NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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TORSION AND BUCKLING OF OPEN SECTIONS*

By Herbert Wagner

Open section members are made of rolled or drawn sheet metal and do not, like the closed or tubular sections, enclose any area. Open sections are applied a great deal in metal constructions because they can so easily be joined to one another or to other plates; in addition, they are accessible at all positions and so lend themselves to easy maintenance and repair.

In contrast to closed sections, however, open sections possess very small torsional rigidity. Thus it is known that the torsional rigidity of an open member whose cross section is not prevented from warping, is only as great as that of the flat metal strip from which it is made. If, however, warping of the section when the member is twisted is prevented, for example, at one end of the section (at least for a relatively short member), then longitudinal stresses arise which offer a considerable resistance to torsion. The computations of this effect of the longitudinal stresses on the torsional rigidity have already been carried out for certain types of sections, especially I beams.** In this paper we shall discuss the general principles for open sections of any shape.

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*"Verdrehung und Knickung von offenen Profilen." From the 25th Anniversary Number of the Technische Hochschule, Danzig, 1904-1929, pp. 329-343. This work appears in somewhat more extended form in the Zeitschrift für Flugtechnik und Motorluftschiffahrt, where also are given the results of tests conducted by the aviation branch of the Danzig Technical High School.

**The work of C. Weber, Zeitschrift für angewandte Mathematik und Mechanik, 1926, on the same subject, came to my attention after I had completed my paper. With respect to the fundamental assumptions, the work of C. Weber agrees with mine except that in my work the effect of the variation in the longitudinal stresses is more accurately taken into account. (Part C_{bdn}, equation 6b, C_{bd} in equation 6.) The work of H. Reissner, Zeitschrift für Flugtechnik und Motorluftschiffahrt, 1926, p. 384 (reference 1), in which the bending accompanying torsion is treated for closed sections, particularly box-shaped sections, should also be mentioned.

Open sections are usually so designed that they are not subject to any torsional stresses. But even where they are applied as compression members, such sections often give way by twisting or tilting long before the Eulerian buckling load or the yield point is reached. Particularly do the compression members used in airplane structures whose ratio of load to length of member (reference 2) is in general small and which are therefore made with very thin walls, have a tendency to twist. In what follows this torsion will be computed and on the basis of the results obtained it will be possible to obtain a proper design of section in each case *

The torsion of buckling members for the case where they are centrally loaded, leads to a problem in pure stability and is similar to that of the bending of stressed beams.

PURE TORSION

Notation

E, G, modulus of elasticity (kg/cm^2)

x, coordinate in direction of axis of member (cm).

φ, angle of twist of member.

M, external torsional moment (kg cm).

GJ_T, torsional rigidity of section when warping of section is not prevented ($kg\ cm^2$).

M_T, torsional moment due to torsional rigidity of section (kg cm).

τ_T, torsional stress corresponding to torsional rigidity (shearing stress) (kg/cm^2).

C_{bd}, resistance to combined bending and twisting of section (cm^6).

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*For centrally loaded angles, this twisting has already been investigated by P. Bryan. See, for example, Rudolf Mayer (reference 3).

M_{bd}, torsio alnmoment of cross section due to combined bending and twisting (kg cm).

ϵ_{bd}, σ_{bd}, τ_{bd}, longitudinal strain, longitudinal stress, and shearing stress due to combined bending and twisting.

The remaining notatioh is indicated on the diagrams.

If a member, the warping of whose cross sectıon is not prevented, ıs twisted, there arise in the cross sectıon shear stresses τ_T, whıch may be computed wıth the aıd of the usual torsıon theory for every shape of cross sectıon. The twisting moment due to these shear stresses ıs of magnıtude $M_T = \varphi' G J_T$ where $\varphi' = d\varphi/dx$ and is ın thıs case equal to tne external twisting moment M. It ıs assumed here that the shape of cross section of the member and the twistıng moment M and therefore also the angle of twıst per unit length φ' ıs constant along the length of the member. In this case there arise no longitudınal stresses ın the member. Furthermore, no shear stresses occur in the surface that is mıdway between the two outer surfaces of the open sectıon ("middle surface") and also ın the planes lying at rıght angles to this surface (the "normal planes").* (See fıg. 1.) Sectıons, the strength of whose walls compared to the developed length of the cross sectıon, is so small that the followıng statements on the cross-sectıonal twisting and the distributıon of the longıtudınal stresses apply wıth sufficient accuracy, we defıne as "open sectıons."

