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No. 793

BENDING OF BEAMS OF THIN SECTIONS

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1. INTRODUCTION

The tendency toward economy of material and lightness of structure has long since led to the increased application of beams having large ratios of moment of area W to cross-sectional area F. According to the elementary theory of bending, within the limits of application of Hooke's law, the critical value of the bending moment M_{crit}, also called briefly, though incorrectly, the breaking moment, is proportional to W (for the same beam material). Since the weight of each unit of length of the beam is proportional to F, the ratio $F:W$ is a measure of the lightness of a beam subject to any given bending moment. As this ratio has the dimension $[L]^{-1}$ it changes with change in unit of length L. This inconvenience is evidently not possessed by the ratio

$$F^{3/2} : W = \beta$$

which is nondimensional and can therefore be used for the comparison of different forms of cross-sectional areas with respect to lightness. For a rectangle of base b and height h, for example,

$$F = b\ h, \qquad W = \frac{b\ h^2}{6}, \qquad \beta = 6\sqrt{\frac{b}{h}}$$

that is, a beam of rectangular cross section is lighter the smaller the ratio $b:h$.

Similarly, we find for an ellipse with semiaxes b and h the ratio

$$\beta = 4\sqrt{\pi}\sqrt{\frac{b}{h}} = 7.09\sqrt{\frac{b}{h}}$$

*"Zginanie belek prostych o przekrojach wiotkich." Instytut Badan Technicznych Lotnictwa, Sprawozdanie Kwartalne, No. 3, pp. 5-13. Warsaw, 1930.

and therefore for the same ratio of b to h an elliptical cross section gives a considerably heavier beam.

These and similar sections belong to the class of close, or compact, cross sections. If the ratio b:h is too small, there may occur a deformation of the beam in the plane of the h axis. For this reason alone we cannot go too far in decreasing the value of this ratio. The magnitude of the critical bending moment M_{crit} would then be decided not by the value of W but by other cross-sectional magnitudes alongside with the elasticity of the material and the distribution of external loads. (See section 15 of "Study of I Beams," Warsaw, 1923, by the author.) In this case we classify the beams as thin or slender beams.

Now the most typical forms of cross section as well as the most important from a practical viewpoint are the double T, the "box" form, and the tubular (figs. 1 to 3).

Let us investigate the lightness of beams of these cross-sectional areas, that is, compute the ratio β, assuming for simplicity that the thickness of the web is very small in comparison with the other dimensions.

a) Double T section of width b and height h measured between centers of flanges of thickness δ. The web is of thickness δ,

$$F = 2\,b\,\delta + h\,\delta_1$$

$$W = b\,\delta\,h + \frac{\delta_1\,h^2}{6}$$

$$I = 2\,b\,\delta\left(\frac{h}{2}\right)^2 + \delta_1\,\frac{h^3}{12}$$

from which it is easily found that

$$\beta = 6\sqrt{\frac{b}{h}}\sqrt{\frac{\delta}{h}}\;\frac{\left(\frac{\delta_1}{\delta}\,\frac{h}{b} + 2\right)^{3/2}}{\frac{\delta_1}{\delta}\,\frac{h}{b} + 6}$$

The same expression evidently holds for the box cross section with the exception that δ_1 is replaced by $2\delta_2$. It clearly shows the advantages of these sections compared with the rectangular. It is clear that these cross sections, too, in the case of too large a decrease in δ_1 and b, and therefore β, undergo deformation and the expressions given by the usual bending theory lose their significance unless deformation is avoided by the use of strengthening ribs.

b) For the tubular cross section of radius r measured to the center of the wall of thickness δ,

$$F = 2\, r\, \pi\, \delta, \qquad I = \pi\, r^3\, \delta, \qquad W = \pi\, r^2\, \delta,$$

$$\beta = \sqrt{8\pi}\sqrt{\frac{\delta}{r}} = \sim 5\sqrt{\frac{\delta}{r}} = 7.09\sqrt{\frac{\delta}{2r}}$$

For this section the lightness ratio is, to be sure, less than that obtained for the others, but on the other hand, the stiffness of the curved tube wall is undoubtedly greater than that of the flat plates and therefore the tubular section can be made much thinner.

To the question, How much thinner? we shall attempt to provide an answer in our present paper.

