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QUANTUM NON DEMOLITION MEASUREMENT OF CYCLOTRON EXCITATIONS IN A PENNING TRAP

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Abstract

The quantum non-demolition measurement of the cyclotron excitations of the electron confined in a Penning trap could be obtained by measuring the resonance frequency of the axial motion, which is coupled to the cyclotron motion through the relativistic shift of the electron mass.

1 Introduction

The process of making a measurement on a quantum mechanical system introduces quantum noise to that system. A quantum non-demolition measurement (QND) scheme seeks to make a measurement of an observable by feeding all the introduced noise into a conjugate variable to that under consideration. An ideal QND observable is one which has always the same values in repeated series of measurements. It means that the total Hamiltonian of the system plus the interaction with the measurement device must commute with the observable to be measured at given times, for a stroboscopic QND, observable or at any times for a continuous QND observable [1].

Recently there has been a number of theoretical papers [2, 3, 4, 5, 6] proposing schemes for QND measurements and fewer experimental realizations mainly in the optical regime [7, 8, 9]. In this paper we present another scheme which could be easily verified because the system is well known and studied. The system is an electron confined in a Penning trap [10]. Penning traps for electrons, protons and ions have been extensively used for high precision measurements of fundamental constants and laws of Nature, like for instance the g-factor of the electron and the CPT invariance [11]. In this paper we will show that it could also be used to give a QND measurement of the excitation number of the cyclotron motion.

2 The Penning trap

A Penning trap consists of a combination of constant magnetic field and quadrupolar electrostatic potential in which a charged particle, for instance an electron, can be confined. It is composed by two end-cap and one ring electrodes to which a static potential V_0 is applied [11]. There is also a homogeneous magnetic field $\vec{B}_0$ along the symmetry axis of the trap assumed as the z-axis. Neglecting the contribution of the spin, which we keep locked, the Hamiltonian for the electron of charge e and rest mass m_0 in the trap is given by the following expression

$$H = \frac{1}{2m_0}\left(\vec{p} - \frac{e}{c}\vec{A}\right) + eV \tag{1}$$

with

$$\begin{aligned} \vec{A} &= \left(-\frac{y}{2}B_0, \frac{x}{2}B_0, 0\right) && (2) \\ V &= \frac{V_0}{2z_0^2}\left(\frac{x^2+y^2}{2} - z^2\right) && (3) \end{aligned}$$

The typical experimental values are

$$\begin{aligned} B_0 &\simeq 58100 \text{ G} \\ V_0 &\simeq 10 \text{ V} \\ z_0 &\simeq 3.3 \times 10^{-3} \text{ m} \end{aligned}$$

where z_0 specifies the dimension of the trap. It is easy to show that in terms of rising and lowering operators the Hamiltonian (1) becomes [10]

$$H = \hbar\omega_c'\left(a_c^\dagger a_c + \frac{1}{2}\right) + \hbar\omega_z\left(a_z^\dagger a_z + \frac{1}{2}\right) - \hbar\omega_m\left(a_m^\dagger a_m + \frac{1}{2}\right) \tag{4}$$

with

$$\begin{aligned} a_c &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m_0\omega_c}{2\hbar}}(x - iy) + \sqrt{\frac{2}{m_0\hbar\omega_c}}(p_y + ip_x)\right] && (5) \\ a_m &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m_0\omega_c}{2\hbar}}(x + iy) - \sqrt{\frac{2}{m_0\hbar\omega_c}}(p_y - ip_x)\right] && (6) \\ a_z &= \sqrt{\frac{m_0\omega_z}{2\hbar}}z + i\sqrt{\frac{1}{2m_0\hbar\omega_z}}p_z. && (7) \end{aligned}$$

The displaced cyclotron angular frequency is

$$\omega_c' \simeq \omega_c\left[1 - \frac{1}{2}\left(\frac{\omega_z}{\omega_c}\right)^2\right] \tag{8}$$

with $\omega_c = |e|B_0/m_0c$ the bare cyclotron frequency.

