


Figure 4.- Lift determina-
tion.

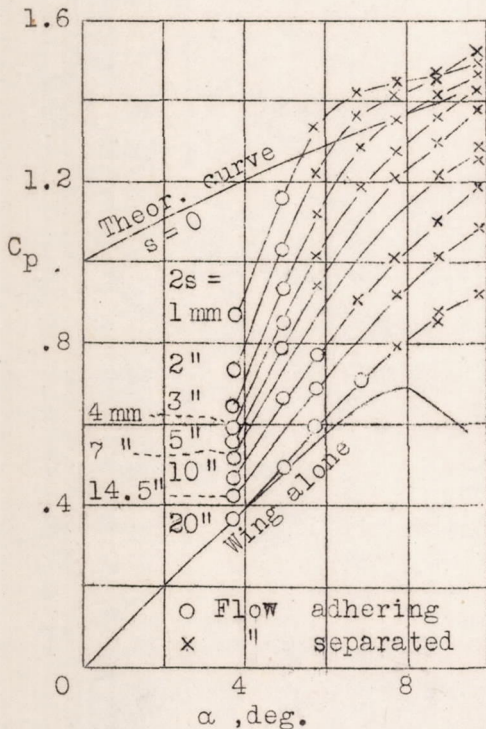


Figure 8.- Zurich experiments
(reflection method)

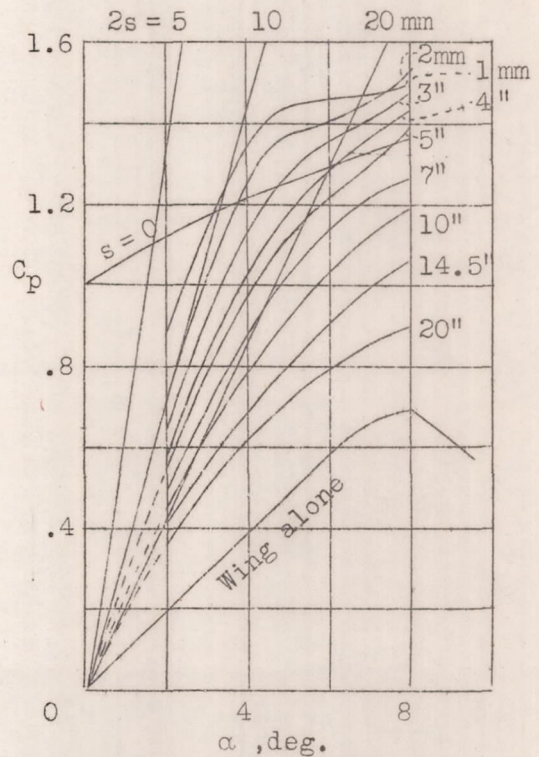


Figure 9.- Comparison between
theory and experiment.

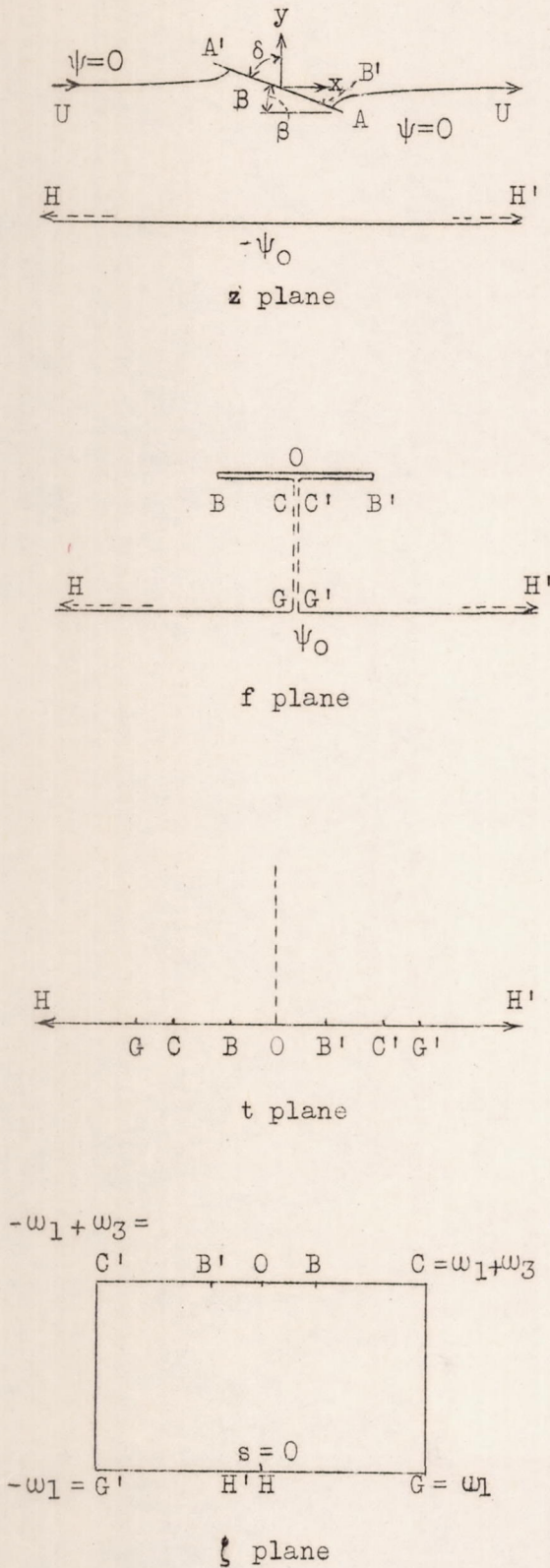


Figure 12.- Tomotika's experiments.