2. THEORY OF "TRANSVERSE" BENDING

It is well known that the exact solution of the equations of the mathematical theory of elasticity is not yet sufficiently developed to explain the behavior of deflected beams for the case where the displacement of its points is of the same order of magnitude as the least of its dimensions. More accurate results are given by the classical theory of the bending of "infinitely thin" elastic beams leading to the well-known equation:

$$\frac{1}{\rho} = \frac{M}{EI}.$$

Here EI is a measure of the stiffness of the beam in the principal plane, M the bending moment at the section, ρ the radius of curvature of the neutral axis. (A more detailed consideration of this subject the reader will find in the author's work, "Criterion for the Steadiness of Equilibrium," published by the Academy of Technical Sciences, 1926, especially in paragraphs 14, 17, and 19.) But it is not difficult to show that this theory, too, is only approximate in all cases where ρ is not very large in comparison with the cross-sectional dimensions.

Let us imagine a "fiber" element of length ds parallel to the neutral axis cut out from the beam, and choose the Z axis as shown in figure 4 in the principal bending plane. (The Y axis on the figure is directed toward the observer.) Then, the unit stress $\sigma = \frac{M}{I} Z$ of the classical theory acting on each of the fiber elements of area $dF = dy\ dz$ gives rise to two forces $\sigma\ dF$ inclined to each other at an angle $d\alpha$ and therefore not fulfilling the conditions of equilibrium for this element. This leads the neighboring elements to exert on each other transverse forces whose resultant $\sigma\ dF\ d\alpha$ is perpendicular to the axis of the fiber element and lies in the plane of bending.

This resultant is evidently determined by the transverse stresses σ' which are functions of z and y and which the classical theory neglects.

The stresses σ' satisfy the equation of equilibrium,

$$\frac{d\sigma'}{dz}\, dz\ ds\ dy = \sigma\ dF\ d\alpha = \sigma\ dz\ dy\ d\alpha$$

or substituting $d\alpha = \frac{ds_o}{\rho}$ in the above equation,

$$\frac{d\sigma'}{dz}\, ds = \sigma\, \frac{ds_o}{\rho}$$

Since

$$\frac{ds}{ds_o} = \frac{\rho + z}{\rho} = 1 + \frac{z}{\rho}$$

$$\frac{d\sigma'}{dz}\left(1 + \frac{z}{\rho}\right) = \frac{\sigma}{\rho}$$

Since the presence of the transverse stress σ' modifies the longitudinal stress σ, we see that we may not, in general, simply put $\frac{1}{\rho} = \frac{M}{EI}$ as given by the classical bending theory. However, we may consider the above as a first approximation and assuming z/ρ to be small compared with 1, obtain the following expression for the transverse stress:

$$\frac{d\sigma'}{dz} = \frac{\sigma M}{EI} = \frac{M^2}{EI^2} z$$

As a result of the presence of the stress σ' the fiber element of longitudinal area $dy\ dx$ is subjected to a pressure which is always directed toward the neutral layer. The stress on each unit of length of fiber is

$$\frac{d\sigma'}{dz} du\ dz = \frac{M^2}{EI^2} z\ du\ dz = \frac{M^2}{EI^2} z\ dF$$

The resultant of the stresses along the chord DD' (fig. 5) exerted by the area F_1 on the remainder has the value

$$P' = \int_{(F)} \frac{d\sigma'}{dz} du\ dz = \frac{M^2}{EI^2} \int_{(F)} z\ dF = \frac{M^2 S}{EI^2}$$

Here S denotes the static moment of the area indicated in the figure. The same dependence on S is shown by the shearing stress per unit length of the beam arising from the shearing stress T, namely, $T' = \frac{TS}{I}$. Hence the shearing stress P' on a deflected beam subject to a bending moment M has its largest value at the neutral layer similar to the shearing stress T'.

The important difference in the two kinds of stresses consists in this, that T' has the same direction over the whole area depending on the direction of T, whereas the stresses P' are oppositely directed on both sides of the neutral axis, always pointing toward this axis.

It is clear that both these types of stresses play a subordinate part in the case of compact beams and may be neglected. In the case of slender beams, however, these stresses may become of first importance and must be taken into consideration to obtain a more accurate evaluation of the strength of such beams.

