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RESISTANCE OF PLATES AND PIPES AT HIGH REYNOLDS NUMBERS

By L. Schiller and R. Hermann

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1. Introduction

Some years ago Prandtl and Karman showed how the resistance of a plate to a flow parallel to its surfaces could be determined from the resistance of a pipe to a fluid flowing through it.** The fundamental principle was to put the frictional resistance at every point of the plate equal to that in a pipe of like Reynolds Number. By the "Reynolds Number" of the plate we mean

$$\frac{\text{flow velocity}}{\text{kinematic viscosity}} \times \text{boundary-layer thickness.}$$

On the other hand the frictional resistance must equal the loss of momentum for the distance considered. The law of velocity distribution is assumed to be the same in both cases. Prandtl and Karman carried out the calculation for the case of the "Blasius Law of Powers," for which they showed the relation between resistance and velocity distribution and obtained, within the range of not excessive R values, close agreement with the experimental results.

*"Widerstand von Platte und Rohr bei hohen Reynoldsschen Zahlen." From Ingenieur-Archiv, Sept., 1930, pp. 391-398.

**Prandtl, "Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen," Report I, p. 136; Report III, p. 1. Von Karman, Z. f. ang. Math. u. Mech., 1921, p. 233.

In the meantime it was learned that the law of resistance for high R values does not follow the simple powers, and that the powers, which can be obtained approximately for the velocity distribution, gradually change.* Since, moreover, very important investigations have recently been made on the resistance of plates at very high R values, it seemed of interest to apply the above line of reasoning to the new general law of resistance.** For this purpose, the resistance and velocity distribution along the plate must always be equal to the values of the pipe flow at the corresponding Reynolds Number. We made two kinds of calculations, of which the one given here is the simpler and more practical and also agrees better with the experimental results.***

According to a friendly communication from Dr. Lerbs, Hamburg, he has also made like calculations,**** but his method and results differ considerably from ours.*****

*Stanton and Pannel, "Similarity of Motion in Relation to the Surface Friction of Liquids." Phil. Trans. Roy. Soc., A, Vol. 214 (1914), p. 199. Jakob and Erk, "Mitteilungen über Forschungsarbeiten," issued by V.D.I. (1924), No. 267. Hermann and Burbach, "Strömungswiderstand und Wärmeübergang in Rohren." Akad. Verlagsges, Leipzig, 1930. L. Schiller, "Rohrwiderstand bei hohen Reynoldsschen Zahlen." Gilles, Hopf, Karman, Aachen lectures, 1929. See also an Aachen paper by Nikuradse.

**G. Kempf, "Neue Ergebnisse der Widerstandsforschung." Werft, Reederei, Hafen, 1929, Nos. 11-12.

***The first, which is not repeated here, I made some time ago with the aid of my former assistant, Mr. Burbach.

****The results of Dr. Lerbs' calculation are summarized by Kempf. See above footnote.

*****Results recently sent us by Dr. Lerbs are considerably more like ours than the published results, but they still show a systematically increasing deviation with increasing R values.

2. Empirical Basis

We designate by u the variable velocity within the boundary layer; by $\bar{u}$ the mean velocity in the pipe of radius a; by U the velocity at the axis of the pipe or the velocity outside the boundary layer of the plate. We further distinguish the Reynolds Number of the pipe, as based on the mean velocity, by

$$\bar{R} = \frac{\bar{u}\, a}{\nu} \left(\nu = \frac{\mu}{\rho}, \quad \mu = \text{viscosity}, \quad \rho = \text{density}\right)$$

and, as based on the velocity at the axis, by

$$R = \frac{U\, a}{\nu}$$

or, putting the pipe radius a equal to the boundary-layer thickness δ, by

$$R = \frac{U\, \delta}{\nu}.$$

Furthermore, the Reynolds Number for a plate of length x is expressed by

$$R_x = \frac{U\, x}{\nu}.$$

As the resistance law of the pipe above $\bar{R} = 50{,}000$, we take for our criterion the empirical equation which, according to our Leipzig experiments in the range of $\bar{R} = 10^4$ to $\bar{R} = 10^6$, yielded

$$\psi = 0.00270 + 0.161\, \bar{R}^{-0.300} \tag{1a}$$

whereby the resistance coefficient ψ is defined by

$$\psi = \frac{dp}{dx} \frac{2\, a}{\rho\, \bar{u}^2} \quad \left(\frac{dp}{dx} = \text{pressure drop per unit length}\right) \tag{1b}$$