The coordinate u (perıpheral coordınate) measured along the middle surface of the sectıon and the coordınate n, measured ın the normal planes, gıve the dıstance of a poınt to the middle surface. We denote by S the shear center of the cross sectıon (reference 4) and the straight lıne whıch ıs the locus of the shear centers we shall call the "shear axıs."

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*The shear stresses at the ends of the section inthe normal planes are confıned to a small regıon for thın-walled sectıon, theır effect on our considerations ıs slight also for the reason that the variatıon ın the longıtudınal stresses σ_{bd} due to the bending accompanyıng the torsion ın the dırectıon of these planes (n dırectıon) ıs only of secondary ımportance. (C_{bdn} ıs small compared to C_{bdu}; see equatıon 6b.)

We now consider a cross section normal to the shear axis. As a result of the cross-sectional warping, the points of this cross section move out of the plane by the amount ξ when twisted. The longitudinal fibers which were straight before the torsion was applied, assume during the torsion the form of helices whose axis is the shear axis. We project two neighboring longitudinal fibers lying in the middle surface onto a plane which is parallel to the shear axis and normal to the line r joining the shear axis with the fibers. The projection of these fibers forms with the projection of the shear axis the angle $r\varphi'$ (fig. 1). The two points of the fibers which were originally in the middle surface are now removed from it by amounts ξ and $\xi + \frac{\partial \xi}{\partial u} du$, respectively. Since there is no shearing stress in the strip of surface lying between the two fibers, we have

$$\frac{\partial \xi}{\partial u} = \sin \alpha \; r \; \varphi' = r_u \cdot \varphi'$$

so that the displacement ξ_u of any point in the middle surface is of the amount

$$\xi_u = \varphi' \int^u r_u \, du$$

This displacement also varies along the normal to the surface n. It may be shown in a similar manner that $\frac{\partial \xi}{\partial n} = r_n \, \varphi'$, so that at any position u, n

$$\xi = \xi_u + \xi_n = \varphi' \left(\int^u r_u \, du + \int_0^n r_n \, dn \right) = \varphi' \, (w_u + w_n) = \varphi' \, w \quad (1)$$

where w denotes the unit increase in the warping, that is, for $\varphi' = 1$, w_u and w_n are the two components of w whose meaning and magnitude are clear from the equation. Since the distance r_n of the normal surface from the shear axis in the second integral is a constant, $w_n = r_n \, n$.

We are still free to determine the lower limit of the first integral, i.e., the longitudinal displacements may still vary by a constant amount. This lower limit we shall now choose so as to make the mean longitudinal displacement of the section equal to zero, that is,

$$\int_F \zeta dF = 0 \quad \text{or} \quad \int_F w \, dF = 0 \tag{1a}$$

Since, as is easily seen, the mean value of w_n is equal to zero, we have

$$\int_F w_u \, dF = \int_0^U w_u \, s \, du = 0 \tag{1b}$$

where U denotes the developed length of the cross section.

COMBINED BENDING AND TORSION

If the unit angle of twist φ' is not constant so that the torsional moment along the length of the section is variable, and if the longitudinal displacement of the section is prevented (for example, at one end), then, in addition to the pure shear stresses τ_T, we also have longitudinal stresses σ_{bd} and torsional stresses τ_{bd} due to the combined bending and torsion.