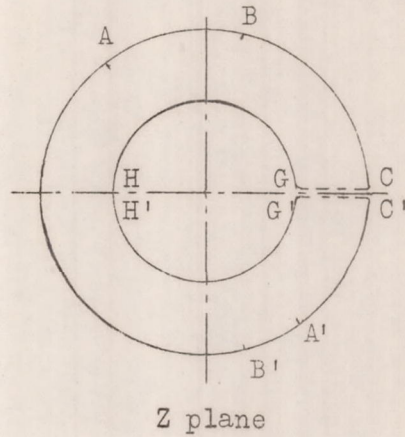


Figure 13.- Tomotika's experiments

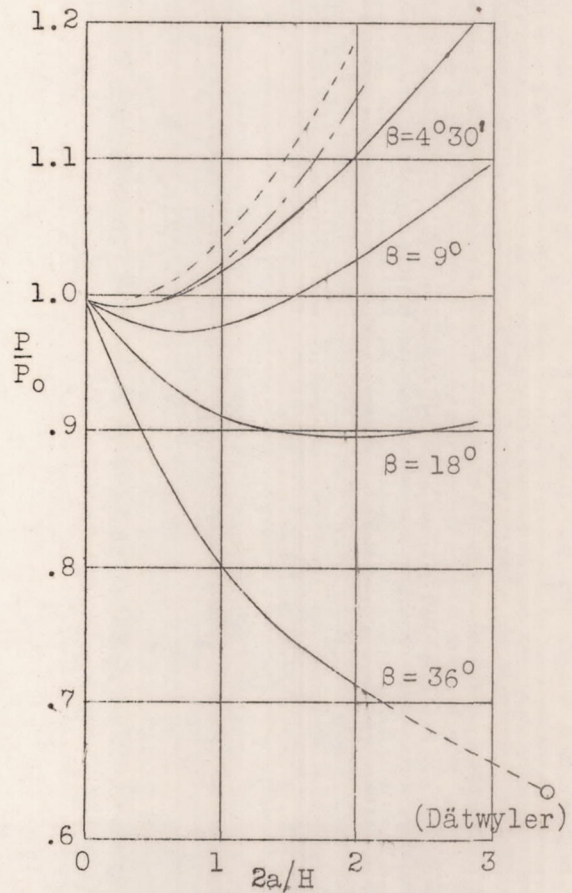


Figure 14.- Tomotika's results compared with those of Pistolesi and Dätwyler.

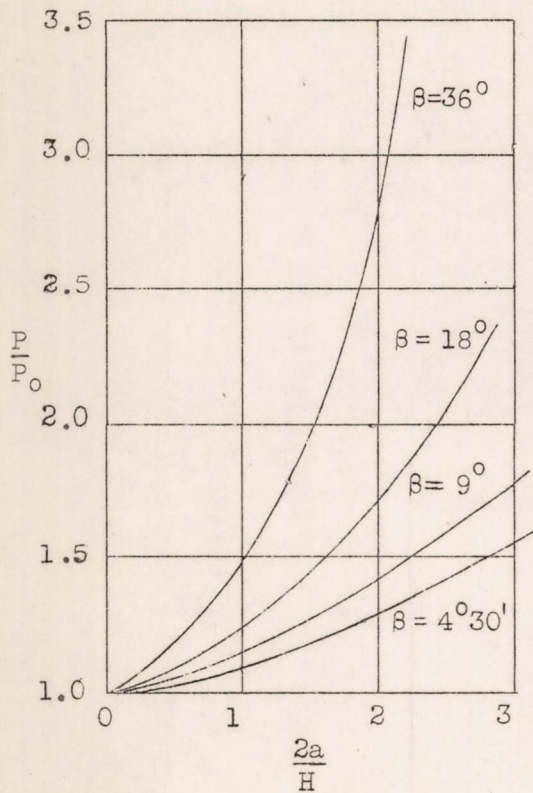


Figure 15.- Tomotika's results for the case of the plane above the wing.

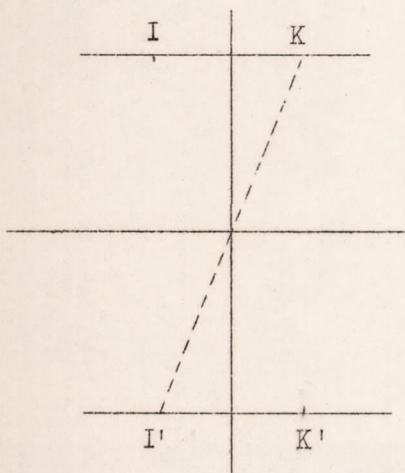


Figure 16.

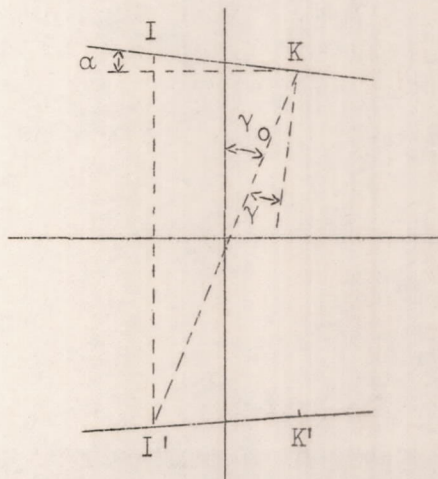


Figure 17.