Prandtl and Karman deduced from the Blasius law of pipe resistance that the velocity increases as the 1/7 power of the distance from the wall. Due to the deviation from the Blasius law, the velocity must generally be put proportional to the $1/n$ power of the distance from the wall. The values used as the basis for the exponents n of the velocity distribution were taken from the results of Nikuradse's experiments in Göttingen and the hitherto unpublished results obtained by Möbius. In accordance therewith, Figure 1 shows n as the function of log $\overline{R}$ in close agreement with both sources. The curve was extended a short distance to $n = 10$ by disregarding the n values at the highest $\overline{R}$ values. In this manner Ψ and n are represented as functions of $\overline{R}$, while the further calculation is to be made with R. The corresponding calculation is made by

$$\frac{\overline{R}}{R} = f = \frac{\overline{u}}{U}, \tag{2}$$

the ratio of the mean velocity to that at the axis, which in turn depends slightly on $\overline{R}$.

The quantity f, under the assumption of the validity of the power law of the velocity distribution up to the axis of the pipe, can be calculated as a function of n. If y denotes the distance of a point from the wall and r the distance of the same point from the axis of the pipe, then the velocity distribution is represented by

$$\frac{u}{U} = \left(\frac{y}{a}\right)^{\frac{1}{n}} = \left(\frac{a - r}{a}\right)^{\frac{1}{n}} \tag{3a}$$

The mean velocity is defined by

$$\pi a^2 \bar{u} = 2\pi \int_0^a r u d r .$$

If we substitute the value of u according to equation (3a) and integrate, we obtain

$$f = \frac{\bar{u}}{U} = \frac{2 n^2}{(n + 1)(n + 2)} \tag{4}$$

Figure 2 shows f as a function of $\bar{R}$. In reality the velocity curve shows certain deviations from the power law at the axis and at the wall. Since, moreover, the values of f (Fig. 2), within their scattering, coincide with the curve calculated according to equation (4), these values will be used in what follows. In their application to the case of the Blasius law, i.e., small R, they occasion slight discrepancies between our results and Karman's *

3. The Differential Equation of Reynolds Numbers for Pipes and Plates

a) Momentum theory.- If we apply the momentum theory to the steady flow along a plate (x component, Fig. 3) in the sense of the boundary-layer theory with increasing boundary-layer thickness, it means that the variation J' in the quantity of fluid per second, whose motion is retarded at the wall by the

*Karman corrected the value f = 0.817, according to the Blasius law, to 0.84, that is, he rounded the profile strongly, which seems too high to us with respect to the recent investigations by Stanton and Pannell and by Schiller and Kirsten, Z. f. tech. Physik 10 (1929), p. 268, who obtained 0.81 and 0.82 with a sufficiently developed profile.

frictional force W, is equal and opposite to the frictional force

$$J' = - W \tag{5a}$$

If we refer the change of momentum on a plate of unit width to the unit length, it then equals the friction per unit area or the internal wall friction τ_0.

$$\frac{dJ'}{d\,x} = - \tau_0 \tag{5b}$$

<u>b) Total momentum change</u>.- We will consider the quantity of retarded fluid $\int_0^\delta \rho\, u\, d\, y$ passing through the cross section at the point x (unit width) per unit of time. Before the beginning of the retardation, this quantity had the full velocity U, hence the momentum $U \int_0^\delta \rho\, u\, d\, y$. In the cross section x the elements of this quantity have the velocity u, and therefore the whole quantity has the momentum $\int_0^\delta \rho\, u^2\, d\, y$.