The axial angular frequency is given by

$$\omega_z = \sqrt{\frac{|e|V_0}{m_0 z_0^2}} \tag{9}$$

and the magnetron frequency by

$$\omega_m \simeq \frac{\omega_c}{2}\left(\frac{\omega_z}{\omega_c}\right)^2 . \tag{10}$$

The ranges of frequency in the experimental situation are:

$$\begin{aligned} \omega_c'/2\pi &\sim 164 \text{ GHz} \\ \omega_z/2\pi &\sim 64 \text{ MHz} \\ \omega_m/2\pi &\sim 11 \text{ kHz.} \end{aligned}$$

Thus each frequency belongs to a very different band of the electromagnetic field.

3 The measurement model

The question one can rise is: how can we measure the various frequencies of oscillation? In order to make a measurement we need to couple the system to what Feynman called the "rest of Universe" [12]. It turns out that the best way of measuring the properties of the various motions of the electron is to measure the current induced by the axial motion of the electron along the z-axis [13]. Indeed, the electric charges induced by the oscillatory motion on the end-cap generate a current which can be measured.
The system plus the measurement device is represented in Fig 1. Here L is the inductance of the measurement device and R its resistance. The induced current dissipates on the resistor R which is in thermal equilibrium at temperature $T \simeq 4\ K$. $u(t)$ represents a stochastic potential which gives the effect of thermal fluctuations or Johnson noise.

The axial motion plus the read-out are described by the following Hamiltonian

$$\begin{aligned} H' &= \frac{p_z^2}{2m_0} + \frac{m_0\omega_z^2}{2}z^2 + \frac{1}{2C}(az+Q)^2 + \frac{\phi^2}{2L} + \\ &+ \int_0^{+\infty} d\Omega \left[(p(\Omega) + k(\Omega)Q)^2 + \Omega^2 q^2(\Omega)\right] \end{aligned} \tag{11}$$

where we have considered a thermal bath with a continuous distribution of modes linearly coupled to the electronic circuit; ϕ is the electric flux in the inductance L, Q is the electric charge on the capacitor C which is the capacity of the trap; az represents the induced charge due to the axial motion of the electron [14] with $a = \alpha e/2z_0$ where $2z_0$ is the distance between the two end-caps and α is a constant of order of unity which takes into account the curvature of the capacitor surfaces.

However, if we wish to measure the properties of the cyclotron motion we need a coupling between the axial motion and the cyclotron motion. In earlier experiments with the Penning trap [10] this coupling was introduced by adding an inhomogeneity on the magnetic field B_0 by means of a "magnetic bottle".

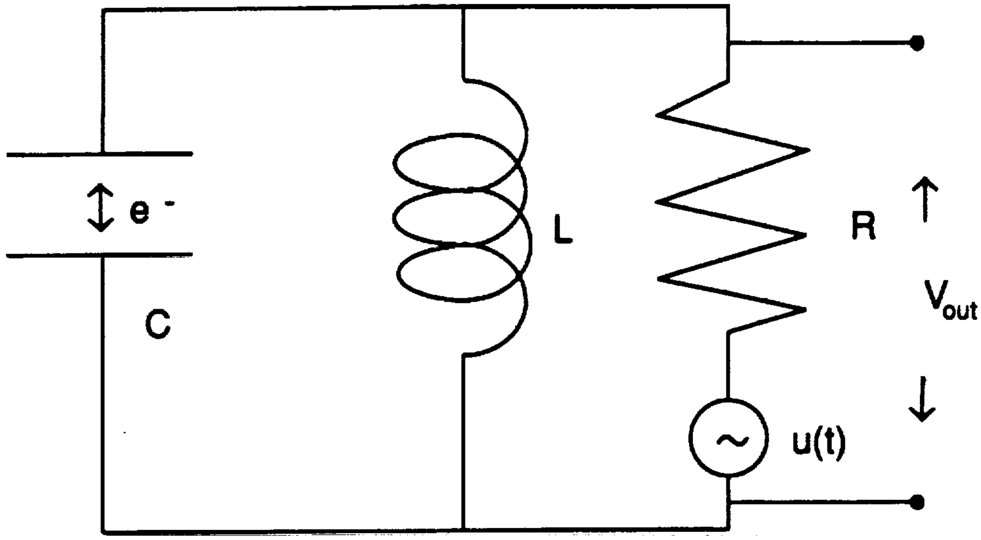

FIG. 1 The axial motion of the electron coupled to the read-out apparatus.