We consider again (fig. 2) the two neighboring longitudinal fibers of the middle surface. The strip now appears bent in the projection and the amount of bending is $r\varphi''$ As a result of this bending, one fiber of the strip must stretch more than the other (similarly to the longitudinal fibers of a bent beam), the difference in the strain*) amounting to $d\epsilon_{bd} = r\varphi'' \, du \sin \alpha$, so that $\frac{\partial \sigma_{bd}}{\partial u} = E\varphi' \, r_u$. The longitudinal stress in the middle surface is therefore

$$\sigma_{bdu} = E\varphi' \int^u r_u \, du$$

The lower limit of the integral we shall later determine.

The stress σ_{bd} also varies in the normal direction n We obtain similarly

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*This equation is valid only when variation in the shearing stresses τ_{bd} in the x-direction due to $\partial\sigma_{bd}/\partial x$ (see equation 6a) may be neglected, that is, for members that are not too short, see also following footnote and references.

$$\frac{\partial \sigma_{bd}}{\partial n} = E\varphi'' r_n, \quad \text{so that} \quad \sigma_{bdn} = E\varphi'' r_n n$$

Here σ_{bdn} denotes the longitudinal stress at the position of the normal plane of any section in addition to the mean longitudinal stress which (on account of the linear variation of σ_{bdn}) is equal to the longitudinal stress σ_{bdu} in the middle surface at this position u. We thus obtain for the total longitudinal stress

$$\sigma_{bd} = \sigma_{bdu} + \sigma_{bdn} = E\varphi'' \left(\int^{u} r_u \, du + r_n \, n \right) \qquad (2)$$

Since there is no longitudinal force acting on the section

$$\int^{F} \sigma_{bd} \, dF = 0 \qquad (2a)$$

Comparing equation (2) with equation (1) and (1a), We see that (2b)

$$\sigma_{bd} = E\varphi'' \, (w_u + w_n) = E\varphi'' \, w \qquad (2b)$$

The longitudinal stresses σ_{bd} should not give a resulting bending moment (since there is no such moment acting on the member). It may easily be shown that this condition may be satisfied if and only if the magnitudes r_u and r_n refer to the shear center, that is, when the section twists about the shear axis, also in the case where longitudinal stresses arise.

These longitudinal stresses arising during torsion set up a resistance against the torsion, which we shall now compute. We consider a member of length L, the end of which are acted on by torsional moments M and, along its length, is also acted on by external moments of magnitude $\frac{dM}{dx} = m$. The internal work of deformation is

$$A_1 = A_{1T} + A_{1bd} = \frac{G \, J_T}{2} \int_0^L \varphi'^2 \, dx + \frac{1}{2E} \int_0^L \int^F \sigma_{bd}^2 \, dx \, dF$$

The first term on the right-hand side gives the work done by the shearing deformation as a result of the stresses τ_T while the second term is the work corresponding to the longitudinal stresses σ_{bd}. The work of deformation

corresponding to the shear stresses τ_{bd} which depend on the variation of σ_{bd} in the x-direction (see below) has been neglected.*

Taking account of equation (2b), we obtain

$$A_1 bd = \frac{1}{2} E \int^{F} w^2 \, dF \int_0^L \varphi''^2 \, dx = \frac{1}{2} E \, C_{bd} \int_0^L \varphi''^2 \, dx \qquad (3)$$

where, for short, we have written C_{bd} in place of one integral.

We now give the deformed condition a variation $\delta\varphi$, where $\delta\varphi$ may vary in any manner along the length of the member except that we assume the boundary conditions

$$\text{for} \quad x = 0 \quad \text{and} \quad x = L \qquad \delta\varphi = 0 \quad \text{and} \quad \delta\varphi' = 0$$

According to the principle of virtual velocities, $\delta A_1 - \delta A_a = 0$, where A_a denotes the work of the external torsional moments, the work at the ends of the member, on account of conditions (4), being zero. We thus obtain

$$\frac{1}{2} GJ_T \int_0^L \delta(\varphi'^2) \, dx + \frac{1}{2} EC_{bd} \int_0^L \delta(\varphi''^2) \, dx - \int_0^L m \, \delta\varphi \, dx = 0$$

Performing the variation and integrating partially (taking account of equation 4) we obtain, after collecting terms under a common integral sign,

$$\int_0^L (- GJ_T \varphi'' + EC_{bd} \varphi^{IV} - m) \, \delta\varphi \, dx = 0$$