The whole momentum change J' from the beginning of the retardation to the point x accordingly amounts to

$$J' = \int_0^\delta \rho\, u^2\, d\, y - U \int_0^\delta \rho\, u\, d\, y.$$

If, according to equation (3a), we write

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{n}} \tag{3b}$$

as the velocity distribution in the boundary layer and integrate, we thus obtain

$$J' = - \rho\, U^2\, \delta\, \frac{n}{(n + 1)\,(n + 2)} \tag{6}$$

c) Momentum change per unit length.- We obtain the momentum change per unit length (equation 5b) by differentiating equation (6) according to x, while taking into account the fact that and n are functions of x. Through extension with U/ν corresponding to the above notation, the differential coefficient $\frac{d\delta}{dx}$ can be written as $\frac{dR}{dR_x}$, and likewise $\delta \frac{dn}{d\delta}$ as $R \frac{dn}{dR}$. If, for abbreviation, we write

$$\frac{n}{(n + 1)\ (n + 2)} - \frac{n^2 - 2}{(n + 1)^2\ (n + 2)^2} R \frac{dn}{dR} = G\ (R), \qquad (7a)$$

we obtain

$$\frac{dJ'}{dx} = - \rho\ U^2\ \frac{dR}{dR_x}\ G\ (R). \qquad (7b)$$

d) Internal wall friction.- The equilibrium equation for a fluid cylinder of unit length

$$\frac{dp}{dx}\ \pi\ a^2 = 2\ a\ \pi\ \tau_0$$

together with equation (1b) yields, for ψ, the relation between the internal wall friction and the resistance coefficient

$$\tau_0 = \frac{\psi}{4}\ \rho\ \bar{u}^2$$

or, with respect to equation (2),

$$\tau_0 = \frac{1}{4}\ \rho\ U^2\ f^2 \psi \qquad (8a)$$

For the case of the Blasius law

$$\psi = 0.1330\ \overline{R}^{-\frac{1}{4}}$$

with $f = 0.817$, we have

$$\tau_0 = 0.0233\ \rho\ U^2\ R^{-\frac{1}{4}} \tag{8b}$$

e) The differential equation.- The introduction of the momentum change per unit length (7b) and the wall friction (8a) into the momentum theorem (5b) yields the differential equation between the Reynolds Number R of the pipe (or the boundary-layer thickness) and the Reynolds Number R_x of the plate length

$$\frac{dR_x}{dR} = \frac{4G}{f^2\,\psi} = F\ (R). \tag{9a}$$

A closed integration of equation (9a) is excluded by the complicated construction of the functions f, ψ and G, and a graphic integration is necessitated by the fact that the empirical functions on the right are already given graphically. Figure 4 shows the differential equation (9a) with R for abscissa and $F\ (R)$ for ordinate.

With n as an independent variable, $\overline{R}$ follows as a definite n from the empirical curve of Figure 1, the corresponding f from equation (4) or Figure 2, and the corresponding R from equation (2). The empirical resistance equation (1a) with $\overline{R}$ yields the desired ψ. Equation (7a), with the values of n and R, gives the desired G, with which the requisite values of R, G, f and ψ in equation (9a) are all determined. The expression $\frac{dn}{dR}$ in equation (7a) was obtained from Figure 1, which is quite possible, since the subtrahend in equation (7a) is always less than 9% of the minuend. For small Reynolds Numbers ($R < 50000$), the Blasius theorem yields equation (8b) and, for $n = 7$, equation (7a) yields $G = 0.0972$, so that

$$F = 4.17\ R^{\frac{1}{4}} \tag{9b}$$

f) Integration of the differential equation.- From equation (9a) we obtain by integration

$$R_x = \int_0^R F\ (R)\ d\ R,$$

in which the integration constant is determined by the fact that, when $R_x = 0$, then $R = 0$ also. The implied disregard of the laminar initial portion is unobjectionable with respect to its brevity for the statement of a resistance law for high values of R_x.