4 The Hamiltonian of the system

The precision of the measurements is, however, so high that we cannot get rid of the relativistic corrections; then, the coupling between the two modes is also given by the relativistic shift of the electron mass [15]. In such a case the system's Hamiltonian we have to consider is

$$H_{sys} = H_{NR} + H_{RC} \tag{12}$$

$$H_{NR} = \left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + eV \tag{13}$$

$$H_{RC} = -\frac{1}{8m_0^3c^2}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^4 \tag{14}$$

Finally we can write the following Hamiltonian of the quantum system:

$$\begin{aligned} H_{sys} &= \hbar\omega_c a_c^\dagger a_c\left[1 - \frac{1}{2}\left(\frac{\omega_z}{\omega_c}\right)^2 - \frac{\hbar\omega_c}{2m_0c^2}\right] - \frac{\hbar^2\omega_c^2}{2m_0c^2}(a_c^\dagger a_c)^2 + \\ &+ \frac{p_z^2}{2m_0}\left[1 - \frac{\hbar\omega_c}{m_0c^2}\left(a_c^\dagger a_c + \frac{1}{2}\right)\right] - \frac{p_z^4}{8m_0^3c^2} + \frac{m_0\omega_z^2}{2}z^2 \end{aligned} \tag{15}$$

where we have completely neglected the magnetron motion which is not coupled to other motions. It is now easily seen that the coupling between the axial motion and the cyclotron motion is due to the relativistic shift of the mass.

5 The QND observable

If we now introduce as before the coupling with the external world, the total hamiltonian becomes

$$
\begin{aligned}
H &= \hbar\omega_c'' a_c^\dagger a_c + \hbar\mu (a_c^\dagger a_c)^2 + \\
&+ \frac{p_z^2}{2m_0}\left[1 - \frac{\hbar\omega_c}{m_0c^2}\left(a_c^\dagger a_c + \frac{1}{2}\right)\right] - \frac{p_z^4}{8m_0^3c^2} + \frac{m_0\omega_z^2}{2}z^2 + \\
&+ \frac{(az+Q)^2}{2C} + \frac{\phi^2}{2L} + \int_0^{+\infty} d\Omega \left[(p(\Omega) + k(\Omega)Q)^2 + \Omega^2 q^2(\Omega)\right]
\end{aligned}
\tag{16}
$$

with

$$
\omega_c'' = \omega_c\left[1 - \frac{1}{2}\left(\frac{\omega_z}{\omega_c}\right)^2 - \frac{\hbar\omega_c}{2m_0c^2}\right] \tag{17}
$$

$$
\mu = -\frac{\hbar\omega_c^2}{2m_0c^2}. \tag{18}
$$

It is evident that $a_c^\dagger a_c = \hat{n}_c$ is a QND observable because

$$
[\hat{n}_c, H] = 0. \tag{19}
$$

The axial motion of the electron represents the probe that enables us to measure the properties of the cyclotron motion. Indeed, the axial frequency now depends on the cyclotron excitation quantum number $\hat{n}_c$, which is a constant of the motion, at least as long as we can neglect the spontaneous emission of the cyclotron motion. It has been measured [10] that the spontaneous emission coefficient is $\gamma_c^{-1} \simeq 1$ s thus, if the measurement is performed in a time much shorter than γ_c^{-1} we can neglect the spontaneous emission of the cyclotron motion and perform a QND measurement of the excitation number $\hat{n}_c$. It has also been shown [16] that γ_c could be reduced by the cavity effect [17]. Indeed, when the characteristic length of the cavity of the trap is shorter than half wavelength of the cyclotron motion, the cyclotron spontaneous emission should be inhibited.