3. DOUBLE T AND BOX SECTIONS
(Figs. 6 and 7)

If we consider the beam divided into sections by planes at unit distance apart, then each section will act as a uniformly loaded frame under the stresses P' (if we neglect for the present the effect of any transverse force). The flanges will act as horizontal beams under a uniform loading of magnitude,

$$p_o = \frac{M^2}{EI^2} \frac{\delta_o h_o}{2}$$

and the web through connection with the flanges will be exposed to longitudinal stresses and bending. The web will be acted upon first by the flange stresses:

$$(a + a_1)\ p_o = \frac{M^2}{EI^2} \frac{h_o}{2} (a + a_1)\ \delta_o$$

secondly, by the stresses:

$$p_z = \frac{M^2}{EI^2} \delta_z\ z$$

distributed continuously along half the depth h_1, and finally, by the constant and statically undetermined moment M_1.

We compute the latter, using the principle of minimum strain energy for the whole frame, considering only the energy of bending, and introduce the following notation:

B_o in kg cm transverse bending modulus of each section of the flange of width, 1 cm.

B_1 the same for the section of the web with a view toward eventual strengthening by stiffening ribs.

M' in kg bending moment of a section of the flange or web of width, 1 cm.

Since the bending moments of the projecting parts of the flanges do not depend on the magnitude of the unknown moment M_1, it is sufficient to express the energy for the middle parts of the flanges and the web. We therefore have for the flange (fig. 8):

$$M' = M_1 + \frac{1}{2}\ p_2 \frac{h_o}{2} x - Q'x + p_o \frac{(x + a_1)^2}{2}$$

in which

$$p_2 = (p_z)_{z = \frac{h_o}{2}} = \frac{M^2 \delta_2}{EI^2} \frac{h_o}{2}$$

$$Q' = \frac{M^2}{EI^2} \frac{h_o}{2} \left[(a + a_1)\, \delta_o + \frac{h_o\, \delta_2}{4} \right]$$

For the web M' is simply equal to M_1 $\frac{\partial M'}{\partial M} = 1$ for both parts and differentiating the strain energy with respect to M_1, the condition is obtained:

$$\int_o^a \frac{M}{B_o} \frac{\partial M}{\partial M_1}\, dx + \int_o^{h_o/2} \frac{M}{B_1} \frac{\partial M}{\partial M_1}\, dz = 0$$

or

$$\int_o^a \left[M_1 + \frac{p_2\, h_o}{4}\, x + p_o \frac{(x + a_1)^2}{2} - Q'\, x\right] dx + \frac{B_o}{B_1}\, M_1 \frac{h_o}{2} = 0$$

Substituting the values and solving for M_1, we find:

$$M_1 = \frac{M^2\, a\, h_o\, \delta_o}{6\, EI^2} \frac{2a^2 - 3a_1^2}{2a + \frac{B_o}{B_1} h_o}$$

It is seen that the most advantageous conditions for the web are obtained when

$$a_1 = a \sqrt{\frac{2}{3}} = 0.816\ a = \text{approximately } 0.8\ a$$

in which case the moment M_1 disappears and the web is under compression only. The longitudinal force at the ends has the value

$$N' = \frac{M^2\, h_o}{2EI^2} (a + a_1)\, \delta_o$$

or about equal to Q'.

Under these conditions the web must be insured against buckling and the critical values of N' and Q' must be computed. We have to deal here with the general problem of Yasinsky, introducing some very laborious calculations in order to obtain an exact solution. In view of the approximate nature of the whole theory, however, we may assume the beam to be loaded at the end sections only to simplify the problem.

The value of the loads can be taken simply:

$$\frac{N' + Q'}{2} = \bar{N}'$$

or

$$\bar{N}' = \frac{M^2 h_o}{2EI^2} \left[(a + a_1)\ \delta_o + \frac{1}{8} h_o\ \delta_2\right]$$

For a given critical value of the principal bending moment M:

$$\bar{N}' \leqq \pi^2 \frac{B_1}{h_o^2} \quad \text{and} \quad Q' \leqq F_1\ \sigma_{crit}$$

where F_1 denotes the area of cross section of a section of the web of 1 cm width, taking into account eventual strengthening by vertical ribs.