Since this integral must vanish for any arbitrary function $\delta\varphi$, the differential equation for the combined torsion and bending reads

$$E \, C_{bd} \, \varphi^{IV} - G \, J_T \varphi'' + m = - \frac{dM}{dx} \qquad (5)$$

Integrating once, we obtain

$$- E \, C_{bd} \varphi''' + G \, J_T \varphi' = M_{bd} + M_T = M \qquad (5a)$$

*This corresponds to the assumption usually made in the bending theory that for a bent beam that is not too short, it is allowable to neglect the deformation due to shear in comparison with the deformation due to bending.

From this equation we see that to the longitudinal stresses σ_{bd} resulting from the bending accompanying the torsion there corresponds an internal torsional moment of magnitude

$$M_{bd} = - E C_{bd} \varphi'''$$

where (see equation 3)

$$C_{bd} = \int_F w^2 \, dF \qquad (6)$$

(C_{bd} = strength in combined bending and torsion)

Having given this purely formal derivation for C_{bd}, we shall next consider the torsional moment M_{bd} resulting from the bending accompanying the torsion. Since the longitudinal stresses σ_{bd} vary in general along the x-direction, there must arise in any cross section shear stresses τ_{bd}, which we split up into two components, τ_{bdu} due to variation in the mean longitudinal stress σ_{bdu}, and τ_{bdn} due to the longitudinal stress σ_{bdn}.

In the same way as for the usual bent be mawe may see from the equilibrium of a strip of width du and length dx (see fig. 2) that, to the variable longitudinal stresses σ_{bdu}, there must correspond the shear stresses τ_{bdu}, which in the normal planes act in the direction of the longitudinal axis and in the cross section act in the direction of the u axis. We have (see equations 2 and 2b)

$$d\,(\tau_{bdu}\, s) = \frac{\partial \sigma_{bdu}}{\partial x} dF = E \varphi''' \, w_u \, dF$$

At any position u, therefore,

$$\tau_{bdu}\, s = E\varphi''' \int_0^u w_u \, s \, du \qquad (6a)$$

The moment taken up by the shear stresses may now be computed as

$$M_{bdu} = \int_F \tau_{bdu}\, r_u \, dF = \int_0^U r_u \, du \, \tau_{bdu}\, s$$

$$= E\varphi''' \int_0^U r_u \, du \int_0^u w_u \, s \, du$$

By integrating partially, taking into account equations (1) and (1b), we obtain

$$M_{bdu} = - E\varphi''' \int^{F} w_u^2 \, dF = - E\varphi''' C_{bdu}$$

The shearing stresses due to the varying σ_{bdn} act in the direction of n and vary parabolically (in the same manner as for the bent beam of rectangular cross section). The corresponding moment may be computed in the same manner as above and is equal to

$$M_{bdn} = - E\varphi''' \int^{F} w_n^2 \, dF = - E\varphi''' C_{bdn}$$

Now $M_{bd} = M_{bdu} + M_{bdn}$ and since, as it is easy to prove, $C_{bd} = C_{bdu} + C_{bdn}$, the results of this consideration agree with equation (6). We may therefore split the resistance C_{bd} due to the bending accompanying the torsion into the two parts

$$C_{bdu} = \int^{F} w_u^2 \, dF \quad \text{and} \quad C_{bdn} = \int^{F} w_n^2 \, dF = \frac{1}{12} \int_0^{U} s^3 \, r_n^2 \, du \tag{6b}$$

It should still be mentioned that the shearing stresses τ_{bd} give zero for a resultant shearing force.

Since the solution of differential equation (5) is also well known for this special problem, we need go no further into it.

TWISTING OF COMPRESSION MEMBERS

If a relatively thin-walled open section (for example, an angle) is put under compression, each leg tends to buckle in a direction normal to its plane (fig. 3). The part of the leg lying against the joining edge supports itself against the other leg which offers a large resistance moment against these stresses and consequently hinders the buckling of the first leg, and conversely.