One can also show that the anharmonicity of the axial motion is very small and can be neglected. It turns out that it is $(\omega_z/\omega_c)^2$ times smaller than the anharmonicity of the cyclotron motion. Thus the equations of motion now are:

$$
\left\{
\begin{aligned}
\dot{z} &= \frac{i}{\hbar}[H, z] \\
\dot{p}_z &= \frac{i}{\hbar}[H, p_z] \\
\dot{Q} &= \frac{\partial H}{\partial \phi} = \frac{\phi}{L} \\
\dot{\phi} &= -\frac{\partial H}{\partial Q} = -\frac{Q}{C} - \frac{az}{C} + \int_0^{+\infty} d\Omega\, [p(\Omega) + k(\Omega)Q]\, k(\Omega).
\end{aligned}
\right.
\tag{20}
$$

By making a Markov approximation in the equation of motion for the variables of the thermal bath, we can write the following equations [18]

$$\left\{\begin{array}{lcl} \dot{z} & = & \frac{p_z}{m_0}\left[1-\frac{\hbar\omega_c}{m_0c^2}\left(\hat{n}_c+\frac{1}{2}\right)\right] \\ \dot{p}_z & = & -m_0\omega_z^2 z-\frac{a}{C}Q \\ \dot{Q} & = & \frac{\phi}{L} \\ \dot{\phi} & = & -\frac{Q}{C}-\frac{az}{C}-\gamma\dot{Q}+\xi(t) \end{array}\right. \tag{21}$$

where γ represents the rate at which the axial motion dissipates its energy due to the coupling with the rest of Universe represented by the read-out apparatus. Of course, in such a case one has to sustain the axial oscillation with an oscillating external potential $V(t)$ tuned at the axial frequency of the electron. In the experimental situation is always

$$\hbar\omega_z << k_B T$$

with k_B the Boltzman's constant. Then, it is possible to show [18] that the statistics of the noise term $\xi(t)$ is that of a white noise with expectations

$$\begin{array}{rcl} \langle\xi(t)\rangle & = & 0 \\ \langle\xi(t)\xi(t')\rangle & = & 2\gamma k_B T\,\delta(t-t'). \end{array} \tag{22}$$

By introducing the Fourier transforms defined by

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega \tilde{f}(\omega)e^{i\omega t} \tag{23}$$

we can write the linear system:

$$\left\{\begin{array}{lllll} i\omega\tilde{z}(\omega) & -\frac{\tilde{p}(\omega)}{m} & & & = 0 \\ m\bar{\omega}_z^2\tilde{z}(\omega) & +i\omega\tilde{p}(\omega) & +\frac{a\tilde{Q}(\omega)}{C} & & = 0 \\ & & i\omega\tilde{Q}(\omega) & -\frac{\tilde{\phi}(\omega)}{L} & = 0 \\ \frac{a\tilde{z}(\omega)}{C} & & +\left(\frac{1}{C}+i\omega\gamma\right)\tilde{Q}(\omega) & +i\omega\tilde{\phi}(\omega) & = \tilde{\xi}(\omega)-\tilde{V}(\omega) \end{array}\right. \tag{24}$$

with

$$m = \frac{m_0}{1-\frac{\hbar\omega_c}{m_0c^2}\left(\hat{n}_c+\frac{1}{2}\right)} \tag{25}$$

$$\bar{\omega}_z = \omega_z\sqrt{1-\frac{\hbar\omega_c}{m_0c^2}\left(\hat{n}_c+\frac{1}{2}\right)}. \tag{26}$$

The determinant of the homogeneous system is

$$\Delta = (\bar{\omega}_z^2 - \omega^2)(\omega_e^2 - \omega^2 + i\gamma_e\omega) - \frac{a^2}{mc}\omega_e^2 \tag{27}$$

with

$$\omega_e = \sqrt{\frac{1}{LC}} \qquad \gamma_e = \frac{\gamma}{L}$$

i.e., the characteristic frequency and the bandwidth of the electronic circuit respectively.