In the general case where $M_1 \neq 0$, it is necessary to consider the web as a rod on whose ends are acting compression forces of magnitude $\bar{N}'$ and moments M_1. In that case the coefficient for the cross section of the web W_1 taken for a width of 1 cm (taking stiffening ribs into consideration) can be computed to a sufficient degree of approximation from the equation:

$$W_1 = \frac{M_1}{\sigma_{crit}\left(1 - \frac{\bar{N}'}{N'_E}\right)}$$

where

$$N'_E = \pi^2 \frac{B_1}{h_o^2}$$

The shearing force T on the given cross section evidently does not affect the character of the transverse bending, but only requires a decrease in the assumed value of σ_{crit} in the above expressions according to some suitable hypothesis. Let the normal stress for an n-point loading computed from the expressions given above, be equal to σ_1 and the continuous stress determined by the transverse force T, be equal to τ_1. If the material is of elastic metal, then according to the most accurate present-day energy hypothesis, the strain of the material is measured by the effective stress:

$$\sigma_{red} = \sqrt{\sigma_1^2 + 3\ \tau_1^2}$$

A certain complication in the computation is introduced by the circumstance that σ_1 is proportional to the square of the load and not to the first power as τ_1 is. In this case the effective stress will not be proportional either to the load or to its square.

In those cases where the load is carried in a given manner on one of the flanges, additional normal stresses arise in the longitudinal cross sections which for the case of thin sections must also be taken into account. On this matter we must refer the reader to the work already mentioned, "Study of I Beams," pointing out, however, that this work still follows the old hypothesis of maximum continuous stress, which in recent years has given way to the theory of maximum strain energy used above.

4. TUBULAR CROSS SECTION
(Figs. 9 and 10)

The ring cut out by two planes separated by 1 cm distance, is subject to transverse bending by the forces perpendicular to the neutral axis and directed toward it. The magnitude of these forces on a fiber element of area $dF = \delta\ r\ d\varphi$ is given by the expression found above, namely,

$$\frac{M^2}{EI^2}\ z\ dF$$

Dividing this by $r\ d\varphi$, we evidently obtain the force per unit area of the cross section of the cylinder of radius r, or

$$p_z = \frac{M^2}{EI^2}\ z\ \delta = C\ z\ (kg/cm^2)$$

if we put $C = \frac{M^2}{EI^2} \delta$ (kg/cm³). Considering half of the ring on one side of the neutral axis and replacing the combined forces by Q and the statically undetermined moments by M_0, we easily find the expression for the bending moment for the section defined by the angle φ to be

$$M = M_0 - Q\, r\, (1 - \cos\varphi) + \int_0^{\varphi} C\, z\, r\, d\psi\, r\, (\cos\psi - \cos\varphi)$$

The force Q is obtained from the projections

$$2Q = \int_0^{\pi} p_z\, r\, d\varphi = \int_0^{\pi} r \sin\varphi\, r\, d\varphi$$

whence $$Q = C\, r^2$$

Putting this value into the expression for M and performing the integration, we obtain:

$$M = M_0 - C\, r^3\, (1 - \cos\varphi) + C\, r^3 \left(\frac{3}{4} + \frac{1}{4}\cos 2\varphi - \cos\varphi\right)$$

Since $\frac{\partial M}{\partial M_0} = 1$, the condition for the determination of M_0 is of the form

$$\int_0^{\pi/2} M\, d\varphi = 0$$

whence $M_0 = \frac{1}{4} C\, r^3$ and $M_1 = (M)_{\varphi = \frac{\pi}{2}} = -\frac{1}{4} C\, r^3$

are the limiting values for the bending moment.

The dangerous section is evidently that which is acted on by M_0, since the largest longitudinal force Q appears here. Computation shows that for a sufficiently small value of δ/r, it is possible for the stress resulting from this transverse bending to exceed considerably the principal bending stress. In this case the tube is reinforced by strengthening members fixed to the tube wall with distances b between them (fig. 11). The presence of these stiffening members causes an increase in the transverse bending stiffness of the tube sections of width b. Under the most advantageous condition where the stiffening members are only weakly joined to the tube wall, the stiffness of this section B_b will at least be equal to the sum of the stiffness of the wall itself; that is,

$$\frac{E}{1 - \mu^2} \frac{b\,\delta^3}{12} \quad \text{(where } \mu \text{ is Poisson's ratio)}$$