By planimetry of the function $F\ (R)$ in Figure 4, we obtain the relation between R_x and R in the form

$$R_x = K\ (R) \quad \text{or} \quad R = H\ (R_x), \tag{10a}$$

as represented in Figure 5. For the Blasius range with the value of F given by equation (9b)

$$R_x = 3.33\ R^{\frac{5}{4}} \quad \text{or} \quad R = 0.382\ R_x^{\frac{4}{5}} \tag{10b}$$

4. Coefficient of Resistance

a) Coefficient of total resistance c_f.- This is ordinarily defined by the one-sided total resistance W_1 or the two-sided resistance W_2 of a plate of area F by

$$c_f = \frac{W_1}{F \frac{\rho U^2}{2}} = \frac{W_2}{F \rho U^2}, \tag{11}$$

On the other hand the total resistance (referred to the unit width) according to equation (5a) equals the total loss of momentum of the flow and, taking equation (6) into account

$$W_2 = - 2 J' = 2 \rho U^2 \delta \frac{n}{(n + 1)(n + 2)} \tag{12}$$

The introduction of equation (12) into equation (11), with $F = x l$, yields

$$c_f = \frac{2 n}{(n + 1)(n + 2)} \frac{\delta}{x}$$

or, by extension with U/ν,

$$c_f = \frac{2 n}{(n + 1)(n + 2)} \frac{R}{R_x} \tag{13a}$$

If we put

$$\frac{2 n}{(n + 1)(n + 2)} = A, \tag{14}$$

then A, on the way to n and $\bar{R}$, is a function of R, and the latter, in conformity with equation (10a), is a function of R_x, so that equation (13a) can also be written

$$c_f = \frac{A (R_x) H (R_x)}{R_x} \tag{13b}$$

The calculation of c_f is made, according to equation (13a), by first determining R, in addition to n chosen as an independent variable (as above in the calculation of F), and then, from Figure 5, the corresponding R_x. For the Blasius range (n = 7), with $A = \frac{14}{72} = 0.194$ according to equation (14) and with R according to equation (10b), formula (13a) yields

$$c_f = 0.074\, R_x^{-\frac{1}{5}} \tag{13c}$$

Figure 6 shows the result of the calculation. As in the corresponding curves of pipe resistance, there is also shown here, with increasing R_x, an increasing deviation of the resistance curve from the extrapolated Blasius-Karman theorem. No suitable comparative data are available.

<u>b) Coefficient of local resistance</u>.- The measurements made by the Hamburg Naval Institute between log R = 6.9 and log R = 8.65 are not measurements of the total resistance, but of the local resistance, as obtained by the installation of short test plates in a long towed body.

The coefficient of local resistance, like equation (11), is defined by

$$c_f' = \frac{d\,W_1}{\frac{\rho}{2}\,U^2\,d\,F} = \frac{\tau_0}{\frac{\rho}{2}\,U^2} \tag{15}$$

The relation of the coefficients c_f and c_f' is then given by

$$c_f = \frac{1}{R}\int_0^R c_f'\,d\,R; \qquad c_f' = c_f + R\,\frac{d\,c_f}{d\,R} \tag{16}$$

The calculation of c_f' is preferably made not according to equation (16) from c_f, but directly from equation (15) while taking account of equation (8a), as derived from the pipe-resistance law, for the internal wall friction, which yields

$$c_f' = \frac{1}{2} f \psi \tag{17a}$$

Here f and ψ over $\overline{R}$ are functions of R and hence, according to equation (10a), also functions of R_x. For n selected as the independent variable, we determine f, ψ and R, for which Figure 5 shows the corresponding R_x. For the Blasius range, on the basis of equation (8b) and with replacement of R by R_x according to (10b), we obtain

$$c_f' = 0.0594\ R_x^{-\frac{1}{5}} \tag{17b}$$

Figure 7 shows c_f' as a function of R_x. Owing to the range of the basic empirical values of the pipe resistance and of the exponent of the velocity distribution, the results of the calculation were extended to $\log R_x = 7.89$. Above this value, the curve was rectilinearly extrapolated, as justified by the fact that the curve, on reaching $\log R_x = 6.9$, is already straight within 1/4%. The equation of this asymptote* is

$$c_f' = 0.0206\ R_x^{-0.1294} \tag{17c}$$

Equation (17b) gives the values for c_f' up to $\log R_x = 6.3$ to within 1/4% and up to $\log R_x = 6.5$ to within 1% deviation