The solution of the algebraic system is easily obtained and we get:

$$\begin{aligned}
\tilde{z}(\omega) &= -\frac{a\omega_e^2}{m\Delta}\left(\tilde{\xi}(\omega) - \tilde{V}(\omega)\right) && (28)\\
\tilde{p}(\omega) &= -\frac{ia\omega_e^2}{\Delta}\left(\tilde{\xi}(\omega) - \tilde{V}(\omega)\right) && (29)\\
\tilde{Q}(\omega) &= \frac{\bar{\omega}_z^2 - \omega^2}{L\Delta}\left(\tilde{\xi}(\omega) - \tilde{V}(\omega)\right) && (30)\\
\tilde{\phi}(\omega) &= \frac{\bar{\omega}_z^2 - \omega^2}{\Delta}\left(\tilde{\xi}(\omega) - \tilde{V}(\omega)\right). && (31)
\end{aligned}$$

We see that at $\omega = \bar{\omega}_z$ both $\tilde{Q}$ and $\tilde{\phi}$ are zero and the current which dissipates energy on the resistor is only due to the induced charge on the end-caps.

6 Output statistics

The signal to be measured by the read-out is the voltage at the extremes of the resistor R which is proportional to the induced current. The induced current is proportional to the axial velocity of the electron through

$$I(t) = a\dot{z}(t) = \frac{ap(t)}{m} \tag{32}$$

thus the fluctuations of the measured potential are directly connected with the fluctuations of the axial momentum of the electron:

$$\tilde{V}_{out}(\omega) = \tilde{I}(\omega)R + \tilde{\xi}(\omega) \tag{33}$$

where $\tilde{\xi}(\omega)$ takes into account the Johnson noise on the resistance R. The spectral density of the output voltage is given by:

$$\begin{aligned}
&\langle \tilde{V}_{out}(\omega)\tilde{V}_{out}(\omega')\rangle - \langle \tilde{V}_{out}(\omega)\rangle\langle \tilde{V}_{out}(\omega')\rangle = \\
&\left(\frac{Ra}{m}\right)^2 \langle \tilde{p}(\omega)\tilde{p}(\omega')\rangle + \frac{Ra}{m}\left(\langle \tilde{p}(\omega)\tilde{\xi}(\omega')\rangle + \langle \tilde{\xi}(\omega)\tilde{p}(\omega')\rangle\right) + \langle \tilde{\xi}(\omega)\tilde{\xi}(\omega')\rangle. \qquad (34)
\end{aligned}$$

For simplicity we take the driving potential $V(t)$ noiseless then we get:

$$\begin{aligned}
\langle \tilde{p}(\omega)\tilde{p}(\omega')\rangle &= -\left(\frac{a}{LC}\right)^2 \frac{\omega\omega'\left[\langle \tilde{\xi}(\omega)\tilde{\xi}(\omega')\rangle + \tilde{V}(\omega)\tilde{V}(\omega')\right]}{\Delta(\omega)\Delta(\omega')} && (35)\\
\langle \tilde{p}(\omega)\tilde{\xi}(\omega')\rangle &= -\frac{a}{LC}\frac{i\omega\langle \tilde{\xi}(\omega)\tilde{\xi}(\omega')\rangle}{\Delta(\omega)} && (36)\\
\langle \tilde{\xi}(\omega)\tilde{\xi}(\omega')\rangle &= 2L\gamma_e k_B T\,\delta(\omega + \omega'). && (37)
\end{aligned}$$