and the stiffness of the member $B_w = EI_w$

or

$$B_b = \frac{E}{1 - \mu^2} \frac{b\,\delta^3}{12} + B_w$$

In this case the bending moment $M_o\,b$ divides itself between the wall and the stiffening member in simple proportion to their stiffness, and therefore the moment on the wall will be

$$M'_o = M_o \frac{1}{1 + \dfrac{12\,B_w\,(1 - \mu^2)}{E\,b\,\delta^3}}$$

As far as the compression force Q is concerned, a certain part, to be sure, is supported by the strengthening member, but on the assumption of very loose joining this part will be very small. We shall therefore assume in what follows that the entire force Q is taken up by the cross section of the tube wall. Under these conditions, we compute the limiting compressive stress in the material of the wall from the expression:

$$\sigma^* = \frac{Q}{1\,\delta} + \frac{\sigma\,M'_o}{1\,\delta^2}$$

Considering that $I = \pi\,r^3\,\delta$ and denoting the area and moment of inertia of a member by F_w and I_w, respectively, we find on substitution of the proper values the following expression for the limiting stress in the tube wall resulting from the transverse deflection caused by the principal bending moment M:

$$\sigma^* = \frac{M^2\,r^2}{EI^2}\left(1 + \frac{3}{2}\frac{r}{\delta}\frac{1}{1 + \alpha}\right); \quad \alpha = \frac{12\,(1 - \mu^2)\,I_w}{b\,\delta^3}$$

or

$$\sigma^* = \frac{M^2}{\pi^2\,E\,r^4\,\delta^2}\left(1 + \frac{3}{2}\frac{r}{\delta}\frac{1}{1 + \alpha}\right)$$

If there are no stiffening members, then obviously, α must be put equal to zero in the above expression.

Let us yet consider the case where the stiffening members are perfectly joined to the tube wall. In this case the transverse stiffness may be measured exactly as for a plate of thickness δ with ribs separated as distance b. This stiffness is equal to that of a cross section formed of the plate together with the rib, having a so-called effective decreased width b_1. This width b_1 differs from b the more, the smaller δ is compared to r, and r^2 compared to b. On the basis of certain theoretical considerations, b_1 may be assumed equal approximately to $\frac{2}{3}\,r$ if $b > \frac{2}{3}\,r$, but $b_1 = b$ if $b < \frac{2}{3}\,r$.

With these assumptions, we obtain another expression for σ^*, namely,

$$\sigma^* = \frac{Q\ b}{F_2} + \frac{M_0\ b}{I_2}\ e_2$$

Here F_2 denotes the section made up of the annular section of the tube of width b and the section of one stiffening member; I_2 is the moment of inertia of the combined sections of effective width b_1; finally e_2 is the distance of the remotest fiber in the section corresponding to I_2. Substituting the values of Q and M_0, we obtain:

$$\sigma^* = \frac{M^2\ r^2\ b\ \delta}{EI^2\ F_2}\left(1 + \frac{1}{4}\ \frac{F_2\ e\ r}{I_2}\right)$$

or

$$\sigma^* = \frac{M^2\ b}{\pi^2\ E\ F_2\ r^4\ \delta}\left(1 + \frac{1}{4}\ \frac{F_2\ e\ r}{I_2}\right)$$

Translation by S. Reiss,
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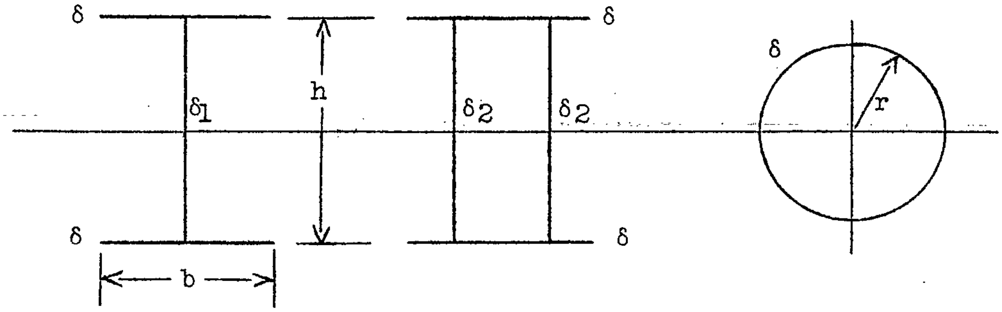

Figures 1 - 3.