*Of course no physical significance can be attached to this extrapolation at the highest R values, especially with respect to the n values of Nikuradse.

from the general equation (17a). Equation (17c) gives the values for c_f' above $\log R_x = 6.5$ to within 1% and above $\log R_x = 6.9$ to within 1/4%.

As experimental results, we have introduced the c_f' values reported by Kempf for his smoothest surface (varnished, polished and waxed). The discrepancies between our calculation and these results lie between -2% and +4%, while the extrapolated Karman equation, within the range of the experiments, lies between 7% and 40% too low. On the whole the experimental curve shows a somewhat sharper curvature than the calculated curve. It should be noted that the plotted curve of the experimental c_f' values represents the mean c_f' values. These were obtained from different plates distributed throughout the length of the towed body and differ among themselves by about 3%. In view of this fact, the agreement between the experimental and calculated results (almost within the accuracy of measurement) must be considered remarkably good.

Translation by Dwight M. Miner,
National Advisory Committee
for Aeronautics.

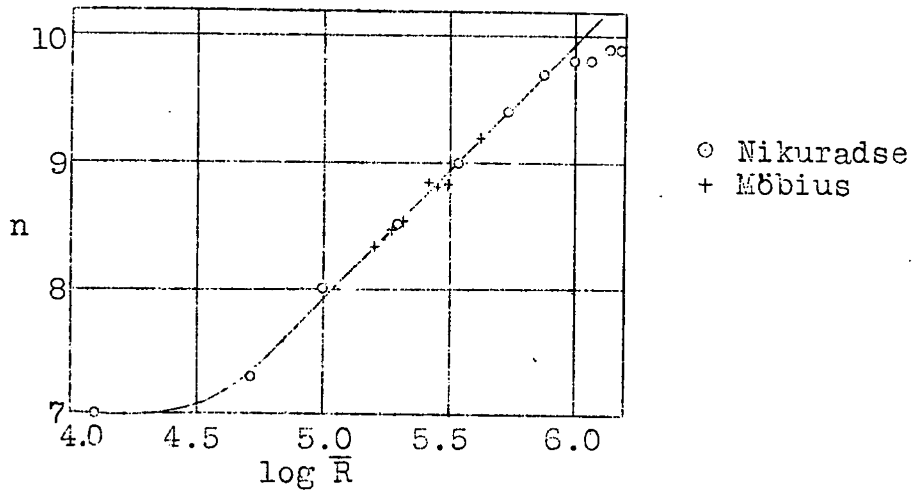

Fig.1 Exponents of velocity distribution in pipe.

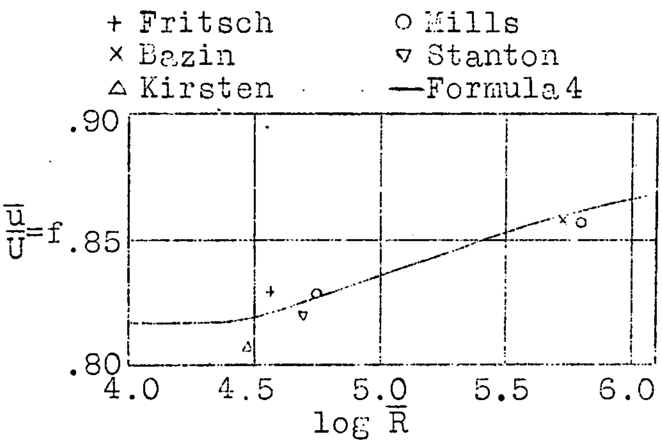

Fig.2 Ratio of mean to maximum velocity in pipe.