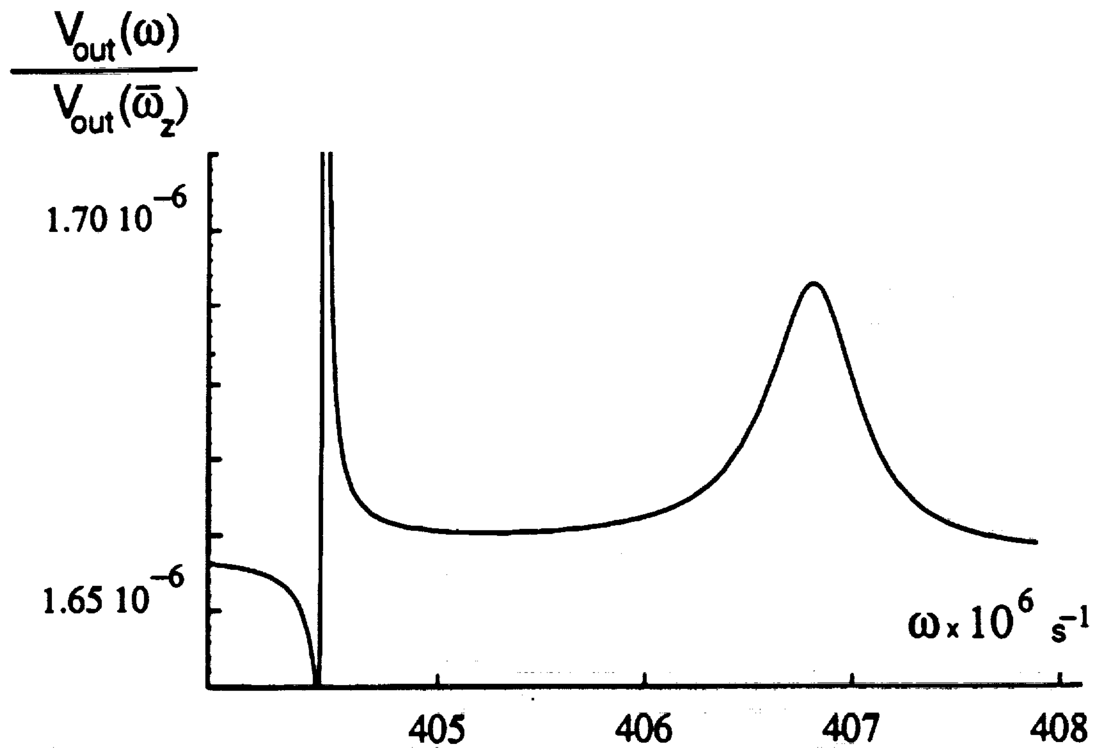

FIG. 2 The two resonances of the normalized output variances of the signal for $\omega_e \neq \bar{\omega}_z$. The value of the maximum at $\omega \simeq \bar{\omega}_z$ is not shown because it goes out of the scale. Its value is 80.

Thus eq. (34) becomes

$$\langle \tilde{V}_{out}(\omega)\tilde{V}_{out}(\omega')\rangle - \langle \tilde{V}_{out}(\omega)\rangle\langle \tilde{V}_{out}(\omega')\rangle = V_{out}(\omega)\,\delta(\omega+\omega') \tag{38}$$

with

$$V_{out}(\omega) = 2L\gamma_e k_B T \left\{ 1 + \frac{(a^2R/m)\omega_e^2\omega^2\left[(a^2R/m)\omega_e^2 - 2\gamma_e(\bar{\omega}_z^2-\omega^2)\right]}{\left[(\omega_e^2-\omega^2)(\bar{\omega}_z^2-\omega^2) - a^2\omega_e^2/mC\right]^2 + \left[\gamma_e\omega(\bar{\omega}_z^2-\omega^2)\right]^2} \right\}. \tag{39}$$

7 Conclusions

In Fig. 2 we plot $V_{out}(\omega)/V_{out}(\bar{\omega}_z)$ versus ω for a given value of $\omega_e \neq \bar{\omega}_z$. We see two maxima for $\omega > 0$; one is for $\omega = \omega_e$ and the other for $\omega \simeq \bar{\omega}_z$. As soon as we tune the electronic frequency ω_e in resonance with $\bar{\omega}_z$, we obtain only one maximum for $\omega = \bar{\omega}_z$ (Fig. 3). From eq. (26) we see that the resonance frequency depends on the quantum number $\hat{n}_c$ of the cyclotron motion. In Fig. 4 we show the top of the curves obtained with $\hat{n}_c = 0$ and $\hat{n}_c = 1$. In order to discriminate between the two maxima we need a sensitivity $\Delta\bar{\omega}_z/\omega_z \sim 7\times 10^{-10}$ which is slightly above the experimental limit, as long as we know, which is extimated to be $\Delta\omega_z/\omega_z \sim 10^{-9}$ [10].