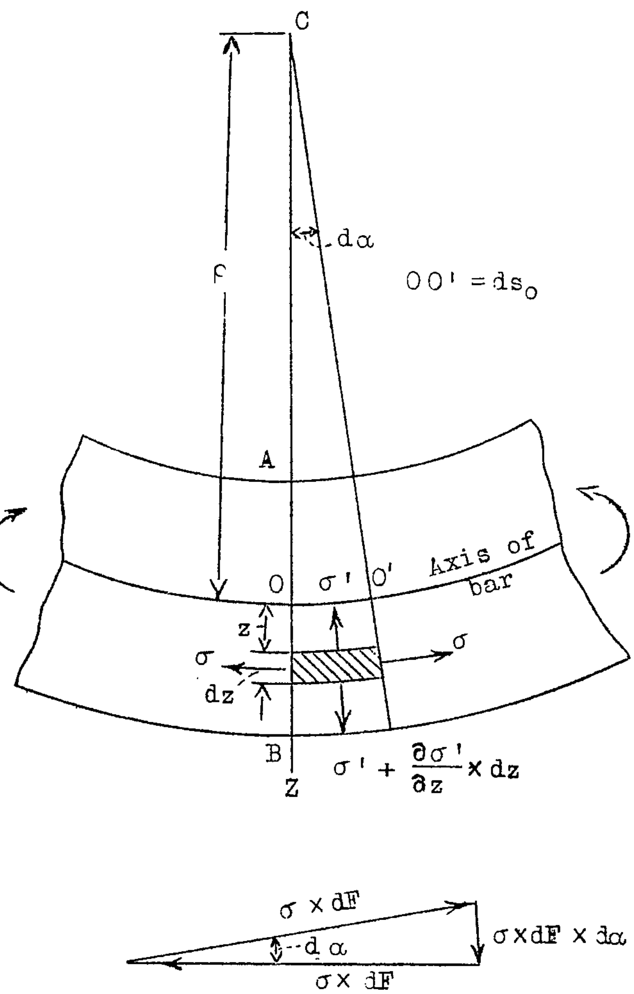

Figure 4.

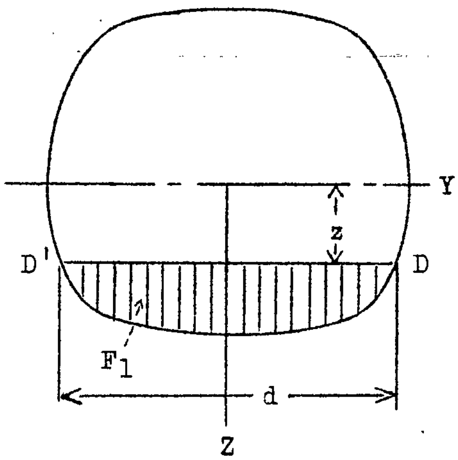

Figure 5.

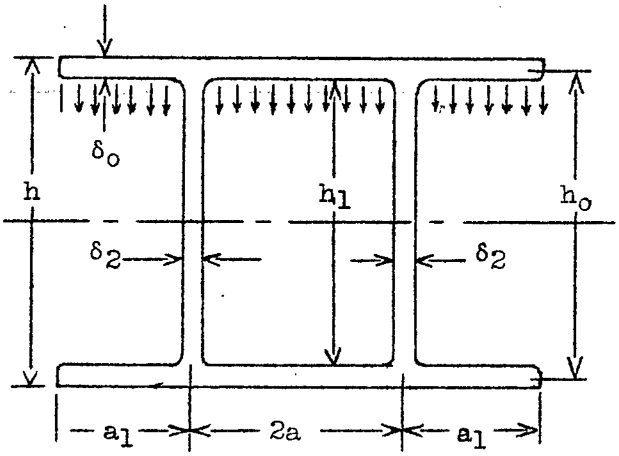

Figure 6.