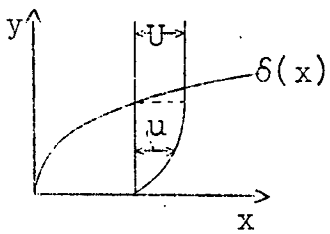

Fig.3 Momentum theory of boundary layer.

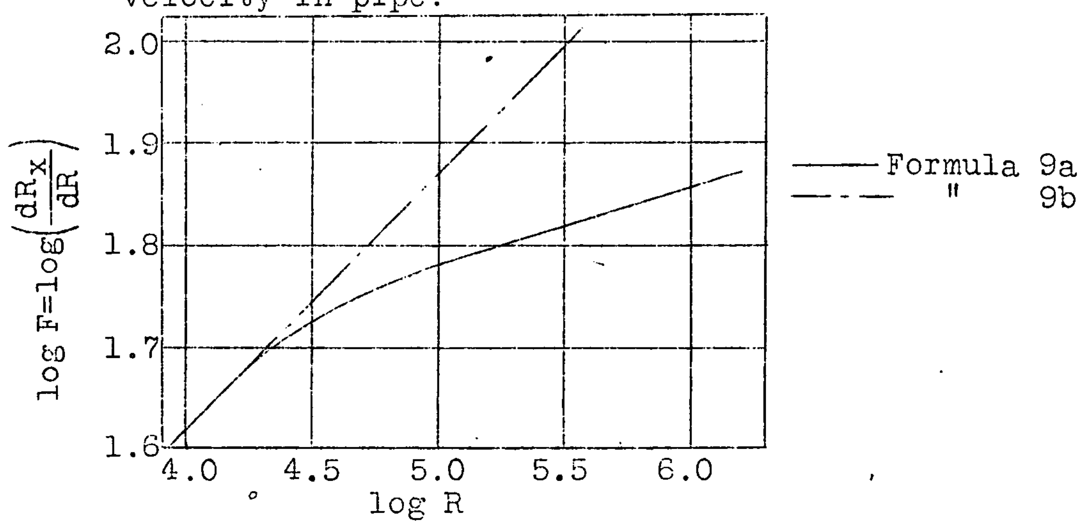

Fig.4 Differential equation of boundary layer.