Two modes of buckling are possible, namely, both legs may buckle in the same direction so that the whole section twists to one side, or the legs may buckle in opposite directions, in which case there is a deformation in the cross section and the work of deformation is therefore much greater in this case. Since, where there are two possible

modes of buckling, the one to occur first is that at which the work of deformation is smaller for the same work of the external loads, the first mode of buckling will occur, i.e., the angle will twist aside

We should now like to discuss the general case. Let an open section which was initially straight be acted on by a compressive and - in general - eccentric load P, whose line of action is parallel to the axis of the member. We shall denote the mean compressive stress by $\sigma_{P_o} = P/F$ and by σ_p, the compressive and bending stress that varies over the cross section, assuming these simply computed stresses as known.

We shall assume further for our computation that the bending due to the eccentricity of P and the angle of twist φ are small in magnitude so that square terms, products and their derivatives may be neglected.

We consider at any section the equilibrium of the external with the internal forces. A necessary consequence of this consideration is that the twisting of the member about the shear axis must take place. (See below.)

Due to the twisting, longitudinal stresses σ_{bd} are set up which give a resultant zero. (See equation 2a.) The shear stresses arising from the twisting give as a resultant a pure torsional moment of magnitude. (See equation 5a.)

$$M_{bd} + M_T = - E C_{bd} \varphi''' + G J_T \varphi'$$

The stresses σ_p are in the direction of the longitudinal fibers of the section so that, due both to the twisting and bending of the section, they are obliquely inclined to the direction of the original axis. We shall now consider more closely the horizontal components of these stresses.

1. Horizontal components of σ_p due to the twisting. The angle of inclination of a fiber as a result of the twisting is $r\varphi'$ so that the horizontal component of σ_p is $\sigma_p r\varphi'$ and its direction is normal to r. These stresses produce a moment about the shear center of magnitude

$$M_{P\varphi} = \varphi' \int^{F} r^2 \sigma_P \, dF = \varphi' \, \sigma_{P_o} \int^{F} \frac{\sigma_P}{\sigma_{P_o}} r^2 \, dF = \varphi' \, P \, i_{SP}^2$$

where for briefness we have set the easily evaluated integral

$$\frac{1}{F} \int^{F} \frac{\sigma_P}{\sigma_{P_o}} r^2 \, dF = i_{SP}^2 \qquad (7)$$

For a centrally acting force P, $M_{P\varphi} = \sigma_{P_o} J_S$ where J_S is the polar moment of inertia of the cross section about the shear axis. The setting up of this moment $M_{P\varphi}$ is the reason for the twisting aside of centrally compressed open sections.

2. Horizontal components of σ_P due to the bending.- We denote the principal moments of inertia of the cross section by J_η and J_ζ, and the coordinates measured from the shear center in the direction of the axes of the principal axis of inertia by η_S and ζ_S. The remaining notation is indicated on figure 4.

We split up the bending moment due to the eccentricity of P into two components about the principal axes of inertia. The bending angle due to the bending about the η axis is

$$\frac{dy_\eta}{dx} = \int^{x} \frac{d^2y}{dx^2} \, dx = - \frac{P \, \zeta_{OP}}{EJ_\eta} \int^{x} dx$$

The limits of this integral we leave undetermined since we shall later differentiate. The horizontal components $\sigma_P \frac{dy_\eta}{dx}$ of σ_P act in the direction of the ζ axis, and produce about the shear center a torsional moment of magnitude

$$\int^{F} \sigma_P \frac{dy_\eta}{dx} \eta_S \, dF = \frac{dy_\eta}{dx} P \, \eta_{SP} = - \frac{P^2}{E} \frac{\zeta_{OP} \, \eta_{SP}}{J_\eta} \int^{x} dx$$

In a similar manner we compute the torsional moment due the bending about the ζ axis. For the total torsional moment due to the bending, we then obtain

$$M_{Pb} = - \frac{P^2}{E} \left(\frac{\zeta_{OP}\, \eta_{SP}}{J_\eta} - \frac{\eta_{OP}\, \zeta_{SP}}{J_\zeta} \right) \int^{\chi} dx = - \frac{P^2 B}{E} \int^{\chi} dx \qquad (8)$$

where the expression in the parentheses is denoted by B.