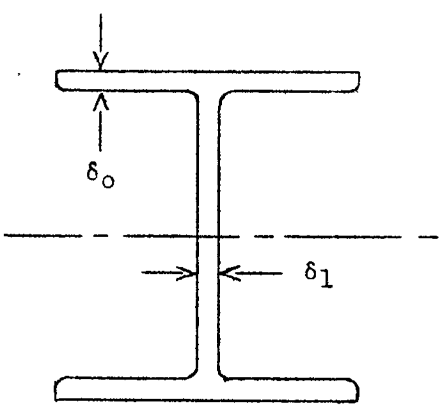

Figure 7.

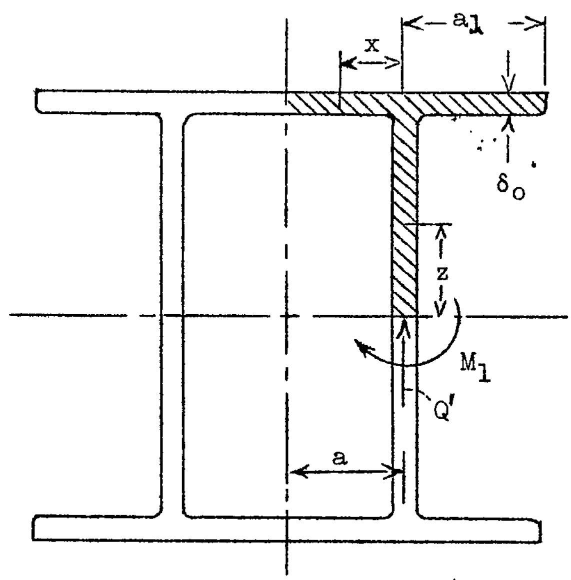

Figure 8.

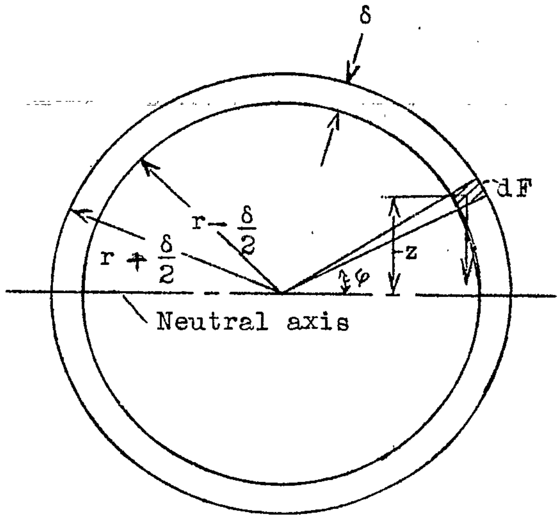

Figure 9.

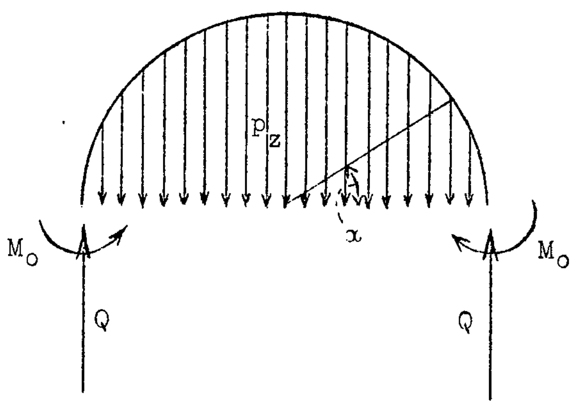

Figure 10.

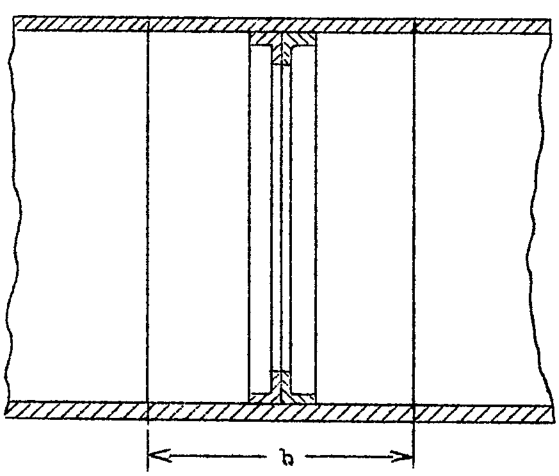

Figure 11.

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