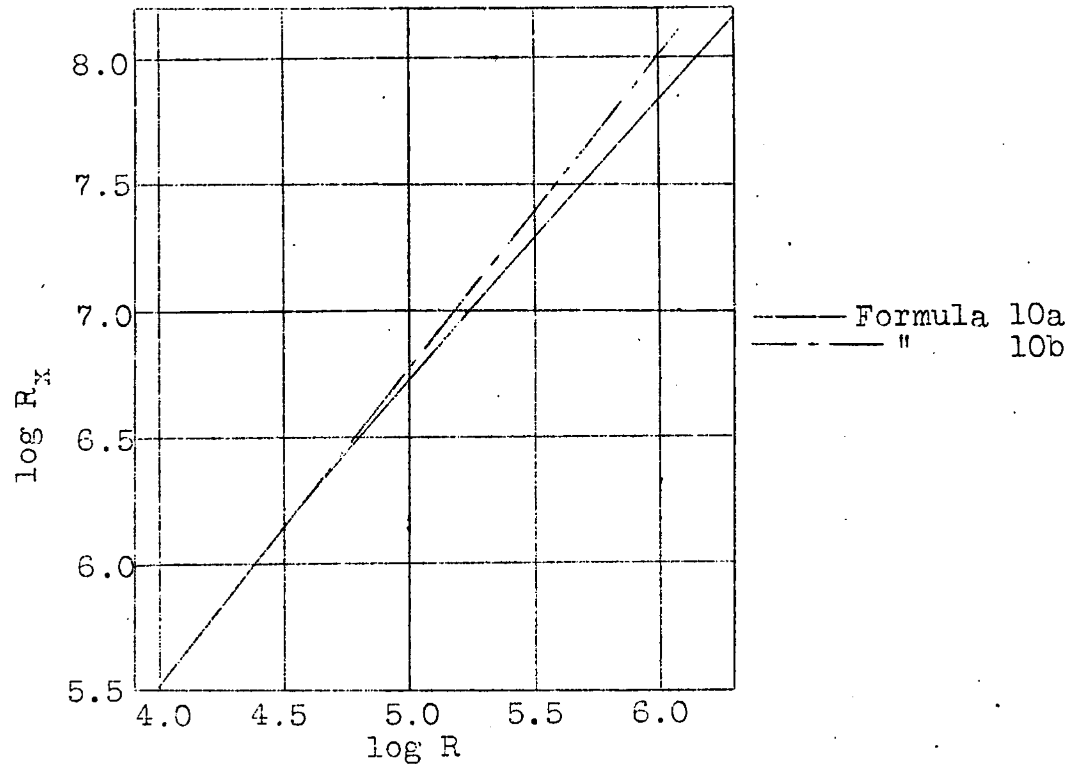

Fig.5 Reynolds number of plate length(R_z)and boundary-layer thickness (R).

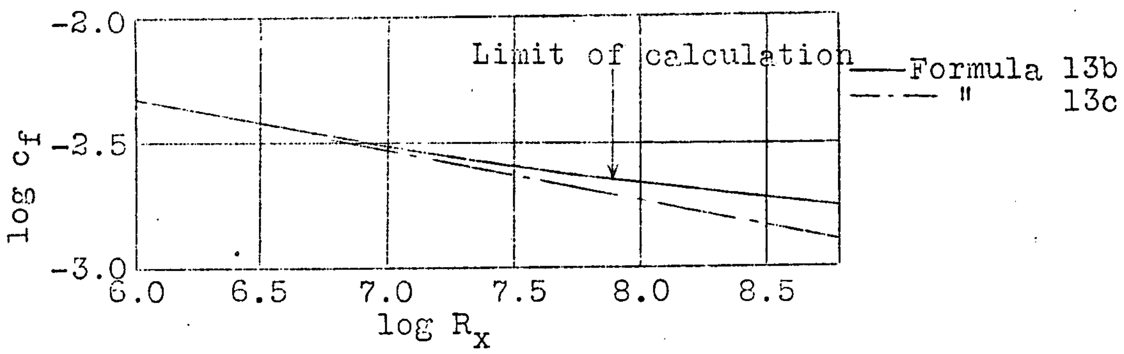

Fig.6 Coefficient of total resistance of plate.

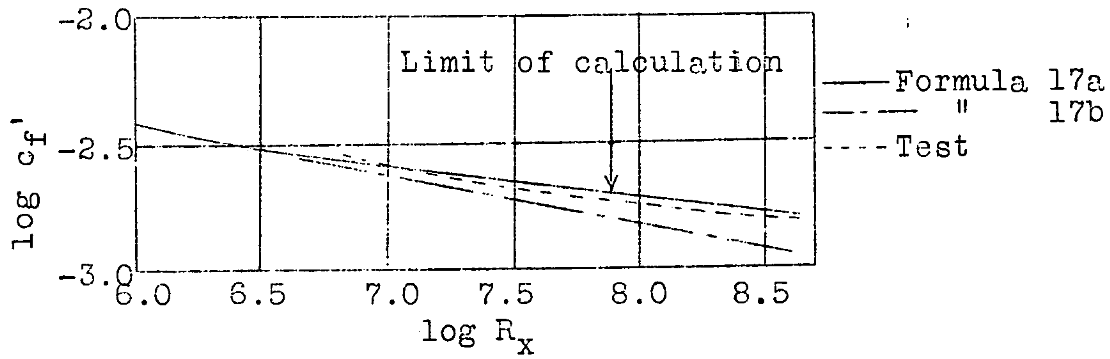

Fig.7 Coefficient of local resistance of plate.