Due to the displacement of the center of gravity of the section as a result of the twisting about the shear center and also due to the bending, the eccentricity of the load P varies along length of the member and thus additional bending moments arise. It may here be stated without proof that no moment about the shear center is produced by those longitudinal stresses arising from the additional bending moments and which correspond to a slight increase in twist and bending, but that the corresponding shear stresses, however, produce cross forces at the shear center which just balance the cross forces of the horizontal components discussed under paragraphs 1 and 2, so that in every respect there is equilibrium with the external forces, provided the condition

$$M_T + M_{bd} = M_{P\varphi} + M_{Pb}$$

is satisfied.

We must still consider the very important case where the section is elastically supported against twisting, so that at every position there is exerted an external torsional moment $M = \frac{M}{\varphi} \varphi$ proportional to the twisting at this position and oppositely directed. Thus let $\frac{M}{\varphi}$ be given and assumed constant along the entire length of the member. Again considering the equilibrium at a cross section, we have the additional torsional moment

$$- M_m = \int^{\chi} \frac{m}{\varphi} \varphi \, dx$$

If we now set

$$M_T + M_{bd} = M_{P\varphi} + M_{Pb} - M_m$$

and differentiate with respect to x we obtain the differential equation of the twisting.

$$\varphi''' E C_{bd} + \varphi'' (P i_{SP}^2 - G J_T) + \varphi \frac{m}{\varphi} = \frac{P^2}{E} B \qquad (8)$$

The solutions of this differential equation are well known and we may therefore confine ourselves to the most important case occurring in practice, namely, where the warping of the end sections of the member is not prevented. Let the boundary condition be

For $x = 0$ and $x = L$ $\varphi = 0$ and $\varphi'' = 0$

We neglect at first the part M_{pb} due to the bending of the section, thus setting $B = 0$. If there is no elastic support $\frac{M}{\varphi} = 0$, the solution of the differential equation is $\varphi = \varphi_0 \sin \frac{\pi x}{L}$ (φ_0 = angle of twist in the middle), and we obtain by substituting the buckling load P into equation (8)

$$P = \frac{1}{i_{SP}^2}\left(G J_T + \frac{\pi^2}{L^2} E C_{bd}\right) \tag{9}$$

Where there is elastic support the computation shows (in a similar manner as for the usual elastically supported buckling rod) that one or more waves of deformation are formed according to the amount of elastic support and the length of the member. If we set the length of a wave equal to L/n, where n is an integer, so that $\varphi = \varphi_0 \sin \frac{n\pi x}{L}$, we obtain the buckling load

$$P = \frac{1}{i_{SP}^2}\left(G J_T + \frac{n^2\pi^2}{L^2} E C_{bd} + \frac{L^2}{n^2\pi^2}\frac{m}{\varphi}\right) \tag{9a}$$

Figure 5 shows the buckling loads for different values of n and L. Since a member always buckles with a number of waves corresponding to the minimum buckling load, the number of waves increases with length of member and for very great lengths the buckling load becomes almost independent of the length of the member. For long members we obtain by differentiation of equation (9a) $\left(\frac{dP}{dL/n} = 0\right)$ the wave lengths L_1, corresponding to the minimum buckling loads

$$\frac{L}{n} = L_1 = \pi \sqrt[4]{\frac{EC_{bd}}{\frac{m}{\varphi}}} \tag{10}$$

By substituting this value in equation (9a) we obtain

the magnitude of the buckling load (or, more accurately, the minimum buckling load) for $L \geqq L_1$.

$$P_{min} = \frac{1}{i_{SP}^2}\left(G J_T + 2\sqrt{\frac{m}{\varphi} E C_{bd}}\right) \tag{10a}$$

For $L \leqq L_1$, only one wave is formed, so that

$$P = \frac{1}{i_{SP}^2}\left(G J_T + \frac{\pi^2}{L^2} E C_{bd} + \frac{L^2}{\pi^2}\frac{m}{\varphi}\right) \tag{10b}$$

If the load acting on the member is smaller than the buckling load given by this equation, there will be no twisting of the member (always assuming that $B \neq 0$). If, however, the load does attain this value, the member collapses abruptly as a result of the twisting (pure stability problem).