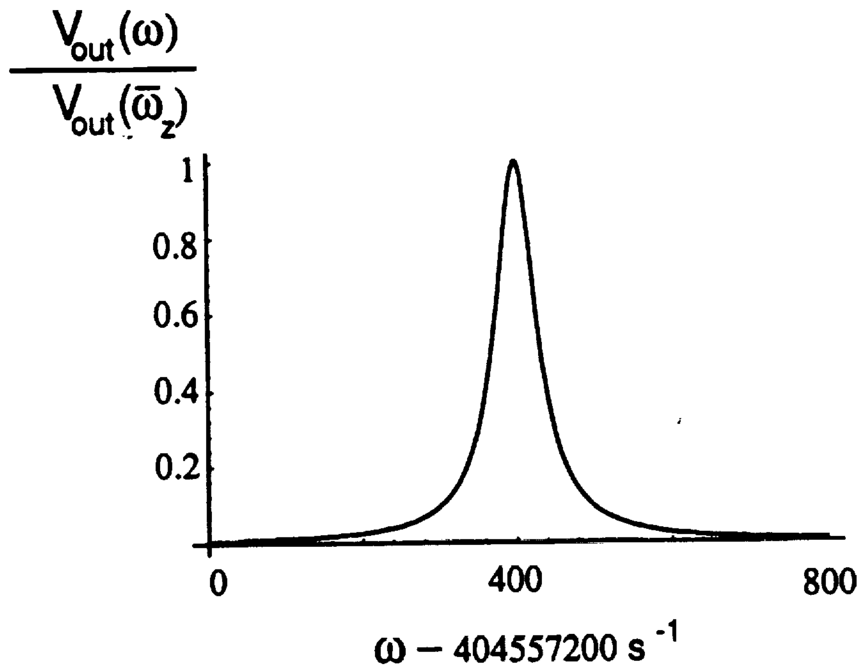

FIG. 3 The resonance of the normalized output variance of the signal for $\omega_e = \bar{\omega}_z$.

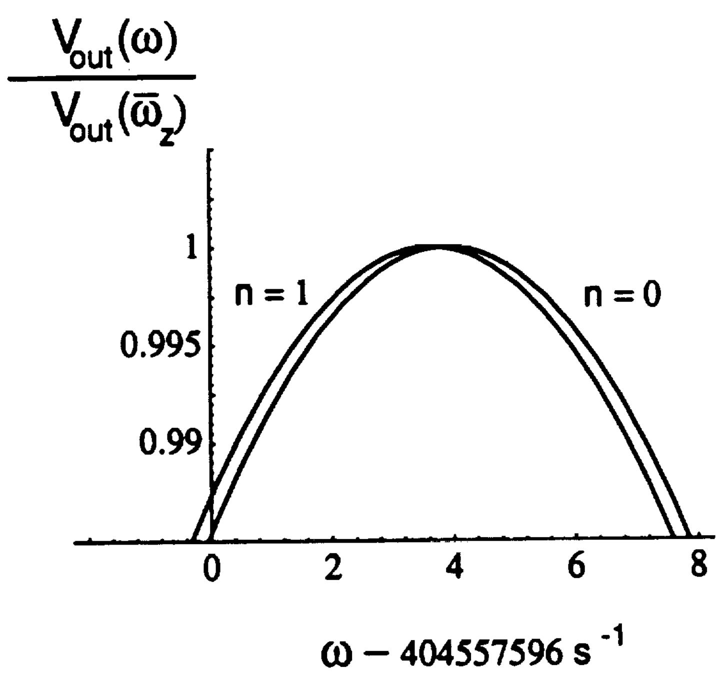

FIG. 4 The amplified top of the resonance of the output variances of the signal for $\hat{n}_c = 1$ and $\hat{n}_c = 0$.

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