If we take into account the lateral bending of the member ($B \neq 0$) we are led to considerably more complicated solutions. We obtain, in fact, four forms for the solution, according to whether or not the member is elastically supported against twisting and whether P is small or already near the buckling load. These solutions show, however, that φ first becomes very large when P attains the value given by the previous equations - which value, therefore, we may also consider in this case as the buckling load. The eccentricity is taken into account by computing the value of i_{SP} from equation (7).

We obtain, for example, for a nonelastically supported member and for a large load P, the solution

$$\varphi = \frac{P^2 B C_{bd}}{(P i_{SP}^2 - G J_T)^2}\left[\frac{1-\cos \omega L}{\sin \omega L}\sin \omega x + \cos \omega x - 1 - \frac{\omega^2 x}{2}(L-x)\right] \tag{11}$$

$$\omega^2 = \frac{P i_{SP}^2 - G J_T}{E C_{bd}} \tag{11a}$$

We see that the buckling load is reached for $\omega L = \pi$, from which we obtain the value of P given in equation (9). For smaller loads P, we may compute from equation (11) the change in form and stress condition and thus determine when the yield point of the material is reached and the member actually collapses.

We shall not consider the solutions for the other cases.

It should be pointed out that for centrally loaded buckling members there is no connection between the Eulerian buckling and the twisting, the section is to be computed either for buckling (Eulerian) or twisting, according to which phenomenon corresponds to the smaller buckling load. For long members eccentrically loaded, it is possible for the bending due to the eccentricity to become so large that the stress distribution in the middle of the member is considerably different from that at the ends. For this case we suggest that the stress distribution in the middle of the member be first computed without taking the twisting into account, and then this stress distribution used as a basis for calculation of the twisting.

In order to give an idea of the magnitude of these effects there are shown in table I the theoretical buckling loads for two duralumin sections of equal cross-sectional area $F = 0.42$ cm^2 and equal width of legs 3 cm. The sections are shown in figure 6.

TABLE I

Form of cross section	1000 J_T	1000 C_{bd}	i_{mP}^2	Theoretical buckling load $\frac{m}{\varphi} = 0$			$\frac{m}{\varphi} = 3$		
	cm^4	cm^6	cm^2	L=20	40	80	20	40	80
⌜	0.695	0.515	3	65	63	62	86	86	86
⌜⌝	.540	35.7	4	188	73	44	218	171	171

If such sections are used as spars for sheet metal beams so that tensile forces act laterally on it (reference 5), back moments are produced during the twisting, the magnitude of which is proportional to the amount of twisting. If, for example, these sections are loaded laterally with a load of 1 kg/cm, $\frac{m}{\varphi} = 3$. The buckling loads of the members are given for this value, not taking the bending moments into account. It may be seen that now the section with the edges turned over is more favorable for all lengths since, according to equation (10a), the value C_{bd} raises the buckling load, particularly when there is elastic support.

In order to show the large effect of eccentric loading on the buckling load, there is indicated in figure 7 the ratio of the buckling loads P for an angle loaded eccentrically in the plane of symmetry (B = 0) to the buckling load P_0 of the centrally loaded section. The nearer to the edge of the angle the force acts, the larger is the buckling load until finally there is no question of twisting.

It should be further noted that in tests of compressively loaded sections set between parallel plates, the twisting likewise begins at the computed loads, but after the twisting no load acts on the legs so that the line of action of the compressive force moves toward the edge and thus a higher compressive force may be borne. The corresponding computation is relatively easy to make.

If very thin-walled angles (or sections of similar cross section) are eccentrically loaded in such a manner that the legs are highly loaded while the edge is less loaded, then the phenomenon occurs of each leg buckling in opposite direction. (The profile thus does not twist.) This phenomenon is easily explained, though the necessary computation is difficult. The buckling load is smaller than would be the case if the section twisted.

It may be remarked, finally, that only such sections tend to twist aside for which the value of C_{bd}/i_{SP}^2 is small compared to the moment of inertia of the cross section. (See equation 9.) For other sections, provided the yield point is not reached first, there occurs either Eulerian buckling or each leg buckles individually without the member as a whole twisting aside. If the legs of a U section are turned over they may be considered elastically fixed, and with the aid of the preceding principles it is possible to compute at least approximately the twisting (buckling) of the leg. For the computation of the buckling of legs that are not turned over, the methods of Timoschenko for rigidly fixed or entirely free plates may be employed (reference 6).

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

REFERENCES

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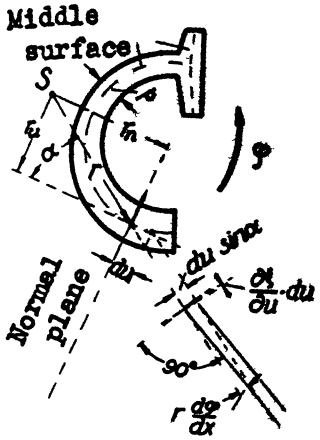

Fig. 1

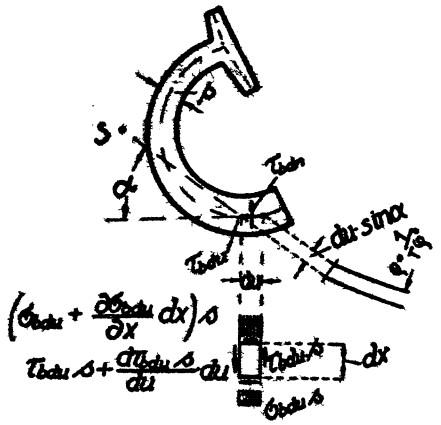

Fig. 2

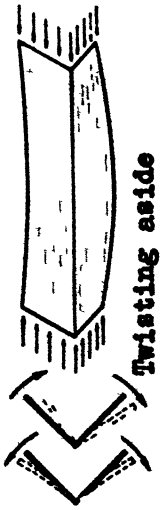

Fig. 3

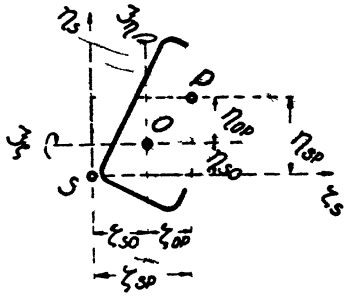

Fig. 4

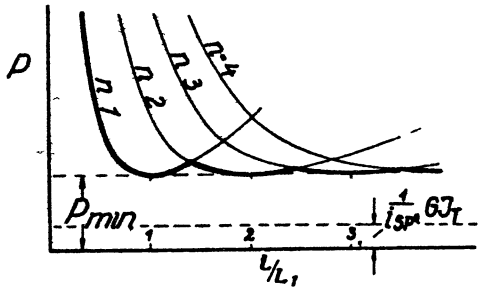

Fig. 5

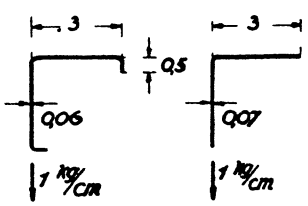

Fig. 6

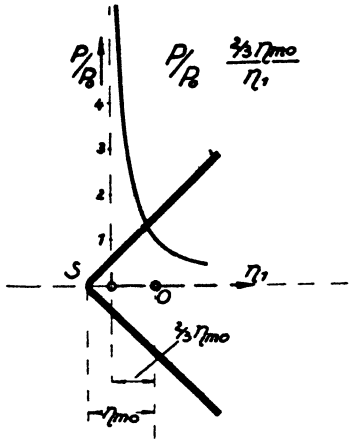

Fig. 7